The Structure of Potentially Semi-Stable Deformation Rings

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The structure of potentially semi-stable deformation rings *

Mark Kisin

Abstract. Inside the universal deformation space of a local Galois representation one has the set of deformations which are potentially semi-stable of given p-adic Hodge and Galois type. It turns out these points cut out a closed subspace of the deformation space. A deep conjecture due to Breuil-Mézard predicts that part of the structure of this space can be described in terms of the local Langlands correspondence. For 2-dimensional representations the conjecture can be made precise. We explain some of the progress in this case, which reveals that the conjecture is intimately connected to the p-adic local Langlands correspondence, as well as to the Fontaine-Mazur conjecture.

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Keywords.

Introduction

The study of deformations of Galois representations was initiated by Mazur [Ma]. Already in that article Mazur considered deformations satisfying certain local conditions formulated in terms of p-adic Hodge theory. The importance of deformations satisfying such conditions became clear with the formulation of the Fontaine-Mazur conjecture [FM], and the spectacular proof of the Shimura-Taniyama conjecture on modularity of elliptic curves over $\mathbb{Q}$ by Wiles, Taylor-Wiles, and their collaborators [Wi], [TW], [BCDT].

The first question which arises concerns the nature of the subspaces cut out by these conditions: Suppose that $K/\mathbb{Q}_p$ is a finite extension with absolute Galois group G_K, let $\mathbb{F}/\mathbb{F}_p$ be a finite extension, and $V_{\mathbb{F}}$ a finite dimensional $\mathbb{F}$-vector space equipped with a continuous, absolutely irreducible G_K-action. Then $V_{\mathbb{F}}$ admits a universal deformation ring $R_{V_{\mathbb{F}}}$. A closed point $x \in \operatorname{Spec} R_{V_{\mathbb{F}}}[1/p]$ gives rise to a deformation L_x of $V_{\mathbb{F}}$, so that $L_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a representation of G_K on a finite dimensional vector space over a finite extension of $W(\mathbb{F})[1/p]$. One can ask whether the points such that $L_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ satisfies the condition are cut out by a closed subspace of $\operatorname{Spec} R_{V_{\mathbb{F}}}[1/p]$.

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Of course the answer depends on the condition one imposes. In [Fo 2] Fontaine suggests (at least implicitly) that the answer should be affirmative if one requires the representations to become semi-stable over a fixed extension K'/K and with Hodge-Tate weights in a fixed interval. Attached to any such representation V is a finite dimensional representation of the inertia subgroup $I_K \subset G_K$, which, in some sense, measures the failure of V to be semi-stable. One can sharpen Fontaine's conjecture by fixing a representation τ of I_K, with open kernel, and requiring $L_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to have fixed Hodge-Tate weights and associated I_K-representation τ. That this refined condition cuts out a closed subspace was conjectured in special cases in the papers of Fontaine-Mazur [FM, p191], Breuil-Conrad-Diamond-Taylor [BCDT, Conj. 1.1.1], and suggested more generally by Breuil-Mézard [BM, Conj. 1.1, p214].

After partial results by several people (see section 1.2.5 below for a more detailed discussion) such a result was proved in general in [Ki 4]. Thus, for some finite normal extension $\mathcal{O}$ of $W(\mathbb{F})$ one obtains a quotient $R^{\mathbf{v},\tau}_{V_{\mathbb{F}}}$ of $R_{V_{\mathbb{F}}} \otimes_{W(\mathbb{F})} \mathcal{O}$ whose points in characteristic 0 correspond precisely to deformations of $V_{\mathbb{F}}$ which become semi-stable over some finite extension of K, have the chosen fixed Hodge-Tate weights and associated I_K-representation τ. [1]

The conjectures of Breuil-Mézard predict a deep connection between the structure of $R^{\mathbf{v},\tau}_{V_{\mathbb{F}}}$ and the representation theory of $\mathrm{GL}_d(\mathcal{O}_K)$, where $d = \dim_{\mathbb{F}} V_{\mathbb{F}}$. [2] This can be made precise when $V_{\mathbb{F}}$ is two dimensional, which we assume for the rest of this introduction. In this case, a result of Henniart attaches to τ a smooth, irreducible, finite dimensional representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$ which is characterized in terms of the local Langlands correspondence. On the other hand, the cocharacter $\mathbf{v}$ gives rise to an algebraic representation $\sigma(\mathbf{v})$ of $\mathrm{GL}_2(\mathcal{O}_K)$. Let $L_{\mathbf{v},\tau} \subset \sigma(\mathbf{v}) \otimes \sigma(\tau)$ be a $\mathrm{GL}_2(\mathcal{O}_K)$ invariant lattice. Then the conjecture predicts the Hilbert-Samuel multiplicity $e(R^{\mathbf{v},\tau}_{V_{\mathbb{F}}}/\pi)$ of $R^{\mathbf{v},\tau}_{V_{\mathbb{F}}}/\pi$ in terms of the multiplicities of the Jordan-Hölder factors of $L_{\mathbf{v},\tau}/\pi L_{\mathbf{v},\tau}$. Here $\pi \in \mathcal{O}$ denotes a uniformizer. Indeed, one can formulate such a conjecture in any dimension assuming an analogue of Henniart's result. When τ is irreducible a higher dimensional analogue of Henniart's result has been proved by Paskunas [Pa].

It is slightly more convenient to work with the quotient $R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}}}$ of $R^{\mathbf{v},\tau}_{V_{\mathbb{F}}}$ which corresponds to deformations having determinant ψ times the cyclotomic character, for some appropriately chosen [3] ψ. The general shape of such a conjecture is then that

$$e(R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}}}/\pi) = \sum_{\bar{\sigma}} a(\bar{\sigma}) \mu_{\bar{\sigma}}(V_{\mathbb{F}}),$$

where $\bar{\sigma}$ runs over irreducible mod p representations of $\mathrm{GL}_2(k)$, k the residue field

[1] Here the symbol $\mathbf{v}$ indicates a conjugacy class of cocharacters corresponding to the choice of Hodge-Tate weights; we refer to section 1.1.3 below for the precise definition. The choice of $\mathcal{O}$ is related to the field of definition of $\mathbf{v}$ and τ.

[2] Strictly speaking [BM] makes this conjecture in detail for two dimensional representations, $K = \mathbb{Q}_p$ and small Hodge-Tate weights. However, the possibility of this connection holding more generally is suggested on p214 of *loc. cit.*

[3] In order that the quotient is non-zero, one needs a condition of compatibility between ψ and $(\mathbf{v}, \tau)$ (see section 2.2 below) which we assume from now on.

of K, $a(\bar\sigma)$ denotes the multiplicity of $\bar\sigma$ as a Jordan-Hölder factor of $L_{\mathbf{v},\tau}/\pi L_{\mathbf{v},\tau}$, and $\mu_{\bar\sigma}(V_{\mathbb{F}})$ is a non-negative integer. This equality can be viewed as a system of infinitely many equations (corresponding to the choices of $\mathbf{v}$ and τ) in the finitely many unknowns $\mu_{\bar\sigma}(V_{\mathbb{F}})$. One can of course also ask for a version of such a conjecture where the $\mu_{\bar\sigma}(V_{\mathbb{F}})$ are given explicitly, as is done in [BM] when $K = \mathbb{Q}_p$.

For two dimensional representations and $K = \mathbb{Q}_p$ most of the Breuil-Mézard conjecture is proved in [Ki 5]. The proof consists of two parts: One uses the p-adic local Langlands correspondence of Breuil and Colmez [Br 1], [Co] to show that $e(R_{V_{\mathbb{F}}}^{\mathbf{v},\tau,\psi}/\pi)$ is bounded above by the expected value. A modified form of the Taylor-Wiles patching argument, introduced in [Ki 1], is then used to prove the other inequality. To do this one uses $L_{\mathbf{v},\tau}$-valued automorphic forms on a totally definite quaternion algebra to construct a module M_∞ which is finite of rank ≤ 1 over a formally smooth $R_{V_{\mathbb{F}}}^{\mathbf{v},\tau,\psi}$-algebra R_∞. Then

$$e(R_{V_{\mathbb{F}}}^{\mathbf{v},\tau,\psi}/\pi) = e(R_\infty/\pi) \geq e(M_\infty/\pi M_\infty)$$

where the final quantity denotes the Hilbert-Samuel multiplicity of the R_∞/π-module $M_\infty/\pi M_\infty$. This multiplicity can in turn be analyzed in terms of the Jordan-Hölder factors of $L_{\mathbf{v},\tau}/\pi L_{\mathbf{v},\tau}$.

The restriction $K = \mathbb{Q}_p$ is used primarily so as to be able to apply the p-adic local Langlands correspondence, which is available for $\mathrm{GL}_2(\mathbb{Q}_p)$ but remains somewhat elusive for $\mathrm{GL}_2(K)$ with $K \neq \mathbb{Q}_p$. Indeed the Breuil-Mézard conjecture may be viewed as an avatar of that correspondence. On the other hand, the modified Taylor-Wiles method can be applied without restrictions on K. It always gives an inequality involving $e(R_{V_{\mathbb{F}}}^{\mathbf{v},\tau,\psi}/\pi)$ with equality being essentially equivalent to a modularity lifting theorem for representations which are of type $(\mathbf{v}, \tau)$ at primes dividing p. Such lifting theorems are predicted by the Fontaine-Mazur conjecture and generalize the results used to prove the Shimura-Taniyama conjecture. They were the main motivation of [Ki 5].

In particular, one can try to *use* modularity lifting theorems to prove cases of the Breuil-Mézard conjecture for $K \neq \mathbb{Q}_p$. We give an example of such a result in §3, using the modularity lifting theorems for potentially Barsotti-Tate representations proved in [Ki 1] and [Ge 1]. The coefficients $\mu_{\bar\sigma}(V_{\mathbb{F}})$ are not made explicit in this case. One can hope to do that when $K/\mathbb{Q}_p$ is unramified, assuming the Buzzard-Diamond-Jarvis conjecture [BDJ] on the weights of automorphic forms giving rise to a given 2-dimensional mod p representation. Most of this has been proved by Gee [Ge 2], but one really needs the whole conjecture to determine all the coefficients. Nevertheless, we explain how to use Gee's result to prove the expected lower bound for $e(R_{V_{\mathbb{F}}}^{\mathbf{v},\tau,\psi}/\pi)$ when $V_{\mathbb{F}}$ is absolutely irreducible and satisfies a mild additional restriction.

The paper is organized as follows: In §1 we recall the definition of the rings $R_{V_{\mathbb{F}}}^{\mathbf{v},\tau,\psi}$ and some of their variants. In §2, we formulate the general form of the Breuil-Mézard conjecture and recall the explicit definition of $\mu_{\bar\sigma}(V_{\mathbb{F}})$ when $K/\mathbb{Q}_p$ is unramified and $V_{\mathbb{F}}$ is absolute irreducible. In this case these integers are all either 0 or 1, and the explicit description is essentially a reformulation of the conjecture of [BDJ]. Finally, in §3 we prove the two theorems on $e(R_{V_{\mathbb{F}}}^{\mathbf{v},\tau,\psi}/\pi)$ mentioned above.

1. Potentially semi-stable deformation rings

1.1. Potentially semi-stable representations. Let $K/\mathbb{Q}_p$ be a finite extension with residue field k, and fix an algebraic closure $\bar{K}/K$. For a subfield $K' \subset \bar{K}$, containing K, we write $G_{K'} = \mathrm{Gal}(\bar{K}/K')$ and $I_{K'} \subset G_{K'}$ for the inertia subgroup of $G_{K'}$. We denote by K'_0 the maximal absolutely unramified subfield of K', and by $\mathcal{O}_{K'}$ the ring of integers of K'.

Recall Fontaine's [Fo 1] period rings

$$B_{\mathrm{cris}} \subset B_{\mathrm{st}} \subset B_{\mathrm{dR}}.$$

The ring B_{st} is a $\bar{K}_0$-algebra, equipped with a Frobenius endomorphism φ and an operator N satisfying $N\varphi = p\varphi N$, and we have $B_{\mathrm{cris}} = B_{\mathrm{st}}^{N=0}$. The ring B_{dR} is a discrete valuation field with residue field $\widehat{\bar{K}}$. In particular, it carries a filtration given by the valuation. The above inclusions induce inclusions

$$B_{\mathrm{cris}} \otimes_{K_0} K \subset B_{\mathrm{st}} \otimes_{K_0} K \subset B_{\mathrm{dR}}.$$

In particular, the rings $B_{\mathrm{cris}} \otimes_{K_0} K$ and $B_{\mathrm{st}} \otimes_{K_0} K$ are equipped with the filtration induced from B_{dR}.

Suppose that V is a finite dimensional $\mathbb{Q}_p$-vector space equipped with a continuous action of G_K. We set

$$D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \quad D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Then $D_{\mathrm{st}}(V)$ is a K_0-vector space of dimension $\leq \dim_{\mathbb{Q}_p} V$ equipped with operators φ and N, with φ a bijection and satisfying $N\varphi = p\varphi N$. We have $D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)^{N=0}$. Moreover,

$$D_{\mathrm{cris}}(V) \otimes_{K_0} K \subset D_{\mathrm{st}}(V) \otimes_{K_0} K \subset D_{\mathrm{dR}}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}. \qquad (1.1.1)$$

So $D_{\mathrm{cris}}(V) \otimes_{K_0} K$ and $D_{\mathrm{st}}(V) \otimes_{K_0} K$ are equipped with a filtration.

A representation V is called *crystalline* (respectively *semi-stable*) if $D_{\mathrm{cris}}(V)$ (resp. $D_{\mathrm{st}}(V)$) has K_0-dimension $\dim_{\mathbb{Q}_p} V$, in which case both (resp. the second) inclusions in (1.1.1) are equalities. We say that V is *potentially crystalline* (resp. *potentially semi-stable*) if $V|_{G_{K'}}$ is crystalline (resp. semi-stable) for some finite extension K'/K.

1.1.2. Fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let $E \subset \bar{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Let V_E be an E-vector space of finite dimension d, equipped with a continuous action of G_K. We assume that V_E is potentially semi-stable (viewed as a $\mathbb{Q}_p$-representation). Then

$$D_{\mathrm{pst}}(V_E) = \varinjlim_{K'} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V_E)^{G_{K'}}$$

is a vector space over $\bar{K}_0$ of dimension $\dim_{\mathbb{Q}_p} V_E$. Note that $D_{\mathrm{pst}}(V_E)$ is a $\bar{K}_0 \otimes_{\mathbb{Q}_p} E$-module equipped with a semi-linear action of G_K, and so with a linear action of I_K. Since φ is a bijection on $D_{\mathrm{pst}}(V_E)$, this is necessarily a free $\bar{K}_0 \otimes_{\mathbb{Q}_p} E$-module, and since the action of φ commutes with that of I_K, we have $\mathrm{tr}(\sigma|D_{\mathrm{pst}}(V_E)) \in E$ for any $\sigma \in I_K$.

Let $\tau : I_K \to \mathrm{GL}_d(\bar{\mathbb{Q}}_p)$ be a representation with open kernel. We say that V_E is of Galois type τ if the I_K-representation $D_{\mathrm{pst}}(V_E)$ is equivalent to τ. That is, $\bar{\mathbb{Q}}_p \otimes_E D_{\mathrm{pst}}(V_E)$, equipped with its I_K action is isomorphic to $\tau \otimes_{\mathbb{Q}_p} \bar{K}_0$. Concretely this means that for any $\sigma \in I_K$, $\mathrm{tr}(\sigma|D_{\mathrm{pst}}(V_E)) = \mathrm{tr}(\tau(\sigma))$.

We can extend this definition to finite local E-algebras B : If V_B is a finite free B-module, equipped with a continuous, potentially semi-stable action of G_K, then $D_{\mathrm{pst}}(V_B)$ gives rise to a representation of I_K on a finite free $\bar{K}_0 \otimes_{\mathbb{Q}_p} B$-module with traces in B. We say that V_B is of Galois type τ if the traces of elements of I_K acting on $D_{\mathrm{pst}}(V_B)$ and τ are equal. If B has residue field E then a potentially semi-stable V_B is of type τ if and only if $V_B \otimes_B E$ is.

1.1.3. Let $\mathbf{v}$ be a conjugacy class of cocharacters of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d$ (defined over $\bar{\mathbb{Q}}_p$). Concretely, $\mathbf{v}$ consists of the data of a d-tuple of integers for each embedding $K \hookrightarrow \bar{\mathbb{Q}}_p$. Let $E_{\mathbf{v}} \subset \bar{E}$ denote the *reflex field* of $\mathbf{v}$. That is, $E_{\mathbf{v}}$ is the fixed field of the group of $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that $\sigma^*(\mathbf{v}) = \mathbf{v}$. Then $\mathbf{v}$ has a representative defined over $E_{\mathbf{v}}$.

Now let V_E be as above, and suppose that $E \supset E_{\mathbf{v}}$. We say that V_E has p-adic Hodge type $\mathbf{v}$, if the filtration on the $K \otimes_{\mathbb{Q}_p} E$-module $D_{\mathrm{dR}}(V_E)$ is induced by the *inverse* of a cocharacter in the conjugacy class $\mathbf{v}$. As in section 1.1.2, we can extend this definition to representations of G_K on finite local E-algebras B.

1.1.4. Suppose that V_E is of p-adic Hodge type $\mathbf{v}$, and Galois type τ. An extension of V_E by V_E in the category of G_K-representations can be regarded as a representation of G_K on a finite free module $V_{E[\epsilon]}$ over the dual numbers $E[\epsilon]$. If $V_{E[\epsilon]}$ is potentially semi-stable it is necessarily of p-adic Hodge type $\mathbf{v}$ and Galois type τ. We can compute the space of such extensions as follows: First observe that

$$\mathrm{ad}D_{\mathrm{pst}}(V_E) \xrightarrow{\sim} D_{\mathrm{pst}}(\mathrm{ad}V_E) \subset D_{\mathrm{dR}}(\mathrm{ad}V_E) \otimes_K \bar{K} \xrightarrow{\sim} \mathrm{ad}D_{\mathrm{dR}}(V_E) \otimes_K \bar{K}$$

where ad denotes the adjoint so that, for example, $\mathrm{ad}V_E = \mathrm{Hom}_E(V_E, V_E)$. Hence

$$(\mathrm{ad}D_{\mathrm{pst}}(V_E))^{G_K} \subset \mathrm{ad}D_{\mathrm{dR}}(V_E). \tag{1.1.5}$$

Suppose for a moment that V_E is potentially crystalline. Then it turns out that the space $\mathrm{Ext}^1_{\mathrm{pcris}}(V_E, V_E)$ of self extensions of V_E which are potentially crystalline is canonically isomorphic to the H^1 of the following complex concentrated in degrees 0 and 1

$$(\mathrm{ad}D_{\mathrm{pst}}(V_E))^{G_K} \overset{(1-\varphi,\mathrm{can})}{\longrightarrow} (\mathrm{ad}D_{\mathrm{pst}}(V_E))^{G_K} \oplus \mathrm{ad}D_{\mathrm{dR}}(V_E)/\mathrm{Fil}^0\mathrm{ad}D_{\mathrm{dR}}(V_E),$$

where the second component of the map is induced by the inclusion (1.1.5). The kernel of this map is canonically isomorphic to $(\mathrm{ad}V_E)^{G_K}$. In particular, we have

$$\dim_E\mathrm{Ext}^1_{\mathrm{pcris}}(V_E, V_E) = \dim_E\mathrm{ad}D_{\mathrm{dR}}(V_E)/\mathrm{Fil}^0\mathrm{ad}D_{\mathrm{dR}}(V_E) + \dim_E(\mathrm{ad}V_E)^{G_K}. \tag{1.1.6}$$

In particular, if V_E is absolutely irreducible, then the right hand side of (1.1.6) depends only on the p-adic Hodge type, and is equal to $1+w_{\mathbf{v}}^{>0}$, where $w_{\mathbf{v}}^{>0}$ is the dimension of the Lie subalgebra of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathfrak{gl}_d$ on which a fixed representative of $\mathbf{v}$ acts with positive weights.

Now suppose that V_E is potentially semi-stable. Then the space $\mathrm{Ext}^1_{\mathrm{pst}}(V_E,V_E)$ of potentially semi-stable self extensions is canonically isomorphic to H^1 of the total complex (concentrated in degrees $0,1,2$) of

$$\begin{array}{ccc} (\mathrm{ad}D_{\mathrm{pst}}(V_E))^{G_K} & \xrightarrow{1-\varphi} & (\mathrm{ad}D_{\mathrm{pst}}(V_E))^{G_K} \\ \downarrow{\scriptstyle N,\mathrm{can}} & & \downarrow{\scriptstyle N} \\ (\mathrm{ad}D_{\mathrm{pst}}(V_E))^{G_K}\oplus \mathrm{ad}D_{\mathrm{dR}}(V_E)/\mathrm{Fil}^0\mathrm{ad}D_{\mathrm{dR}}(V_E) & \xrightarrow{p\varphi-1,0} & (\mathrm{ad}D_{\mathrm{pst}}(V_E))^{G_K} \end{array}$$

If V_E is absolutely irreducible, we deduce that the dimension of $\mathrm{Ext}^1_{\mathrm{pst}}(V_E,V_E)$ is again $1+w_{\mathbf{v}}^{>0}$ provided the H^2 of the above total complex vanishes. In general, this H^2 contains obstructions for the deformation theory of V_E as a potentially semi-stable representation.

1.2. Deformation rings.

Now let $\bar{\mathbb{F}}_p$ be the residue field of $\bar{\mathbb{Q}}_p$, and $\mathbb{F}\subset\bar{\mathbb{F}}_p$ a finite extension of $\mathbb{F}_p$. Let $V_{\mathbb{F}}$ be an $\mathbb{F}$-vector space of dimension d equipped with a continuous action of G_K. Let $\mathfrak{AR}_{W(\mathbb{F})}$ denote the category of Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$. If A is in $\mathfrak{AR}_{W(\mathbb{F})}$, a *deformation* of $V_{\mathbb{F}}$ to A is a finite free A-module equipped with a continuous action of G_K and a G_K-equivariant isomorphism $V_A\otimes_A\mathbb{F}\xrightarrow{\sim}V_{\mathbb{F}}$. We denote by $D_{V_{\mathbb{F}}}(A)$ the set of isomorphism classes of deformations of $V_{\mathbb{F}}$ to A.

If we fix a basis for $V_{\mathbb{F}}$, then a *framed deformation* is a deformation V_A of $V_{\mathbb{F}}$ to A, together with a lifting to V_A of the chosen basis of $V_{\mathbb{F}}$. We denote by $D^{\square}_{V_{\mathbb{F}}}(A)$ the set of isomorphism classes of framed deformations of $V_{\mathbb{F}}$ to A.

The functor $D^{\square}_{V_{\mathbb{F}}}$ is always pro-representable by a complete local $W(\mathbb{F})$-algebra $R^{\square}_{V_{\mathbb{F}}}$. If $\mathrm{End}_{\mathbb{F}[G_K]}V_{\mathbb{F}}=\mathbb{F}$ then the functor $D_{V_{\mathbb{F}}}$ is pro-representable by a complete local $W(\mathbb{F})$-algebra $R_{V_{\mathbb{F}}}$ [Ma]. In this case the canonical morphism $R_{V_{\mathbb{F}}}\to R^{\square}_{V_{\mathbb{F}}}$ is formally smooth.

Now let $E\subset\bar{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ as before, and assume that the residue field of E contains $\mathbb{F}$. Fix a representation $\tau: I_K\to \mathrm{GL}_d(E)$ with open kernel, and a p-adic Hodge type $\mathbf{v}$ such that $E_{\mathbf{v}}\subset E$. The main result of [Ki 4] is that $R^{\square}_{V_{\mathbb{F}}}$ and $R_{V_{\mathbb{F}}}$ (when it is defined) admit quotients which parameterize potentially semi-stable deformations of $V_{\mathbb{F}}$ of Galois type τ and p-adic Hodge type $\mathbf{v}$.

Theorem 1.2.1. *There exists a p-torsion free quotient $R^{\square,\tau,\mathbf{v}}_{V_{\mathbb{F}}}$ of $R^{\square}_{V_{\mathbb{F}}}\otimes_{W(\mathbb{F})}\mathcal{O}$ such that for any finite local E-algebra B, and any homomorphism $\xi: R^{\square}_{V_{\mathbb{F}}}\to B$, the B-representation of G_K induced by ξ is potentially semi-stable of Galois type τ and p-adic Hodge type $\mathbf{v}$ if and only if ξ factors through $R^{\square,\tau,\mathbf{v}}_{V_{\mathbb{F}}}$.*

The irreducible components of $\mathrm{Spec}\,R^{\square,\tau,\mathbf{v}}_{V_{\mathbb{F}}}[1/p]$ *are generically reduced and of dimension $d^2+w_{\mathbf{v}}^{>0}$.*

If $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$, *then there exists an analogous quotient* $R^{\tau,\mathbf{v}}_{V_{\mathbb{F}}}$ *of* $R_{V_{\mathbb{F}}}$, *except that the components of* $\mathrm{Spec}\, R^{\tau,\mathbf{v}}_{V_{\mathbb{F}}}[1/p]$ *have dimension* $1 + w^{>0}_{\mathbf{v}}$.

We have a completely analogous statement for potentially crystalline representations, except that one can then make a more precise statement about the local structure of the generic fibres of the corresponding rings:

Theorem 1.2.2. *There exists a p-torsion free quotient* $R^{\square,\tau,\mathbf{v}}_{V_{\mathbb{F}},\mathrm{cr}}$ *of* $R^{\square}_{V_{\mathbb{F}}} \otimes_{W(\mathbb{F})} \mathcal{O}$ *such that for any finite local E-algebra B, and any homomorphism* $\xi : R^{\square}_{V_{\mathbb{F}}} \to B$, *the B-representation of* G_K *induced by* ξ *is potentially crystalline of Galois type* τ *and p-adic Hodge type* $\mathbf{v}$ *if and only if* ξ *factors through* $R^{\square,\tau,\mathbf{v}}_{V_{\mathbb{F}},\mathrm{cr}}$.

The irreducible components of $\mathrm{Spec}\, R^{\square,\tau,\mathbf{v}}_{V_{\mathbb{F}},\mathrm{cr}}[1/p]$ *are formally smooth of dimension* $d^2 + w^{>0}_{\mathbf{v}}$.

If $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$, *then there exists an analogous quotient* $R^{\tau,\mathbf{v}}_{V_{\mathbb{F}},\mathrm{cr}}$ *of* $R_{V_{\mathbb{F}}}$, *except that the components of* $\mathrm{Spec}\, R^{\tau,\mathbf{v}}_{V_{\mathbb{F}},\mathrm{cr}}[1/p]$ *have dimension* $1 + w^{>0}_{\mathbf{v}}$.

Note that it is clear that, if the above quotients exist, then they are unique. The reason for taking B a finite local E-algebra, rather than just a finite field extension of E, was to ensure this uniqueness.

1.2.3. For τ trivial, the above results were previously known in special cases: In each of those cases what was actually shown were special cases of the following conjecture of Fontaine [Fo 2]:

Conjecture 1.2.4. *(Fontaine) Let* $a \leq b$ *be integers and* V *a continuous representation of* G_K *on a finite free* $\mathbb{Z}_p$*-module. Suppose that for* $n \geq 1$ V/p^nV *is a subquotient of a* G_K*-stable lattice in a semi-stable (resp. crystalline) representation* V_n *whose Hodge-Tate weights are in* $[a, b]$. *Then* $V \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is semi-stable (resp. crystalline) with Hodge-Tate weights in* $[a, b]$.

1.2.5. For crystalline deformations this was shown by Ramakrishna [Ra] when $[a, b] = [0, 1]$, using results of Raynaud, [4] by Fontaine-Lafaille [FL] when $K = K_0$ and $[a, b] = [0, p - 2]$, and by Berger [Be] whenever $K = K_0$. For semi-stable representations with $[K : K_0]|b - a| < p - 1$ this is a result of Breuil [Br 2].

The results of [Ki 4], are not proved via Fontaine's conjecture. Rather the quotients $R^{\square,\mathbf{v},\tau}_{V_{\mathbb{F}}}$ are constructed more directly using the results of [Ki 2] on Galois stable lattices in semi-stable representations. On the other hand, T. Liu has also used the theory of [Ki 2] to prove Fontaine's conjecture in general [Li].

2. The Breuil-Mézard conjecture

2.1. Local Langlands and I_K-representations. From now on we fix a normalization of local class field theory so that the restriction of the cyclotomic

[4] Actually, what Ramakrishna shows is that if V_n arises from a p-divisible group then so does V. It was a later result of Breuil that V arises from a p-divisible group if and only if it is crystalline with Hodge-Tate weights in $[0, 1]$.

character $\chi_{\mathrm{cyc}} : G_K \to \mathbb{Z}_p$ to $\mathcal{O}_K^\times \subset G_K$ is given by the norm $N_{K/\mathbb{Q}_p}$. This corresponds to the normalization of global class field theory which takes uniformizers to geometric Frobenii.

Consider a representation $\tau : I_K \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ with open kernel as in section 1.1.2 We will assume that τ is the restriction to I_K of a 2-dimensional representation of the Weil-Deligne group WD_K of K.

If $\tilde{\tau}$ is any continuous, Frobenius semi-simple 2-dimensional representation of WD_K, we denote by $\pi(\tilde{\tau})$ the representation of $\mathrm{GL}_2(K)$ attached to $\tilde{\tau}$ by the local Langlands correspondence [5] , normalized so that $\pi(\tilde{\tau})$ has central character $\det \tilde{\tau}|_{K^\times}|\cdot|^{-1}$. We have the following result [BM, Appendix].

Theorem 2.1.1. *(Bushnell-Kutzko, Henniart) There is a finite dimensional, irreducible $\bar{\mathbb{Q}}_p$-representation $\sigma(\tau)$ (resp. $\sigma_{\mathrm{cr}}(\tau)$) of $\mathrm{GL}_2(\mathcal{O}_K)$ such that for any 2-dimensional, Frobenius semi-simple representation $\tilde{\tau}$ of WD_K, $\pi(\tilde{\tau})|_{\mathrm{GL}_2(\mathcal{O}_K)}$ contains $\sigma(\tau)$ (resp. $\sigma_{\mathrm{cr}}(\tau)$) if and only if $\tilde{\tau}|_{I_K} \sim \tau$ (resp. $\tilde{\tau}|_{I_K} \sim \tau$ and $N = 0$ on $\tilde{\tau}$).*

The representation $\sigma(\tau)$ (resp. $\sigma_{\mathrm{cr}}(\tau)$) is uniquely determined by this property except possibly [6] when $|k| = 2$.

2.1.2. Let $\mathbf{v}$ be a cocharacter of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$ and suppose that E contains the image of all embeddings $K \hookrightarrow \bar{\mathbb{Q}}_p$. In particular, $E_{\mathbf{v}} \subset E$. Concretely, $\mathbf{v}$ consists of the data of a pair of integers $(w_\iota, k_\iota + w_\iota)$ with $k_\iota \geq 0$, for each embedding $\iota : K \hookrightarrow \bar{\mathbb{Q}}_p$. We say that $\mathbf{v}$ is *regular* if $k_\iota \geq 1$ for all ι. For a regular $\mathbf{v}$ we set

$$\sigma(\mathbf{v}) = \otimes_{\iota:K\hookrightarrow E}\iota^*(\mathrm{Sym}^{k_\iota-1}K^2 \otimes \det{}^{w_\iota})$$

Now suppose that τ, $\sigma(\tau)$ and $\sigma_{\mathrm{cr}}(\tau)$ are defined over E. We again denote by $\sigma(\tau)$ and $\sigma_{\mathrm{cr}}(\tau)$ the corresponding E-vector spaces. Then we set $\sigma(\mathbf{v},\tau) = \sigma(\tau) \otimes_E \sigma(\mathbf{v})$, and $\sigma_{\mathrm{cr}}(\mathbf{v},\tau) = \sigma_{\mathrm{cr}}(\tau) \otimes_E \sigma(\mathbf{v})$.

2.2. Formulation of the conjecture.

Let ϖ be a uniformizer of K, and χ_ϖ the Lubin-Tate character attached to ϖ. For $\mathbf{v}$ as above we set

$$\chi_{\mathbf{v}} = \prod_{\iota:K\hookrightarrow E} (\iota \circ \chi_\varpi)^{k_\iota+2w_\iota-1}.$$

Now fix τ as in section 2.1 and $\mathbf{v}$ as above. Let $\psi : G_K \to \mathcal{O}^\times$ be a continuous character such that $\psi|_{I_K} = \chi_{\mathbf{v}}|_{I_K} \cdot \det \tau$.

Let $\mathbb{F} \subset \bar{\mathbb{F}}_p$ be the residue field of E, and let $V_{\mathbb{F}}$ be a two dimensional $\mathbb{F}$-vector space equipped with a continuous action of G_K such that the determinant of $V_{\mathbb{F}}$ is equal to the reduction of $\psi\chi_{\mathrm{cyc}}$.

[5] If $\tilde{\tau} \sim \chi \oplus \chi|\cdot|$ for some character χ of WD_K, then we take $\pi(\tilde{\tau})$ to be the reducible principal series representation $\chi \circ \det \otimes \mathrm{Ind}_B^{\mathrm{GL}_2(K)}\mathbf{1}$ where $B \subset \mathrm{GL}_2(K)$ is a Borel, rather than the more classical choice of the one dimensional representation $\chi \circ \det$.

[6] More precisely, if $|k| = 2$ and $\tau \sim \chi \oplus \chi\varepsilon_0$ with ε_0 a ramified character then there are two such representations. In this case, we take $\sigma(\tau) = \sigma_{\mathrm{cr}}(\tau)$ to be $\chi \circ \det$ times the representation denoted by $u_{N_0}(\varepsilon_0)$ in [He, A.2.2]. A more adventurous conjecture below would be to allow $\sigma(\tau)$ and $\sigma_{\mathrm{cr}}(\tau)$ to be either of the two representations having the property in the theorem.

We denote by $R^{\square,\mathbf{v},\tau,\psi}_{V_{\mathbb{F}}}$ the quotient of the ring $R^{\square,\mathbf{v},\tau}_{V_{\mathbb{F}}}$ introduced in Theorem 1.2.1 corresponding to deformations with determinant (the image of) $\psi\chi_{\mathrm{cyc}}$. Similarly we have the ring $R^{\square,\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}$ and, when $\mathrm{End}_{\mathbb{F}[G_K]}V_{\mathbb{F}} = \mathbb{F}$, the rings $R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}}}$ and $R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}$

Let $\pi \subset \mathcal{O}$ be a uniformizer. We want to relate the Hilbert-Samuel multiplicity of the ring $R^{\square,\mathbf{v},\tau,\psi}_{V_{\mathbb{F}}}/\pi$ and its variants to the reduction mod π of a $\mathrm{GL}_2(\mathcal{O}_K)$-stable $\mathcal{O}$-lattice $L_{\mathbf{v},\tau} \subset \sigma(\mathbf{v},\tau)$. To do this we need to recall the irreducible mod p representations of $\mathrm{GL}_2(k)$ [BL].

2.2.1. Let $\underline{n} = \{n_{\bar\iota}\}$ and $\underline{m} = \{m_{\bar\iota}\}$ be tuples of integers indexed by the embeddings $\bar\iota : k \hookrightarrow \mathbb{F}$, with $0 \le n_{\bar\iota}, m_{\bar\iota} \le p-1$ and not all $m_{\bar\iota} = p-1$. Then the representations

$$\sigma_{\underline{n},\underline{m}} = \otimes_{\bar\iota}\bar\iota^*(\mathrm{Sym}^{n_{\bar\iota}}k^2 \otimes \det{}^{m_{\bar\iota}})$$

are irreducible and pairwise distinct, and any irreducible mod p representation of $\mathrm{GL}_2(k)$ is isomorphic to one of the $\sigma_{\underline{n},\underline{m}}$. These are also the irreducible mod p representations of $\mathrm{GL}_2(\mathcal{O}_K)$.

2.2.2. Recall that the Hilbert-Samuel multiplicity is an invariant which measures the complexity of a Noetherian, local ring A. If A has dimension d and maximal ideal $\mathfrak{m} \subset A$ then, for sufficiently large n, the function $n \mapsto \ell(A/\mathfrak{m}^{n+1})$ is a polynomial of degree d, where ℓ denotes length. Then the Hilbert-Samuel multiplicity $e(A)$ is defined as $d!$ times the coefficient of X^d in this polynomial. It is necessarily an integer.

More generally, if M is a finite A-module, then for n sufficiently large, $n \mapsto \ell(M/\mathfrak{m}^{n+1})$ is a polynomial of degree at most d. The coefficient of X^d has the form $e_A(M)/d!$ for a non-negative integer $e_A(M)$ which is called the Hilbert-Samuel multiplicity of M.

The following is a natural generalization of the Breuil-Mézard conjecture which is, to some extent, already hinted at in [BM, p214].

Conjecture 2.2.3. *There exist integers $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}})$ such that for any τ and $\mathbf{v}$, and ψ as above, with $\mathbf{v}$ regular, we have*

$$e(R^{\square,\mathbf{v},\tau,\psi}_{V_{\mathbb{F}}}/\pi) = \sum_{\underline{n},\underline{m}} a(\underline{n},\underline{m})\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}),$$

where

$$(L_{\mathbf{v},\tau} \otimes_{\mathcal{O}} \mathbb{F})^{\mathrm{ss}} \xrightarrow{\sim} \oplus_{\underline{n},\underline{m}}\sigma_{\underline{n},\underline{m}}^{a(\underline{n},\underline{m})}.$$

Similarly, if $L^{\mathrm{cr}}_{\mathbf{v},\tau}$ is a $\mathrm{GL}_2(\mathcal{O}_K)$-stable lattice in $\sigma_{\mathrm{cr}}(\mathbf{v},\tau)$ then

$$e(R^{\square,\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}/\pi) = \sum_{\underline{n},\underline{m}} a(\underline{n},\underline{m})\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}),$$

where

$$(L^{\mathrm{cr}}_{\mathbf{v},\tau} \otimes_{\mathcal{O}} \mathbb{F})^{\mathrm{ss}} \xrightarrow{\sim} \oplus_{\underline{n},\underline{m}}\sigma_{\underline{n},\underline{m}}^{a(\underline{n},\underline{m})}.$$

2.2.4. Note that when $V_{\mathbb{F}}$ has trivial endomorphisms, the morphism $R_{V_{\mathbb{F}}}^{\mathbf{v},\tau,\psi} \to R_{V_{\mathbb{F}}}^{\square,\mathbf{v},\tau,\psi}$ (resp. $R_{V_{\mathbb{F}},\mathrm{cr}}^{\mathbf{v},\tau,\psi} \to R_{V_{\mathbb{F}},\mathrm{cr}}^{\square,\mathbf{v},\tau,\psi}$) is formally smooth, so the Hilbert-Samuel multiplicities of these two rings are equal.

The equalities in Conjecture 2.2.3 can be viewed as an infinite number of equations (corresponding to the choices of $\mathbf{v}$ and τ) in the finitely many unknowns $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}})$. If these equalities hold, then the $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}})$ may be determined by taking τ trivial, and selecting $\mathbf{v}$ as follows: Choose a subset L of the set of embeddings $K \hookrightarrow E$ such that L maps bijectively onto the set of embeddings $k \hookrightarrow \mathbb{F}$. Define $\mathbf{v}$ by $k_\iota = n_{\bar{\iota}} + 1$ and $w_\iota = m_\iota$ if $\iota \in L$ and $k_\iota = 1, w_\iota = 0$ otherwise. Here $\bar{\iota}$ denotes the reduction of ι. Then $\sigma_{\mathrm{cr}}(\tau)$ is the trivial representation of $\mathrm{GL}_2(\mathcal{O}_K)$ and any $\mathrm{GL}_2(\mathcal{O}_K)$-stable lattice $L_{\mathbf{v},\tau}^{\mathrm{cr}}$ in $\sigma_{\mathrm{cr}}(\mathbf{v},\tau)$, has reduction isomorphic to $\sigma_{\underline{n},\underline{m}}$. So Conjecture 2.2.3 predicts

$$\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}) = e(R_{\mathrm{cr}}^{\square,\mathbf{v},\tau,\psi}/\pi). \tag{2.2.5}$$

2.3. The case of an unramified extension.

When $K/\mathbb{Q}_p$ is unramified, the integers on the right hand side of (2.2.5) can be determined in almost all cases, and are usually in $\{0,1,2\}$. In this case, the condition that $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}) \neq 0$ is closely related to the Buzzard-Diamond-Jarvis conjecture on when a given two dimensional, mod p *global* Galois representation is modular of weight $\sigma_{\underline{n},\underline{m}}$.

2.3.1. Suppose now that $K/\mathbb{Q}_p$ is unramified. We will give the explicit values of $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}})$ when $V_{\mathbb{F}}$ is absolutely irreducible.

Let K'/K be the unramified extension of degree 2, so that $I_K = I_{K'} = I_{\mathbb{Q}_p}$. Let k' denote the residue field of K'. Let $n = [K : \mathbb{Q}_p]$ and $\omega_{2n} : I_{\mathbb{Q}_p} \to k'^{\times}$ the fundamental character of level $2n$ and $\omega_n = \omega_{2n}^{p^n+1}$ the fundamental character of level n. We will assume that E contains all embeddings of K' into $\bar{\mathbb{Q}}_p$.

Let J be a subset of the embeddings $k' \hookrightarrow \mathbb{F}$ which bijects onto the set of all embeddings $k \hookrightarrow \mathbb{F}$. We set

$$\omega_J = \prod_{\bar{\iota} \in J} \iota \circ (\omega_{2n}^{n_\iota + 1} \cdot \omega_n^{m_\iota}),$$

where for $\iota \in J$ we again denote by ι the restriction of ι to k. Thus ω_J is a character $I_K \to \mathbb{F}^\times$. Similarly, if J' denotes the compliment of J in the set of embeddings $\bar{\iota} : k' \hookrightarrow \mathbb{F}$, we have the character $\omega_{J'}$.

Conjecture 2.3.2. *Suppose $V_{\mathbb{F}}$ is absolutely irreducible. Then Conjecture 2.2.3 holds with $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}) = 0$ unless there exists J as above such that*

$$V_{\mathbb{F}}|_{I_K} \sim \begin{pmatrix} \omega_J & 0 \\ 0 & \omega_{J'} \end{pmatrix},$$

in which case $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}) = 1$.

3. Theorems

3.1. Statements.

We will review some cases when Conjecture 2.2.3 is known as well as sketching some of the arguments. We assume from now on that $p > 2$.

Most of the conjecture is known when $K = \mathbb{Q}_p$. In this case each of $\underline{n}$, $\underline{m}$ consist of a single integer which we denote by n and m respectively, and we write $\mu_{n,m}(V_{\mathbb{F}})$ for $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}})$. The explicit value of $\mu_{n,m}(V_{\mathbb{F}})$ is known in all cases, except when $n = p-2$ and $V_{\mathbb{F}}$ is scalar. One has the following result [Ki 5], which, in particular includes (most of) the original conjecture stated by Breuil-Mézard (here ω denotes the mod p cyclotomic character).

Theorem 3.1.1. *Suppose that $K = \mathbb{Q}_p$, that $V_{\mathbb{F}} \nsim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$ for any character χ, and that if $V_{\mathbb{F}}$ has scalar semi-simplification then it is scalar.*

Then Conjecture 2.2.3 holds for any regular $\mathbf{v}$ and any τ.

3.1.2. The proof uses the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ to prove that the left hand side in the equalities in Conjecture 2.2.3 is bounded above by the right hand side. To each two dimensional E-representation V_E of $G_{\mathbb{Q}_p}$, this correspondence attaches a certain representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a p-adic Banach space $\Pi(V)$. A key ingredient in the proof is the fact that the p-adic local Langlands correspondence is compatible with the usual local Langlands correspondence, in the sense that, if V_E is potentially semi-stable with p-adic Hodge type $\mathbf{v}$ and Galois type τ, then the locally algebraic vectors in $\Pi(V)$ contain a copy of the $\mathrm{GL}_2(\mathbb{Z}_p)$-representation $\sigma(\mathbf{v},\tau)$. This was proved by Colmez and Berger-Breuil [Co 2], [BB] when τ arises from an *abelian* representation of the Weil group, and by Colmez [Co] in general, using Emerton's work on the local-global compatibility of the p-adic Langlands correspondence [Em].

The opposite inequality is proved by a Taylor-Wiles style patching argument. Indeed, this patching argument shows that Conjecture 2.2.3 is very closely related to the conjecture of Fontaine-Mazur on the modularity of geometric Galois representations. One can attempt to run this argument in reverse and deduce Conjecture 2.2.3 from a modularity lifting theorem. For potentially Barsotti-Tate representations such a theorem was proved in [Ki 1] and generalized by Gee [Ge 1]. Using it one can show that for any $K/\mathbb{Q}_p$ we have

Theorem 3.1.3. *Denote by $\mathbf{v}_0$ the cocharacter corresponding to $k_\iota - 1 = w_\iota = 0$ for all ι. If $V_{\mathbb{F}}$ is absolutely irreducible, then there exist non-negative integers $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}})$ such that for any τ,*

$$e(R_{\mathrm{cr}}^{\square,\mathbf{v}_0,\tau,\psi}/\pi) = \sum_{\underline{n},\underline{m}} a(\underline{n},\underline{m})\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}),$$

where

$$(L^{\mathrm{cr}}_{\mathbf{v}_0,\tau} \otimes_{\mathcal{O}} \mathbb{F})^{\mathrm{ss}} \xrightarrow{\sim} \oplus_{\underline{n},\underline{m}} \sigma_{\underline{n},\underline{m}}^{a(\underline{n},\underline{m})}.$$

3.1.4. Now return to the case where $K/\mathbb{Q}_p$ is unramified. We assume that $V_{\mathbb{F}}$ is absolutely irreducible, and we now take $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}})$ to be defined as in Conjecture 2.3.2, so that $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}})$ is non-zero if and only if there exists J such that $V_{\mathbb{F}}|_{I_K} \sim \begin{pmatrix} \omega_J & 0 \\ 0 & \omega_{J'} \end{pmatrix}$ in which case $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}) = 1$.

We will say that $\mathbf{v}$ is *paritious* if the integers $k_\iota + 2w_\iota$ are independent of ι. We will say that $V_{\mathbb{F}}$ is regular, if there exists $(\underline{n},\underline{m})$ with $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}) \neq 0$ and $2 \leq n_\iota \leq p-4$ for all ι.

Theorem 3.1.5. *Suppose that $K/\mathbb{Q}_p$ is unramified, that $\mathbf{v}$ is paritious and that $V_{\mathbb{F}}$ is absolutely irreducible and regular. Then*

$$e(R^{\mathbf{v},\tau,\psi}/\pi) \geq \sum_{\underline{n},\underline{m}} a(\underline{n},\underline{m})\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}),$$

where

$$(L_{\mathbf{v},\tau} \otimes_{\mathcal{O}} \mathbb{F})^{\mathrm{ss}} \xrightarrow{\sim} \oplus_{\underline{n},\underline{m}} \sigma_{\underline{n},\underline{m}}^{a(\underline{n},\underline{m})},$$

and similarly for $e(R_{\mathrm{cr}}^{\mathbf{v},\tau,\psi}/\pi)$.

3.2. A sketch of the proofs.

We now give a sketch of some of the methods which are used to prove Theorems 3.1.3 and 3.1.5. These involve relating the Hilbert-Samuel multiplicities in the conjectures to those of certain spaces of automorphic forms.

It ought to be possible to extend these methods to prove Conjecture 2.2.3 for $e(R_{\mathrm{cr}}^{\square,\mathbf{v},\tau,\psi}/\pi)$ with an explicit collection of integers $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}})$, when $\mathbf{v} = \mathbf{v}_0$ and $K/\mathbb{Q}_p$ is unramified. This is work in progress with Toby Gee.

3.2.1. Let F be a totally real number field and D a totally definite quaternion algebra over F, which is unramified at all primes $v|p$ of F. Denote by $\mathbb{A}_F^f \subset \mathbb{A}_F$ the finite adeles. For each finite place v of F we will denote by $\pi_v \in F_v$ a uniformizer. Fix a maximal order $\mathcal{O}_D \subset D$, and an isomorphism $(\mathcal{O}_D)_v \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$ for each finite place where D is unramified. Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup contained in $\prod_v (\mathcal{O}_D)_v^\times$. We assume that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ for $v|p$.

For each $v|p$, we fix a continuous representation $\sigma_v : U_v \to \mathrm{Aut}(W_{\sigma_v})$ on a finite $\mathcal{O}$-module. Write $W_\sigma = \otimes_{v|p,\mathcal{O}} W_{\sigma_v}$ and denote by $\sigma : \prod_{v|p} U_v \to \mathrm{Aut}(W_\sigma)$ the corresponding representation. We regard σ as being a representation of U by letting U_v act trivially if $v \nmid p$. Finally, assume there exists a continuous character $\psi : (\mathbb{A}_F^f)^\times/F^\times \to \mathcal{O}^\times$ such that σ on $U \cap (\mathbb{A}_F^f)^\times$ is given by multiplication by ψ. Fix such a ψ, and extend the action of U on W_σ to $U(\mathbb{A}_F^f)^\times$, by letting $(\mathbb{A}_F^f)^\times$ act via ψ.

Let $S_{\sigma,\psi}(U)$ denote the set of continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \to W_\sigma$$

such that for $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have $f(gu) = \sigma(u)^{-1} f(g)$ for $u \in U$, and $f(gz) = \psi^{-1}(z) f(g)$ for $z \in (\mathbb{A}_F^f)^\times$.

We consider the left action of $(D \otimes_F \mathbb{A}_F^f)^\times$ on W_σ-valued functions on $(D \otimes_F \mathbb{A}_F^f)^\times$ given by the formula $(gf)(z) = f(zg)$. Then for any finite prime v, the double cosets of U_v in $(D \otimes_F \mathbb{A}_F^f)^\times$ act naturally on $S_{\sigma,\psi}(U)$. Denote by $\mathbb{T}_{\sigma,\psi}(U)$ the $\mathcal{O}$-algebra generated by the endomorphisms S_v and T_v of $S_{\sigma,\psi}(U)$ corresponding to $U_v \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} U_v$ and $U_v \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} U_v$ respectively, where $v \nmid p$ runs over primes at which D is unramified. If U_v is maximal compact in $(D \otimes_F F_v)^\times$, then these operators do not depend on the choice of π_v.

3.2.2. Now fix an algebraic closure $\bar{F}$ of F and let S be a finite set of primes of F, containing the infinite primes, the primes dividing p, the primes where D is ramified, and the primes where U_v is not maximal compact in $(D \otimes_F F_v)^\times$. Let $F_S \subset \bar{F}$ be the maximal extension of F unramified outside S, and set $G_{F,S} = \mathrm{Gal}(F_S/F)$.

Let $\mathfrak{m} \subset \mathbb{T}_{\sigma,\psi}(U)$ be a maximal ideal. Such an ideal is called *Eisenstein* if $T_v - 2 \in \mathfrak{m}$ for all but finitely many primes $v \notin S$ which split completely in some fixed abelian extension of F. After possibly replacing $\mathcal{O}$ by an extension we may assume that $\mathfrak{m}$ has residue field $\mathbb{F}$. If $\mathfrak{m}$ is a non-Eisenstein ideal, then the work of Carayol [Ca] and Taylor [Ta], together with the Jacquet-Langlands correspondence, implies that there exists a unique representation

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}})$$

such that if $v \notin S$ is a prime of F, and Frob_v denotes an arithmetic Frobenius at v then $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ has trace T_v. We denote by $\bar{\rho}_{\mathfrak{m}}$ the reduction of $\rho_{\mathfrak{m}}$ modulo $\mathfrak{m}$. As $\mathfrak{m}$ is non-Eisenstein $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

3.2.3. Now suppose we are given $\mathbf{v}$ and τ as in section 2.1.2 with $\mathbf{v}$ paritious and an absolutely irreducible representation $V_{\mathbb{F}}$ of G_K. Then we choose F such that there is a unique prime $\mathfrak{p}|p$ of F and $F_{\mathfrak{p}} \xrightarrow{\sim} K$. Fix an embedding $\bar{F} \hookrightarrow \bar{K}$, extending this isomorphism. We choose the character $\psi : (\mathbb{A}_F^f)^\times/F^\times \to \mathcal{O}^\times$ so that $\psi|_{I_K} = \chi_{\mathbf{v}}|_{I_K} \det \tau$, and we apply the above constructions with σ a $\mathrm{GL}_2(\mathcal{O}_K)$-stable $\mathcal{O}$-lattice $L^{\mathrm{cr}}_{\mathbf{v},\tau}$ in $\sigma_{\mathrm{cr}}(\mathbf{v},\tau)$.

Using CM forms, one can find $\mathfrak{m}$ such that $\bar{\rho}_{\mathfrak{m}}|_{G_K} \sim V_{\mathbb{F}}$, and we again denote by $V_{\mathbb{F}}$ the underlying $\mathbb{F}$-vector space of $\bar{\rho}_{\mathfrak{m}}$.

Let $R_{F,S}$ and $R_{\mathfrak{p}}$ denote the the universal deformation rings of $V_{\mathbb{F}}$ and $V_{\mathbb{F}}|_{G_K}$ respectively. We denote by $R^{\psi}_{F,S}$ the quotient of $R_{F,S}$ which parameterizes deformations of determinant $\psi\chi_{\mathrm{cyc}}$, where χ_{cyc} now denotes the p-adic cyclotomic character on $G_{F,S}$. Set

$$R^{\mathbf{v},\tau,\psi}_{F,S} = R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}} \otimes_{R_{\mathfrak{p}}} R^{\psi}_{F,S}.$$

The map

$$R_{F,S} \to \mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}},$$

induced by $\rho_{\mathfrak{m}}$, factors through $R^{\mathbf{v},\tau,\psi}_{F,S}$. (See for example [Ki 4, §4].)

Under some technical restrictions on the choice of F, D and U, which can always be arranged for a given representation $V_{\mathbb{F}}$ of G_K, a Taylor-Wiles patching argument, as modified by Diamond [Di] and Fujiwara, and in [Ki 1, §3], [Ki 5, §2], shows that there exist an $\mathcal{O}$-algebra R_∞, maps of $\mathcal{O}$-algebras

$$\mathcal{O}[\![y_1, \dots, y_h]\!] \to R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}[\![x_1, \dots, x_{h-d}]\!] \twoheadrightarrow R_\infty, \tag{3.2.4}$$

and an R_∞-module M_∞ satisfying the following properties:

(1) $h \geq d = \dim R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}/\pi = [K : \mathbb{Q}_p]$.

(2) There is an isomorphism of $R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}$ algebras $R_\infty/(y_1, \dots, y_h) \xrightarrow{\sim} R^{\mathbf{v},\tau,\psi}_{F,S}$.

(3) M_∞ is a finite free $\mathcal{O}[\![y_1,\dots,y_h]\!]$-module and has rank at most 1 on any irreducible component on $\operatorname{Spec} R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}[\![x_1,\dots,x_{h-d}]\!]$.

(4) There is an isomorphism of $R^{\mathbf{v},\tau,\psi}_{F,S}$-modules

$$M_\infty/(y_1,\dots,y_h)M_\infty \xrightarrow{\sim} S_{\sigma,\psi}(U)_{\mathfrak{m}}.$$

Now let

$$\{0\} = M^0 \subset M^1 \subset \cdots \subset M^s = L^{\mathrm{cr}}_{\mathbf{v},\tau}/\pi$$

be a filtration such that M^{i+1}/M^i is an irreducible representation of $\mathrm{GL}_2(k)$. Then we can enhance the above construction (see [Ki 5, 2.2.9]) in such a way that there exists a filtration

$$\{0\} = M^0_\infty \subset M^1_\infty \subset \cdots \subset M^s_\infty = M_\infty/\pi M_\infty$$

by R_∞-modules such that

(5) $M^i_\infty/M^{i-1}_\infty$ is a finite free $\mathbb{F}[\![y_1,\dots,y_h]\!]$-module.

(6) If $M^i/M^{i-1} \xrightarrow{\sim} \sigma_{\underline{n},\underline{m}}$ then the isomorphism in (4) above induces an isomorphism

$$M^i_\infty/M^{i-1}_\infty \otimes_{R_\infty} R_\infty/(y_1,\dots,y_h) \xrightarrow{\sim} S_{\sigma_{\underline{n},\underline{m}},\psi}(U)_{\mathfrak{m}}.$$

Moreover this construction can be made so that, as an $R_{\mathfrak{p}}[\![x_1,\dots,x_{h-d}]\!]$-module, $M^i_\infty/M^{i-1}_\infty$ depends only on $\sigma_{\underline{n},\underline{m}}$ and $\mathfrak{m}$, and not on the choice of $\mathbf{v}$ and τ. More precisely this module is made by an analogous patching argument but with $\sigma_{\underline{n},\underline{m}}$ in place of $L^{\mathrm{cr}}_{\mathbf{v},\tau}$. We denote this module by $M^{\underline{n},\underline{m}}_\infty$.

Set $R'_\infty = R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}[\![x_1,\dots,x_{h-d}]\!]$, and let $a(\underline{n},\underline{m})$ be the multiplicity with which $\sigma_{\underline{n},\underline{m}}$ appears as a Jordan-Hölder factor in $L^{\mathrm{cr}}_{\mathbf{v},\tau}/\pi$. Using (3) and (5) and standard facts about Hilbert-Samuel multiplicities one obtains

$$e(R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}/\pi) = e(R'_\infty/\pi R'_\infty) \geq e_{R'_\infty/\pi}(M_\infty/\pi M_\infty) = \sum_{\underline{n},\underline{m}} a(\underline{n},\underline{m}) e_{R'_\infty/\pi}(M^{\underline{n},\underline{m}}_\infty). \tag{3.2.5}$$

with equality if and only if $\operatorname{Spec} R'_\infty[1/p]$ is contained in the support of the R'_∞-module M_∞ (cf. [Ki 5, Lem. 2.2.11]). Note that the freeness condition in (3) implies that this support is a union of irreducible components of $\operatorname{Spec} R'_\infty[1/p]$ as the dimensions of $\mathcal{O}[\![y_1,\dots,y_h]\!]$ and R'_∞ coincide by (1). This also implies that $e_{R'_\infty/\pi}(M^{\underline{n},\underline{m}}_\infty)$ depends only on the image of $R_{\mathfrak{p}}[\![x_1,\dots,x_{h-d}]\!]$ in $\operatorname{End} M^{\underline{n},\underline{m}}_\infty$ and not on R'_∞, and is therefore independent of $\mathbf{v}$ and τ.

3.2.6. *Proof of Theorem 3.1.5.* In this case $K/\mathbb{Q}_p$ is unramified and $V_{\mathbb{F}}$ is assumed regular. We have to show that

$$e_{R'_\infty/\pi}(M^{\underline{n},\underline{m}}_\infty) \geq \mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}). \tag{3.2.7}$$

By definition, the term on the right is 0 or 1, and in the former case there is nothing to prove. Suppose $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}) = 1$. As above, the condition (5) implies that the support of $M_\infty^{\underline{n},\underline{m}}$ has dimension equal to $\dim R'_\infty/\pi$. Hence it suffices to show that $M_\infty^{\underline{n},\underline{m}} \neq \{0\}$. By (6) it suffices to show that $S_{\sigma_{\underline{n},\underline{m}},\psi}(U)_{\mathfrak{m}} \neq \{0\}$. This follows from Gee's proof [Ge 2] of the Buzzard-Diamond-Jarvis conjecture for regular weights. Namely our condition on the regularity of $V_{\mathbb{F}}$ implies that any $\sigma_{\underline{n},\underline{m}}$ such that $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}) \neq 0$ is regular in the sense of [Ge 2].

This completes the proof of Theorem 3.1.5 for $R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}$ and the proof for $R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}}}$ is identical, replacing $L^{\mathrm{cr}}_{\mathbf{v},\tau}$ by a $\mathrm{GL}_2(\mathcal{O}_K)$-invariant lattice in $\sigma(\mathbf{v},\tau)$. □

3.2.8. *Proof of Theorem 3.1.3:* Let $\mathbf{v} = \mathbf{v}_0$, and set $\mu_{\underline{n},\underline{m}}(V_{\mathbb{F}}) = e_{R'_\infty}(M_\infty^{\underline{n},\underline{m}})$. To prove the theorem we have to show that the inequality in (3.2.5) is an equality. It is enough to show that M_∞ is a faithful R'_∞-module.

The following lemma will be useful.

Lemma 3.2.9. *The following are equivalent*

(1) The support of $S_{\sigma,\psi}(U)_{\mathfrak{m}}$ contains $\mathrm{Spec}\, R^{\mathbf{v},\tau,\psi}_{F,S}[1/p]$ *and $R^{\mathbf{v},\tau,\psi}_{F,S}$ is a finite $\mathcal{O}$-algebra.*

(2) M_∞ is a faithful R'_∞-module.

Proof. (2) $\Longrightarrow$ (1): If M_∞ is a faithful R'_∞-module then $R'_\infty = R_\infty$ and both are finite over $\mathcal{O}[\![y_1,\dots,y_h]\!]$. Then (1) follows from conditions (2) and (4) in (3.2.3).

(1) $\Longrightarrow$ (2): One can use an argument of Khare-Wintenberger [KW 2, Cor. 4.7] to show that the second condition in (1) implies that the image of $\mathrm{Spec}\, R^{\mathbf{v},\tau,\psi}_{F,S}[1/p]$ in $\mathrm{Spec}\, R^{\mathbf{v},\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}[1/p]$ meets every irreducible component of the latter scheme. Hence the first condition implies that the support of $S_{\sigma,\psi}(U)_{\mathfrak{m}}$ meets every irreducible component of R'_∞. Since the support of M_∞ is a union of irreducible components of $\mathrm{Spec}\, R'_\infty[1/p]$, it must contain all of $\mathrm{Spec}\, R'_\infty[1/p]$ by condition (4) in (3.2.3). Finally as R'_∞ is flat over $\mathcal{O}$ with formally smooth (so in particular reduced) generic fibre, this implies that M_∞ is a faithful R'_∞-module. □

3.2.10. We return to the proof of Theorem 3.1.3. Since $\mathbf{v} = \mathbf{v}_0$ the main result of [Ki 1] and [Ge 1] shows that the support of $S_{\sigma,\psi}(U)_{\mathfrak{m}}$ contains $\mathrm{Spec}\, R^{\mathbf{v},\tau,\psi}_{F,S}[1/p]$.

Moreover the proof in *loc. cit* (cf. also [Ki 3, §1]) together with an argument of Khare-Wintenberger [KW 1, Prop. 3.8] shows that that $R^{\mathbf{v},\tau,\psi}_{F,S}$ is a finite $\mathcal{O}$-algebra. More precisely, the argument in [Ki 1, §3.4] carries out a patching argument analogous to the one sketched here, but over a finite, solvable, totally real extension F'/F. In that situation the analogue of the ring $R^{\mathbf{v}_0,\tau,\psi}_{V_{\mathbb{F}},\mathrm{cr}}$ turns out to be a domain. This implies that the analogue of the condition (2) in Lemma 3.2.9 is automatically satisfied, and hence so is the condition (1). This is enough to imply the finiteness of $R^{\mathbf{v},\tau,\psi}_{F,S}$ itself. □

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Department of Mathematics, Harvard,
1 Oxford st, Cambridge MA 02139, USA.
E-mail: kisin@math.harvard.edu