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General Bound on the Rate of Decoherence

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(Dated: May 3, 2010)

We establish the necessary and sufficient conditions for a quantum system to be stable under any general system-environment interaction. Quantum systems are stable when the time-derivative of their purity is zero. This stability provides a dynamical explanation of the classicality of measurement apparatus. We also propose a protocol to detect global quantum correlations using only local dynamical information. We show how quantum correlations to the environment provide bounds to the purity rate, which in turn can be used to estimate dissipation rates for general non-Markovian open quantum systems.

Understanding how to preserve a quantum system from interacting with its environment via decoherence [1, 2] is one of the main challenges in quantum computation. A straightforward measure that quantifies the degree of decoherence is the purity, $\mathbf{P} := \mathrm{tr}_S \left\{ \rho^2 \right\}$. In this paper we answer the following question: are there any universal characteristics of the loss of purity of the system ($\mathcal{S}$) that are independent of the details of the system-environment ($\mathcal{SE}$) coupling?

A common assumption in the theory of open quantum systems is that it is reasonable to consider a system state initially uncorrelated from its environment, but weakly coupled to it. Perhaps it is more realistic to think of decoherence as arising from correlations developed as a consequence of the $\mathcal{SE}$ coupling. Kimura *et al.* [3, 4] showed that $\mathcal{SE}$ correlations were a necessary condition for the change of purity under any interaction: if the system was uncorrelated from its environment , $\rho_\tau^\mathcal{S} \otimes \rho^\mathcal{E}$, the purity rate of change was zero. Under this simple condition, the system is said to be stable under decoherence.

In this paper we generalize the result in [3, 4] by showing the necessary and sufficient conditions for which the purity rate of the system is zero at a time τ, for *any* type of interaction with the environment H_{int}. More precisely, assuming that the total state, $\rho_\tau^\mathcal{SE}$, evolves according to the von Neumann equation, then the purity of the system $\mathbf{P}_t^\mathcal{S}$ has the property

$$\left[\frac{d}{dt}\mathbf{P}_t^\mathcal{S} \right]_{t=\tau} = 0 \ \Leftrightarrow \ \left[\rho_\tau^\mathcal{S} \otimes \mathbb{I}^\mathcal{E}, \rho_\tau^\mathcal{SE} \right] = 0, \qquad (1)$$

for any time-dependent total Hamiltonian H_{tot} with any interaction H_{int}, (see Theorem 1). The result of Eq. (1) does not require any assumptions about the strength of the coupling, or Markovianity (it is non-Markovian [5]).

We also prove that the norm of the commutator between the system-environment state $\rho^\mathcal{SE}$ and its subsystem $\rho^\mathcal{S}$ provides a general bound for the system's purity rate of change of the form:

$$\left| \frac{d}{dt}\mathbf{P}_t^\mathcal{S} \right|_{t=\tau} \leq 2 \|H_{int}\| \ \left\| \left[\rho_\tau^\mathcal{S} \otimes \mathbb{I}^\mathcal{E}, \rho_\tau^\mathcal{SE} \right] \right\|_1. \qquad (2)$$

This is a universal feature of decoherence for *any* system-environment interaction H_{int} of arbitrary strength (see Theorem 2). Our result suggests how to estimate the rate of decoherence, as measured by the change of purity at time τ, from some knowledge of the total $\mathcal{SE}$ state in relationship to $\mathcal{S}$ and some information about the strength of the $\mathcal{SE}$ interaction.

We refer to the class of states that satisfy the relationship $\left[\rho^\mathcal{S} \otimes \mathbb{I}^\mathcal{E}, \rho^\mathcal{SE} \right] = 0$ as Stable System States (SSS). Here, "stable" signifies that the derivative of the purity is zero. The purity of SSS can be preserved under any environmental interaction by means of suitable local measurements: a condition for the realization of the quantum Zeno effect [6, 7]. This sense of stability is independent of the particular local dynamics and of the total $\mathcal{SE}$ interaction, making it different from decoherence-free subspaces [8, 9] which appear as a consequence of the specific symmetries of the decoherence dynamics. By deciding to focus on the structure of $\mathcal{SE}$ states, we have defined a form of stability that is valid for all types of interactions with the environment.

Ferraro *et al.* [10] showed that SSS are sparse in the space of density matrices, in both the sense of volume and topology, having measure zero in the whole Hilbert space and nowhere dense. We discuss the physical meaning of such class of states and its connection to open systems dynamics with environmental correlations. The set of SSS, although sparse, has the property of being dynamically stable. These states include uncorrelated states, maximally entangled states and states with zero quantum discord [11–15], and any other states that satisfy $\left[\rho^\mathcal{S} \otimes \mathbb{I}^\mathcal{E}, \rho^\mathcal{SE} \right] = 0$. First, we will describe how SSS can

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be interpreted as a generalization of the concept of zero quantum discord.

Quantum discord is a useful measure that assigns a degree of quantumness to $\mathcal{SE}$ correlations. When it is zero, the state is said to have only classical correlations. More explicitly, quantum discord quantifies the difference between the quantum mutual information of $\mathcal{S}$ and $\mathcal{E}$ and the mutual information after an optimal set of measurements $\{|j\rangle\langle j|^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}\}$. A state $\rho^{\mathcal{SE}}$ is classically correlated (has zero discord) if and only if it has the form $\rho^{\mathcal{SE}} = \sum_j p_j |j\rangle\langle j|^{\mathcal{S}} \otimes \rho_j^{\mathcal{E}}$, where $\{|j\rangle\}$ form a one-dimensional orthonormal basis of $\mathcal{S}$, where $\{p_j\}$ are the corresponding probabilities, and $\rho_j^{\mathcal{E}}$ are density matrices. This condition of classicality is equivalently expressed by an invariance under the set of measurements $\{|j\rangle\langle j|^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}\}$ such that $\rho^{\mathcal{SE}} = \sum_j |j\rangle\langle j|^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} \rho^{\mathcal{SE}} |j\rangle\langle j|^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}$ [31].

According to the concept of quantum discord, classical correlations are characterized by states stable under a set of measurements given by one-dimentional projectors. By generalizing this concept to a complete set of m-dimensional projectors, we can give the SSS an alternative physical interpretation, on which we elaborate below.

Proposition 1 *Let $\rho^{\mathcal{SE}}$ be an arbitrary bipartite state on the $\mathcal{SE}$ space. Then, $\rho^{\mathcal{SE}}$ is a SSS if and only if*

$$\rho^{\mathcal{SE}} = \sum_j \Pi_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} \, \rho^{\mathcal{SE}} \Pi_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \qquad (3)$$

where $\rho^{\mathcal{S}} = \sum_j p_j \Pi_j^{\mathcal{S}}$, $\{\Pi_j^{\mathcal{S}}\}$ are orthonormal projectors of m-dimensions, and SSS is the set of states $\rho^{\mathcal{SE}}$ such that $[\rho^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \rho^{\mathcal{SE}}] = 0$.

Proof Assume that $[\rho^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \rho^{\mathcal{SE}}] = 0$. It follows that any eigenspace of $\rho^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}$ is an invariant subspace of $\rho^{\mathcal{SE}}$. Since $\Pi_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}$ is a projector to the eigenspace of $\rho^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}$, we have $\Pi_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} \, \rho^{\mathcal{SE}} \Pi_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} = \rho^{\mathcal{SE}} \Pi_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} = \Pi_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} \, \rho^{\mathcal{SE}}$. By the completeness $\sum_j \Pi_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} = \mathbb{I}^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}$, we have that $\sum_j \Pi_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} \, \rho^{\mathcal{SE}} = \rho^{\mathcal{SE}}$. The converse can be seen from direct calculation. $\square$

The right side of Eq. (3) is the post measurement state under local projectors $\{\Pi_j\}$. We can interpret the set of SSS as being undisturbed by that set of measurements.

The set of SSS arise naturally as states with the dynamical property given by Eq. (1). To prove this result, we consider general $\mathcal{SE}$ evolution of the density operator [32] given by the von Neumann equation, $\frac{d}{dt}\rho_t^{\mathcal{SE}}|_{t=\tau} = -i[H_{tot}, \rho_\tau^{\mathcal{SE}}]$. The total Hamiltonian is $H_{tot} = H^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} + \mathbb{I}^{\mathcal{S}} \otimes H^{\mathcal{E}} + H_{int}$, which consists of the system, the environment and the interaction Hamiltonians [33]. The time dependence of H_{tot} is implicit, but without loss of generality we write H_{tot} in the picture where H_{int} is time-independent. With this at hand, we derive Eq. (1).

Theorem 1 *The necessary and sufficient condition for the time derivative of the reduced purity to be zero under*

any Hamiltonian is that the bipartite state $\rho^{\mathcal{SE}}$ is a SSS, as in Eq. (1).

Proof By taking the trace with respect of the environment in the von Neumann equation, and taking the time derivative of the reduced purity, we obtain

$$\left[\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right]_{t=\tau} = \text{tr}_{\mathcal{S}} \left\{ 2\rho_\tau^{\mathcal{S}} \left(-i \, \text{tr}_{\mathcal{E}} \left[H_{tot}, \rho_\tau^{\mathcal{SE}} \right] \right) \right\}$$
$$= -2i \, \text{tr}_{\mathcal{SE}} \left\{ \rho_\tau^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} \left[H_{int}, \rho_\tau^{\mathcal{SE}} \right] \right\},$$

where the cyclic property of the trace was used [34]. Note that the dependence on $H^{\mathcal{S}}$ and $H^{\mathcal{E}}$ vanishes [35]; the system-environment interaction H_{int} plays the role of changing the purity. Using the cyclic property of the trace once more, we reach

$$\left[\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right]_{t=\tau} = 2i \, \text{tr}_{\mathcal{SE}} \left\{ H_{int} \left[\rho_\tau^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \rho_\tau^{\mathcal{SE}} \right] \right\}. \qquad (4)$$

Thus, the sufficient and necessary condition for the purity rate in Eq. (4) to be zero for any H_{int} is that $[\rho_\tau^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \rho_\tau^{\mathcal{SE}}] = 0$. $\square$

Theorem 1 defines the class of SSS as a natural consequence of dynamical stability of the purity measure. This directly connects dynamical properties of reduced systems to the structure of the $\mathcal{SE}$ state [16–21].

Corollary 1 *Under the presence of any interaction with its environment, a quantum system is stable if the system has classical correlations to the environment. Previous results for uncorrelated states [3, 4] fall in this category.*

Corollary 2 *Experiments can be designed to detect global bipartite quantum correlations by monitoring only the dynamics of a subsystem. If the purity of the local subsystem has a non-zero time derivative, then the total state is not a SSS, and it has a non-zero quantum discord.*

This can be related to Corollary 1 by noting that the purity of the subsystem is a locally detectable quantity. To derive this, we combine Proposition 1 with Eq. (1). This constitutes an experimental protocol that can detect quantum discord between $\mathcal{S}$ and $\mathcal{E}$ by monitoring the purity of $\mathcal{S}$, without knowledge of any environmental properties or of the total Hamiltonian. This generalizes the result from [3, 4].

Corollary 3 *The stability of states provides a dynamical explanation for the robustness of a classical measurement apparatus.*

The orthonormal states that define the measurement apparatus $\{|\mu_i\rangle\langle\mu_i|\}$ are called pointer states and uniquely specify the measured quantity. Theorem 1 provides a dynamical argument in favor of the stability of pointer states [22]. Let $\mathcal{Q}$ be a quantum state to be measured and $\mathcal{M}$ be the macroscopic measurement apparatus. The act of a measurements correlates

the measurement apparatus with the quantum state into $\rho^{\mathcal{MQ}} = \sum_i p_i |\mu_i\rangle\langle\mu_i|^{\mathcal{M}} \otimes \rho_i^{\mathcal{Q}}$. This is a classically correlated state from the apparatus' point of view. Corollary 1 provides a dynamical explanation for the robustness of these pointer states because stability is a defining characteristic of a measurement.

So far, we have focused on the importance of Eq. (4) without commenting on how to estimate its magnitude. This equation can be approximated by the upper bound Eq. (2) which depends on parameters that can be estimated from knowledge of the $\mathcal{SE}$ coupling strength and partial information about the structure of the $\mathcal{SE}$ state with respect of $\mathcal{S}$. The proof follows.

Theorem 2 *The magnitude of the purity rate is bounded from above by:* $2\|H_{int}\| \, \|[\rho_\tau^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \rho_\tau^{\mathcal{SE}}]\|_1$.

Proof Consider a bounded operator A and a trace class operator σ on a Hilbert space. For them it follows that $|\text{tr}\,[A\sigma]| \leq \|A\sigma\|_1 \leq \|A\|\|\sigma\|_1$ [23]. Combine this with Eq. (4) to obtain Eq. (2). $\square$

The trace norm of the commutator in Eq. (2) is essential for our analysis, and we will refer to it as

$$\Delta := \|\,[\rho^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \rho^{\mathcal{SE}}]\,\|_1. \tag{5}$$

The quantity Δ measures how "far" is the eigenbasis of the total system-environment matrix from commuting with the eigenbasis of its reduced system. Also, Δ quantifies the deviation from the equality of Eq. (2). Conceptually, Eq. (2) establishes that the purity rate is bound by the product of the interaction strength and Δ. In the weak-coupling limit, $\|H_{int}\|$ is very small and the purity rate will be small in general. High purity rates can only occur when Δ is large and the coupling to the environment is also strong.

To gain intuition of the meaning of Δ, we will discuss its relationship to different measures of $\mathcal{SE}$ correlations. The bounds provided by Δ can in turn be expressed by bounds in terms of $\mathcal{SE}$ correlations.

It is easy to show that for pure states all the notions of uncorrelated states, classically correlated states, and separable (not entangled) states coincide. This suggests that we can have quantitative estimations for the purity rate using any measure of those notions. To show this, we will now consider only $\mathcal{SE}$ states that are pure, $|\chi\rangle\langle\chi|^{\mathcal{SE}}$. We can show how this is not a restrictive class by invoking the Church of the Larger Hilbert Space [24]. By defining an ancillary space $\mathcal{A}$, any state $\rho^{\mathcal{SE}}$ can be purified into $|\chi\rangle\langle\chi|^{\mathcal{SEA}}$ [25]. Since we are interested in properties of the evolution of the system $\mathcal{S}$, we can refer to the rest of the Hilbert space $\mathcal{EA}$ simply as a new environment, and for simplicity relabel it as $\mathcal{EA} \to \mathcal{E}$. Similarly, the total Hamiltonian can be thought of trivially acting on the ancilla, $H_{tot} \to H_{tot} \otimes \mathbb{I}^{\mathcal{A}}$.

By considering $\varrho^{\mathcal{SE}} = |\chi\rangle\langle\chi|^{\mathcal{SE}}$, we can establish connections between purity rates and the amount of $\mathcal{SE}$ correlations. We consider the measures of robustness of entanglement, negativity, entropy of entanglement, quan-

tum mutual information, and quantum discord. Before we proceed, we need the following proposition.

Proposition 2 *Let* $\varrho^{\mathcal{SE}} = |\chi\rangle\langle\chi|$ *be a pure state, with Schmidt decomposition* $\chi = \sum_{i=1}^{s} \sqrt{p_i}\psi_i \otimes \phi_i$ $(p_i > 0)$, *where the elements of* $\{\sqrt{p_i}\}$ *are Schmidt coefficients and* $s \leq \min[d_s, d_e]$ *is the Schmidt rank. Then,* $\varrho^{\mathcal{SE}} = |\chi\rangle\langle\chi|$ *is a SSS if and only if* $p_i = \frac{1}{s}$.

Proof Using $\varrho^{\mathcal{S}} = \sum_{i=1}^{s} p_i|\psi_i\rangle\langle\psi_i|$ to explicitly calculate $[\varrho^{\mathcal{S}} \otimes \mathbb{I}, \varrho^{\mathcal{SE}}] = 0$, we have $p_i^{\frac{3}{2}}p_k^{\frac{1}{2}} = p_k^{\frac{3}{2}}p_i^{\frac{1}{2}}$ for all $i, k = 1, \ldots, s$. This is satisfied if and only if $p_i = \frac{1}{s}$ for all $i = 1, \ldots, s$. $\square$

Corollary 4 *Proposition 2 includes maximally-entangled state, where the commutative condition of SSS is trivially satisfied.*

Now, we can show how Δ_χ for $\varrho^{\mathcal{SE}} = |\chi\rangle\langle\chi|$ is bound by the robustness of entanglement [26] and the negativity [27], and thus these quantities provide bounds for the purity rate.

Proposition 3 *For* $\varrho^{\mathcal{SE}} = |\chi\rangle\langle\chi|$, *the purity rate of the system is bound by* $|\frac{d}{dt}\mathbf{P}^{\mathcal{S}}| \leq 2\|H_{int}\| \, R(\chi) = 4\|H_{int}\| \, N(\chi)$, *where* $R(\chi)$ *is the robustness of entanglement and* $N(\chi)$ *is negativity of the total state* $|\chi\rangle\langle\chi|$, *respectively.*

Proof By direct computation one has $[\varrho^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \varrho^{\mathcal{SE}}] = \sum_{i \neq k} \sqrt{p_i p_k}(p_i - p_k)|\psi_i\rangle\langle\psi_k| \otimes |\phi_i\rangle\langle\phi_k|$. Taking the trace norm, using the triangle inequality and $\||\psi\rangle\langle\psi|\|_1 = 1$ for a unit vector ψ, we obtain $\Delta_\chi \leq \sum_{i \neq k} \sqrt{p_i p_k}|p_i - p_k|$. The inequality is generally strict but if the Schmidt number is 2, the equality holds. Since $0 \leq p_i \leq 1$, one has $|p_i - p_k| \leq 1$, which gives the bound: $\Delta_\chi \leq \sum_{i \neq k} \sqrt{p_i p_k} = (\sum_i \sqrt{p_i})^2 - 1$. The right hand side is the robustness of entanglement $R(\chi)$ for pure states [26], which coincides with $2N(\chi)$, where $N(\chi)$ is the negativity of χ [27]. $\square$

Next, we show that the purity rate is bounded by different measures of correlations: the quantum mutual information, the entropy of entanglement [28], and the quantum discord.

Proposition 4 *For* $\varrho^{\mathcal{SE}} = |\chi\rangle\langle\chi|$, *the purity rate of the system is bound by* $|\frac{d}{dt}\mathbf{P}^{\mathcal{S}}| \leq 4\|H_{int}\|\sqrt{2I(\chi)} = 8\|H_{int}\|\sqrt{E(\chi)} = 8\|H_{int}\|\sqrt{\delta_{\mathcal{S}\to\mathcal{E}}(\chi)}$, *where* $I(\chi) := S(\varrho^{\mathcal{S}}) + S(\varrho^{\mathcal{E}}) - S(\varrho^{\mathcal{SE}})$ *is the quantum mutual information,* $E(\chi) := S(\varrho^{\mathcal{S}})$ *is the entropy of entanglement and* $\delta_{\mathcal{S}\to\mathcal{E}}(\chi) := I(\chi) - \sup_{|i\rangle\langle i|^{\mathcal{S}}}(S(\rho^{\mathcal{E}}) - \sum_j p_j S(\varrho_j^{\mathcal{E}}))$ *is the quantum discord.*

Proof The first inequality was previously obtained in [3]. Since ϱ is pure, it follows that $I(\chi) = 2S(\rho^{\mathcal{S}}) = 2E(\chi)$, and we have the second equality. The third equality is obtained by noting that the quantum discord for a pure state χ coincides with $E(\chi)$. $\square$

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In conclusion, we found the class of system states that are stable under any general open quantum systems evolution without making any assumptions about the nature of the coupling to the environment. We defined the Stable System States (SSS) to have the property $\left[\rho^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \rho^{\mathcal{S}\mathcal{E}}\right] = 0$, and prove that this is a sufficient and necessary condition for the time-derivative of the purity to be zero. This sense of stability if more general than the uncorrelated $\mathcal{S}\mathcal{E}$ states considered in dynamical-decoupling techniques. This result can be used to explain the dynamical stability of quantum measurement apparata. We proposed an experimental protocol for detecting global quantum correlations from local observables. Finally, we showed how the time-derivative of the purity is bound by the amount of system-environment correlations, establishing that bipartite correlations not only restrict the purity of a subsystem, but also its rate of change. An open question remains: are there tighter bounds coming from other measures of system-environment correlations? Finally, the structure of SSS may suggest new methods for decoherence protection that focus on engineering the structure of system-environment states but are independent of the system-environment coupling.

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[31] See [17, 20], Lemma 8.12 in [29] and Theorem 2 in [30].
[32] A linear operator lives on a complex Hilbert space $\mathcal{H}$ where the inner product is given by $\langle\psi|\phi\rangle$ with $\psi, \phi \in \mathcal{H}$. The norm is denoted by $||\psi|| := \sqrt{\langle\psi|\psi\rangle}$. For a bounded operator A, the operator norm is denoted by $||A|| := \sup_{0\neq\psi\in\mathcal{H}} ||A\psi||/||\psi||$. For a trace class operator σ, the trace norm is $||\sigma||_1 := \operatorname{tr}|\sigma|$ ($|\sigma| := \sqrt{\sigma^\dagger\sigma}$). Density matrices ρ are trace class positive operators with $||\rho||_1 = \operatorname{tr}\rho = 1$.
[33] For technical simplicity, we assume that $H^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}}, \mathbb{I}^{\mathcal{S}} \otimes H^{\mathcal{E}}$ and H_{int} are bounded operators on $\mathcal{H}^{\mathcal{S}} \otimes \mathcal{H}^{\mathcal{E}}$, *i.e.*, self-adjoint operators whose spectrum is bounded. Similar qualitative results can be easily generalized to an unbounded Hamiltonian if one carefully deals with an operator domain, imposing the assumption of the finite variance of energy, as shown in [4].
[34] For any bounded operator A and for any trace class operator σ on a Hilbert space $\mathcal{H}$, it follows that $A\sigma, \sigma A$ are trace class operators and $\operatorname{tr}\{A\sigma\} = \operatorname{tr}\{\sigma A\}$.
[35] $H^{\mathcal{S}}$ vanishes since $\operatorname{tr}_{\mathcal{S}\mathcal{E}}\left\{\rho^{\mathcal{S}\mathcal{E}}\left[\rho^{\mathcal{S}}\otimes\mathbb{I}^{\mathcal{E}}, H^{\mathcal{S}}\otimes\mathbb{I}^{\mathcal{E}}\right]\right\} = \operatorname{tr}_{\mathcal{S}}\left\{[\rho^{\mathcal{S}}, H^{\mathcal{S}}]\operatorname{tr}_{\mathcal{E}}\rho^{\mathcal{S}\mathcal{E}}\right\}$. Using the cyclic property again, $\operatorname{tr}_{\mathcal{S}}\left\{[\rho^{\mathcal{S}}, H^{\mathcal{S}}]\operatorname{tr}_{\mathcal{E}}\rho^{\mathcal{S}\mathcal{E}}\right\} = \operatorname{tr}_{\mathcal{S}}\left\{H^{\mathcal{S}}[\rho^{\mathcal{S}}, \rho^{\mathcal{S}}]\right\} = 0$. The term $H^{\mathcal{E}}$ vanishes in a similar manner.