Deformations of GQp and GL2(Qp) Representations

DEFORMATIONS OF $G_{\mathbb{Q}_p}$ AND $\mathrm{GL}_2(\mathbb{Q}_p)$ REPRESENTATIONS.

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Introduction

The purpose of this appendix is to prove that Colmez's functor $\mathbf{V}$ from $\mathrm{GL}_2(\mathbb{Q}_p)$-representations to $G_{\mathbb{Q}_p}$-representations produces essentially all two dimensional representations of $G_{\mathbb{Q}_p}$. Here $G_{\mathbb{Q}_p}$ denotes the absolute Galois group of $\mathbb{Q}_p$. More precisely, let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and uniformiser $\pi_{\mathcal{O}}$. For a continuous representation of $G_{\mathbb{Q}_p}$ on a 2-dimensional E-vector space V, and $L \subset V$ a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}$-lattice, we denote by $\bar{V}$ the semi-simplification of $L/\pi_{\mathcal{O}}L$. This does not depend on L.

We denote by $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ the cyclotomic character and by $\omega : G_{\mathbb{Q}_p} \to \mathbb{F}_p^\times$ its mod p reduction. Finally we denote by ω_2 a fundamental character of level 2 of $I_{\mathbb{Q}_p}$.

Then our main result is the following

Theorem (0.1). *Suppose that $p > 2$ and if $p = 3$ assume that $\bar{V}$ is not of the form $\left(\begin{smallmatrix} \omega & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \chi$ and $\bar{V}|_{I_{\mathbb{Q}_p}}$ is not of the form $\left(\begin{smallmatrix} \omega_2^2 & 0 \\ 0 & \omega_2^6 \end{smallmatrix}\right) \otimes \chi$.*

(1) *If V is irreducible, then there exists an admissible $\mathcal{O}$-lattice Π with central character $\det V \cdot \chi_{\mathrm{cyc}}^{-1}$ such that*

$$\mathbf{V}(\Pi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V.$$

(2) *If $\bar{V} \not\sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$ then for any $G_{\mathbb{Q}_p}$-stable $\mathcal{O}$-lattice $L \subset V$, then there exists an admissible $\mathcal{O}$-lattice Π with central character $\det V \cdot \chi_{\mathrm{cyc}}^{-1}$ such that $\mathbf{V}(\Pi) \xrightarrow{\sim} L$.*

Here by an admissible $\mathcal{O}$-lattice we mean a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a p-torsion free, p-adically complete and separated $\mathcal{O}$-module Π such that for $n \geq 1$ the quotient $\Pi/p^n\Pi$ is a smooth, finite length representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. These are exactly the representations to which Colmez's functor applies, and we can then extend it to admissible $\mathcal{O}$-lattices, so that $\mathbf{V}(\Pi)$ is a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}$-lattice in V.

To explain the idea of the argument, let $\mathbb{F}$ be the residue field of $\mathcal{O}$ and $V_{\mathbb{F}}$ a two dimensional representation of $G_{\mathbb{Q}_p}$. Suppose that $V_{\mathbb{F}} \not\sim \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$. Then Colmez

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shows that there is a smooth, finite length representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on an $\mathbb{F}$-vector space $\bar{\pi}$, having central character $\bar{\psi} = \det V_{\mathbb{F}}\chi_{\mathrm{cyc}}^{-1}$ and such that $\mathbf{V}(\bar{\pi}) \xrightarrow{\sim} V_{\mathbb{F}}$.

Now fix a continuous character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ lifting $\bar{\psi}$. For simplicity of notation we will assume that $V_{\mathbb{F}}$, and hence $\bar{\pi}$ has only scalar endomorphisms.[1] Then one can define three deformation problems over the category of finite, local, Artinian $\mathcal{O}$-algebras: The first one, $D_{\bar{\pi},\psi}$, parameterizes deformations of $\bar{\pi}$ with central character ψ. The other two $D_{V_{\mathbb{F}}}$, (resp. $D_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$) parameterize deformations of $V_{\mathbb{F}}$ (resp. deformations of $V_{\mathbb{F}}$ with determinant $\psi\chi_{\mathrm{cyc}}$). Each of these deformation problems is pro-representable by a complete local $\mathcal{O}$-algebra which we denote by $R_{\bar{\pi},\psi}$, $R_{V_{\mathbb{F}}}$ and $R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$ respectively. Colmez's functor produces a map

$$\operatorname{Spec} R_{\bar{\pi},\psi} \to \operatorname{Spec} R_{V_{\mathbb{F}}} \tag{0.2}$$

and one of the main results of [Co 2, §VII] is that (0.2) induces an injection on tangent spaces. Hence it is a closed embedding.

One can sometimes show that this embedding factors through $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$, but this does not always hold.[2] On the other hand, results of Colmez and Berger-Breuil allow one to show that any crystalline point with distinct Hodge-Tate weights and determinant $\psi\chi_{\mathrm{cyc}}$ is in the image of (0.2). By imitating the "infinite fern" argument of Gouvêa-Mazur [GM], we are able to show that the set of crystalline points is dense in $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}[1/p]$:

Theorem (0.3). *Suppose that $p > 2$, and that $V_{\mathbb{F}} \nsim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$ and $V_{\mathbb{F}}|_{I_{\mathbb{Q}_p}} \nsim \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega_2^6 \end{pmatrix}$. Then the set of closed points $x \in \operatorname{Spec} R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}[1/p]$ such that the corresponding $G_{\mathbb{Q}_p}$-representation is crystalline is Zariski dense.*

In fact it is technically simpler to work with crystalline points satisfying some mild non-degeneracy conditions, so the results in the text refer to "benign" or "twisted benign" points. Such points are, in particular, crystalline.

As an immediate consequence of (0.3), one sees that the image of (0.2) contains $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$ and this leads to Theorem (0.1). The restrictions on $V_{\mathbb{F}}$ in (0.3) arise because in these cases $R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$ is not formally smooth over $\mathcal{O}$, and we know of no way to check that every component of $R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}[1/p]$ contains a crystalline point.

The result (0.3) is a local analogue of a theorem of Gouvêa-Mazur [GM], extended by Böckle [Bö] which says that for a two dimensional $\mathbb{F}$-representation of the absolute Galois group of $\mathbb{Q}$, the generic fibre of the universal deformation space has a Zariski dense set of points corresponding to cusp forms on $\Gamma_1(N)$ (of various weights) where N is a suitable integer not divisible by p. The original argument of Gouvêa-Mazur uses the *eigencurve* [CM], which is a kind of p-adic interpolation of these cusp forms. In particular, one can interpolate the global Galois representations attached to cusp forms into a family of Galois representations over the eigencurve.

In [Ki 1], we showed that the Galois representation attached to a point of the eigencurve admits at least one crystalline period. This local property (up to twist) was later dubbed *trianguline* by Colmez [Co 1]. One of the results of [Ki 1] shows

[1] Below this condition will be avoided by using framings.

[2] However see the remark at the end of this introduction.

that 2-dimensional representations of $G_{\mathbb{Q}_p}$ with a crystalline period can be interpolated into a p-adic analytic space X_{fs}, which is a kind of local analogue of the eigencurve. Using it, one can imitate the arguments of Gouvêa-Mazur for local Galois representations, and show a statement about density of crystalline representations. This has also been carried out by Colmez, using his theory of Vector Spaces [Co 1].

Finally let us mention that Paskunas [Pa] has shown that, when $\bar{\pi}$ is supersingular (that is, $V_{\mathbb{F}}$ is absolutely irreducible), then the surjection $R_{\bar{\pi},\psi} \to R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$ is an isomorphism. To prove this he shows directly that the dimension of the tangent space of the left hand side is at most 3, which is the dimension of the tangent space of the right hand side.[3] As a consequence one sees that, in this case, if π_A is a deformation of $\bar{\pi}$ with central character ψ, then $\det \mathbf{V}(\pi_A) = \psi\chi_{\mathrm{cyc}}$. Paskunas has also pointed out that this formula does not hold if $V_{\mathbb{F}}$ is unipotent.

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§1 Density of crystalline representations

(1.1) Let $\bar{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. We will write $G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, and we denote by $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character. As in the introduction, we denote by ω the mod p cyclotomic character, by $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ the inertia subgroup, and by ω_2 a fundamental character of level 2 of $I_{\mathbb{Q}_p}$.

Let $E/\mathbb{Q}_p$ be a finite extension. We will consider pairs (V, λ) consisting of a continuous representation of $G_{\mathbb{Q}_p}$ a two dimensional E-vector space V and $\lambda \in E^\times$ such that

(1) $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V, V) = E$.
(2) V is crystalline and the action of φ on
$$D_{\mathrm{cris}}(V^*) = \mathrm{Hom}_{E[G_{\mathbb{Q}_p}]}(V, B_{\mathrm{cris}}^+ \otimes_{\mathbb{Q}_p} E)$$
has eigenvalues λ, λ' with $\lambda' \neq \lambda, p^{\pm 1}\lambda$.
(3) V has Hodge-Tate weights $0, k$ with k a positive integer.

A pair (V, λ) satisfying the above condition will be called a *benign pair*. If V is a continuous representation of $G_{\mathbb{Q}_p}$ on a 2-dimensional E-vector space, then we say that V is benign if there exists a finite extension E'/E and $\lambda \in E'$ such that $(V \otimes_E E', \lambda)$ is benign. In particular, the condition (1) implies that V admits a universal deformation ring R_V.

Fix an E-basis of V. We denote by $R_V^\square$ the universal framed deformation ring of V. That is, if $\mathfrak{AR}_E$ denotes the category of Artinian local E-algebras with residue field E, then $R_V^\square$ represents the functor which to B in $\mathfrak{AR}_E$ assigns the set of isomorphism classes of pairs (V_B, β), where V_B is a deformation of V_E to B and β is a B-basis of V_B lifting the chosen basis of V_E. Recall that there is a natural map $R_V \to R_V^\square$ which is easily seen to be formally smooth of relative dimension 3.

[3] Of course when $p = 3$ we continue to exclude the case $\begin{pmatrix} \omega_2^2 & 0 \\ 0 & \omega_2^6 \end{pmatrix} \otimes \chi$.

Let (V,λ) be a benign pair. We denote by $D_V^{h,\varphi}$ the functor on $\mathfrak{AR}_E$ which assigns to B the set of isomorphism classes of deformations V_B of V_E to B such that, if

$$h : V \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi=\lambda}$$

is any non-zero E-linear, $G_{\mathbb{Q}_p}$-equivariant map, then h lifts to a map

$$\tilde{h} : V_B \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} B)^{\varphi=\tilde{\lambda}}$$

where $\tilde{\lambda} \in B^\times$ lifts λ. Note that (2) above implies that the set of maps h forms a torsor under $E^\times$. If V_B is in $D_V^{h,\varphi}(B)$, then the map $\tilde{h}$ is determined up to a unit in $B^\times$ and $\tilde{\lambda}$ is uniquely determined by λ [Ki 1, 8.12].

Let $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ denote the inertia subgroup. We have the following

Proposition (1.1.1). *The functor $D_V^{h,\varphi}$ is pro-represented by a quotient $R_V^{h,\varphi}$ of R_V, which is formally smooth over E of dimension 3. The composite*

$$I_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p} \to R_V^{h,\varphi\times}$$

given by the determinant of the universal deformation, does not factor through $E^\times$.

Proof. This is [Ki 1, 10.2]. □

Proposition (1.1.2). *Let $h \in D_{\mathrm{cris}}(V^*)^{\varphi=\lambda}$ and $h' \in D_{\mathrm{cris}}(V^*)^{\varphi=\lambda'}$ be non-zero. Then the closed subschemes* $\mathrm{Spec}\, R_V^{h,\varphi}$ *and* $\mathrm{Spec}\, R_V^{h',\varphi}$ *of* $\mathrm{Spec}\, R_V$ *are distinct. More precisely,* $\mathrm{Spec}\, R_V^{h,\varphi} \otimes_{R_V} R_V^{h',\varphi}$ *is a smooth subscheme of* $\mathrm{Spec}\, R_V$ *of dimension* 2.

Proof. Let B be in $\mathfrak{AR}_E$ and V_B in $D_V(B)$. It is not hard to check that V_B is in $D_V^{h,\varphi}(B)$ and $D_V^{h',\varphi}(B)$ if and only if V_B is crystalline. For example, use [Ki 1, 8.9]. The proposition now follows from [Ki 3, Thm. 3.3.8], which shows that the preimage of $\mathrm{Spec}\, R_V^{h,\varphi} \otimes_{R_V} R_V^{h',\varphi}$ in $\mathrm{Spec}\, R_V^{\square}$ is formally smooth and 5-dimensional. □

(1.2) Let $\mathbb{F}$ be a finite field of characteristic p and $V_{\mathbb{F}}$ a two dimensional $\mathbb{F}$-vector space equipped with a continuous action of $G_{\mathbb{Q}_p}$. We fix an $\mathbb{F}$-basis of $V_{\mathbb{F}}$.

Let $\mathfrak{AR}_{W(\mathbb{F})}$ denote the category of local Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$. We denote by $R^{\square}_{V_{\mathbb{F}}}$ the universal framed deformation ring of $V_{\mathbb{F}}$. That is, $R^{\square}_{V_{\mathbb{F}}}$ is the complete local $W(\mathbb{F})$-algebra which prorepresents the functor assigning to A in $\mathfrak{AR}_{W(\mathbb{F})}$ the set of isomorphism classes of pairs (V_A, β), where V_A is a deformation of the $G_{\mathbb{Q}_p}$-representation $V_{\mathbb{F}}$ to A, and β is an A-basis of V_A lifting the chosen $\mathbb{F}$-basis of $V_{\mathbb{F}}$. We set $Z = \mathrm{Spec}\, R^{\square}_{V_{\mathbb{F}}}[1/p]$.

If $E/W(\mathbb{F})[1/p]$ is a finite extension and $x : R^{\square}_{V_{\mathbb{F}}} \to E$ an E-valued point, then x gives rise to a two dimensional E-representation of $G_{\mathbb{Q}_p}$, equipped with an E-basis. Let $\hat{R}^{\square}_{V_{\mathbb{F}},x}$ denote the completion of $R^{\square}_{V_{\mathbb{F}}}[1/p]$ at the maximal ideal generated by the kernel of x. Then $\hat{R}^{\square}_{V_{\mathbb{F}},x} \otimes_{\kappa(x)} E$ is canonically isomorphic to $R^{\square}_{V_x}$, the framed deformation ring of V_x [Ki 2, 2.3.5]. Here $\kappa(x)$ denotes the residue field of x.

We call x benign if V_x is benign. We say that $(x,\lambda) \in (Z \times \mathbb{G}_m)(E)$ is benign if (V_x, λ) is a benign pair.

Let S denote the universal deformation ring of $\det V_{\mathbb{F}}$, thought of as a representation of the inertia subgroup of the maximal abelian quotient of $G_{\mathbb{Q}_p}$. Then S

is formally smooth over $W(\mathbb{F})$ of relative dimension 1 if $p > 2$ and isomorphic to $W(\mathbb{F})[\![Y, Z]\!]/((1+Z)^2 - 1)$ if $p = 2$. We have an obvious map

$$\operatorname{Spec} R^{\square}_{V_{\mathbb{F}}} \overset{V_A \mapsto \det V_A|_{I_{\mathbb{Q}_p}}}{\longrightarrow} \operatorname{Spec} S.$$

.

In the following we will use the construction of the p-adic analytic space attached to $\operatorname{Spec} R[1/p]$ where R is a complete local $W(\mathbb{F})$-algebra with finite residue field [deJ, §7]. In particular, we write $\mathcal{W} = \operatorname{Spec} S[1/p]$ and we denote by $\mathcal{W}^{\mathrm{an}}$ the associated p-adic analytic space.[4]

Proposition (1.2.1). *There exists a reduced, Zariski closed, analytic subspace $X_{fs} \subset (Z \times \mathbb{G}_m)^{\mathrm{an}}$ with the following properties.*

(1) *If $E/W(\mathbb{F})[1/p]$ is a finite extension and $(x, \lambda) \in X_{fs}(E)$, then there exists a non-zero, E-linear, $G_{\mathbb{Q}_p}$-equivariant map*

$$V_x \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi=\lambda}.$$

(2) *If $(x, \lambda) \in (Z \times \mathbb{G}_m)(E)$ is benign then $(x, \lambda) \in X_{fs}(E)$.*

(3) *If (x, λ) is benign and its image in X_{fs} has residue field E, then the complete local ring $\hat{\mathcal{O}}_{X_{fs},(x,\lambda)}$ at (the image of) (x, λ) satisfies*

$$\hat{\mathcal{O}}_{X_{fs},(x,\lambda)} \xrightarrow{\sim} R^{\square}_{V_x} \otimes_{R_{V_x}} R^{h,\varphi}_{V_x}.$$

In particular $\hat{\mathcal{O}}_{X_{fs},(x,\lambda)}$ is formally smooth and the composite

$$X_{fs} \to Z^{\mathrm{an}} \to \mathcal{W}^{\mathrm{an}}$$

is flat at (x, λ).

(4) *If $(x, \lambda) \in X_{fs}$ then V_x has (at least) one Hodge-Tate weight equal to* 0.

Proof. The space X_{fs} is constructed in [Ki 1, 10.3]. It satisfies (1) (2) and (4) by [Ki 1, 10.4] and (3) by [Ki 1, 10.6]. The final claim in (3) follows from (1.1.1). More precisely the results of [Ki 1] apply with the versal deformation ring $R^{\mathrm{ver}}_{V_{\mathbb{F}}}$ in place of $R^{\square}_{V_{\mathbb{F}}}$, however the construction goes over verbatim. Alternatively, one can deduce the results for $R^{\square}_{V_{\mathbb{F}}}$ from the analogue for $R^{\mathrm{ver}}_{V_{\mathbb{F}}}$ by choosing a morphism $R^{\mathrm{ver}}_{V_{\mathbb{F}}} \to R^{\square}_{V_{\mathbb{F}}}$ which induces the universal deformation over $R^{\square}_{V_{\mathbb{F}}}$. $\square$

(1.2.2) In fact using the results of Colmez [Co 1] one can determine the local structure of X_{fs} at essentially all points, and not just at potentially semi-stable points as was done in [Ki 1]. This has been carried out by Bellaiche-Chenevier [BC, §2.3].

Corollary (1.2.3). *If $E/\mathbb{Q}_p$ is finite, $(x, \lambda) \in X_{fs}(E)$ and V_x has Hodge-Tate weights $0, k$ with k a positive integer satisfying $v_p(\lambda) < k$, then V_x is potentially semi-stable.*

Proof. We first remark that since $v_p(\lambda) < k$, $\mathrm{Fil}^k(B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi=\lambda} = 0$. To see this, suppose that $s \in \mathrm{Fil}^k(B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi=\lambda}$ and let r be a positive integer such that

[4] $\mathcal{W}$ and $\mathcal{W}^{\mathrm{an}}$ are what is usually referred to as "weight space".

$m = rv_p(\lambda)$ is an integer. Then $\varphi(s^r) = \lambda^r s^r = p^m w s^r$ for $w \in \mathcal{O}_E^\times$. Choosing $u \in (W(\bar{\mathbb{F}}_p) \otimes E)^\times$ (here $\bar{\mathbb{F}}_p$ is the residue field of $\bar{\mathbb{Q}}_p$) such that $\varphi(u) = wu$ we see that $\varphi(s^r u^{-1}) = p^m s^r u^{-1}$ so

$$s^r u^{-1} \in \mathrm{Fil}^{rk}(B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi = p^m} = \mathrm{Fil}^{rk} B^{+,\varphi=p^m}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E = 0$$

since $m = rv_p(\lambda) < rk$.

It follows that the non-zero $G_{\mathbb{Q}_p}$-equivariant map

$$V_x \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi=\lambda}$$

given by (1.2.1)(1) cannot factor through $\mathrm{Fil}^k B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E$. Since V_x has a Hodge-Tate weight equal to k there exists a non-zero E-linear $G_{\mathbb{Q}_p}$-equivariant map $V_x \to \mathbb{C}_p(k) \otimes E$. As the other Hodge-Tate weight of V_x is $0 < k$, and $H^1(G_{\mathbb{Q}_p}, \mathbb{C}_p(i)) = 0$ for $i > 0$ by a result of Tate, this map necessarily lifts to $\mathrm{Fil}^k B_{\mathrm{dR}} \otimes E$ (cf. [Ki 1, 2.5]). Hence V_x is de Rham, and therefore potentially semi-stable by Berger's theorem [Be 1, Thm. 5.19]. □

Proposition (1.2.4). *Let $(x, \lambda) \in X_{fs}(E)$ be a benign point. Then there exists a quasi-compact admissible open subset $U \subset X_{fs}$ containing (x, λ), such that if $(x', \lambda') \in U$ is a closed point whose image in $\mathcal{W}^{\mathrm{an}}$ is χ^k_{cyc} with k a positive integer, then (x', λ') is benign.*

Proof. Choose $U = \mathrm{Sp}\,\mathcal{R}$ an affinoid neighbourhood of (x, λ) such that if $(x', \lambda') \in U$, then $V_{x'}$ satisfies (1.1)(1), and if $(x', \lambda') \neq (x, \lambda)$ and has image χ^k_{cyc} in $\mathcal{W}^{\mathrm{an}}$ with k a positive integer, then $k > 2v_p(\lambda') + 1$.

Then $V_{x'}$ is potentially semistable by (1.2.3). The condition (1.2.1)(1) implies that the associated Weil group representation is reducible and of the form $\chi_1 \oplus \chi_2$ with χ_1 an unramified character. Since $\det V_{x'}|_{I_{\mathbb{Q}_p}} = \chi^k_{\mathrm{cyc}}$, $\det V_{x'}$ is crystalline so $\chi_1\chi_2$ is unramified, and χ_2 is unramified. It follows that $V_{x'}$ is semi-stable. The inequality $k > 2v_p(\lambda') + 1$ implies that $V_{x'}$ is crystalline and that $(V_{x'}, \lambda')$ satisfies (1.1)(2). Hence (x', λ') is benign. □

Corollary (1.2.5). *Let $Y \subset X_{fs}$ be the smallest Zariski closed analytic subspace which contains all benign points $(x, \lambda) \in X_{fs}$. Then Y is a union of irreducible components of X_{fs}.*

Proof. Let $(x, \lambda) \in X_{fs}(E)$ be a benign point. We have to show that the irreducible component of X_{fs} passing through (x, λ) is contained in Y. Note that this irreducible component is unique by (1.2.1)(3).

Let U be an open admissible subspace of X_{fs} containing (x, λ) and satisfying the conclusion of (1.2.4). We may assume that U is connected and smooth. It follows by (1.2.6) below, applied with $T = Y \cap U$ and $I \subset \mathcal{W}^{\mathrm{an}}$ the set of points of the form χ^k_{cyc} with k a positive integer, that $U \subset Y$. Hence Y contains the irreducible component of X_{fs} passing through (x, λ). □

Lemma (1.2.6). *Let U be a quasi-compact, irreducible, reduced rigid space over $\mathbb{Q}_p$, and $\pi : U \to \mathcal{W}^{\mathrm{an}}$ a flat morphism. Let $T \subset U$ be a Zariski closed subspace and $I \subset \pi(U)$ an infinite set of points such that for $x \in I$, $\pi^{-1}(x) \subset T$. Then $T = U$.*

Proof. Since T is quasi-compact the set of points of $\mathcal{W}^{\mathrm{an}}$ at which $\pi|_T$ is not flat is finite. Hence there exists $x \in I$ such that T is flat over $\mathcal{W}^{\mathrm{an}}$ at x, and

$$\dim T \geq \dim \pi^{-1}(x) + 1 = \dim U.$$

It follows that $\dim T = \dim U$, and hence $T = U$ as U is reduced and irreducible. $\square$

(1.3) Let Γ denote the maximal pro-p quotient of $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$. Then Γ is canonically a subgroup of $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ and we may, for example, regard χ_{cyc} as a character of Γ.

We denote by S_0 the universal deformation ring of the trivial representation $\mathbb{F}$ of Γ, and we write $\mathcal{W}_0 = \mathrm{Spec}\, S_0[1/p]$. We will denote by $\eta_0 \in \mathcal{W}_0$ the point corresponding to the trivial character of Γ, and by $\hat{\mathcal{O}}_{\mathcal{W}_0,\eta_0}$ the complete local ring at this point.

We call a closed point $x \in R^{\square}_{V_{\mathbb{F}}}[1/p]$ *twisted benign* if $V_x \xrightarrow{\sim} V \otimes \eta$ where V is benign and $\eta = \chi^k_{\mathrm{cyc}}$ with k an integer.

Lemma (1.3.1). *If (V, λ) is a benign pair, and $h \in D_{\mathrm{cris}}(V^*)^{\varphi=\lambda}$ and $h' \in D_{\mathrm{cris}}(V^*)^{\varphi=\lambda'}$ are non-zero, then*

(1) *The map*

$$R_V \to R^{h,\varphi}_V \widehat{\otimes}_{W(\mathbb{F})[1/p]} \hat{\mathcal{O}}_{\mathcal{W}_0,\eta_0} \tag{1.3.2}$$

which for $B \in \mathfrak{AR}_E$ induces the map $(V_B, \eta) \mapsto V_B \otimes \eta$ on B-points, is surjective.

(2) *If $\tilde{R}^{h,\varphi}_V$ denotes the image of (1.3.2), then $\tilde{R}^{h,\varphi}_V$ is formally smooth over E of dimension 4. The subspaces of $\mathrm{Spec}\, R_V$ corresponding to $\tilde{R}^{h,\varphi}_V$ and $\tilde{R}^{h',\varphi}_V$ are distinct. More precisely their intersection is formally smooth of dimension 3.*

Proof. To prove (1) is enough to show that for any B in $\mathfrak{AR}_E$ (1.3.2) induces an injection on B-points. To do this we use the theory of the Sen polynomial [Ki 1, §2.2], which attaches a monic polynomial $P_\phi(T) \in B[T]$ to any continuous representation of $G_{\mathbb{Q}_p}$ on a finite free B-module M. Let (V_B, η) and (V'_B, η') be two B-points of the right hand side of (1.3.2) with $V_B \otimes \eta \xrightarrow{\sim} V'_B \otimes \eta'$. We may assume that $\eta = \eta_0$. The Sen polynomials of V_B and V'_B have the form $T(T-k+b)$ and $T(T-k+b')$ respectively where b, b' are in the maximal ideal of B. Fix a topological generator $\gamma \in \Gamma$ (or a topological generator of a pro-cyclic open subgroup of Γ if $p = 2$), and let $a = \log\eta'(\gamma)/\log\chi_{\mathrm{cyc}}(\gamma)$. Then the Sen polynomial of $V'_B \otimes \eta'$ is $(T+a)(T-k+b'+a)$. Since $k \neq 0$, and $a, b, b' \in \mathrm{rad}(B)$ we must have $a = 0$. It follows that η is trivial and $V_B \xrightarrow{\sim} V'_B$.

The first claim of (2) now follows from (1.1.1). Moreover the argument in the previous paragraph shows that

$$\tilde{R}^{h,\varphi}_V \otimes_{R_V} \tilde{R}^{h',\varphi}_V \xrightarrow{\sim} R^{h,\varphi}_V \otimes_{R_V} R^{h',\varphi}_V \otimes_{W(\mathbb{F})[1/p]} \hat{\mathcal{O}}_{\mathcal{W}_0,\eta_0}$$

so (2) follows from (1.1.2). $\square$

Proposition (1.3.3). *Let $Y \subset \mathrm{Spec}\, R^{\square}_{V_{\mathbb{F}}}[1/p]$ denote the Zariski closure of the twisted benign points of $R^{\square}_{V_{\mathbb{F}}}$. If $Y \neq \emptyset$, then $\dim Y \geq 8$ at every point of Y.*

Proof. After replacing $V_{\mathbb{F}}$ by a twist, we may assume that Y contains a benign point. Let $\tilde{Y}$ be the preimage of Y under

$$X_{fs} \times \mathcal{W}^{\mathrm{an}}_0 \to Z^{\mathrm{an}}; \quad (x, \lambda, \eta) \mapsto V_x \otimes \eta.$$

Then $\tilde{Y}$ contains all the irreducible components of $X_{fs}\times\mathcal{W}_0^{\mathrm{an}}$ which contain a point of the form (x,λ,η) with (x,λ) a benign pair, by (1.2.5). Combining (1.3.1) and (1.2.1)(3) one sees that for any benign point $x\in Z$, $\mathrm{Spf}\,\hat{\mathcal{O}}_{Y,x}$ contains two distinct, 7-dimensional, formally smooth subspaces of $\mathrm{Spf}\,\hat{\mathcal{O}}_{Z,x}$, namely the preimages in $\mathrm{Spf}\,R^{\square}_{V_x}\xrightarrow{\sim}\mathrm{Spf}\,\hat{\mathcal{O}}_{Z,x}$ of the subspace $\mathrm{Spf}\,\tilde{R}^{h,\varphi}_{V_x}$, $\mathrm{Spf}\,\tilde{R}^{h',\varphi}_{V_x}$ of $\mathrm{Spf}\,R_{V_x}$ introduced in (1.3.1).

Hence any irreducible component of Y has dimension at least 7, and any point at which Y has dimension 7 is a singular point. Let $Y'\subset Y$ be the union of the irreducible components of dimension 7. Then any benign point in Y' is a singular point of Y. Hence, any twisted benign point of Y' is also a singular point, since if $V_{\mathbb{F}}\otimes\omega^k\sim V_{\mathbb{F}}$ then twisting by χ^k_{cyc} induces an automorphism of Y'. It follows that the singular locus of Y' is equal to Y', which is impossible if Y' is non-empty. Hence Y has dimension ≥ 8 at every point. $\square$

Corollary (1.3.4). *The closure of the twisted benign points in* $\mathrm{Spec}\,R^{\square}_{V_{\mathbb{F}}}[1/p]$ *is non-empty and a union of irreducible components. In particular, if* $R^{\square}_{V_{\mathbb{F}}}[1/p]$ *is irreducible then the set of twisted benign points in* $\mathrm{Spec}\,R^{\square}_{V_{\mathbb{F}}}[1/p]$ *is dense.*

Proof. A standard obstruction argument (cf. [Maz, Prop 2, §1.6]) shows that $R^{\square}_{V_{\mathbb{F}}}$ is a quotient of a power series ring over $W(\mathbb{F})$ in g generators by at most r relations, where $g-r=8$. It follows that all the irreducible components of $R^{\square}_{V_{\mathbb{F}}}[1/p]$ are at least 8-dimensional [Ma, Thm. 13.5]. On the other hand, one easily sees that the reducible locus in $\mathrm{Spec}\,R^{\square}_{V_{\mathbb{F}}}[1/p]$ is at most 7-dimensional. This shows that the irreducible locus in $\mathrm{Spec}\,R^{\square}_{V_{\mathbb{F}}}[1/p]$ is a dense subspace. The deformation theoretic description of a complete local ring at a closed point x of $\mathrm{Spec}\,R^{\square}_{V_{\mathbb{F}}}[1/p]$ such that V_x is irreducible shows that this scheme is formally smooth and 8-dimensional at such a point. It follows that all components of $R^{\square}_{V_{\mathbb{F}}}[1/p]$ are 8-dimensional. In particular, it follows from (1.3.3) that the closure in $\mathrm{Spec}\,R^{\square}_{V_{\mathbb{F}}}[1/p]$ of the twisted benign points is a union of irreducible components. It remains to show that this closure is non-empty.

The explicit description of reductions of crystalline representations of small weight shows that, after replacing $V_{\mathbb{F}}$ by $V_{\mathbb{F}}\otimes\omega^s$, for some integer s, there exists a closed point x on $\mathrm{Spec}\,R^{\square}_{V_{\mathbb{F}}}[1/p]$ such that V_x is crystalline and satisfies (1) and (3) of (1.1).[5]

By [Ki 3, 2.5.5], the set of closed points $x'\in\mathrm{Spec}\,R^{\square}_{V_{\mathbb{F}}}[1/p]$ such that $V_{x'}$ is crystalline with the same Hodge-Tate weights as V_x, is parameterized by a Zariski closed subspace $Z^{\mathrm{cr}}\subset Z$. Moreover, Z^{cr} is equipped with a vector bundle $D_{Z^{\mathrm{cr}}}$, together with an automorphism φ of $D_{Z^{\mathrm{cr}}}$, which realizes the weakly admissible module attached to $V_{x'}$ at every closed point $x'\in Z^{\mathrm{cr}}$. It follows easily from [Ki 3, 3.3.1] that there is a non-empty open subset $U\subset Z^{\mathrm{cr}}$ such that $V_{x'}$ satisfies (1)-(3) for $x'\in U$. $\square$

(1.3.5) We will write $\chi: G_{\mathbb{Q}_p}\to\mathbb{F}^{\times}$ for an arbitrary character.

Corollary (1.3.6). *Let* $p>2$. *Suppose that* $V_{\mathbb{F}}$ *is not of the form* $\begin{pmatrix}1 & *\\ 0 & \omega\end{pmatrix}\otimes\chi$, *and if* $p=3$ *assume also that* $V_{\mathbb{F}}|_{I_{\mathbb{Q}_p}}$ *is not of the form* $\begin{pmatrix}\omega_2^2 & 0\\ 0 & \omega_2^6\end{pmatrix}\otimes\chi$. *Then the set of twisted benign points is dense in* $\mathrm{Spec}\,R^{\square}_{V_{\mathbb{F}}}[1/p]$.

[5] In fact one can show that such an x exists without twisting, but the argument is more difficult.

Proof. Under the conditions of the corollary $H^2(G_{\mathbb{Q}_p}, \mathrm{ad}_{\mathbb{F}} V_{\mathbb{F}}) = 0$, so $R^{\square}_{V_{\mathbb{F}}}$ is smooth, and $\operatorname{Spec} R^{\square}_{V_{\mathbb{F}}}[1/p]$ is irreducible. The corollary follows from (1.3.4). □

(1.3.7) The irreducibility in (1.3.4) should always hold. The author knows how to prove this when $V_{\mathbb{F}} \sim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$, and the extension class corresponding to $*$ is non-trivial. The proof requires some auxiliary constructions and would take us too far afield here.

Now let $\mathcal{O}$ be the ring of integers in a finite extension of $W(\mathbb{F})[1/p]$ and fix a continuous character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ which lifts $\det V_{\mathbb{F}}$, and such that $\psi|_{I_{\mathbb{Q}_p}}$ is an integral power of the cyclotomic character. We set $R^{\square}_{V_{\mathbb{F}},\mathcal{O}} = R^{\square}_{V_{\mathbb{F}}} \otimes_{W(\mathbb{F})} \mathcal{O}$ and we denote by $R^{\psi,\square}_{V_{\mathbb{F}},\mathcal{O}}$ the quotient of $R^{\square}_{V_{\mathbb{F}},\mathcal{O}}$ corresponding to deformations with determinant ψ.

Corollary (1.3.8). *The closure of the twisted benign points in* $\operatorname{Spec} R^{\psi,\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p]$ *is a union of irreducible components. .*

Proof. Let T_0 denote the universal deformation of the trivial $\mathbb{F}$-representation of $G_{\mathbb{Q}_p}$. Then T_0 is a power series ring over $W(\mathbb{F})$ in two variables unless $p = 2$ in which case $T_0 \xrightarrow{\sim} W(\mathbb{F})[\![X, Y, Z]\!]/((1 + Z)^2 - 1)$. We have a natural map

$$R^{\square}_{V_{\mathbb{F}},\mathcal{O}} \to R^{\psi,\square}_{V_{\mathbb{F}},\mathcal{O}} \widehat{\otimes}_{W(\mathbb{F})} T_0 \tag{1.3.9}$$

which for an Artinian $\mathcal{O}$-algebra A with residue field $\mathbb{F}$ induces the map $(V, \eta) \mapsto V \otimes \eta$ on A-valued points. One checks easily that (1.3.9) is a finite map and a surjection if $p > 2$. The induced map

$$\operatorname{Spec} R^{\psi,\square}_{V_{\mathbb{F}},\mathcal{O}} \times \operatorname{Spec} T_0[1/p] \to \operatorname{Spec} R^{\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p]; \quad (V_x, \eta) \mapsto V_x \otimes \eta. \tag{1.3.10}$$

is finite étale, an immersion if $p > 2$, and a covering of its image with group $\operatorname{Hom}(G_{\mathbb{Q}_p}, \mathbb{Z}/2\mathbb{Z})$ if $p = 2$.

Let $Y_\psi \subset \operatorname{Spec} R^{\psi,\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p]$ denote the closure of the twisted benign points. If $x \in \operatorname{Spec} R^{\square}_{V_{\mathbb{F}},\mathcal{O}}$ is a twisted benign point, such that $\det V_x \cdot \psi^{-1}|_{I_{\mathbb{Q}_p}} \sim \chi_{\mathrm{cyc}}^{j(p-1)}$ with j an even integer, then x is in the image of

$$Y_\psi \times \operatorname{Spec} T_0[1/p] \to \operatorname{Spec} R^{\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p]; \quad (V_x, \eta) \mapsto V_x \otimes \eta. \tag{1.3.11}$$

If $p > 2$, then by (1.2.1)(3) any twisted benign point can be approximated (even in the naive p-adic topology on the $\bar{\mathbb{Q}}_p$-points of $\operatorname{Spec} R^{\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p]$) by twisted benign points with this property. Hence the image of (1.3.11) is a union of irreducible components of $\operatorname{Spec} R^{\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p]$ by (1.3.4), so Y_ψ is a union of components of $\operatorname{Spec} R^{\psi,\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p]$.

If $p = 2$, then the image of $I_{\mathbb{Q}_2} \to G^{\mathrm{ab}}_{\mathbb{Q}_2}$ is isomorphic to $\mathbb{Z}_2^\times$, and the character $\det V_x \cdot \psi^{-1}|_{\pm 1}$ is locally constant on $\operatorname{Spec} R^{\square}_{V_{\mathbb{F}},\mathcal{O}}$. In particular if a twisted benign point x satisfies $\det V_x \cdot \psi^{-1}|_{I_{\mathbb{Q}_2}} \sim \chi^j_{\mathrm{cyc}}$ with j an even integer, then every twisted benign point in the same component of x has this property. This shows that the image of (1.3.11) is a union of irreducible components in this case also, which implies the corollary, as before. □

Corollary (1.3.12). *Let $p > 2$. Suppose $V_{\mathbb{F}} \not\sim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix} \otimes \chi$ and $V_{\mathbb{F}}|_{I_{\mathbb{Q}_p}} \not\sim \begin{pmatrix} \omega_2^2 & 0 \\ 0 & \omega_2^6 \end{pmatrix} \otimes \chi$ if $p = 3$. Then the set of twisted benign points in $\operatorname{Spec} R^{\psi,\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p]$ is dense.*

Proof. Under the conditions of the corollary, both sides of (1.3.9) are formally smooth of the same dimension. Hence this map is an isomorphism and the corollary follows from (1.3.8) $\square$

§2 Deformation theory

(2.1) Let $E/\mathbb{Q}_p$ be a finite extension, as in the previous section, $\mathcal{O}$ the ring of integers of E, and $\mathbb{F}$ its residue field. As above, we denote by $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ the p-adic cyclotomic character. We will regard characters of $G_{\mathbb{Q}_p}$ as character of $\mathbb{Q}_p^\times$ via class field theory, normalized to take uniformisers to geometric Frobenii.

We set $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and we denote[6] by $\mathrm{Rep}_{\mathcal{O}} G$ the category of representations of G on torsion $\mathcal{O}$-modules, which are smooth, admissible, finite length and admit an $\mathcal{O}^\times$-valued central character. Recall that admissibility means that the invariants under any compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ are a finite $\mathcal{O}$-module. An $\mathcal{O}$-representation which admits a central character satisfies these conditions if and only if it is a successive extension of a finite number of smooth, irreducible $\mathbb{F}$-representations of G. This may be seen using the classification of smooth irreducible $\mathbb{F}$-representations, admitting a central character, due to Bartel-Livné and Breuil, from which one easily deduces that any such representation is admissible.

We denote by $\mathrm{Rep}_{\mathcal{O}} G_{\mathbb{Q}_p}$ the category of representations of $G_{\mathbb{Q}_p}$ on finite length $\mathcal{O}$-modules. The following summarizes some of the main results of [Co 2].

Theorem (2.1.1). *(Colmez) There is an exact functor*

$$\mathbf{V} : \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Rep}_{\mathcal{O}} G_{\mathbb{Q}_p}$$

with the following properties:

If $\pi \in \mathrm{Rep}_{\mathcal{O}} G$ then $\mathbf{V}(\pi) = 0$ if and only if π is a finite $\mathcal{O}$-module.

If $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ is a continuous character then

$$\mathbf{V}(\Pi \otimes \psi \circ \det) \xrightarrow{\sim} \mathbf{V}(\Pi) \otimes \psi.$$

If $V_{\mathbb{F}}$ is a two dimensional $\mathbb{F}$-representation in $\mathrm{Rep}_{\mathcal{O}} G_{\mathbb{Q}_p}$ with $V_{\mathbb{F}} \not\sim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix} \otimes \chi$ then there exists $\bar{\pi}$ in $\mathrm{Rep}_{\mathcal{O}} G$, with the following properties.

1. $\mathbf{V}(\bar{\pi}) \xrightarrow{\sim} V_{\mathbb{F}}$ *and $\bar{\pi}$ has central character $(\det V_{\mathbb{F}})\chi_{\mathrm{cyc}}^{-1}$.*
2. *The natural map*

$$\mathrm{Ext}^1_G(\bar{\pi}, \bar{\pi}) \to \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(V_{\mathbb{F}}, V_{\mathbb{F}})$$

is an injection. Here the left hand side means extensions in the category of $\mathbb{F}$-representations having a ($\mathbb{F}^\times$-valued) central character.

3. $\bar{\pi}^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0.$

Proof. The first claim is [Co 2, Thm. IV.2.14], while the second can be seen directly from the definition of $\mathbf{V}$. The properties in (1) follow from [Co 2, §VII.4], while (2) is [Co 2, Thm. VII.5.2]. $\square$

[6] In the final version of [Co 2] this category is denoted $\mathrm{Rep}_{\mathrm{tors}} G$, while $\mathrm{Rep}_{\mathcal{O}} G$ denotes the category of admissible $\mathcal{O}$-lattices - see (2.3) below. Here we retain the notation of earlier versions of [Co 2].

Lemma (2.1.2). *Let $V_{\mathbb{F}}$ and $\bar{\pi}$ be as above. The canonical map*

$$\mathrm{Hom}_G(\bar{\pi}, \bar{\pi}) \xrightarrow{\sim} \mathrm{Hom}_{G_{\mathbb{Q}_p}}(V_{\mathbb{F}}, V_{\mathbb{F}}) \tag{2.1.3}$$

is an isomorphism.

Proof. To see that the map is injective suppose that $f : \bar{\pi} \to \bar{\pi}$ satisfies $\mathbf{V}(f) = 0$. Then $\mathbf{V}(\mathrm{Im}(f)) = 0$, which implies that $\mathrm{Im}\,(f)$ is finite dimensional. Hence $\mathrm{Im}\,(f) \subset \bar{\pi}^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$, so $f = 0$.

For the surjectivity there is nothing to show if $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V_{\mathbb{F}}, V_{\mathbb{F}}) = \mathbb{F}$. If $V_{\mathbb{F}}$ is a sum of two distinct (resp. equal) characters then $\bar{\pi}$ is a sum of two distinct (resp. equal) representations [Co 2, §VII.4], so in this case both sides of (2.1.3) are two dimensional. Similarly if $V_{\mathbb{F}}$ is a self extension of a character, then $\bar{\pi}$ is a self extension of an G-representation, and $\bar{\pi}$ is a split extension if and only if $V_{\mathbb{F}}$ is a split extension. $\square$

(2.2) Now fix $V_{\mathbb{F}}$ and $\bar{\pi}$ as in (2.1.1). We will denote by $\mathfrak{AR}_{\mathcal{O}}$ the category of finite local Artinian $\mathcal{O}$-algebras. Let A be in $\mathfrak{AR}_{\mathcal{O}}$. A *deformation* of $\bar{\pi}$ to A is a representation of G on a flat A-module π_A having an $\mathcal{O}$-valued central character, together with an $\mathbb{F}$-linear isomorphism of G-modules $\pi_A \otimes_A \mathbb{F} \xrightarrow{\sim} \bar{\pi}$.

Lemma (2.2.1). *Let π_A be a deformation of $\bar{\pi}$ to A. Then*

1. *π_A is a free A-module.*
2. *As an $\mathcal{O}$-representation of G, π_A is in $\mathrm{Rep}_{\mathcal{O}}G$.*

Proof. For (1), pick a (countable) basis for $\bar{\pi}$ indexed by a set I, and lift it to π_A. Let K be the kernel of $A^I \to \pi_A$, and C the cokernel. If $\mathfrak{m}_A$ denotes the radical of A, then we have $C/\mathfrak{m}_A C = 0$, and since π_A is A-flat, $K/\mathfrak{m}_A K = 0$. Since $\mathfrak{m}_A$ is nilpotent this implies $K = C = 0$.

In particular, (1) shows that π_A is a successive extension of copies of $\bar{\pi}$ which shows (2). $\square$

Lemma (2.2.2). *Let $A \to A'$ be a morphism in $\mathfrak{AR}_{\mathcal{O}}$ and π_A a deformation of $\bar{\pi}$ to A. Then $\mathbf{V}(\pi_A)$ is a flat A-module and there is a canonical isomorphism*

$$\mathbf{V}(\pi_A) \otimes_A A' \xrightarrow{\sim} \mathbf{V}(\pi_A \otimes_A A').$$

In particular, $\mathbf{V}(\pi_A) \otimes_A \mathbb{F} \xrightarrow{\sim} \mathbf{V}(\bar{\pi}) \xrightarrow{\sim} V_{\mathbb{F}}$, and $\mathbf{V}(\pi_A)$ is a finite free A-module of rank 2.

Proof. This is a formal consequence of the exactness of the functor $\mathbf{V}$ (cf. [Ki 2, 1.2.7]). $\square$

(2.2.3) Fix a character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$, such that (the reduction of) ψ is equal to the central character of $\bar{\pi}$.

We define two groupoids over $\mathfrak{AR}_{\mathcal{O}}$. (See the appendix in [Ki 2] for basic notions involving groupoids). Denote by $D_{V_{\mathbb{F}}}$ the groupoid such that $D_{V_{\mathbb{F}}}(A)$ is the category of deformations of $V_{\mathbb{F}}$ to a representation of $G_{\mathbb{Q}_p}$ on a finite free A-module. Let $D_{\bar{\pi},\psi}$ be the groupoid such that $D_{\bar{\pi},\psi}(A)$ is the category of deformations of $\bar{\pi}$ to A with central character ψ.

Lemma (2.2.4). *The functor* $\mathbf{V}$ *induces a morphism of groupoids*

$$D(\mathbf{V}) : D_{\bar{\pi},\psi} \to D_{V_{\mathbb{F}}}; \quad \pi_A \mapsto \mathbf{V}(\pi_A)$$

which is fully faithful.

Proof. That $\mathbf{V}$ induces a morphism of groupoids follows from (2.2.2).

To show that $D(\mathbf{V})$ is fully faithful, let A be in $\mathfrak{AR}_{\mathcal{O}}$ with radical $\mathfrak{m}_A$. Let π_A and π'_A be in $D_{\bar{\pi},\psi}(A)$. First we remark that π_A and π'_A are isomorphic if $\mathbf{V}(\pi_A)$ and $\mathbf{V}(\pi'_A)$ are isomorphic. To see this, we proceed by induction on the length of A. When $A = \mathbb{F}$, there is nothing to prove. Let $I \subset A$ be a non-zero ideal with $\mathfrak{m}_A \cdot I = 0$. By induction, we may assume that $\pi_{A/I} = \pi_A \otimes_A A/I$ and $\pi'_{A/I} = \pi'_A \otimes_A A/I$ are isomorphic. A standard argument shows that the set of isomorphism classes of objects of $D_{\bar{\pi}}(A)$ (resp. $D_{V_{\mathbb{F}}}(A)$) lifting $\pi_{A/I}$ (resp. $\mathbf{V}(\pi_{A/I})$) is a torsor under $\mathrm{Ext}^1_G(\bar{\pi},\bar{\pi})\otimes I$ (resp. $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{V}(\bar{\pi}),\mathbf{V}(\bar{\pi}))\otimes I$). Hence the induction step follows from (2.1.1)(2).

It follows that to prove full faithfulness it suffices to show that the map

$$\mathrm{Hom}_G(\pi_A, \pi_A) \to \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathbf{V}(\pi_A), \mathbf{V}(\pi_A))$$

is an isomorphism. Again, we prove this by induction on the length of A. When $A = \mathbb{F}$ this is (2.1.2). In general let I be as above. Then the set of endomorphisms in $\mathrm{Hom}_G(\pi_A, \pi_A)$ (resp. $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathbf{V}(\pi_A), \mathbf{V}(\pi_A))$) lifting a fixed endomorphism of $\pi_{A/I}$ (resp. $\mathbf{V}(\pi_{A/I})$) is a torsor under $\mathrm{Hom}_G(\bar{\pi},\bar{\pi})\otimes I$ (resp. $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V_{\mathbb{F}}, V_{\mathbb{F}})\otimes I$). Hence the induction step follows from (2.1.2). $\square$

(2.2.5) Let A be in $\mathfrak{AR}_{\mathcal{O}}$ and V_A in $D_{V_{\mathbb{F}}}(A)$. We will denote by ξ the groupoid on $\mathfrak{AR}_{\mathcal{O}}$ corresponding to V_A. An object of ξ consists of an A-algebra A' in $\mathfrak{AR}_{\mathcal{O}}$ and an object $V_{A'}$ of $D_{V_{\mathbb{F}}}(A')$ together with an isomorphism $V_{A'} \xrightarrow{\sim} V_A \otimes_A A'$ in $D_{V_{\mathbb{F}}}(A')$.

Set

$$D_{\bar{\pi},\xi} = D_{\bar{\pi},\psi} \times_{D_{V_{\mathbb{F}}}} \xi.$$

An object of $D_{\bar{\pi},\xi}$ consists of an A-algebra A' in $\mathfrak{AR}_{\mathcal{O}}$ together with a pair $(\pi_{A'}, \iota_{A'})$, where $\pi_{A'}$ is in $D_{\bar{\pi},\psi}(A')$ and $\iota_{A'}$ is an isomorphism $\iota_{A'} : \mathbf{V}(\pi_{A'}) \xrightarrow{\sim} V_A \otimes_A A'$ in $D_{V_{\mathbb{F}}}(A')$.

For any groupoid D over $\mathfrak{AR}_{\mathcal{O}}$ we denote by $|D|(A)$ the set of isomorphism classes of $D(A)$.

Lemma (2.2.6). *The morphism* $D(\mathbf{V}) : D_{\bar{\pi},\psi} \to D_{V_{\mathbb{F}}}$ *is relatively representable. That is for any* ξ *as above, the functor* $|D_{\bar{\pi},\xi}|$ *is representable.*

Proof. Let B, B', B'' be in $\mathfrak{AR}_{\mathcal{O}}$ equipped with maps $B \to B'$ and $B \to B''$. By Schlessinger's criterion $|D_{\bar{\pi},\xi}|$ is representable if and only if for any such B, B', B'' with $B \to B''$ surjective[7] , the natural map

$$(2.2.7) \qquad |D_{\bar{\pi},\xi}|(B \times_{B''} B') \to |D_{\bar{\pi},\xi}|(B) \times_{|D_{\bar{\pi},\xi}|(B'')} |D_{\bar{\pi},\xi}|(B')$$

is a bijection. Set $C = B \times_{B''} B'$.

[7]In fact Schlessinger's criterion involves an even more restricted class of triples (B, B', B'') but the surjectivity of $B \to B'$ is all we will need.

Since $D_{\bar{\pi},\xi} \to \xi$ is fully faithful, the injectivity of (2.2.7) follows from the corresponding property for $|\xi|$. To check surjectivity, let (π_B, ι_B) (resp. $(\pi_{B'}, \iota_{B'})$) be in $D_{\bar{\pi},\xi}(B)$ (resp. $D_{\bar{\pi},\xi}(B')$) with image in $D_{\bar{\pi},\xi}(B'')$ isomorphic to some object $(\pi_{B''}, \iota_{B''})$. Fix such an isomorphism and set

$$\pi_C = \pi_B \times_{\pi_{B''}} \pi_{B'}.$$

Then π_C is a free C-module. To produce a C-basis, choose a basis for $\pi_{B'}$, take its image in $\pi_{B''}$ and lift the resulting B''-basis to a $\pi_{B'}$. Hence π_C is in $D_{\bar{\pi},\psi}(C)$. Moreover, using the exactness of $\mathbf{V}$, one sees that the maps $\iota_B, \iota_{B'}$ induce an isomorphism

$$\iota_C : \mathbf{V}(\pi_C) \xrightarrow{\sim} \mathbf{V}(\pi_B) \times_{\mathbf{V}(\pi_{B''})} \mathbf{V}(\pi_{B'}) \xrightarrow{\sim} V_A \otimes_A C.$$

This produces the required element in the left hand side of (2.2.7) $\square$

(2.3) By an *admissible $\mathcal{O}$-lattice* we mean a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a p-adically separated and complete, p-torsion free $\mathcal{O}$-module Π, such that Π has an $\mathcal{O}$-valued central character and $\Pi/p^n\Pi$ is a smooth finite length representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. We extend the functor $\mathbf{V}$ to $\mathcal{O}$-lattices by setting

$$\mathbf{V}(\Pi) = \varprojlim_n \mathbf{V}(\Pi/p^n\Pi).$$

We denote by $\pi_{\mathcal{O}}$ a uniformiser for $\mathcal{O}$.

Lemma (2.3.1). *Let $V_{\mathbb{F}}$ and $\bar{\pi}$ be as in (2.1.1). Suppose that V is a deformation of $V_{\mathbb{F}}$ to a representation of $G_{\mathbb{Q}_p}$ on a finite free $\mathcal{O}$-module. Let Π' be an admissible $\mathcal{O}$-lattice such that $\bar{\pi}$ and $\Pi'/\pi_{\mathcal{O}}\Pi'$ have isomorphic semi-simplifications, and suppose that there exists an injective $G_{\mathbb{Q}_p}$-equivariant map $V \to \mathbf{V}(\Pi')$ with finite cokernel.*

Then there is an admissible $\mathcal{O}$-sublattice $\Pi \subset \Pi'$ such that $\mathbf{V}(\Pi) \subset \mathbf{V}(\Pi')$ is identified with V. Moreover $\Pi/\pi_{\mathcal{O}}\Pi$ is isomorphic to $\bar{\pi}$.

Proof. We begin by proving the first claim. Let $\Pi \subset \Pi'$ be a minimal admissible $\mathcal{O}$-sublattice, such that $V \subset \mathbf{V}(\Pi)$. Then $V \subsetneq \pi_{\mathcal{O}}\mathbf{V}(\Pi)$, so the image $W_{\mathbb{F}} \subset \mathbf{V}(\Pi)/\pi_{\mathcal{O}}\mathbf{V}(\Pi)$ of V is non-zero. If this image is two dimensional then $V = \mathbf{V}(\Pi)$ and we are done. Thus we may assume that $W_{\mathbb{F}}$ is 1-dimensional and, in particular, that $V_{\mathbb{F}}$ is reducible.

It will suffice to show that there exists a subrepresentation $\sigma \subset \Pi/\pi_{\mathcal{O}}\Pi$ such that $\mathbf{V}(\sigma)$ is identified with $W_{\mathbb{F}}$, for if $\Sigma \subset \Pi$ denotes the preimage of σ, then $V \subset \mathbf{V}(\Sigma)$ which contradicts the minimality of Π. If $\mathbf{V}(\Pi)/\pi_{\mathcal{O}}\mathbf{V}(\Pi)$ is not a direct sum of two characters then we may take for σ any submodule of $\Pi/\pi_{\mathcal{O}}\Pi$ which has infinite dimension and codimension. Thus we may assume $\mathbf{V}(\Pi)/\pi_{\mathcal{O}}\mathbf{V}(\Pi)$ is a sum of two characters. If $V_{\mathbb{F}} \nsim \left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \chi$, then $\bar{\pi}$ has two Jordan-Hölder factors, and by [Co 2, Prop. VII.4.16] $\Pi/\pi_{\mathcal{O}}\Pi$ is a sum of two infinite dimensional representations, so we are done in this case also.

It remains to consider the case $V_{\mathbb{F}} \sim \left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \chi$, with $*$ a non-trivial extension class,[8] and after twisting, we may assume $\chi = 1$. Then using the notation of [Co 2], the Jordan-Hölder factors of $\bar{\pi}$ consist of one copy of the trivial representation $\mathbf{1}$, and two infinite dimensional representations St and $B(1,\omega)$ with image under

[8]Note in particular, that this implies $p \geq 5$ since we are assuming $V_{\mathbb{F}} \nsim \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$.

$\mathbf{V}$ equal to ω and 1 respectively. If $\Pi/\pi_{\mathcal{O}}\Pi$ contains a subrepresentation of infinite dimension and codimension, which has $B(1,\omega)$ as a Jordan-Hölder factor, then we are done. Suppose no such subrepresentation exists. Since there are no non-trivial extensions of $B(1,\omega)$ by St [Co 2, Prop. VII.4.25], $\Pi/\pi_{\mathcal{O}}\Pi$ must have socle St and cosocle $B(1,\omega)$. But then $\mathbf{V}(\Pi/\pi_{\mathcal{O}}\Pi)$ is a non-trivial extension of 1 by ω, by [Co 2, Thm. VII.4.21], so $V_{\mathbb{F}}$ admits ω as a quotient, contradicting our assumptions.

It remains to show that $\Pi/\pi_{\mathcal{O}}\Pi$ is isomorphic to $\bar{\pi}$. Since these two representations have the same Jordan-Hölder factors, this follows from the results of Colmez cited above. $\square$

(2.3.2) Suppose now that R is a complete local $\mathcal{O}$-algebra with residue field $\mathbb{F}$, and radical $\mathfrak{m}_R$. Let V_R be a deformation to R of the $G_{\mathbb{Q}_p}$-representation $V_{\mathbb{F}}$. For $n \geq 1$, let ξ_n denote the groupoid associated to $V_R \otimes_R R/\mathfrak{m}_R^n$. Then (2.2.6) produces a quotient $R_{\bar{\pi},n}$ for $R/\mathfrak{m}_R^n$ which represents $D_{\bar{\pi},\xi_n}$. Passing to the limit we obtains a quotient $R_{\bar{\pi}}$ of R.

In particular, applying this construction with $R = R^{\square}_{V_{\mathbb{F}},\mathcal{O}}$, we obtain a quotient $R^{\square}_{\bar{\pi},\psi}$ of $R^{\square}_{V_{\mathbb{F}},\mathcal{O}}$.

Proposition (2.3.3). *Suppose $V_{\mathbb{F}} \nsim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix} \otimes \chi$. Then*

$$\operatorname{Spec} R^{\psi\chi_{\mathrm{cyc}},\square}_{V_{\mathbb{F}},\mathcal{O}} \cap \operatorname{Spec} R^{\square}_{\bar{\pi},\psi} \subset \operatorname{Spec} R^{\square}_{V_{\mathbb{F}},\mathcal{O}}.$$

contains every irreducible component of $\operatorname{Spec} R^{\psi\chi_{\mathrm{cyc}},\square}_{V_{\mathbb{F}}}[1/p]$ *which contains a twisted benign point.*

Proof. Let $\psi' : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a character such that $\psi'|_{I_{\mathbb{Q}_p}}$ is an integral power of χ_{cyc} and ψ' has the same reduction as ψ. Suppose also that $\psi'\psi^{-1}$ admits an $\mathcal{O}^\times$-valued square root ε whose reduction is trivial (a condition which is automatic unless $p=2$). Twisting by ε induces an automorphism of $R^{\square}_{V_{\mathbb{F}}}$. Hence by the second claim in (2.1.1), we may replace ψ by ψ' and assume that $\psi|_{I_{\mathbb{Q}_p}}$ is a power of the cyclotomic character.

Let $x \in \operatorname{Spec} R^{\psi\chi_{\mathrm{cyc}},\square}_{V_{\mathbb{F}},\mathcal{O}}$ be a twisted benign point. We will show that x is in $\operatorname{Spec} R^{\square}_{\bar{\pi},\psi} \subset \operatorname{Spec} R^{\square}_{V_{\mathbb{F}},\mathcal{O}}$. To prove this we may extend the base ring $\mathcal{O}$, and assume that x has residue field E. Let $V = V_x$ be the deformation of $V_{\mathbb{F}}$ to $\mathcal{O}$ corresponding to x.

The results of Colmez [Co 1], Berger-Breuil [BB] and Breuil-Emerton [BE] imply that there exists an admissible $\mathcal{O}$-lattice Π' with central character ψ, such that $\mathbf{V}(\Pi') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as $G_{\mathbb{Q}_p}$-representations (see [Ki 4, 1.2.8]). Moreover, by a result of Berger [Be 2], $\Pi'/\pi_{\mathcal{O}}\Pi'$ and $\bar{\pi}$ have the same Jordan-Hölder factors. Dividing this map by a power of $\pi_{\mathcal{O}}$ we may assume that $V \subset \mathbf{V}(\Pi')$. Applying (2.3.1), there is an admissible $\mathcal{O}$-sublattice $\Pi \subset \Pi'$ such that $\mathbf{V}(\Pi) \subset \mathbf{V}(\Pi')$ is identified with V, and $\Pi/\pi_{\mathcal{O}}\Pi \xrightarrow{\sim} \bar{\pi}$. Hence Π corresponds to an $\mathcal{O}$-valued point of $\operatorname{Spec} R^{\square}_{\bar{\pi},\psi}$ which maps to x.

It follows that any element in $\ker(R^{\square}_{V_{\mathbb{F}},\mathcal{O}} \to R^{\square}_{\bar{\pi},\psi})$ vanishes at a twisted benign point of $R^{\psi\chi_{\mathrm{cyc}},\square}_{V_{\mathbb{F}},\mathcal{O}}$. Hence the lemma follows from (1.3.8). $\square$

Corollary (2.3.4). *Suppose $p > 2$, $V_{\mathbb{F}} \nsim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix} \otimes \chi$ and $\bar{V}|_{I_{\mathbb{Q}_p}} \nsim \begin{pmatrix} \omega_2^2 & 0 \\ 0 & \omega_2^6 \end{pmatrix} \otimes \chi$ if $p=3$. Then the image of*

$$\operatorname{Spec} R^{\square}_{\bar{\pi},\psi} \hookrightarrow \operatorname{Spec} R^{\square}_{V_{\mathbb{F}},\mathcal{O}}$$

contains Spec $R^{\psi\chi_{\mathrm{cyc}},\square}_{V_{\mathbb{F}},\mathcal{O}}$.

Proof. This follows from (2.3.3) and (1.3.12). $\square$

(2.3.5) Note that (2.3.4) produces a map

$$R^{\square}_{\bar{\pi},\psi} \to R^{\psi\chi_{\mathrm{cyc}},\square}_{V_{\mathbb{F}},\mathcal{O}}. \tag{2.3.6}$$

If one assumes that for any A in $\mathfrak{AR}_{\mathcal{O}}$ and π_A in $D_{\bar{\pi}}(A)$ with central character ψ, one has $\det_A \mathbf{V}(\pi_A) = \psi\chi_{\mathrm{cyc}}$, one finds that (2.3.6) is an isomorphism.

As mentioned in the introduction, by bounding the dimension of the tangent space of $R^{\square}_{\bar{\pi},\psi}$, Paskunas has shown that (2.3.6) is an isomorphism when $\bar{\pi}$ is supersingular. Hence in this case one does have $\det_A \mathbf{V}(\pi_A) = \psi\chi_{\mathrm{cyc}}$. On the other hand the formula $\det_A \mathbf{V}(\pi_A) = \psi\chi_{\mathrm{cyc}}$ does not always hold. For example it fails, in general, if $V_{\mathbb{F}}$ is unipotent. We are grateful to Paskunas for pointing this out to us.

For V a finite dimension E-representation of $G_{\mathbb{Q}_p}$, let $L \subset V$ be a $G_{\mathbb{Q}_p}$-stable lattice. We denote by $\bar{V}$ the semi-simplification of $L/\pi_{\mathcal{O}}L$. This does not depend on the choice of L.

Corollary (2.3.7). *Suppose $p > 2$. Let L be a representation of $G_{\mathbb{Q}_p}$ on a free $\mathcal{O}$-module of rank 2, and suppose that $L/\pi_{\mathcal{O}}L \nsim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix} \otimes \chi$ and $L/\pi_{\mathcal{O}}L|_{I_{\mathbb{Q}_p}} \nsim \begin{pmatrix} \omega_2^2 & 0 \\ 0 & \omega_2^6 \end{pmatrix} \otimes \chi$ if $p = 3$. Then there exists an admissible $\mathcal{O}$-lattice Π with central character $\det L \cdot \chi_{\mathrm{cyc}}^{-1}$, such that $\mathbf{V}(\Pi)$ is isomorphic to L.*

Proof. This is an immediate consequence of (2.3.4). $\square$

Corollary (2.3.8). *Suppose $p > 2$. Let V be an irreducible, 2-dimensional E-representation of $G_{\mathbb{Q}_p}$. If $p = 3$ assume that $\bar{V} \nsim \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \otimes \chi$ and $\bar{V}|_{I_{\mathbb{Q}_p}} \nsim \begin{pmatrix} \omega_2^2 & 0 \\ 0 & \omega_2^6 \end{pmatrix} \otimes \chi$. Then there exists an admissible $\mathcal{O}$-lattice Π with central character $\det V \cdot \chi_{\mathrm{cyc}}^{-1}$, such that*

$$\mathbf{V}(\Pi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V.$$

Proof. Let $L \subset V$ be a $G_{\mathbb{Q}_p}$-stable lattice. Our assumptions on $\bar{V}$ imply that L satisfies the conditions of (2.3.7), expect possibly when $p \geq 5$, and $\bar{V} \sim \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \otimes \chi$. In this case, since V is irreducible L can be chosen so that $L/\pi_{\mathcal{O}}L \sim \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \chi$ with $*$ a non-trivial extension class. In particular, such an L satisfies the conditions of (2.3.7), and the corollary follows. $\square$

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