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Abstract

A simple characterization of the equilibrium conditions required to compute Vickrey payments in the Combinatorial Allocation Problem leads to an ascending price Generalized Vickrey Auction. The ascending auction, *i*Bundle Extend & Adjust (*i*BEA), maintains non-linear and perhaps non-anonymous prices on bundles of items, and terminates with the efficient allocation and the Vickrey payments in *ex post* Nash equilibrium. Crucially, *i*BEA is able to implement the Vickrey outcome even when the Vickrey payments are not supported in a single competitive equilibrium. The auction closes with Universal competitive equilibrium prices, which provide enough information to compute individualized discounts to adjust the final prices and implement Vickrey payments.

1 Introduction

In a combinatorial allocation problem (CAP) there is a set of discrete heterogeneous items to allocate to a group of agents, and each agent may have quite general preferences across items, including both complements and substitutes. The CAP has many interesting applications: in logistics [32, 50]; in job-shop scheduling [53]; for airport slot allocation [47]; bandwidth allocation [21]; multi-agent planning [27]; class registration [22]; and resource allocation [31]. Combinatorial auctions allow agents to bid for bundles of items, and express complex preferences that may include both synergies and complements.

This paper proposes an allocatively-efficient ascending-price combinatorial auction. The proposed auction, *i*Bundle Extend&Adjust (*i*BEA), implements the outcome of the Vickrey-Clarke-Groves mechanism for the CAP, the so-called Generalized Vickrey auction (GVA). Myopic best-response, or *straightforward bidding*, is an *ex post* Nash equilibrium of the auction, maximizing the payoff to an agent given MBR strategies of every other agent over all feasible strategies. In particular, the free-riding problem that characterizes equilibrium strategies in other ascending combinatorial auctions [8, 12, 36] is *not* a problem in *i*BEA. A number of efficient ascending price auctions have been proposed in the literature for special cases of the combinatorial allocation problem; for example, by Demange et al. [19] for the *unit-demand* case, by Ausubel [5] for homogeneous items with diminishing marginal returns, and recently by Parkes [42] and Ausubel & Milgrom [4] for an *agents are substitutes* condition. In this paper we place only a *free-disposal* requirement on agent preferences. *i*BEA extends *i*Bundle [40, 44], that was efficient for straightforward agent strategies, but for which straightforward bidding was not in equilibrium.

Prices in *i*BEA are both non-linear and non-anonymous, with non-anonymous prices introduced dynamically as necessary to compute the outcome of the GVA. The bidding language allows agents to submit *exclusive-or* bids across bundles of items, e.g. "I only want bundle S_1 or bundle S_2 but not both". The provisional allocation in each round is selected to maximize revenue given agent bids, and a simple rule determines ask price increases across rounds based on unsuccessful bids. *i*BEA terminates in competitive equilibrium, and elicits enough preference information from agents to compute individualized discounts at the end of the auction and implement Vickrey payments. If Vickrey payments are supported in a single set of competitive equilibrium prices then *i*BEA terminates as soon as competitive equilibrium is achieved; otherwise, *i*BEA remains open long enough to elicit the additional information about agent preferences that is required to make the auction incentive-compatible.

The theoretical properties of *i*BEA are established via linear-programming duality theory. The auction maintains a feasible primal (the provisional allocation) and a feasible dual (the ask prices) solution to an extended linear program formulation of CAP [11]. Termination in competitive equilibrium is equivalent, for agents with myopic best-response strategies, to termination with an allocation that satisfies *complementary slackness* conditions with the prices, which implies allocative efficiency. Crucially, we also characterize necessary and sufficient conditions for competitive equilibrium (CE) prices to provide enough information to compute the Vickrey payments, in addition to the efficient allocation. Vickrey payments can be computed from CE prices if and only if they are *Universal* CE prices, which means that the prices are CE in the combinatorial auction with all agents and in the subproblems induced by removing each agent from the auction in turn.

Ascending-price auctions can avoid the high cost of information revelation that is required in efficient sealed-bid mechanisms [41, 44, 6, 14]. In many interesting problems there is a cost associated with determining the value for a set of items [4]: perhaps the bidder in the wireless spectrum auction must determine a new business plan to understand the value of any particular combination of licenses; perhaps the bidder in the procurement problem must run a new optimization problem to determine an optimal manufacturing plan given a particular delivery schedule [50]. Ascending-price auctions allow bidders to compute optimal strategies with approximate preference information [42, chapter 7]. Secondary reasons to prefer ascending-price over sealed-bid auctions include reduced information revelation [48] and better performance in common-value settings [16, 46].

As a general methodology, *i*BEA suggests a new paradigm for the design of efficient ascending-price Vickrey auctions. The method requires a suitable extended linear program formulation of the underlying allocation problem, such as that provided by Bikchandani & Ostroy [11] for the CAP, and a characterization of the information required to compute Vickrey payments. Interestingly, even though the extended formulation itself can be very (perhaps exponentially large), the primal-dual method only explicitly constructs as much of the formulation as is necessary to solve the problem at hand.

1.1 Outline

Section 2 contains preliminaries. We present assumptions about agent preferences; introduce the combinatorial allocation problem (CAP) and the Generalized Vickrey auction; and present an extended linear program formulation due to Bikchandani & Ostroy [11], with a dual solution that computes non-linear and non-anonymous competitive equilibrium prices. We define Universal-, Quasi-, and Quasi-Universal CE prices, and individual- and group-minimal CE prices, and introduce sufficient and almost necessary conditions on agent preferences for Vickrey payments to be supported in competitive equilibrium.

Section 3 characterizes the relationship between minimal CE prices and Vickrey payments. First, in Section 3.1 we show that Universal CE prices are both necessary and sufficient to adjust to individual-minimal CE prices, and therefore Vickrey payments. Then, in Section 3.2 we show that Quasi-CE prices on a chain of subproblems are sufficient to adjust prices to group-minimal CE prices, and derive a useful characterization for the special-case that Vickrey payoffs to every agent are supported in a single competitive equilibrium, which at the group-minimal CE prices. Section 3.3 gives some illustrative examples.

Section 4 introduces the *i*BEA auction, and presents an extended example. We also propose a variation in which price increases are implemented based on a subset of bids from agents. In Section 5 we prove that *i*BEA is allocatively-efficient, and that myopic best-response is an *ex post* Nash equilibrium of the auction, which terminates with the VCG outcome. Section 6 places *i*BEA in the context of existing ascending auction theory, and discusses the incentive properties of *i*BEA in comparison with the GVA, and in comparison to a related non-VCG ascending-price auction design. Section 7 concludes.

2 Preliminaries

Combinatorial auctions, in which bidders can submit bids on bundles of items, are important when items are complements and/or substitutes and agents have quite general preferences [49]. Our primary goal is allocative efficiency, to allocate items across agents to maximize total utility. In some cases this goal is compatible with revenue maximization, for example when there is a perfect resale market [7], but in general it is well-known that revenue maximization is incompatible with efficiency [37]. A brief discussion of the revenue properties of *i*BEA *vis-a-vie* competing ascending-price combinatorial auction designs is made in Section 6.

In this section we first define the combinatorial allocation problem (CAP), and state our assumptions about agent preferences. Then, we define Quasi-, Universal, and Quasi-Universal CE prices, and relate minimal CE prices to Vickrey payments.

2.1 The Combinatorial Allocation Problem

We consider a finite set, $\mathcal{G}$, of discrete items to allocate to a set $\mathcal{I}$ of agents. Let $m = |\mathcal{G}|$ and $n = |\mathcal{I}|$. Each agent $i \in \mathcal{I}$ has a valuation function, $v_i : 2^G \rightarrow \mathbb{R}_+$, that defines its value $v_i(S) \geq 0$ for every bundle of items, $S \subseteq \mathcal{G}$. We make the following assumptions about agent preferences:

(i) *private values*, each bidder has a method to determine its own value for bundles of items, and this value does not depend on the values of other agents

(ii) *quasi-linear utility*, the net payoff or utility to agent i for bundle S is $u_i(S, p) = v_i(S) - p$, where it pays price p

(iii) *no externalities*, an agent's utility does not depend on the items allocated to other agents

(iv) *free-disposal*, $v_i(S) \leq v_i(S')$ for all $S \subseteq S'$.

(v) *zero seller values*, the seller has no value for the items.

Beyond this, we make no additional assumptions about agent preferences; for example, we do not need to assume that agents values are *submodular*, which is a generalized form of decreasing marginal-returns.

Assumption (i) rules out common-value preferences, in which agents learn about the value of items from the revealed preferences of other agents. Assumption (ii), quasi-linear utilities, is tantamount to assuming risk-neutral agents. Assumption (iii), no externalities, rules out auctions in which agents interact over an extended period of time, for example in a long-term competitive relationship [2, 28]. Assumption (iv) is quite reasonable in most settings. It is impossible to assumption (v) and maintain efficiency and budget-balance in equilibrium, unless one is willing to accept either a budget deficit or a loss of interim individual-rationality [3, 17, 38].

Let $S = (S_1, \ldots, S_n)$ define an allocation of items across agents, such that agent i receives bundle $S_i \subseteq \mathcal{G}$. The *ex post* efficient allocation, S^*, maximizes the total value over all agents. Let Γ denote the set of feasible allocations, such that $k \in \Gamma$ defines a partition of items and an assignment of bundles in the partition to agents. We write $[i, S] \in k$ to indicate that bundle S is assigned to agent i in allocation k. The CAP for $\mathcal{I}$ agents, is:

$$V(\mathcal{I}) = \max_{k \in \Gamma} \sum_{[i,S] \in k} v_i(S) \qquad [\textsc{cap}(\mathcal{I})]$$

Later, $\textsc{cap}(\mathcal{I} \setminus i)$, is used to denote the combinatorial allocation problem without agent i, with $V(\mathcal{I} \setminus i)$ the value of the efficient allocation to the subproblem.

The computational properties of CAP are quite well-understood. The general problem is NP-hard [49], equivalent to the maximal weighted clique problem, but a number of tractable special-cases have been identified by considering special properties of linear program formulations [49, 18, 33]. In addition, fast algorithms are known for particular distributions of problem instances, using variants on branch-and-bound search [51, 1], while fast linear-program based approximation algorithms are also proposed [56].

In Bikchandani & Ostroy [10] a hierarchy of linear program (LP) formulations are introduced for CAP. The formulations are interesting because their duals correspond to linear prices, non-linear prices, and non-linear and non-anonymous prices as we move up the hierarchy. Each successive formulation introduces additional valid inequalities and auxiliary variables to lift the LP formulations and better approximate the optimal solution to the underlying CAP. The strongest formulation, LP_3, has the *integrality* property, such that all solutions are integral and solve the CAP. *i*BEA implements a primal-dual algorithm, in equilibrium with agent best-response strategies, for LP_3. Although the extended formulations introduce an exponential number of additional variables and constraints and are of limited value for centralized computation, the duality properties are invaluable to prove formal properties in ascending-price combinatorial auction design.

2.1.1 The Extended LP Formulation

Let $x_i(S) \geq 0$ denote the weight with which bundle S is allocated to agent i, and variable, $y(k) \geq 0$, indicate the level with which allocation $k \in \Gamma$ is selected, where as before Γ is the set of all feasible allocations. Formulation LP_3 is integral for the CAP.

$$\max_{x_i(S), y(k)} \sum_{S \subseteq \mathcal{G}} \sum_{i \in \mathcal{I}} x_i(S) v_i(S) \qquad [\text{LP}_3]$$

$$\text{s.}t. \quad \sum_{S \subseteq \mathcal{G}} x_i(S) \leq 1, \quad \forall i \in \mathcal{I} \qquad (\text{LP}_3\text{-1})$$

$$x_i(S) \leq \sum_{k \in \Gamma, [i,S] \in k} y(k), \quad \forall i \in \mathcal{I}, S \subseteq \mathcal{G} \qquad (\text{LP}_3\text{-2})$$

$$\sum_{k \in \Gamma} y(k) \leq 1 \qquad (\text{LP}_3\text{-3})$$

$$x_i(S), y(k) \geq 0, \quad \forall i \in \mathcal{I}, S \subseteq \mathcal{G}, k \in \Gamma$$

Constraints (LP_3-1) ensure that each agent receives at most one bundle, constraints (LP_3-3) ensure that the total weight allocated to allocations is at most one, and constraints (LP_3-2) ensure that the bundles allocated to agents are consistent with the selected allocations.

In the dual, we associate each constraint (LP_3-1) with variable π_i, which can be interpreted as agent i's surplus. Each constraint (LP_3-2) is associated with variable $p_i(S)$, which can be interpreted as the ask price to agent i for bundle S. Constraint (LP_3-3) is associated with variable π^s, which can be interpreted as the seller's surplus. The dual objective is to compute a set of non-linear and non-anonymous prices to minimize the sum of agent surplus and seller surplus.

$$\min_{\pi_i, p_i(S), \pi^s} \sum_{i \in \mathcal{I}} \pi_i + \pi^s \qquad [\mathrm{DLP}_3]$$

$$\text{s.}t. \quad \pi_i + p_i(S) \geq v_i(S), \quad \forall i \in \mathcal{I}, S \subseteq \mathcal{G} \qquad (\mathrm{DLP}_3\text{-}1)$$

$$\pi^s - \sum_{[i,S] \in k} p_i(S) \geq 0, \quad \forall k \in \Gamma \qquad (\mathrm{DLP}_3\text{-}2)$$

$$\pi_i, p_i(S), \pi^s \geq 0, \quad \forall i \in \mathcal{I}, S \subseteq \mathcal{G}$$

Given a set of prices $p_i(S)$, then the optimal dual solution is $\pi_i = \max_{S \subseteq \mathcal{G}} \pi_i(S)$ and $\pi^s = \max_{k \in \Gamma} \pi^s(k)$, where $\pi_i(S) = [v_i(S) - p_i(S)]^+$ and $\pi^s(k) = \sum_{[i,S] \in k} p_i(S)$, and $v_i(\emptyset) - p_i(\emptyset) = 0$ and $\pi^s(\emptyset) = 0$ by definition. Throughout the paper we use x^+ to denote $\max(0, x)$.

This extended formulation is integral, essentially by formulating away the problem with the introduction of an exponential number of constraints, one associated with each element of Γ.

2.2 Competitive, Quasi-Competitive, and Universal-Competitive Equilibrium Prices

By strong LP duality, an optimal dual solution defines competitive equilibrium prices. A feasible primal solution, S^*, and a feasible dual solution, p^*, is optimal if and only if they satisfy *complementary slackness* (CS) conditions. The non-trivial CS conditions[1] for primal-dual formulation LP_3/DLP_3 are:

[1]We ignore $p_i(S) > 0 \Rightarrow x_i(S) = \sum_{k \in \Gamma, [i,S] \in k} y(k)$ because the RHS holds for all feasible and optimal solutions of LP_3. We also reformulate $\pi_i > 0 \Rightarrow \sum_i x_i(S) = 1$ as $\sum_i x_i = 0 \Rightarrow \pi_i = 0$, taking advantage of the integrality property of optimal solutions. Finally, we ignore $\pi^s > 0 \Rightarrow \sum_k y(k) = 1$ because this is trivially satisfied in any integral feasible solution to LP_3 in which some allocation is selected.

$$x_i(S) > 0 \Rightarrow \pi_i + p_i(S) = v_i(S), \quad \forall i \in \mathcal{I}, S \subseteq \mathcal{G} \tag{CS1a}$$

$$\sum_{i \in \mathcal{I}} x_i(S) = 0 \Rightarrow \pi_i = 0 \tag{CS1b}$$

$$y(k) > 0 \Rightarrow \pi^s - \sum_{[i,S] \in k} p_i(S) = 0, \quad \forall k \in \Gamma \tag{CS2}$$

Equivalently, CS conditions are nothing more than a statement of competitive equilibrium, such that both the agents and the seller maximize surplus given the prices and the allocation. Formally, allocation S^* and prices p^* are in competitive equilibrium if and only if:

- (CS1a and CS1b) Bundle S_i^* solves $\max_{S \subseteq \mathcal{G}} \pi_i(S)$ for every agent i, with $\pi_i(S) = [v_i(S) - p_i(S)]^+$.
- (CS2) Allocation S^* solves $\max_{k \in \Gamma} \pi^s(k)$ for the seller, with $\pi^s(k) = \sum_{[i,S] \in k} p_i(S)$.

From the integrality of LP_3, competitive equilibrium (CE) prices (perhaps non-linear, and perhaps non-anonymous) exist in the CAP, and always support the efficient allocation. In special cases, the general non-linear and non-anonymous prices can be represented as linear-additive prices, p_j on each item $j \in \mathcal{G}$, with $p_i(S) = \sum_{j \in S} p_j$ for all $i \in \mathcal{I}$. Gross-substitutes preferences [29] defines the largest set of preferences that contain unit-demand preferences for which the existence of linear competitive equilibrium prices can be shown [25]. In other cases, prices can be represented as anonymous but non-linear prices, $p(S)$ for all bundles $S \subseteq \mathcal{G}$, with $p_i(S) = p(S)$ for all $i \in \mathcal{I}$. In all cases, whenever prices support an allocation in equilibrium, then the allocation is efficient.

Definition 1 (Universal CE prices). *Prices, p_{uce}, are Universal competitive equilibrium (UCE) prices if and only if they are in competitive equilibrium in problem* $\textsc{cap}(\mathcal{I})$ *and in each subproblem* $\textsc{cap}(\mathcal{I} \setminus i)$ *induced by removing agent $i \in \mathcal{I}$.*

Universal CE prices exist in every CAP instance. Consider prices $p_i(S) = v_i(S)$ for every agent i and bundle S, these prices are UCE by construction. In Section 3.1, we show that UCE prices provide necessary and sufficient information to compute Vickrey payments in CAP. Later, in Section 5, we prove that *i*BEA terminates with prices that are Universal CE prices, and use this to derive the VCG outcome.

Quasi-CE prices, defined with respect to a subproblem $\textsc{cap}(\mathcal{K})$, allow a characterization of a set of weaker conditions on prices that are sufficient to compute group-minimal CE prices. Let $\Gamma(\mathcal{K})$ denote the set of allocations restricted to agents $\mathcal{K}$, such that $\Gamma(\mathcal{K}) = \{k \in \Gamma \ : \ [i,S] \in k \Rightarrow i \in \mathcal{K}\}$.

Definition 2 (Quasi-CE Prices). *Prices, p_{qce}, are Quasi-CE prices in subproblem* $\textsc{cap}(\mathcal{K})$*, for some $\mathcal{K} \subseteq \mathcal{I}$, if and only if there is an allocation, S_{qce}, that satisfies:*

(CS1a) for every $i \in \mathcal{K}$, if $S_{\mathrm{qce},i} \neq \emptyset$ then $S_{\mathrm{qce},i}$ solves $\max_{S \subseteq \mathcal{G}} \pi_i(S)$, with $\pi_i(S) = [v_i(S) - p_i(S)]^+$.

(CS2) for the seller, allocation S_{qce} solves $\max_{k \in \Gamma(\mathcal{K})} \pi^s(k)$, with $\pi^s(k) = \sum_{[i,S] \in k} p_i(S)$.

Notice that Quasi-CE prices do not need to satisfy (CS1b), i.e. some agents that receive no bundle might have positive surplus. This weaker requirement differentiates Quasi- from full CE prices.

Definition 3 (Universal Quasi-CE Prices). *Prices, p, are Universal Quasi-CE prices if and only if they are CE prices in problem $\textsc{cap}(\mathcal{I})$ and quasi-CE prices in each subproblem $\textsc{cap}(\mathcal{I} \setminus i)$ induced by removing agent i.*

Since CE prices in subproblem $\textsc{cap}(\mathcal{I} \setminus i)$ are also Quasi-CE prices, clearly Universal CE prices are also Universal Quasi-CE prices.

Definition 4 (individual-minimal CE prices). *The individual-minimal CE prices to agent j, $p^{\min,j}$, maximize the surplus across all third-order CE prices.*

Given individual-minimal prices, $p^{\min,j}$, the prices to agent i are denoted, $p_i^{\min,j}$. Let $\overline{\overline{\pi}}_j$ denote the surplus to agent j in the individual-minimal CE for agent j. The following restriction of dual LP, [DLP$_3$], computes this individual-minimal equilibrium.

$$
\begin{aligned}
\overline{\overline{\pi}}_j = \max_{\pi_i, p_i(S), \pi^s} \ & \pi_j && [\mathrm{RD}(j)] \\
\text{s.t.} \quad & \pi^s + \sum_{i \in \mathcal{I}} \pi_i = V(\mathcal{I}) && (*) \\
& \pi_i + p_i(S) \geq v_i(S), \quad \forall i \in \mathcal{I}, S \subseteq \mathcal{G} && (\text{RD-1}) \\
\pi^s - & \sum_{[i,S] \in k} p_i(S) \geq 0, \quad \forall k \in \Gamma && (\text{RD-2}) \\
& \pi_i, p_i(S), \pi^s \geq 0, \quad \forall i \in \mathcal{I}, S \subseteq \mathcal{G}
\end{aligned}
$$

Constraint (*) ensures that the solution to this restricted dual is an optimal solution to [DLP$_3$], with $V(\mathcal{I})$ equal to the value of the efficient allocation. Constraints (RD-1) and (RD-2) ensure that the solution is feasible in [DLP$_3$].

Definition 5 (group-minimal CE prices). *Group-minimal CE prices, $p^{\min}$, are third-order CE prices that maximize the total surplus to the agents.*

Let $(\overline{\pi}_1, \ldots, \overline{\pi}_n)$ denote the payoff to each agent in a group-minimal CE price solution. The following restriction of dual LP [DLP$_3$] computes the group-minimal equilibrium.

$$(\overline{\pi}_1, \ldots, \overline{\pi}_n) = \arg \max_{\pi_i, p_i(S), \pi^s} \sum_{i \in \mathcal{I}} \pi_i \quad \text{[RD]}$$

$$\text{s.t.} \quad \pi^s + \sum_{i \in \mathcal{I}} \pi_i = V(\mathcal{I}) \quad (*)$$

$$\pi_i + p_i(S) \geq v_i(S), \quad \forall i \in \mathcal{I}, S \subseteq \mathcal{G} \quad \text{(RD-1)}$$

$$\pi^s - \sum_{[i,S] \in k} p_i(S) \geq 0, \quad \forall k \in \Gamma \quad \text{(RD-2)}$$

$$\pi_i, p_i(S), \pi^s \geq 0, \quad \forall i \in \mathcal{I}, S \subseteq \mathcal{G}$$

The constraints in [RD] are identical to the constraints in [RD(j)], it is only the objective function that is different. We revisit these formulations in Section 3, presenting alternative formulations in terms of complementary slackness conditions, from which we derive methods to adjust CE prices to minimal CE prices and Vickrey payments.

2.3 Vickrey Payoffs

In this section we demonstrate that the payoff to an agent in the VCG mechanism, its *Vickrey payoff*, is always supported at the minimal CE prices defined for that agent. In comparison, CE prices will only simultaneously support the Vickrey payoff to every agent when agents are more like substitutes than complements. We care about this relationship between CE prices and Vickrey payoffs because CE prices provide a useful certificate that the outcome of an ascending-price auction is efficient, and implementing Vickrey payoffs allows ascending-price auctions to inherit useful incentive-compatibility properties from VCG mechanisms.

The Vickrey-Clarke-Groves [52, 13, 24] mechanism for the CAP, sometimes called the Generalized Vickrey Auction (GVA), is a strategyproof, *ex post* individual-rational, and efficient direct-revelation mechanism.

Let $\hat{v}_i : 2^{\mathcal{G}} \rightarrow \mathbb{R}_+$, denote the reported valuation function of agent i, $\hat{V}(\mathcal{I})$, denote the reported value of the allocation that maximizes the reports, and $\hat{V}(\mathcal{I} \setminus i)$ denote the reported value of the allocation that maximizes the reports without agent i. The VCG computes allocation, $\hat{S}$, to solve $\hat{V}(\mathcal{I}) = \max_{k \in \Gamma} \sum_{[i,S] \in k} \hat{v}_i(S)$ and also computes $\hat{V}(\mathcal{I} \setminus i) = \max_{k \in \Gamma(\mathcal{I} \setminus i)} \sum_{[i,S] \in k} \hat{v}_i(S)$ for every i with $\hat{S}_i \neq \emptyset$. Allocation, $\hat{S}$, is implemented, and agent i makes payment $\hat{v}_i(S_i^*) - (\hat{V}(\mathcal{I}) - \hat{V}(\mathcal{I} \setminus i))$ to the auctioneer, which is exactly zero for agents not in the final allocation.

The dominant strategy, for every agent i, is truth-revelation, with $\hat{v}_i = v_i$ [24]. Given this, the VCG mechanism is a strategyproof and truthful implementation of the efficient allocation, S^*. In equilibrium, agent i makes payment $p_{\text{vick},i} = v_i(S_i^*) - (V(\mathcal{I}) - V(\mathcal{I} \setminus i))$, and receives its Vickrey payoff.

Definition 6 (Vickrey payoff). *The Vickrey payoff,*

$$\pi_{\text{vick},i} = V(\mathcal{I}) - V(\mathcal{I} \setminus i)$$

is the utility to agent i in the equilibrium outcome of the GVA.

The VCG mechanism is an interesting special case of the family of Groves mechanisms, which includes all strategyproof mechanisms for the class of efficient allocation problems with quasi-linear preferences [23].

Proposition 1. *[30] The VCG mechanism maximizes surplus to the seller out of all efficient, strategy-proof, and* interim *individual-rational mechanisms.*

To demonstrate the correspondence between individual-minimal CE prices and Vickrey payments, let $\mathcal{T}^*$ denote the set of agents in the efficient allocation, and restate LP, [RD(j)], as a combinatorial optimization problem, [RD(j)'], defined over payoffs $(\pi_1, \ldots, \pi_n)$. In constructing [RD(j)'], π^s is expressed as $\pi^s = V(\mathcal{I}) - \sum_i \pi_i$, which satisfies (*) in [RD(j)], and prices $p_i(S) = [v_i(S) - \pi_i]^+$ satisfy (RD-1), which by substitution into (RD-2) and simplification gives constraints (RD-3) in [RD(j)'].[2]

$$\overline{\overline{\pi}}_j = \max_{(\pi_1,\ldots,\pi_n)} \pi_j \qquad \text{[RD}(j)\text{']}$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{L}} \pi_i \leq V(\mathcal{I}) - V(\mathcal{I} \setminus \mathcal{L}), \quad \forall \mathcal{L} \subseteq \mathcal{T}^* \qquad \text{(RD-3)}$$

$$\pi_i = 0, \quad \forall i \notin \mathcal{T}^*, \pi_i \geq 0, \quad \forall i \in \mathcal{T}^*$$

The individual-minimal CE prices are computed from the solution, $\overline{\overline{\pi}}_j$, as:

$$p_i^{\min,j}(S) = \begin{cases} v_j(S) - \overline{\overline{\pi}}_j & \text{, if } i = j \\ v_i(S) & \text{, otherwise.} \end{cases}$$

Proposition 2. *[45, Theorem 6] The Vickrey payoff to agent j is supported at individual-minimal CE prices for that agent.*

Proof. Solution $\overline{\overline{\pi}}_j = V(\mathcal{I}) - V(\mathcal{I} \setminus j) = \pi_{\text{vick},j}$, with $\pi_i = 0$ for all $i \neq j$, is feasible and optimal. Feasible, because constraint (RD-3) holds with equality for $\mathcal{K} = \{j\}$, and constraint (RD-3) holds with weak inequality for all $\mathcal{K} \supset \{j\}$, since the value $V(\mathcal{I} \setminus \mathcal{K})$ is weak-monotonic decreasing as agents are added to $\mathcal{K}$. Optimal, because constraint $\pi_j \leq V(\mathcal{I}) - V(\mathcal{I} \setminus j)$ is tight. □

As an immediate implication, we have that the payoff to agent j in any CE outcome is weakly bounded-above by its payoff at the Vickrey outcome.

In comparison, the Vickrey payoff is only supported *simultaneously* to every agent in competitive equilibrium when agents are substitutes.

Definition 7 (agents are substitutes).

$$V(\mathcal{I}) - V(\mathcal{K}) \geq \sum_{l \in \mathcal{I} \setminus \mathcal{K}} [V(\mathcal{I}) - V(\mathcal{I} \setminus l)] \quad \forall \mathcal{K} \subseteq \mathcal{I}$$

[2] Constraints (RD-3) have an equivalent interpretation as constraints for a vector of payoffs to be in the core [4]. Computing individual-minimal CE prices is equivalent to computing a buyer-optimal core outcome.

where $V(\mathcal{K})$ denotes the value of the optimal solution to CAP$(\mathcal{K})$*, the combinatorial allocation problem defined only over the set $\mathcal{K} \subseteq \mathcal{I}$ of agents.*

Proposition 3. *[11] The Vickrey payoff can be supported simultaneously to every agent in a group-minimal competitive equilibrium if and only if agents are substitutes.*

The substitutes condition was introduced by Ausubel [6], as a necessary and sufficient condition to support Vickrey payoffs in the core of the coalitional game defined by the combinatorial allocation problem, and by Bikchandani & Ostroy [11] in the present context of minimal CE prices and Vickrey payments.

Perhaps the easiest way to understand the agents are substitutes condition is through a combinatorial formulation of the restricted dual, [RD], for the group-minimal CE prices, that parallels formulation [RD(j)'] of the individual-minimal restricted dual, [RD(j)]:

$$(\overline{\pi}_1, \ldots, \overline{\pi}_n) = \arg \max_{(\pi_1, \ldots, \pi_n)} \sum_{i \in \mathcal{I}} \pi_i \qquad \text{[RD']}$$

$$\text{s.t.} \sum_{i \in \mathcal{L}} \pi_i \leq \quad V(\mathcal{I}) - V(\mathcal{I} \setminus \mathcal{L}), \quad \forall \mathcal{L} \subseteq \mathcal{T}^* \qquad \text{(RD-3)}$$

$$\pi_i = 0, \quad \forall i \notin \mathcal{T}^*, \pi_i \geq 0, \quad \forall i \in \mathcal{T}^*$$

Group-minimal prices, $p^{\min}(S)$, are computed as $p_i^{\min}(S) = v_i(S) - \overline{\pi}_i$ for all $i \in \mathcal{I}$, given solution $(\overline{\pi}_1, \ldots, \overline{\pi}_n)$. Proposition 3 immediately follows, with substitution $\pi_{\text{vick},i} = V(\mathcal{I}) - V(\mathcal{I} \setminus i)$, since agents are substitutes is equivalent to constraint set (RD-3).

In fact, agents are substitutes exactly captures a condition on preferences required for the *uniqueness* of group-minimal CE prices. Uniqueness is defined with respect to the payoffs of agents in equilibrium, prices p_{ce} and p'_{ce} are equivalent if they provide the same surplus, π_i, to every agent i in equilibrium.

Proposition 4. *[42, 4] Group-minimal CE prices are unique if and only if they support Vickrey payoffs simultaneously to every agent.*

Proof. ($\Leftarrow$) Assume that $\overline{\pi}_i = \pi_{\text{vick},i}$ for every i, but that group-maximal CE payoffs are not unique. Any alternative group-maximal payoff vector must provide more payoff to at least one agent, but $\overline{\overline{\pi}}_i = \pi_{\text{vick},i}$. ($\Rightarrow$) Assume that $\overline{\pi}_j \neq \pi_{\text{vick},j}$ for some agent, j. Then, the constraint $\pi_j \leq V(\mathcal{I}) - V(\mathcal{I} \setminus j)$ in [RD'] must not be binding, and some constraint involving j and at least one other agent, say k, must be binding. From this, we can construct multiple equivalent group-maximal payoff vectors from different distributions of marginal product across agents j and k. $\square$

Intuitively, if agents are substitutes then all individual-minimal CE prices intersect and agents can all agree on the same equilibrium, which supports the Vickrey payoff to every agent.

The substitutes condition is a little hard to interpret because it is defined over the characteristic function, $V(\mathcal{I})$, of the coalitional game and not in terms of individual agent preferences. One *necessary* condition is that the total value to the buyers of the efficient allocation must be *at least* the sum of the marginal products of each agent; i.e., $V(\mathcal{I}) \geq \sum_{i\in\mathcal{I}} V(\mathcal{I}) - V(\mathcal{I} \setminus i)$.

In recent analysis, Ausubel & Milgrom [4], relate agents are substitutes to a disaggregated condition on agent preferences, and show that *gross-substitutes* (GS) preferences is a sufficient condition for agents are substitutes. GS is defined in terms of the demand function of an agent given a set of linear prices, and essentially states that an agent that demands bundle S at prices p continues to demand the items in S that do not increase in price if some of the prices on items in S increase [29]. GS preferences are submodular, but there are preferences that are submodular but not GS.

In fact, if agent preferences also include all linear-additive preferences then GS preferences are also *necessary* for agents are substitutes [4]. This is quite a negative result, because GS preferences are the largest set of preferences for which linear CE prices exist [25]. This implies that Vickrey outcomes are not supported in competitive equilibrium by non-linear and non-anonymous prices in the very problems for which non-linear and non-anonymous prices are necessary to support efficient outcomes in competitive equilibrium.

Finally, it is useful to consider the relationship between agents are substitutes and a typical characteristic of CAP problem instances in which free-riding is observed in ascending combinatorial auctions [35, 12]. Let $\mathcal{T}^* \subseteq \mathcal{I}$ denote the winning coalition of agents, the agents that receive items in that allocation, and let $V(\mathcal{I} \setminus \mathcal{T}^*)$ denote the value of the efficient allocation computed for the agents in the losing coalition. We say that an agent in the winning coalition is *critical* if the losing coalition's allocation is efficient without the agent.

Definition 8 (critical). *An agent l in the winning coalition is critical if $V(\mathcal{I}) > V(\mathcal{I} \setminus l)$, and $V(\mathcal{I} \setminus l) = V(\mathcal{I} \setminus \mathcal{T}^*)$, where $\mathcal{T}^*$ are the agents in the winning coalition.*

Proposition 5. *Agents are substitutes fails whenever at least two agents in the winning coalition, $\mathcal{T}^*$, are critical, and $\mathcal{I} \setminus \mathcal{T}^* \neq \emptyset$.*

Proof. Let $\mathcal{K} = (\mathcal{I} \setminus \mathcal{T}^*)$, the losing coalition of agents. Let $b = V(\mathcal{I})$ and $c = V(\mathcal{K})$. Then, we have $V(\mathcal{I}) - V(\mathcal{K}) = b - c < N(b-c) = \sum_{l\in\mathcal{I}\setminus\mathcal{K}} V(\mathcal{I}) - V(\mathcal{I} \setminus l)$, where $N = |\mathcal{T}^*| > 1$ is the number of critical agents in the winning coalition. □

Free-riding occurs when there are two or more agents that must coordinate their bids to out-bid another coalition of agents, and the minimal group CE prices are not unique. This non-uniqueness creates a bargaining problem, each agent wants to implement the CE outcome in which their individual price is minimized. In turn, this leads to free-riding, with each agent preferring the other agents to bid up the price on the bundles that will fit with its own bundle. It is a reasonable hypothesis that typical ascending combinatorial auctions will tend to suffer from free-riding when agents are substitutes fails. By implementing the Vickrey outcome, *i*BEA is protected against this efficiency-reducing coordination game.

3 Adjusting to Vickrey Payoffs

The novel and fundamental connections between equilibrium solutions and Vickrey payoffs presented in this section are a crucial step in the development of an ascending-price GVA. The process of computing Vickrey payments from equilibrium information without requiring that the payments correspond to any single equilibrium represents the main departing point from earlier studies [11, 10].

Assuming that equilibrium prices and the efficient allocation represent the *only* information available about the preferences of agents, we demonstrate that Universal CE prices are necessary and sufficient to compute individual-minimal CE prices. A simple method is proposed to compute individual-minimal CE prices, and Vickrey payments, from Universal CE prices. This method is used to adjust prices toward Vickrey payments after *i*BEA terminates.

3.1 Adjusting CE Prices to Individual Minimal-CE Prices

The first step is to reformulate [RD(j)] to remove explicit information about agent values, and replace with complementary-slackness conditions to ensure allocative-efficiency. Let $p_{\mathrm{ce},i}(S)$ denote competitive equilibrium prices, and introduce variables $\Delta_i \geq 0$ to denote a *discount* to agent i from its current CE prices. Consider LP formulation [RD-CS(j)], that computes discounts to adjust CE prices towards the individual-minimal CE prices for agent j while maintaining CS conditions with the efficient allocation, S^*.

$$\Delta_{\mathrm{adjust},j} = \max_{p_i(S),(\Delta_1,\ldots,\Delta_n)} \Delta_j \qquad \text{[RD-CS}(j)\text{]}$$

$$\text{s.t.} \quad p_i(S_i^*) \leq p_{\mathrm{ce},i}(S_i^*) - \Delta_i, \quad \forall i \in \mathcal{T}^* \qquad \text{(rdcs-1)}$$

$$p_i(S) \geq p_{\mathrm{ce},i}(S) - \Delta_i, \quad \forall i \in \mathcal{I}, \forall S \subseteq \mathcal{G} \qquad \text{(rdcs-2)}$$

$$\Delta_i = 0, \quad \forall i \notin \mathcal{T}^* \qquad \text{(rdcs-3)}$$

$$\sum_i p_i(S_i^*) \geq \sum_{[i,S]\in k} p_i(S), \quad \forall k \in \Gamma \qquad \text{(rdcs-4)}$$

$$p_i(S) \geq 0, \forall i, \forall S, \quad \Delta_i \geq 0, \forall i$$

Without explicit information about the *value* of the efficient allocation, $V(\mathcal{I})$, but with information about the *identity* of the efficient allocation, constraint (*) in [RD(j)] is replaced with constraints (rdcs-1,rdcs-2,rdcs-3) that maintain (CS1a) and (CS1b) and constraints (rdcs-4) maintain (CS2). Dual variables π_i and π^s are implicit in this formulation; agent surplus, π_i, increases by $\Delta_{\mathrm{adjust},i}$, and seller surplus, π^s, falls by $\sum_i \Delta_{\mathrm{adjust},i}$, but remains positive because $\Delta_{\mathrm{adjust},i} \leq p_{\mathrm{ce},i}(S_i^*)$ for each agent.

In turn, it is useful to reformulate [RD-CS(j)] as a combinatorial optimization problem, [RD-CS(j)']:

$$\begin{aligned}\Delta_{\text{adjust},j} = &\max_{(\Delta_1,\ldots,\Delta_n)} \Delta_j && \text{[RD-CS}(j)\text{']}\\ &\text{s.t.} \quad \sum_{i\in\mathcal{L}} \Delta_i \leq P(\mathcal{I}) - P(\mathcal{I}\setminus\mathcal{L}), \quad \forall \mathcal{L}\subseteq\mathcal{T}^*\\ &\quad \Delta_i \geq 0, \forall i, \qquad \Delta_i = 0, \quad \forall i \notin \mathcal{T}^*\end{aligned}$$

with $P(\mathcal{K}) = \max_{k\in\Gamma(\mathcal{K})} \sum_{[S,i]\in k} p_i(S_i)$, the maximal surplus to the seller given prices, $p_i(S)$, over allocations restricted to agents in set $\mathcal{K}\subseteq\mathcal{I}$.

The adjusted prices are computed as $p_{\text{adjust},j}(S) = p_{\text{ce},j} - \Delta_{\text{adjust},j}$ to agent j, with the prices unchanged to other agents.

Proposition 6 (upper-bound). *Price adjustment, $\Delta_{\text{adjust},j} = P(\mathcal{I}) - P(\mathcal{I}\setminus j)$, solves [RD-CS($j$)'], computing the maximal price adjustment to agent j that maintains conditions for CE prices.*

Proof. $\Delta_{\text{adjust},j}$ is a feasible solution, because $P(\mathcal{I}\setminus\mathcal{L})$ is monotonically decreasing in $\mathcal{L}$, and optimal because constraint $\Delta_j \leq P(\mathcal{I}) - P(\mathcal{I}\setminus j)$ is tight. Again, $\Delta_{\text{adjust},j} = P(\mathcal{I}) - P(\mathcal{I}\setminus j) \leq p_{\text{ce},j}(S_j^*)$. □

Discount $\Delta_{\text{adjust},j}$ is an information-theoretic upper-bound on the additional surplus that agent j can receive in competitive equilibrium. After the price adjustment, (CS1a) continues to hold for agent j, and (CS2) is binding for the seller, with $P(\mathcal{I}) - \Delta_{\text{adjust},j} = P(\mathcal{I}\setminus j)$. Implementing a smaller price to agent j while maintaining (CS2) requires a change in price to one of the other agents that might violate either (CS1a) or (CS1b) to that agent.

Theorem 1 (sufficient). *Given prices, p_{ce}, and efficient allocation, S^*, then it is sufficient that prices are Universal to compute Vickrey payments.*

Proof. Consider agent j. Universal CE prices are CE in subproblem CAP($\mathcal{I}\setminus j$), and therefore $V(\mathcal{I}\setminus j)+\sum_{i\neq j}\pi_i = P(\mathcal{I}\setminus j)$, and the adjusted surplus, $\pi_{\text{adjust},j} = v_j(S_j^*) - p_{\text{ce},j}(S_j^*) + (P(\mathcal{I}) - P(\mathcal{I}\setminus j)) = v_j(S_j^*) - p_{\text{ce},j}(S_j^*) + [V(\mathcal{I}) - \sum_i \pi_i] - [V(\mathcal{I}\setminus j) - \sum_{i\neq j}\pi_i] = \pi_{\text{vick},j}$, because $v_j(S_j^*) - p_{\text{ce},j}(S_j^*) = \pi_j$ and all terms except $V(\mathcal{I}) - V(\mathcal{I}\setminus j)$ cancel. □

The efficient allocation, S^*, is used to know how to break ties in the case of two allocations with the same surplus to the seller. Notice that the standard VCG payment is recovered as a special case because equating prices to agent valuation functions is a trivial method to generate Universal CE prices. Notice also that $p_{\text{vick},j} = 0$ iff $P(\mathcal{I}) = P(\mathcal{I}\setminus j) + p_{\text{ce},j}(S_j^*)$, which corresponds with $V(\mathcal{I}) = V(\mathcal{I}\setminus j) + v_j(S_j^*)$.

Theorem 2 (necessary). *If the Vickrey payments can be computed from CE prices and the efficient allocation, S^*, then prices must be Universal CE.*

Proof. Given Proposition 6, this is proved with respect to adjusted prices computed with discount, $\Delta_{\text{adjust},j}$. Without Universal CE prices there must always remain residual uncertainty in the maximal discount that could be computed across all CE prices. Let p_{ce} denote the initial CE prices, and assume a violation of either (CS1a,CS1b) or (CS2) in subproblem $\textsc{cap}(\mathcal{I} \setminus j)$. Case (a), in which (CS1a) fails for an agent $i \neq j$ that is in $\mathcal{T}^*$ but not in $\mathcal{T}^*_{-j}$. Agent i still has positive surplus for some bundle, S', $p_{\text{ce},i}(S') = v_i(S') - \delta$, for some $\delta > 0$; then, since prices are CE, it must be the case that $p_{\text{ce},i}(S_i^*) \leq v_i(S_i^*) - \delta$, and that equilibrium surplus $\pi_i \geq \delta > 0$. Then, higher prices $p'_{\text{ce},i}(S) = p_{\text{ce},i}(S) + \epsilon$, for some $\epsilon > 0$, maintain (CS1a) for agent i, while also increasing the discount, $\Delta_{\text{adjust},j}$ computed to agent j. Finally, there is not enough information to determine the maximal increase, ϵ, presenting a contradiction. Similar arguments can be constructed when (CS1a) or (CS1b) fails in subproblem $\textsc{cap}(\mathcal{I} \setminus j)$ for an agent $i \neq j$ that is: (b) in $\mathcal{T}^*$ and in $\mathcal{T}^*_{-j}$; (c) not in $\mathcal{T}^*$ but in $\mathcal{T}^*_{-j}$. A trivial contradiction follows in the case that (CS1a) and (CS1b) hold for all agents $i \neq j$, but (CS2) fails. $\square$

Proposition 7 (anonymous prices). *If anonymous CE prices are Universal then agents are substitutes.*

Proof. Immediate, because $P(\mathcal{I}) = P(\mathcal{I} \setminus i) = 0$ and $\Delta_{\text{adjust},i} = 0$ for all agents i, and therefore the CE prices already support Vickrey payoffs. $\square$

In situations in which agents are not substitutes, then only non-anonymous CE prices can extract enough information about agent preferences to implement the Vickrey outcome.

3.2 Adjusting CE Prices to Group-Minimal CE Prices

In the special case that agents are substitutes, we can characterize simpler conditions to compute Vickrey payments in terms of the conditions required to adjust to group-minimal CE prices. In addition, recent analysis suggests that auctions that terminate in group-minimal CE prices with straightforward bidding have interesting incentive and reputation properties [4], even when these prices do not support Vickrey payments.

Again, we reformulate the restricted dual LP, [RD], to compute the group-minimal CE prices in terms of complementary-slackness conditions, and then as a combinatorial optimization problem, [RD-CS'].

$$(\Delta^g_{\text{adjust},1}, \ldots, \Delta^g_{\text{adjust},I}) = \arg \max_{(\Delta_1,\ldots,\Delta_n)} \sum_{i \in \mathcal{I}} \Delta_i \qquad \text{[RD-CS']}$$
$$\text{s.t.} \quad \sum_{i \in \mathcal{L}} \Delta_i \leq P(\mathcal{I}) - P(\mathcal{I} \setminus \mathcal{L}), \quad \forall \mathcal{L} \subseteq \mathcal{T}^*$$
$$\Delta_i \geq 0, \forall i, \qquad \Delta_i = 0, \quad \forall i \notin \mathcal{T}^*$$

It is useful to define prices that are a simple translation of an agent's valuation function.

Definition 9 (negative translation). *Prices, $p_{\rm ntv}$, are negative translation of agent value (NTV) prices if and only if $p_{{\rm ntv},i}(S) = [v_i(S) - c_i]^+$ for all $S \subseteq \mathcal{G}$ and $i \in \mathcal{I}$, for some constant, $c_i \geq 0$.*

Let $c_{{\rm ce},i} \geq 0$ denote the parameterization of a set of NTV and CE prices, with $c_{{\rm ce},i} = 0$ for all i not in the efficient allocation. These prices must satisfy $P(\mathcal{I}) = V(\mathcal{I}) - \sum_{i \in \mathcal{I}^*} c_{{\rm ce},i}$ from basic duality.

Lemma 1 (ntv 1). *If CE prices are NTV then there is enough information to adjust prices to group-minimal CE prices.*

Proof. The special structure provided by NTV prices allows a simplification of the combinatorial constraints in [RD-CS'], from which the connection between adjusted CE prices and Vickrey payments is immediate. The full proof is in the Appendix. □

This explains why *i*Bundle(3) [41, 44] and Ausubel & Milgrom's ascending-proxy auction can compute the Vickrey outcome when agents are substitutes. In both auctions agents' bid prices are approximate NTV prices, and both auctions terminate in competitive equilibrium.

The surplus from NTV prices is related to the surplus from Quasi-CE prices, which leads sufficient conditions for Quasi-CE prices to provide enough information to adjust to group-minimal CE prices. Let $\mathcal{C}$ denote a set of subsets of agents.

Lemma 2 (ntv 2). *For any CE prices $p_{\rm ce}$ that are Quasi-CE in subproblems* CAP$(\mathcal{K})$ *defined over sets of agents $\mathcal{K} \in \mathcal{C}$, there is a set of NTV and CE prices, $p_{\rm ntv}$, such that the seller has the same maximal surplus in problem* CAP$(\mathcal{I})$ *and all subproblems* CAP$(\mathcal{K})$*, for $\mathcal{K} \in \mathcal{C}$, and prices are unchanged on bundles in the efficient allocation.*

Proof. See the appendix. □

The following property is used in the analysis of *i*BEA to show that the auction terminates immediately at the end of Phase I, and as soon as a single set of CE prices have been discovered, when agents are substitutes.

Proposition 8. *If agents are substitutes, then Universal Quasi-CE prices are also Universal CE prices, and provide enough information to compute Vickrey payments.*

Proof. Consider a set of NTV and CE prices, $p_{\rm ntv}$. By Lemma 1, these prices provide enough information to compute group-minimal CE prices. Because agents are substitutes, the group-minimal CE prices are also individual-minimal CE prices, and can be computed with discount $\Delta_{{\rm adjust},j} = P(\mathcal{I}) - P(\mathcal{I} \setminus j)$. Finally, by Lemma 2, given Universal Quasi-CE prices, that are Quasi-CE for each subproblem CAP$(\mathcal{I} \setminus j)$, one can construct NTV prices with the same $\Delta_{{\rm adjust},j}$ and adjusted prices as would be computed directly with the Universal Quasi-CE prices. Universal Quasi-CE prices provide enough information to adjust to Vickrey payments, and must also be Universal CE prices by Theorem 2. □

We isolate another interesting special case for group-minimal CE prices, when *agents are complements*. Agents are complements is slightly stronger than the negation of agents are substitutes.

Definition 10 (agents are complements).

$$V(j \cup \mathcal{T}) - V(\mathcal{T}) \geq V(j \cup \mathcal{S}) - V(\mathcal{S})$$

for every $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{T}^*$, and agent $j \notin \mathcal{S}$, where $\mathcal{T}^*$ is the set of agents in the efficient allocation.

If agents are complements, the combinatorial formulation, [RD'], of the group-minimal CE price problem is a submodular optimization problem, with constraints (RD-3) written as $\sum_{i \in \mathcal{L}} \pi_i \leq f(\mathcal{L}), \quad \forall \mathcal{L} \subseteq \mathcal{T}^*$, for submodular $f(\mathcal{L}) = V(\mathcal{I}) - V(\mathcal{I} \setminus \mathcal{L})$. In turn, this special structure allows a greedy algorithm to compute a set of payoffs compatible with group-minimal CE prices [54]:

(i) Select an arbitrary order for the agents in the efficient allocation, w.o.l.g. 1, 2, …, m where $m = |\mathcal{T}^*|$.

(ii) Set $\overline{\pi}_1 = V(\mathcal{I}) - V(\mathcal{I} \setminus 1)$, $\overline{\pi}_2 = V(\mathcal{I} \setminus 1) - V(\mathcal{I} \setminus \{1,2\})$, $\overline{\pi}_3 = V(\mathcal{I} \setminus \{1,2\}) - V(\mathcal{I} \setminus \{1,2,3\})$, etc. and $\overline{\pi}_i = 0$ for all $i \notin \mathcal{T}^*$.

Proposition 9. *The Vickrey payoff to any agent i is supported in some set of group-minimal CE prices when agents are complements.*

Proof. Constructive. The adjusted payoff, $\overline{\pi}_1 = V(\mathcal{I}) - V(\mathcal{I} \setminus 1) = \pi_{\text{vick},1}$, for the first agent, and any agent can be selected as the first agent in the greedy algorithm. □

In the special case of NTV and CE prices and agents are complements, the corresponding price-based formulation of the group-minimal CE prices problem is also submodular (see [RD-NTV] in the Appendix). The same greedy algorithm can be used to compute group-minimal CE prices from NTV prices; this time computing $\Delta^g_{\text{adjust},1} = P(\mathcal{I}) - P(\mathcal{I} \setminus 1), \Delta^g_{\text{adjust},2} = P(\mathcal{I} \setminus 1) - P(\mathcal{I} \setminus \{1,2\}), \ldots$, for some order of agents in $\mathcal{T}^*$, with $\Delta^g_{\text{adjust},i} = 0$ for $i \notin \mathcal{T}^*$. From this, we can derive a sufficient condition to compute group-minimal CE prices when agents are complements, in terms of Quasi-CE prices.

Let $\xi \in \Xi(\mathcal{K})$ define some permutation $\xi = (\xi_1, \ldots, \xi_m)$, or order, over agents in set $\mathcal{K}$, $m = |\mathcal{K}|$, with set $\Xi(\mathcal{K})$ used to denote all possible permutations. Let $\mathcal{S}_j(\xi) = \{\xi_1, \ldots, \xi_j\} \subseteq \mathcal{K}$ contain the first j agents in an order ξ.

Definition 11 (chain). *Given order $\xi \in \Xi(\mathcal{K})$ define a chain of sets of agents as $\mathcal{C}_\xi = \{(\mathcal{I} \setminus \mathcal{S}_j(\xi)) \mid j \in \{1, \ldots, m\}\ \}$, where $m = |\mathcal{K}|$, and $\mathcal{S}_j(\xi)$ is the first j agents in order ξ.*

For example, $\xi = (1,2,3)$, is a valid ordering for set $\{1,2,3\} \subseteq \mathcal{I}$, with chain $\mathcal{C}_\xi = \{(\mathcal{I} \setminus \{1\}), (\mathcal{I} \setminus \{1,2\}), (\mathcal{I} \setminus \{1,2,3\})\}$ and $S_1(\xi) = \{1\}, S_2(\xi) = \{1,2\}$, etc.

	A	B	AB
Agent 1	1	1	6^*
Agent 2	4	4	5

Example 1.

	A	B	AB
Agent 1	8^*	9	12
Agent 2	6	8^*	14

Example 2.

Figure 1: Motivating examples for Combinatorial Auctions. In Example 1, no linear CE prices exist, while in Example 2, no linear CE prices exist that support Vickrey payments. The efficient allocation is indicated *.

Theorem 3 (sufficient). *If agents are complements, any CE prices that are also Quasi-CE on subproblems* $\textsc{cap}(\mathcal{K})$ *defined on sets* $\mathcal{K} \in \mathcal{C}_\xi$*, where chain* $\mathcal{C}_\xi$ *is defined for some ordering,* $\xi \in \Xi(\mathcal{T}^*)$*, of the agents,* $\mathcal{T}^*$*, in the efficient allocation are sufficient to compute group-minimal CE prices.*

Proof. Immediate from Lemma 2, as long as the greedy algorithm is used in the same order as the chain of subproblems for which Quasi-CE holds. □

As an example, suppose $\mathcal{T}^* = \{1,2,3\}$ and the set $\mathcal{I} = \{1,2,3,4,5\}$. Then, if agents $\mathcal{T}^*$ are complements it is sufficient that CE prices are also Quasi-CE for subproblems $\textsc{cap}(\{2,3,4,5\})$, $\textsc{cap}(\{3,4,5\})$ and $\textsc{cap}(\{4,5\})$, with $\Delta^g_{\text{adjust},1} = P(\mathcal{I}) - P(\{2,3,4,5\})$, $\Delta^g_{\text{adjust},2} = P(\{2,3,4,5\}) - P(\{3,4,5\})$, etc.

3.3 Examples: Competitive Equilibrium Analysis and Vickrey Payoffs

The following examples are selected to illustrate the competitive equilibrium analysis and price-adjustment methods introduced in this section.

Examples 1 and 2 motivate the combinatorial auction problem. In Example 1 there are no single-item (linear) CE prices, and non-combinatorial auctions suffer from the *exposure problem* [36]. In Example 2, while linear CE prices exist, the prices cannot support Vickrey payments, and straightforward bidding is not an equilibrium [26]. Not only do non-linear (perhaps non-anonymous) CE prices always exist, but in these problems agents are substitutes, and group-minimal CE prices support Vickrey payments.

In Example 3, which is used to motivate *i*BEA, the agents are substitutes property fails, and Vickrey payments are not supported in CE, even with non-linear and non-anonymous prices. Bykowsky et al. [12] present an analysis of the free-riding problem for Example 3, in standard ascending-price combinatorial auction designs. Finally, we introduce Examples 4(a) and 4(b), which will be used in Section 4.3 to illustrate *i*BEA. Agents are substitutes fails in Example 4(a), but holds in a simple variation, Example 4(b).

3.3.1 Examples 1 and 2: Motivating Combinatorial Auctions

Consider a problem with 2 goods, $\{A, B\}$, and 2 agents, $\{1, 2\}$. Two examples of agent valuation functions are illustrated in Figure 1. The efficient allocations

	A	**B**	**C**	**AB**	**BC**	**AC**	**ABC**
Agent 1	60	30	30	100	60	100*	156
Agent 2	30	62	20	90	82	94	170
Agent 3	40	75*	20	115	95	60	161

Example 3

Figure 2: Free-riding can occur in standard ascending combinatorial auctions in this problem because agents 1 and 3 must coordinate their bids to out-bid agent 2. Agents are not substitutes and Vickrey payments are not supported in competitive equilibrium.

are indicated *; i.e. $(AB, \emptyset)$ in Example 1 and (A, B) in Example 2. The Vickrey payments are $(5, 0)$ in Example 1, and $(6, 4)$ in Example 2.

First, consider Example 1. Linear CE prices do not exist in this problem: linear CE prices require $p(A) \geq 4$, $p(B) \geq 4$, and $p(A) + p(B) \leq 6$, which is impossible. The essential problem is that the items are synergies to agent 1 but complements to agent 2. The absence of linear CE prices leads to an exposure problem in a simultaneous ascending auction; by bidding straightforwardly agent 1 risks exposure to receiving just a single item, and a negative payoff. To avoid exposure, the agent should bid at most 1 for A and 1 for B, which results in inefficiency. A combinatorial auction can readily handle this synergy, since agent 1 can express its contingency, "*A* only if *B*", through a bid on bundle AB.

Second, consider Example 2. Linear CE prices do exist in this problem, but the group-minimal CE prices, $p(A) = 6, p(B) = 7$, do not support Vickrey payments. However, in this problem, and in Example 1, agents are substitutes holds and non-linear and non-anonymous prices can support Vickrey payments in competitive equilibrium. Group-minimal CE prices are $p_1 = (6, 7, 10)$ and $p_2 = (2, 4, 10)$, with $p_1(A) = p_{\text{vick},1}$ and $p_2(B) = p_{\text{vick},2}$. Universal CE prices, for example $p_1 = (7, 8, 11)$ and $p_2 = (3, 5, 11)$, provide enough information to compute Vickrey payments: $p_{\text{vick},1} = p_1(A) - \Delta_{\text{adjust},1} = p_1(A) - (P(12) - P(2)) = 7 - (12 - 11) = 6$ and $p_{\text{vick},2} = p_2(B) - \Delta_{\text{adjust},2} = 5 - (12 - 11) = 4$.

3.3.2 Example 3: A Problem with Free-riding

Now, consider the problem illustrated in Figure 2, with 3 goods, $\{A, B, C\}$, and 3 agents, $\{1, 2, 3\}$. In this problem, the efficient allocation, $S^* = (AC, \emptyset, B)$, and Vickrey payments are $(95, 70)$.

The free-rider problem characterizes standard ascending combinatorial auction designs for this example, because agents 1 and 3 must coordinate their bids to defeat agent 2 but each agent prefers to wait while the other agent bids and accepts more of the cost of out-bidding agent 2. As we might expect, both agents 1 and 3 are *critical* (Def 8), and agents are substitutes fails to hold, because $V(123) - V(23) = 175 - 70 > V(12) - V(2) = 170 - 170$, and the marginal product of agent 1 is greater with agent 3 than without agent 3. Group minimal CE prices require $p_1(AC) + p_3(B) = 170$, $94 \leq p_1(AC) \leq 100$, and $62 \leq p_3(B) \leq 75$, with $p_2(S) = v_2(S)$, while Vickrey payments require

	A	B	AB
Agent 1	30*	0	30
Agent 2	0	40*	40
Agent 3	0	20	40

Example 4(a).

	A	B	AB
Agent 1	30*	0	30
Agent 2	0	40*	40
Agent 3	0	20	40
Agent 4	25	0	25
Agent 5	0	25	25

Example 4(b).

Figure 3: Example 4(a): agents are not substitutes, and Vickrey payments are not supported in competitive equilibrium. Example 4(b): with the addition of agents 4 and 5, agents are substitutes. *i*BEA terminates with the Vickrey outcome in both problems.

$p_1(AC) + p_3(B) = 165 < 170$. Notice that individual minimal CE prices equal Vickrey payments: $p_{\text{ind},1}(AC) = 170 - 75 = p_{\text{vick},1}$ and $p_{\text{ind},2}(B) = 170 - 100 = p_{\text{vick},2}$.

In this example, Universal prices require that agents 1 and 3 face prices $p_1(AC) = v_1(AC) = 100$ and $p_3(B) = v_3(B) = 75$. Neither agent is in the efficient allocation in the subproblem induced by removing the other agent from the auction, and must face (at least) its full value for an empty allocation to maximize its surplus.

3.3.3 Example 4: *i*BEA Problem Instances

Finally, consider Example 4(a) and 4(b) in Figure 3. Example 4(b) is the same as 4(a), with the addition of agents 4 and 5. In both problems the efficient allocation is to allocate A to agent 1 and B to agent 2, however Vickrey payments in Example 4(a) are $(0, 20, 0)$ while Vickrey payments in Example 4(b) are $(25, 25, 0, 0, 0)$. In Section 4.3 we illustrate *i*BEA on both Example 4(a) and Example 4(b), and demonstrate that it terminates with the Vickrey outcome in both cases.

Figure 4 illustrates the space of CE prices in Example 4(a), in which agents are not substitutes, and Vickrey payments are not supported in competitive equilibrium. Group minimal CE prices require $p_1(A)+p_2(B) \geq 40$, $p_1(A) \leq 30$, and $20 \leq p_2(B) \leq 40$, and cannot support Vickrey payments. Universal CE prices require, $p_1(A) \geq 20$, to provide equilibrium for agent 1 without agent 2, and $p_2(B) = 40$, to provide equilibrium for agent 2 without agent 1. For instance, $p_1 = (20, 0, 20)$, $p_2 = (0, 40, 40)$, and $p_3 = (0, 20, 40)$ are UCE prices.

In comparison, in Example 4(b) agents 4 and 5 act as substitutes for agents 1 and 2, the agents are substitutes condition holds, and Vickrey payments can be priced. In this example, the group-minimal CE prices across all non-linear and non-anonymous prices, are *anonymous*, $p_{\text{anon}} = (25, 25, 25)$, to agents $\{1,2,4,5\}$, and individualized, $p_3 = (0, 20, 40)$, to agent 3, and support Vickrey payments.

The *agents are complements* condition holds for agents 1 and 2, in both Examples 4(a) and 4(b). It follows, from the analysis in Section 3.2, that we

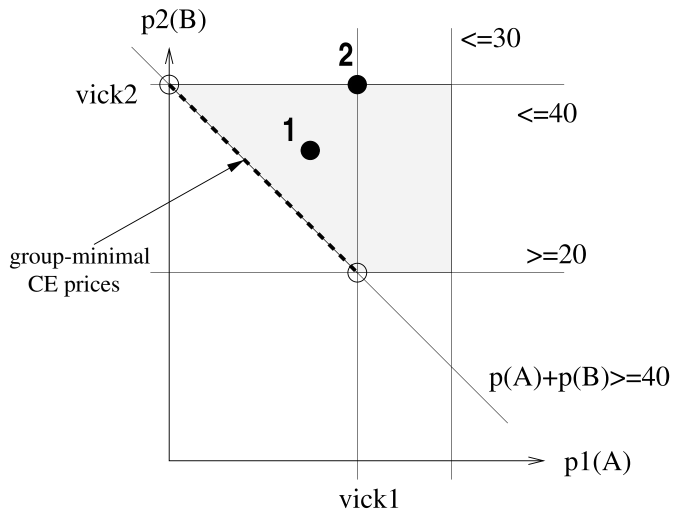

Figure 4: Example 4(a). The shaded region is the space of feasible CE prices, the open circles represent individual-minimal CE prices, and the dashed line is the set of group-minimal CE prices. Moving from prices 1 to 2, we construct Universal CE prices, from which individual-minimal prices can be computed with discounts $\Delta_{\text{adjust},j}$.

can compute group-minimal CE prices with CE prices that are Quasi-CE on a chain of subproblems induced by removing agent 1, and then agent 1 and 2. For instance, in Example 4(a), prices $p_1 = (20, 0, 20)$, $p_2 = (0, 30, 30)$, and $p_3 = (0, 20, 40)$ satisfy this property; e.g. $p_2 = (0, 30, 30)$, $p_3 = (0, 20, 40)$ satisfy quasi-CE in the subproblem without agent 1, because the preferred allocation to the seller is $(\emptyset, \emptyset, AB)$, which is consistent with best-response bids from agent 3. These are not, however *Universal* CE prices, because agent 2 has surplus for item B but receives no allocation. With these prices, the greedy method first computes discount $\Delta_1^* = P(123) - P(23) = 50 - 40 = 10$ to agent 1, and then $\Delta_2^* = P(23) - P(3) = 40 - 40 = 0$ to agent 2, providing group-minimal CE prices, $p_1 = (10, 0, 10)$, $p_2 = (0, 30, 30)$ and $p_3 = (0, 20, 40)$.

4 *i*Bundle Extend & Adjust

In this section we introduce *i*BEA, an efficient ascending-price combinatorial auction. *i*BEA is an indirect mechanism for the direct VCG mechanism for the combinatorial allocation problem. *i*BEA terminates with Universal CE prices, that are adjusted after termination to Vickrey payments, and myopic best-response is an *ex post* Nash equilibrium, with agents submitting truthful demand sets in response to ask prices in each round.

At the center of *i*BEA is a price-update, bidding, and winner-determination rule very much like that introduced in *i*Bundle [40, 44], an earlier auction that terminates with the efficient allocation with straightforward bidding strategies, but only implements Vickrey payments when agents are substitutes [42], and for which the straightforward bidding strategy is not always in Nash equilibrium.

The auction has two distinct phases, that are designed to be indistinguishable to participants. In *Phase I* prices are increased until they are in competitive equilibrium, and maintained in Quasi-CE in each subproblem induced by dropping one agent from the auction. In *Phase II* prices are increased until they are in Universal CE, i.e. in CE for all subproblem induced by dropping one agent, at which point the auction terminates and final prices are adjusted to Vickrey payments.

The analysis of *i*BEA, presented in the next section, uses linear-programming duality theory to prove the theoretical properties of the auction. *i*BEA, with straightforward agent bids, implements a primal-dual algorithm for the VCG mechanism.

4.1 Auction Specification

*i*Bundle Extend & Adjust is an ascending-price combinatorial auction. The auction proceeds in rounds, $t \geq 1$, and maintains both ask prices and a provisional allocation. In general, ask prices can be both non-linear and non-anonymous. Agents can submit *exclusive-or* bids for bundles in each round, to indicate a demand for at most one of the bundles. The provisional allocation is computed to maximize the revenue of the seller given the current bids, with the bids from one agent ignored during each round of Phase II.

It is useful to define weak-dominance:

Definition 12 (weak dominance). *Bid (S', p') weakly-dominates bid (S, p), written $(S', p') \succeq (S, p)$, if and only if $S' \subseteq S$ and $p(S') \geq p(S)$.*

By analogy, bid (S', p') *strongly*-dominates bid (S, p), if and only if $S' \subseteq S$ and $p(S') > p(S)$.

4.1.1 Prices and Bidding Rules

In general the ask prices in any round can be both non-linear and non-anonymous. However, non-anonymous (or discriminatory) pricing is introduced incrementally.

In any particular round, t, a subset, $anon^t \subseteq \mathcal{I}$, of agents still face anonymous ask prices, while the remaining agents face individualized ask prices. Initially, $anon^1 = \mathcal{I}$. Let $p^t_{\text{anon}}(S)$ denote the anonymous price for bundle S in round t, and $p^t_{\text{ind},j}(S)$ denote the individualized price on bundle S to agent $j \in (\mathcal{I} \setminus anon^t)$. It is convenient to define a combined ask price, $p^t_{\text{ask},i}(S)$, to agent i in round t:

$$p^t_{\text{ask},i}(S) = \begin{cases} p^t_{\text{anon}}(S) & \text{, if } i \in anon^t \\ p^t_{\text{ind},i}(S) & \text{, otherwise.} \end{cases}$$

At all times ask prices are maintained to be *consistent* with free-disposal, such that:

$$p^t_{\text{ask},i}(S') \geq p^t_{\text{ask},i}(S) \quad \forall S' \supseteq S, \forall t \geq 1 \tag{1}$$

The basic auction provides bidders with an *exclusive-or* (XOR) bidding language, such that a bid, $\mathcal{B}_i^t$, from agent i in round t can include a request for multiple bundles. The XOR interpretation restricts the auctioneer to selecting at most one bundle from each agent's bid set for the current provisional allocation in each round.[3]

Let $p_{\mathrm{bid},i}^t(S)$ denote the bid price for bundle S in round t from agent i, and write $S \in \mathcal{B}_i^t$ if bids contain bundle S. We rule out *jump bids*, and require instead that an agent bids no more than the ask price for a bundle. Let ϵ denote the minimal bid increment, which is the rate with which prices are increased across rounds.

Bidding rules:

- The bid price must equal the ask price, except if submitting a *last-and-final* bid, or repeating a bid that was successful in the previous round for a bundle that has increased in price. In both cases an agent can bid at ϵ below the ask price.
- An agent must bid for any bundle that it received in the provisional allocation in the previous round.
- Once an agent has made a *last-and-final* bid is made on a bundle, no future bid from the agent can strongly-dominate this bid; i.e., the agent cannot submit a bid with a higher (or higher implied) bid price on this bundle in a future round.

The purpose of the last-and-final bid is to allow an agent to continue to bid for a bundle while the bid price is narrowly above its value. The purpose of the ϵ-discount when repeating a bid is to allow a winning agent to continue to bid at the same price until another agent actually bids and wins at a higher price.

In a practical implementation, an auctioneer might maintain the set of last-and-final bids submitted by agents, and throw out any future bids that violate the strict-dominance rule. An auctioneer could also automatically resubmit winning bundles on behalf of participants. In Section 6.1 we discuss a proxy-bidding extension of *i*BEA in which bidders communicate with automated bidding agents that follow straightforward bidding strategies.

4.1.2 Winner Determination

In each round, t, the auctioneer selects a provisional allocation to maximize revenue given the current bids. During Phase I, bids from all agents are considered, while during Phase II, the winner determination problem is solved without bids from one of the agents.

[3]Although XOR bids are completely expressive, other languages can be more compact in special cases [39]. We expect to identify interesting future extensions of *i*BEA for alternative bid description languages, including restrictions to bundles for which the winner-determination problem is worst-case polynomial solvable [18].

Procedure: SelectPivot

1) Initialize, $\delta^t = \emptyset$, select an agent $j \in (open^t \setminus \delta^t)$ at random, and compute $[\text{WD}^t(\mathcal{I} \setminus j)]$.

2) From this, if $unhappy^t(\mathcal{I} \setminus j) = \emptyset$, then add j to δ^t and select another agent.

3) Continue, until either (a) $open^t \setminus \delta^t = \emptyset$, or (b) $unhappy^t(\mathcal{I} \setminus j) \neq \emptyset$ for some $j \in open^t$, which becomes the new pivotal agent.

Figure 5: Selecting the Pivotal Subproblem in each round of Phase II.

Definition 13 (Pivotal Subproblem). *The bids from agents in the pivotal subproblem, $\mathcal{K}^t_{\text{pivot}} \subseteq \mathcal{I}$, in round t, are used to determine the provisional allocation.*

During Phase I there is no pivotal agent, and the winner-determination problem is solved with bids from all agents. During Phase II, at the end of each round and once bids have been received, the auctioneer selects a pivotal subproblem for which the provisional allocation leaves at least one agent still bidding without a bundle, or terminates if no such subproblem exists.

Suppose that bids, $\mathcal{B}^t$, are submitted by agents in round t. The winner-determination problem, $[\text{WD}^t(\mathcal{K})]$, defined with respect to agents in set $\mathcal{K}$ is:

$$\max_{k \in \Gamma(\mathcal{B}^t)} \sum_{i \in \mathcal{K}, [i,S] \in k} p^t_{\text{bid},i}(S) \qquad [\text{WD}^t(\mathcal{K})]$$

where $\Gamma(\mathcal{B}^t)$ is the set of feasible allocations consistent with the set of bids, and respects the XOR bid language.

Definition 14 (competitive bid). *Given bid, $\mathcal{B}^t_i$, from agent i, and ask prices $p^t_{\text{ask},i}(S)$, agent i's competitive bid, $\mathcal{B}^{t+}_i$, contains only bundles not weakly-dominated and priced at the current ask price.*

Let $S^t(\mathcal{K})$ denote the provisional allocation that solves $[\text{WD}^t(\mathcal{K})]$. Agent $i \in \mathcal{K}$ is said to be *unhappy* with allocation $S^t(\mathcal{K})$ if $\mathcal{B}^{t+}_i \neq \emptyset$ but $S^t_i(\mathcal{K}) = \emptyset$. Let $unhappy^t(\mathcal{K}) \subseteq \mathcal{K}$ denote the set set of unhappy agents.

Call the agent missing from the pivotal subproblem the *pivotal agent*, denoted $pivot^t$. At the start of Phase II, initialize a candidate set of pivotal agents, $open^t = \mathcal{T}^*$, where $\mathcal{T}^*$ are the agents in the provisional allocation at the end of Phase I. Figure 5 summarizes the method to select the pivotal subproblem in each round of Phase II. In termination case (a), the entire auction will terminate (see Section 4.1.4). Otherwise, in termination case (b), then $\mathcal{K}^t_{\text{pivot}} = (\mathcal{I} \setminus j)$, the solution to $[\text{WD}^t(\mathcal{I} \setminus j)]$ becomes the provisional allocation, and $open^{t+1} = open^t \setminus \delta^t$.

A simple variation continues with the same pivotal subproblem until all agents are happy, and then switches to another subproblem. We use this method in the worked example of *i*BEA in Section 4.3. The actual method used to select the next pivotal agent is not critical to our results as long as a pivotal subproblem is selected to contain at least one unhappy agent.[4]

Breaking ties when solving winner-determination can be an issue. We propose to break ties as follows: (i) first, in favor of bids included in the previous provisional allocation; (ii) then, in favor of bids that are at the current ask price; (iii) then in favor of including more agents; (iv) then at random. In some applications, such as the FCC wireless spectrum auction, there may be legal issues that make it particularly important to break ties at random. In this case we suggest (i) and then (iv); step (i) remains useful to prevent cycles.

Example: Pivotal Subproblem Selection

Consider the pivotal subproblem selection problem in rounds 15 and 16 of *i*BEA, on Example 4(a), see Section 4.3.

Phase I ends in round 15, and the auction selects a pivotal subproblem. Bids are $\{(A, 15)\}$ from agent 1, $\{(B, 25)\}$ from agent 2, and $\{(B, 20), (AB, 40)\}$ (last-and-final bids) from agent 3. The efficient allocation is $(A, B, \emptyset)$, and $\mathcal{T}^* = \{1, 2\}$ at the end of around 15. Initialize $open^{15} = \{1, 2\}$, and suppose agent $j = 2 \in open^{15}$ is selected, with corresponding $\mathcal{K} = \{1, 3\}$. Solving $\text{WD}^{15}(\{1, 3\})$, we have allocation $(\emptyset, \emptyset, AB)$, and agent 1 is unhappy. Therefore, set $pivot^{15} = 2$, $\mathcal{K}^{15}_{\text{pivot}} = \{1, 3\}$, and $open^{16} = \{1, 2\}$.

In round 16, the new bids are $\{(A, 20)\}$, $\{(B, 25)\}$ and agent 2, and $\{(B, 20), (AB, 40)\}$, and allocation $(A, \emptyset, B)$ solves $\text{WD}^{15}(\{1, 3\})$. Therefore, $\delta^{16} = \{2\}$, and select $j = 1$. Allocation $(\emptyset, \emptyset, AB)$ solves $\text{WD}^{16}(\{2, 3\})$, and agent 2 is unhappy. Therefore, set $pivot^{16} = 1$, $\mathcal{K}^{16}_{\text{pivot}} = \{2, 3\}$, and $open^{17} = \{1\}$.

4.1.3 Price Increases

At the end of round t, after the provisional allocation and pivotal subproblem has been determined, prices are increased based on bids from unhappy agents. The price increase rule generalizes the price-update rule in the English auction, increasing the price on a bundle to ϵ above the highest bid price from an unsuccessful agent. In Section 5 we bound the accuracy of the final allocation and Vickrey payments with respect to the minimal bid increment ϵ.[5]

The general price increase rule is designed to minimize price discrimination during the auction. We also describe a simplified rule in which every agent faces individualized prices throughout the auction, for which the theoretical analysis continues to hold. We expect the variation with dynamic price discrimination to have better information revelation and performance in terms of the number of rounds to termination because the information exchange is more efficiently

[4]The proposed method provides quite robust termination as it does not allow cycles, once there are no unhappy agents in a subproblem in some round that subproblem is never considered again. Theoretical results show that with myopic BR bids, once a subproblem is in CE then it will remain in CE for all future bids (Lemma 4) and will not need to be reconsidered.

[5]In earlier work, we reported the results of computational experiments in *i*Bundle, that demonstrate order-of-magnitude speed-increases over computation in the VCG as the bid increment is increased and less rounds are required.

coupled between participants when price increases are on anonymous prices that everyone faces.

In round t, a subset $anon^t$ of agents face anonymous prices. This set weakly monotonically increases across the auction. The price increase rule has two components: (i) determine $anon^{t+1}$, and (ii) determine price increases given $anon^{t+1}$. For now, let us assume $anon^{t+1}$ and present the rules to compute price increases with this information.

Definition 15 (bid safety). *A bid, $\mathcal{B}_i^t$, from agent i is safe if all bundles in competitive, $\mathcal{B}_i^{t+}$, are mutually disjoint.*

Let $safe^t \subseteq \mathcal{I}$ denote the set of agents submitting safe bids in round t. The importance of bid safety *vis-a-vie* anonymous price increases in the CAP was first identified in Parkes [40]. In the following definitions, remember that the unhappy agents for a particular pivotal subproblem never include the pivot agent itself.

Definition 16 (anonymous price increase). *For every unhappy agent, i, submitting safe bids that continues to face anonymous prices in round $t+1$, increase the price on bundle $S \in \mathcal{B}_i^{t+}$ to $p_{\text{anon}}^{t+1}(S) = p_{\text{anon}}^t(S) + \epsilon$. Finally, adjust the anonymous ask prices to make them consistent with weak-dominance.*

The only bids that can increase non-anonymous prices to agent i are bids from agent i itself, and then only if the agent is unhappy and submitted competitive bids.

Definition 17 (non-anonymous price increase). *For every unhappy agent, i, that faces non-anonymous prices in round $t+1$, increase the price on bundle $S \in \mathcal{B}_i^{t+}$ to $p_{\text{ind}}^{t+1}(S) = p_{\text{ind}}^t(S) + \epsilon$. Then, adjust the individualized ask prices to make them all self-consistent with respect to weak-dominance.*

In the special case of an agent that faces individualized prices for the first time in round $t+1$, its prices $p_{\text{ind},i}^{t+1}$ are first initialized to p_{anon}^t, and then increased.

The only agents that can begin to face non-anonymous prices in the next round are those which are unhappy. To continue to face anonymous prices, we must consider the effect that the price increases due to their bids would have on anonymous prices.

It is useful to introduce the idea of a *redundant* set of bids, with respect to another set of bids.

Definition 18 (redundant). *Bid, $\mathcal{B}_i$, from agent i is said to be redundant, given a set of bids from $\mathcal{B}_j$ from agents $j \in \mathcal{K}$, written $redundant(i, \mathcal{K})$, if every bundle in bid $\mathcal{B}_i$ is weakly-dominated by some bundle one of the bids from an agent in $\mathcal{K}$.*

For a particular pivotal subproblem, $\mathcal{K}_{\text{pivot}}^t$, the method used to determine, $anon^{t+1}$, is described in Figure 6. Set $\mathcal{L}_{\text{anon}}$ is the set of all unhappy agents that submitted safe bids and will continue to face anonymous prices in the

Procedure: SelectDrop

Notation: $\mathit{unhappy}^t = \mathit{unhappy}^t(\mathcal{K}^t_{\text{pivot}})$; $\mathcal{L}_{\text{anon}} = \mathit{unhappy}^t \cap \mathit{anon}^{t+1} \cap \mathit{safe}^t$; $\mathcal{L}_{\text{extra}} = \{i \,:\, i \notin \mathit{unhappy}^t, i \in \mathit{anon}^{t+1}, i \in \mathit{safe}^t, \mathit{redundant}(i, \mathcal{L}_{\text{anon}})\}$.

1) initialize $\mathit{anon}^{t+1} = \mathit{anon}^t$; compute $\mathcal{L}_{\text{anon}}$.

2) $\mathit{ok} \leftarrow$ true; compute $\mathcal{L}_{\text{extra}}$.

3) for each $j \in \mathit{unhappy}^t \cap \mathit{anon}^{t+1}$ but $j \notin \mathit{safe}^t$, if $\neg \mathit{redundant}(j, \mathcal{L}_{\text{anon}})$ then $\mathit{anon}^{t+1} \leftarrow \mathit{anon}^{t+1} \setminus \{j\}$.

4) for each $j \in \mathcal{L}_{\text{anon}}$, if $\neg \mathit{redundant}(j, (\mathcal{L}_{\text{anon}} \setminus j) \cup \mathcal{L}_{\text{extra}})$ then $\mathit{anon}^{t+1} \leftarrow \mathit{anon}^{t+1} \setminus \{j\}$; $\mathcal{L}_{\text{anon}} \leftarrow \mathcal{L}_{\text{anon}} \setminus \{j\}$; $\mathit{ok} \leftarrow$ false.

5) if $\neg(\mathit{ok})$ then goto 2) else stop.

Figure 6: Dynamically Introducing Additional Price Discrimination

next round. Set $\mathcal{L}_{\text{extra}}$ is the set of agents that submitted safe bids, will continue to face anonymous prices in the next round, are happy, and for which $\mathit{redundant}(i, \mathcal{L}_{\text{anon}})$ holds. At the end of the procedure, the dropped agents, $\mathit{drop}^t(\mathcal{K}^t_{\text{pivot}})$, are computed as $\mathit{anon}^{t+1} \setminus \mathit{anon}^t$.

Step 3) checks unhappy agents that submit unsafe bids. Such an agent can remain in the anonymous price set and increase anonymous prices if the price increases are redundant given the anonymous price increases due to safe bids from agents, $\mathcal{L}_{\text{anon}}$.

Step 4) checks that the anonymous price increases will continue to make progress towards CE in every subproblem. This additional check is the key departure from the price-update rules in *i*Bundle, and is essential to make progress towards Universal CE prices. In particular, the concern is to check that anonymous price increases due to an agent j submitting safe bids are consistent with price increases due to bids from agents in the pivotal subproblem without j. Agent j's bids must be redundant with respect to bids from a subset of agents except j with safe bids that will continue to face anonymous prices, the set $(\mathcal{L}_{\text{anon}} \setminus j) \cup \mathcal{L}_{\text{extra}}$.

Steps 2), 3) and 4) repeat whenever one or more unhappy agents that submit safe bids are removed from the anonymous price set, and drop out of $\mathcal{L}_{\text{anon}}$. The decision is reconsidered on every unhappy agent that is currently assigned to face anonymous prices in the next round.

Example: Dynamic Price Discrimination

Consider a problem with 4 goods, $\{A, B, C, D\}$, and 7 agents. Suppose that bids $\{(AB)\}, \{(CD)\}, \{(AB), (CD)\}, \{(AB)\}, \{(CD), (AC)\}, \{(AC)\}, \{(A), (CD)\}$ are received from the 7 agents in round t. In addition, suppose that $\mathit{anon}^t = \{1, 2, 3, 4, 5, 6, 7\}$, all bids are at the current anonymous ask prices, the auction is in Phase I so that $\mathcal{K}^t_{\text{pivot}} = \{1, 2, 3, 4, 5, 6, 7\}$, and the provisional allocation gives AB to agent 4 and CD to agent 5. We have $\mathit{unhappy}^t = \{1, 2, 3, 6, 7\}$, $\mathit{anon}^{t+1} = \mathit{anon}^t$, and $\mathcal{L}_{\text{anon}} = \{1, 2, 6\}$ because the bids from agents 3 and 7 are not safe.

In step 2), the candidate agents for $\mathcal{L}_{\text{extra}}$ are agents 4 and 5, because they are both in the provisional allocation and submitted safe bids. Checking $redundant(4, \mathcal{L}_{\text{anon}})$, this holds because AB is bid by agent 1. Checking $redundant(5, \mathcal{L}_{\text{anon}})$, this also holds because CD is bid by agent 2 and AC is bid by agent 6. So, $\mathcal{L}_{\text{extra}} = \{4, 5\}$.

In step 3), we check $redundant(j, \mathcal{L}_{\text{anon}})$ for agents $j \in \{3, 7\}$. This redundancy check holds for agent 3, because bundle AB is also bid by agent 1 and bundle CD is also bid by agent 2. However, this redundancy check fails for agent 7 because no agent in $\mathcal{L}_{\text{anon}}$ bids for item A. Therefore, $anon^{t+1} \leftarrow anon^{t+1} \setminus \{7\}$.

In step 4), we check $redundant(j, (\mathcal{L}_{\text{anon}} \setminus j) \cup \mathcal{L}_{\text{extra}})$ for each $j \in \{1, 2, 6\}$. The result is that $redundant(1, \{2, 4, 5, 6\})$, $redundant(2, \{1, 4, 5, 6\})$, and $redundant(6, \{1, 2, 4, 5\})$ all hold. The method terminates, with $anon^{t+1} = anon^{t} \setminus \{7\}$, and anonymous price increases based on bids from agents {1,2,6}. Even though agent 3's bid is unsafe and unsuccessful, agent 3 continues to face anonymous prices because it has a redundant effect on anonymous prices given increases due to agents {1,2,6}.

4.1.4 Termination and Vickrey Price Adjustment

Phase I terminates as soon as the provisional allocation computed with bids from all agents assigns a bundle to all agents submitting competitive bids. Phase II terminates as soon as, for each pivotal subproblem, a provisional allocation has been computed that assigns a bundle to all agents in the subproblem that submit competitive bids. With respect to the method proposed to select the pivotal subproblem, the auction terminates as soon as there are no *open* pivotal subproblems.[6] Notice that it is not necessary to satisfy bids from agents that submit only last-and-final bids at ϵ-discount.

Let S^* denote the allocation computed in the last round of Phase I, and $\mathcal{I}^*$ denote the agents that receive a bundle in allocation S^*. In addition, let $S^*(\mathcal{I} \setminus j)$ denote the provisional allocation computed in the round in which there were no unhappy agents for pivotal subproblem $\text{CAP}(\mathcal{I} \setminus j)$.[7]

Let $p^T_{\text{ask},i}(S)$ and $p^T_{\text{bid},i}(S)$ denote the ask and bid price for bundle S and agent i at the end of Phase II. Of course, agent i will not bid for every bundle; define $p^T_{\text{bid},i}(S) = \infty$ when bundle S receives neither a bid nor a weakly-dominating bid. Let $p^*_i(S) = \min\{p^T_{\text{ask},i}(S), p^T_{\text{bid},i}(S)\}$, denote the *combined* price, for bundle S to agent i.[8]

[6] As a stronger termination condition, that is equivalent to this termination condition with straightforward bidding strategies, we can require the auction to terminate in a round in which each pivotal subproblem *simultaneously* allocates bundles to all agents with competitive bids.

[7] In fact, with straightforward bidding strategies, allocations S^* and $S^*(\mathcal{I} \setminus j)$ continue to be optimal for the masterproblem and the respective subproblems in all future rounds. The method stated here is suggested to provide additional robustness against alternative agent strategies.

[8] If agents follow MBR strategies, then every bundle allocated to agent i in S^* and $S^*(\mathcal{I} \setminus j)$, for all $j \in \mathcal{I}^*$, will receive a bid from agent i in the last round of Phase II. The combined price is constructed to provide robustness in the case that agents do not follow MBR strategies. It is possible that agents are not even still bidding for the bundles in the allocation at the end

Procedure: UnhappyCertain
1) $\mathcal{L}_{\text{safe}} = \mathcal{K}' \cap \textit{safe}^t$; $\mathcal{L}_{\text{unhappy}} = \emptyset$.
2) while $\exists i \in \mathcal{L}_{\text{safe}}$ s.t. $\textit{redundant}(i, L_{\text{safe}} \setminus \{i\})$, then $\mathcal{L}_{\text{safe}} \rightarrow \mathcal{L}_{\text{safe}} \setminus \{i\}$; $\mathcal{L}_{\text{unhappy}} \leftarrow \mathcal{L}_{\text{unhappy}} \cup \{i\}$;

Figure 7: Method to Determine Unhappy-Certain Agents.

At the end of Phase II, **compute discounts**, $\Delta_{\text{iBEA},j}$, **to agents as**:

$$\Delta_{\text{ibea},j} = \begin{cases} [\pi^s(p^*, S^*) - \pi^s(p^*, S^*(\mathcal{I} \setminus j))]^+ & \text{, if } j \in \mathcal{T}^* \\ 0 & \text{, otherwise.} \end{cases}$$

where $\pi^s(p, S)$ is defined as the surplus to the seller from allocation S at prices p. The auction terminates with allocation S^* and agent payments $p_{\text{ibea},j} = p_j^*(S_j^*) - \Delta_{\text{ibea},j}$ for agents $j \in \mathcal{T}^*$, and $p_{\text{ibea},j} = 0$ otherwise.

4.1.5 Example: Price-Adjustment

Consider the price adjustment at the end of Phase II of *i*BEA, on Example 4(b), see Section 4.3. Phase II ends with provisional allocation, $S^* = (A, B, \emptyset, \emptyset, \emptyset)$, and pivotal allocations, $S^*(\{2,3,4,5\}) = (\emptyset, B, \emptyset, A, \emptyset)$, and $S^*(\{1,3,4,5\}) = (A, \emptyset, \emptyset, \emptyset, B)$. All bundles in these allocations receives bids in the last round, with $p_1^*(A) = 30, p_2^*(B) = 30, p_4^*(A) = 25, p_5^*(B) = 25$. The price adjustments are $\Delta_{\text{ibea},1} = \pi^s(p^*, S^*) - \pi^s(p^*, S^*(\{2,3,4,5\})) = 60 - 55 = 5$, and similarly, $\Delta_{\text{ibea},2} = 60 - 55 = 5$. Finally, allocation S^* is implemented, and agent 1's payment is $p_{\text{ibea},1} = 30 - \Delta_{\text{ibea},1} = 25$, similarly $p_{\text{ibea},2} = 30 - 5 = 25$. This is the outcome of the VCG mechanism on Example 4(b).

4.2 An Asynchronous Auction Variation

In one important variation, *i*BEA allows *asynchronous* bids and price increases without updated bids from all agents. This allows the auction to make progress towards termination with less information from agents, and can significantly boost the information revelation and valuation cost advantages of *i*BEA in comparison to the one-shot VCG. As a special case, this asynchronous price update rule implements the English auction, in which the price is increased while at least two agents are bidding at the current price.

We make no assumptions about agent preferences in stating this asynchronous variation. Instead, we identify conditions on a partial set of bids for which particular price increases must occur for *any* completion of the bids. The provisional allocation from the last full round is retained in any round in which an asynchronous price-update is used, because the partial bids provide insufficient information to determine the new provisional allocation.

Definition 19 (unhappy-certainty). *Given bids from agents $\mathcal{K}'$, we say that bid $\mathcal{B}_i$ from agent $i \in \mathcal{K}'$, is unhappy-certain if, for all bids from agents $(\mathcal{I} \setminus \mathcal{K}')$,*

of Phase I by the end of the auction.

Procedure: AnonCertain
Notation: $\mathcal{L}^*_{\text{extra}} = \{i \ : \ i \in \mathcal{L}_{\text{safe}} \cap \mathit{anon}^t, \mathit{redundant}(i, \mathcal{L}^*_{\text{anon}})\}$;
1) Initialize $\mathcal{L}^*_{\text{anon}} = \mathcal{L}_{\text{unhappy}} \cap \mathit{anon}^t$;
2) compute $\mathcal{L}^*_{\text{extra}}$; $ok \leftarrow$ true;
3) for each $j \in \mathcal{L}^*_{\text{anon}}$, if $\neg \mathit{redundant}(j, (\mathcal{L}^*_{\text{anon}} \setminus j) \cup \mathcal{L}^*_{\text{extra}})$, then $\mathcal{L}^*_{\text{anon}} \leftarrow \mathcal{L}^*_{\text{anon}} \setminus \{j\}$; $ok \leftarrow$ false;
4) if $\neg(ok)$ then goto 2), else stop.

Figure 8: Method to Determine Anon-Certain Agents.

the winner-determination problem, WD($\mathcal{I}$), has an optimal solution that does not accept any bids from agent i.

Notice that if agent i is unhappy-certain with respect to all possible solutions to the winner-determination problem with all agents, it is all also unhappy-certain with respect to all possible solutions to the winner-determination problem for any pivotal subproblem.

In Figure 7 we describe a simple method to determine agents, $\mathcal{L}_{\text{unhappy}}$, that are unhappy-certain given bids from agents $\mathcal{K}'$. The method checks for agents with bids that are redundant given bids from other agents, considering both agents that face anonymous and non-anonymous prices. Recall that agent i's bids are *redundant* with respect to bids from agents in some set $\mathcal{K}$ if every bid from i is weakly-dominated by some bid from an agent in $\mathcal{K}$.

Proposition 10 (unhappy test). *All agents, $\mathcal{L}_{\text{unhappy}}$, satisfy unhappy-certainty.*

Proof. Given final $\mathcal{L}_{\text{safe}}$, for all $i \in \mathcal{L}_{\text{unhappy}}$, then $\mathit{redundant}(i, \mathcal{L}_{\text{safe}})$, and consider a feasible allocation, S, over all agents in $\mathcal{K}'$. Suppose that $S_i \neq \emptyset$, then allocation S can be transformed into a feasible allocation, S', with at least as much surplus by substituting a bid from an agent, i', in $\mathcal{L}_{\text{safe}}$ that weakly-dominates the bid from agent i. Because agent i' submits safe bids and weakly-dominates agent i's bid, agent i' must receive no bundle in S, and allocation S' is feasible. □

Definition 20 (anon-certainty). *Given bids from agents $\mathcal{K}'$, we say that bid $\mathcal{B}_i$ from agent $i \in \mathcal{K}' \cap \mathit{anon}^t$, is anon-certain if, for all bids from agents $(\mathcal{I} \setminus \mathcal{K}')$, agent k will continue to face anonymous prices in the next round.*

Figure 8 describes a simple method to determine agents, $\mathcal{L}^*_{\text{anon}}$, that are anon-certain given bids from agents $\mathcal{K}'$. Taking the agents, $\mathcal{L}_{\text{unhappy}}$, that submitted safe bids and are unhappy-certain, and the agents $\mathcal{L}_{\text{safe}}$, that submitted safe bids and may or may not be unhappy in the next round, the method computes a subset of those agents that will remain in the anonymous price set in the next round once all bids have been received.

Proposition 11 (anonymous test). *All agents that satisfy $\mathcal{L}_{\text{anon}}$ are anonymous-certain.*

Proof. The proof is by comparison with the Proc. SelectDrop, which is used in the standard *i*BEA auction to determine which agents will continue to face anonymous prices. First, notice that the set $\mathcal{L}^*_{\text{anon}} \subseteq \mathcal{L}_{\text{anon}}$, where $\mathcal{L}_{\text{anon}}$ is the initial set of unhappy, anonymous, and safe bids in the complete information method (Figure 6). Similarly, an agent in $\mathcal{L}^*_{\text{extra}}$, that submits safe bids and currently faces anonymous prices will be in $\mathcal{L}_{\text{anon}}$ or $\mathcal{L}_{\text{extra}}$ with complete information, depending on whether the agent is eventually happy or unhappy, because $\mathit{redundant}(i, \mathcal{L}^*_{\text{anon}})$ implies $\mathit{redundant}(i, \mathcal{L}_{\text{anon}})$. Finally, if $\mathit{redundant}(j, (\mathcal{L}^*_{\text{anon}} \setminus j) \cup \mathcal{L}^*_{\text{extra}})$ then $\mathit{redundant}(j, (\mathcal{L}_{\text{anon}} \setminus j) \cup \mathcal{L}_{\text{extra}})$, and j will face anonymous prices for any completion. □

Pulling this together, a simple *anonymous-price asynchronous update* rule, for pivotal subproblem, $\mathcal{K}^t_{\text{pivot}}$, is:

compute $\mathcal{L}_{\text{unhappy}}$ *and* $\mathcal{L}_{\text{safe}}$*; compute* $\mathcal{L}^*_{\text{anon}}$*; use bids from agents* $\mathcal{L}^*_{\text{anon}}$ *to increase anonymous ask prices;*

As a special case, it is interesting to consider the effect of the asynchronous price-update rules in the special case that two agents that face anonymous prices submit *identical* safe bids.

Proposition 12. *If two agents that face anonymous prices submit identical safe bids then the anonymous ask prices can be increased based on the bids from one of the agents.*

Proof. Consider agents {1,2}, and suppose without loss of generality that agent 1 is selected first in Proc. UnhappyCertain. Then, $1 \in \mathcal{L}_{\text{unhappy}}$, because $\mathit{redundant}(1, 2)$. Agent 2 may end-up in $\mathcal{L}_{\text{unhappy}}$ or in $\mathcal{L}_{\text{safe}}$. Then, in Proc. AnonCertain, set $\mathcal{L}^*_{\text{anon}}$ initially contains agent 1. Either $2 \in \mathcal{L}^*_{\text{anon}}$ (if $2 \in \mathcal{L}_{\text{unhappy}}$), or $2 \in \mathcal{L}^*_{\text{extra}}$ because $\mathit{redundant}(2, 1)$. Then, $\mathit{redundant}(1, (\mathcal{L}^*_{\text{anon}} \setminus 1) \cup \mathcal{L}^*_{\text{extra}})$, because $2 \in \mathcal{L}^*_{\text{anon}} \cup \mathcal{L}^*_{\text{extra}}$. Finally, at least agent $1 \in \mathcal{L}^*_{\text{anon}}$, and will increase the anonymous ask price. □

Notice that the asynchronous anonymous price-update variation of *i*BEA *reduces to the English auction for a single item*, increasing the ask price whenever at least two bids are received at the current price.

Turning to *individualized* price increases, the problem is to determine from a subset of bids whether an agent is unhappy-certain *and* will face non-anonymous prices in the next round. Clearly this holds if an agent is unhappy-certain and currently faces anonymous prices. Rather than propose a complex rule to determine special cases in which *i*BEA will drop an agent from the anonymous prices for the first time, we propose to implement this simple rule.

compute $\mathcal{L}_{\text{unhappy}}$*; increase non-anonymous ask prices to agents* $(\mathcal{L}_{\text{unhappy}} \setminus \mathit{anon}^t)$*;*

agent	Bids		
Agent 1	$(AB, 20)$	$(ABC, 30)$	
Agent 2	$(AB, 20)$	$(ABD, 40)$	
Agent 3	$(ABC, 30)$	$(CD, 30)$	
Agent 4	$(CD, 30)$	$(BD, 20)$	$(DEF, 50)$
Agent 5	$(ABD, 40)$	$(BD, 20)$	
Agent 6	$(CD, 30)$	$(ABD, 40)$	

Figure 9: A partial set of bids in a particular round of *i*BEA.

In implementing asynchronous price updates, the auctioneer has a choice about when to increase prices, waiting for additional bids beyond the first instance in which the minimal conditions hold for some asynchronous prices increases. The unhappy-certain, $\mathcal{L}_{\text{unhappy}}$, and anon-certain, $\mathcal{L}^*_{\text{anon}}$, sets of agents increase monotonically as more bids are received. The appropriate timing decision represents a tradeoff between the cost of interrupting the decision-processes of bidders, and the benefits of providing new information to guide decisions just as soon as possible.

Example: Asynchronous Price Updates

Consider a particular round of *i*BEA, in which the bids in Figure 9 have been received by the auctioneer in round t.

Variation 1. Suppose that $anon^t \supseteq \{1,2,3,4,5,6\}$. First, compute $\mathcal{L}_{\text{unhappy}}$, the set of unhappy-certain agents. Initialize $\mathcal{L}_{\text{safe}} = \{1,2,3,4,5,6\}$, and $\mathcal{L}_{\text{unhappy}} = \emptyset$. Agent 1 is added to $\mathcal{L}_{\text{unhappy}}$, because $redundant(1, \{2,3,4,5,6\})$ (by agents 2 and 3). Then, agent 5 is added to $\mathcal{L}_{\text{unhappy}}$, because $redundant(5, \{2,3,4,6\})$ (by agents 2 and 4). Then, agent 6 is added to $\mathcal{L}_{\text{unhappy}}$, because $redundant(6, \{2,3,4\})$ (by agents 2 and 3). At this stage, $\mathcal{L}_{\text{safe}} = \{2,3,4\}$, and none of agents $j \in \{2,3,4\}$ are redundant with respect to $\mathcal{L}_{\text{safe}} \setminus \{j\}$. The method terminates, with $\mathcal{L}_{\text{unhappy}} = \{1,5,6\}$ representing the unhappy-certain agents.

Second, initialize $\mathcal{L}^*_{\text{anon}} = \mathcal{L}_{\text{unhappy}} \cap anon^t = \{1,5,6\}$. Initialize $\mathcal{L}^*_{\text{extra}} = \emptyset$. Then, for each $i \in \{2,3,4\}$, check $redundant(i, \{1,5,6\})$. This holds for {2,3} but not agent 4, and $\mathcal{L}^*_{\text{extra}} = \{2,3\}$. Then, for each $j \in \mathcal{L}^*_{\text{anon}}$, check $redundant(j, (\mathcal{L}^*_{\text{anon}} \setminus j) \cup \mathcal{L}^*_{\text{extra}})$. Testing, we have $redundant(1, \{5,6,2,3\})$ and $redundant(6, \{1,5,2,3\})$, but not $redundant(5, \{1,6,2,3\})$. Looping back to 2), the set $\mathcal{L}^*_{\text{extra}}$ is unchanged, and $redundant(1, \{6,2,3\})$ and $redundant(6, \{1,2,3\})$ continue to hold. The method terminates with $\mathcal{L}^*_{\text{anon}} = \{1,6\}$, and the anonymous ask prices on bundles $\{(AB,20),(ABD,40),(CD,30),(ABD,40)\}$ can be increased.

Variation 2. As a simple non-anonymous variation, suppose that the same bids are received but that agents {1,5} are not in $anon^t$. Agents {2,3,4,6} remain in $anon^t$. Again, $\mathcal{L}_{\text{unhappy}} = \{1,5,6\}$ and $\mathcal{L}_{\text{safe}} = \{2,3,4\}$. This time we initialize $\mathcal{L}^*_{\text{anon}} = \mathcal{L}_{\text{unhappy}} = \{1,5,6\} \cap anon^t = \{6\}$. Computing $\mathcal{L}^*_{\text{extra}}$, none of agents {2,3,4} are redundant given $\mathcal{L}^*_{\text{anon}}$, and $\mathcal{L}^*_{\text{extra}} = \emptyset$. Finally, agent 6 fails *redundant*$(6, \emptyset)$, and $\mathcal{L}^*_{\text{anon}} = \emptyset$. Even though agents {1,5,6} are definitely unhappy, we cannot increase the anonymous prices based on their bids without more information about the bids from the other agents. However, we can increase non-anonymous prices to agent 1 based on bids $\{(AB, 20), (ABC, 30)\}$ and non-anonymous prices to agent 5 based on bids $\{(ABD, 40), (BD, 20)\}$.

4.3 *i*BEA Example

Figure 10 illustrates the progress of *i*BEA on Example 4(a), with the minimal bid increment, $\epsilon = 5$, in which agents are not substitutes. For comparison, Figure 11 illustrates the progress of *i*BEA on Example 4(b), in which two additional agents are introduced to act as close substitutes for agents 1 and 2, and agents are substitutes. Comparing *i*BEA on 4(a) and 4(b), we notice that: (i) much less price-discrimination is required in 4(b), which in turn allows the auction to terminate in fewer rounds; and (ii) *i*BEA terminates at the end of Phase I in 4(b), the first set of CE prices provide enough information to compute Vickrey payments.

Example 4(a). In Figure 10, all strict-positive individualized ask prices for an agent are indicated in each agent's column, and all strict-positive anonymous ask prices are indicated in the righthand column. Most bids are at the ask price, but when a bid is placed for bundle S at ϵ below the ask price, this is indicated as $S_{-\epsilon}$. The provisional allocation in each round is denoted with *. Special tags, **I**, and, **II**, in the *label* column indicate the last rounds of Phase I and II respectively. During Phase II, between rounds 16–19, the tag (e.g. {1,3}) indicates the agents in the current pivotal subproblem. For example, label {1,3} in round 15 indicates that the provisional allocation, $(\emptyset, \emptyset, AB)$, corresponds to the solution to the winner-determination problem without bids from agent 2. In any round in which multiple pivotal subproblems are considered, each subproblem is represented as an additional row (although this cycling through subproblems, checking for CE, is hidden from agents). For example, in round 16, it is first determined that the auction is in competitive equilibrium for subproblem {1,3}, and the next subproblem {2,3} is considered.

Agent 3 is removed from the anonymous price set at the end of round 1. Stepping through the dynamic price-discrimination method in Figure 6, initially set $\mathcal{L}_{\text{safe}} = \{3\}$, $anon^2 = \{1, 2, 3\}$, and there are no unhappy agents submitting unsafe bids in step 2). In step 3), neither agent 1 not agent 2 are in $\mathcal{L}_{\text{extra}}$ because their bids are not redundant with respect to the bid from agent 3. Then, agent 3's bid fails *redundant*$(3, \emptyset)$, and $anon^2 = \{1, 2\}$. Based on agent 3's bid, $\mathcal{B}^1_3 = \{(AB, 0)\}$, its individualized ask price in round 2 is $p^2_3(AB) = 5$, the minimal bid increment. Similarly, agents 1 and 2 are removed from the anonymous-price set because the bids of agents 1 and 2 are for different items, and not mutually-redundant.

The auction terminates in round 19, with Universal CE prices. The final row, labeled **II**, repeats the allocation, $S^* = (A, B, \emptyset)$, computed in round 15 at the end of Phase I. This is the allocation implemented at the end of the auction. The efficient allocations for each pivotal subproblem are $S^*(\{2, 3\}) = (\emptyset, B, \emptyset)$

round	label	agent 1		agent 2		agent 3		anon
		prices	bids	prices	bids	prices	bids	prices
1			A^*		B^*		AB	
2			A		B	$(AB,5)$	AB^*	
3		$(A,5)$	A^*	$(B,5)$	B^*	$(AB,5)$	AB	
4		$(A,5)$	A^*	$(B,5)$	B^*	$(AB,10)$	AB	
5		$(A,5)$	A	$(B,5)$	B	$(AB,15)$	B, AB^*	
6		$(A,10)$	A^*	$(B,10)$	B^*	$(AB,15)$	B, AB	
7		$(A,10)$	A^*	$(B,10)$	B^*	$(B,5),(AB,20)$	B, AB	
8		$(A,10)$	A	$(B,10)$	B	$(B,10),(AB,25)$	B, AB^*	
9		$(A,15)$	A^*	$(B,15)$	B^*	$(B,10),(AB,25)$	B, AB	
10		$(A,15)$	A^*	$(B,15)$	B^*	$(B,15),(AB,30)$	B, AB	
11		$(A,15)$	A^*	$(B,15)$	B	$(B,20),(AB,35)$	B^*, AB	
12		$(A,15)$	A^*	$(B,20)$	B	$(B,20),(AB,35)$	B^*, AB	
13		$(A,15)$	A^*	$(B,25)$	B^*	$(B,20),(AB,35)$	B, AB	
14		$(A,15)$	A^*	$(B,25)$	B^*	$(B,25),(AB,40)$	$B_{-\epsilon}, AB$	
15	**I**	$(A,15)$	A^*	$(B,25)$	B^*	$(B,25),(AB,45)$	$B_{-\epsilon}, AB_{-\epsilon}$	
	{1,3}	$(A,15)$	A	$(B,25)$	B	$(B,25),(AB,45)$	$B_{-\epsilon}, AB^*_{-\epsilon}$	
16	{1,3}	$(A,20)$	A^*	$(B,25)$	B	$(B,25),(AB,45)$	$B^*_{-\epsilon}, AB_{-\epsilon}$	
	{2,3}	$(A,20)$	A	$(B,25)$	B	$(B,25),(AB,45)$	$B_{-\epsilon}, AB^*_{-\epsilon}$	
17	{2,3}	$(A,20)$	A	$(B,30)$	B	$(B,25),(AB,45)$	$B_{-\epsilon}, AB^*_{-\epsilon}$	
18	{2,3}	$(A,20)$	A	$(B,35)$	B	$(B,25),(AB,45)$	$B_{-\epsilon}, AB^*_{-\epsilon}$	
19	{2,3}	$(A,20)$	A	$(B,40)$	B^*	$(B,25),(AB,45)$	$B_{-\epsilon}, AB_{-\epsilon}$	
	II	$(A,20)$	A^*	$(B,40)$	B^*	$(B,25),(AB,45)$	$B_{-\epsilon}, AB_{-\epsilon}$	

Figure 10: Progress of *i*BEA on Example 4(a). Minimal bid increment, $\epsilon = 5$. Label **I** indicates the end of Phase I, labels such as {1,3} indicate the current pivotal subproblem. Label **II** indicates the end of Phase II. The final allocation is $S^* = (A, B, \emptyset)$, with $\Delta_{\text{ibea},1} = 20, \Delta_{\text{ibea},2} = 20$, and $p_{\text{ibea},1} = 20 - 20 = 0, p_{\text{ibea},2} = 40 - 20 = 20$. The set of non-zero anonymous ask prices is empty in this example because price-discrimination is quickly introduced.

round	label	agent 1 pr bids	agent 2 pr bids	agent 3 pr	agent 3 bids	agent 4 pr bids	agent 5 pr bids	anon prices A	B	AB
1		A^*	B^*		AB	A	B			
2		$A_{-\epsilon}$	$B_{-\epsilon}$		AB	A^*	B^*	5	5	5
3		A	B	$(5,5,10)$	AB	A^*	B^*	5	5	5
4		A^*	B^*	$(5,5,15)$	AB	$A_{-\epsilon}$	$B_{-\epsilon}$	10	10	10
5		A^*	B^*	$(5,5,20)$	B, AB	A	B	10	10	10
6		$A_{-\epsilon}$	$B_{-\epsilon}$	$(5,10,25)$	B, AB	A^*	B^*	15	15	15
7		A	B	$(5,15,30)$	B, AB	A^*	B^*	15	15	15
8		A^*	B^*	$(5,20,35)$	B, AB	$A_{-\epsilon}$	$B_{-\epsilon}$	20	20	20
9		A^*	B^*	$(5,25,40)$	$B_{-\epsilon}, AB$	A	B	20	20	20
10		$A_{-\epsilon}$	$B_{-\epsilon}$	$(5,25,45)$	$B_{-\epsilon}, AB_{-\epsilon}$	A^*	B^*	25	25	25
11		A	B	$(5,25,45)$	$B_{-\epsilon}, AB_{-\epsilon}$	A^*	B^*	25	25	25
12	**I**	A^*	B^*	$(5,25,45)$	$B_{-\epsilon}, AB_{-\epsilon}$	$A_{-\epsilon}$	$B_{-\epsilon}$	30	30	30
	{2,3,4,5}	A	B^*	$(5,25,45)$	$B_{-\epsilon}, AB_{-\epsilon}$	$A^*_{-\epsilon}$	$B_{-\epsilon}$	30	30	30
	{1,3,4,5}	A^*	B	$(5,25,45)$	$B_{-\epsilon}, AB_{-\epsilon}$	$A_{-\epsilon}$	$B^*_{-\epsilon}$	30	30	30
	II	A^*	B^*	$(5,25,45)$	$B_{-\epsilon}, AB_{-\epsilon}$	$A_{-\epsilon}$	$B_{-\epsilon}$	30	30	30

Figure 11: Progress of *i*BEA on Example 4(b). Minimal bid increment, $\epsilon = 5$. Phase I ends in round 12, and Phase II ends immediately, after cycling through subproblems defined over agents {2,3,4,5} and {1,3,4,5} to check that that prices are CE in the pivotal subproblems. The final allocation, with label **II**, is $(A, B, \emptyset, \emptyset, \emptyset)$, with $\Delta_{\text{ibea},1} = 60 - 55 = 5, \Delta_{\text{ibea},2} = 60 - 55 = 5$, and $p_{\text{ibea},1} = 30 - 5 = 25, p_{\text{ibea},2} = 30 - 5 = 25$.

and $S^*(\{1,3\}) = (A, \emptyset, B)$. The final combined prices, p^*, are equal to agent bid prices on the bundles in allocations $S^*, S^*(\{2,3\})$ and $S^*(\{1,3\})$. Agent 1 receives bundle A, discount $\Delta_{\text{ibea},1} = \pi^s(p^*, S^*) - \pi^s(p^*, S^*(\{2,3\})) = 60 - 40 = 20$, and makes payment $p_{\text{ibea},1} = p_1^*(A) - \Delta_{\text{ibea},1} = 20 - 20 = 0$. Agent 2 receives bundle B, discount $\Delta_{\text{ibea},2} = 60 - (20 + 20) = 20$, and makes payment $p_{\text{ibea},2} = p_2^*(B) - 20 = 20$. This is the VCG outcome for Example 4a).

Example 4(b). In Example 4(b), in which agents are substitutes, only agent 3 faces individualized prices and price increases are propagated more quickly across agents, with the auction terminating after 12 rounds. Phase II is unnecessary and the auction terminates at the end of Phase I, with Universal CE prices and Vickrey payments.

To illustrate dynamic price-discrimination, consider the end of round 3, in which agents 1 and 2 continue to face anonymous prices are unhappy and currently face anonymous prices. First, set $\mathcal{L}_{\text{anon}} = \{1, 2\}$, and $anon^4 = \{1, 2, 4, 5\}$. Agents 4 and 5 are introduced to $\mathcal{L}_{\text{extra}}$, because the bids of agents 4 and 5 are redundant with respect to the bids of agents 1 and 2. Step 3) is skipped, because there are no unhappy agents submitting unsafe bids. In step 4), both agents 1 and 2 remain in $\mathcal{L}_{\text{anon}}$ because $redundant(j, \{1, 2\} \setminus \{j\} \cup \{4, 5\})$ is satisfied for $j \in \{1, 2\}$, with agent 4 making agent 1's bid redundant and agent 5 making agent 2's bid redundant. Agents 1 and 2 remain in the anonymous price set, and price increases $p^4_{\text{anon}}(A) = p^4_{\text{anon}}(B) = p^4_{\text{anon}}(AB) = 10$ are implemented in the next round.

The auction terminates with allocation, $S^* = (A, B, \emptyset, \emptyset, \emptyset)$, and agent payments, $p_{\mathrm{ibea},1} = p_1^*(A) - (\pi^s(p^*, S^*) - \pi^s(p^*, S^*(\{2,3,4,5\}))) = 30 - ((30+30) - (30+25)) = 25$, and $p_{\mathrm{ibea},2} = p_2^*(B) = (\pi^s(p^*, S^*) - \pi^s(p^*, S^*(\{1,3,4,5\}))) = 30 - ((30+30) - (30+25)) = 25$. This is the VCG outcome for Example 4(b).

4.3.1 Information Revelation

We propose to provide only minimal information to bidders during the auction to minimize strategic opportunities that are not available in the one-shot VCG mechanism. One concern, discussed in Section 6, is to minimize opportunities for collusion during Phase II, by making it difficult for an agent to determine that the auction is in Phase II. We propose the following:

- Agents are provided with information about the prices that they face, i.e. anonymous prices or their own set of non-anonymous prices, and informed about any bundle allocated in the provisional allocation.

- Agents are *not* provided with information about the prices faced by other agents, bids place by other agents, bundles allocated to other agents, or the current revenue from the provisional allocation.

A further complication arises for pivotal agents during Phase II. For example, consider agent 2 in round 15. Agent 2 is not in the provisional allocation (because 2 is the pivotal agent), but neither does agent 2 face higher prices. Taken together, and because the agent submitted competitive bids, this would by itself indicate that the auction is in Phase II. One approach could report item B to agent 2, which was its allocation in round 14, although this has the additional problem that agents 2 and 3 could now determine together that the auction is in Phase II, because they are both apparently allocated B. A better solution may be to hide information about the *rounds* in the auction, and instead provide a more asynchronous interface such that agents do not necessarily know when a round passes in which they were the pivotal agent. Essentially, an agent is just aware of its current allocation and the current ask prices. Proxy bidding agents provide yet another level of information hiding, as discussed in Section 6.1.

5 Theoretical Properties

We prove that *i*BEA is efficient, and terminates with the outcome of the GVA, when agents follow myopic best-response (MBR) bidding strategies, which is an *ex post* Nash equilibrium of the auction. The methodology used to establish the theoretical properties of *i*BEA builds on Bertsekas' primal-dual analysis an an auction-based algorithm for the basic assignment problem [9]. Related approaches are also taken in Demange et al. [19], and more recently Parkes&Ungar [44] and Bikchandani et al. [10]. In equilibrium, *i*BEA implements a primal-dual algorithm for the GVA. In Section 5.1 we show that the auction maintains complementary-slackness conditions (CS1a) between ask prices and the provisional allocation for all agents, and maintains complementary-slackness condition (CS2) with respect to the provisional allocation in each round. At the end of Phase I, condition (CS1b) also holds for all agents, the prices are in competitive equilibrium, and the allocation is efficient.

In Section 5.2 we show that the auction maintains Quasi-CE prices in every pivotal subproblem in all rounds, and terminates at the end of Phase II with Universal CE prices. From the analysis in Section 3.1, the adjusted prices implement the Vickrey payments. We establish error-bounds on the efficiency of the final allocation and the distance between agent payoffs and Vickrey payoffs in terms of the minimal bid increment ϵ, and show that as $\epsilon \to 0$, the auction implements the VCG mechanism. In Section 5.3, we prove that myopic best-response is an ex-post Nash equilibrium of *i*BEA.

5.1 Efficient Allocation

First, we show that with MBR strategies, *i*BEA maintains a relaxation of (CS1a), ϵ-CS1a, and maintains a relaxation of (CS2), ϵ-CS2. At the end of Phase I, condition (CS1b) also holds, and prices are approximately in competitive equilibrium with the provisional allocation.

Associating ask prices, $p_{\mathrm{ask},i}$, with dual prices, $p_i(S)$, in [DLP$_3$], and with $\pi_i = [\max_S(v_i(S) - p_{\mathrm{ask},i}(S))]^+$, then ($\epsilon$-CS1a) is defined as:

$$x_i(S) > 0 \Rightarrow \pi_i + p_{\mathrm{ask},i}(S) \leq v_i(S) + 2\epsilon \qquad (\epsilon\text{-CS1a})$$

where $x_i(S) = 1$ iff agent i is assigned bundle S in the provisional allocation. In words, any bundle allocated to agent i in the provisional allocation should approximately maximize its surplus.

Let $p^t_{\mathrm{br},i}(S)$ denote the *effective* ask price for agent i, and let $\pi^t_{\mathrm{br},i} = [\max_S v_i(S) - p^t_{\mathrm{br},i}(S)]^+$. This effective price is equal to ask price, $p^t_{\mathrm{ask},i}(S)$, except in the special cases that an agent repeats a bundle from the provisional allocation in the previous round that has increased in price, or has value within ϵ of the ask price, in which case $p^t_{\mathrm{br},i}(S) = p^t_{\mathrm{ask},i}(S) - \epsilon$. A myopic best-response (MBR), or straightforward bidding strategy, submits bids, $\mathcal{B}^t_{\mathrm{BR},i}$, that ϵ-maximize the agent's surplus at the current prices, with no regard to the effect of a bid on future prices or the bidding strategies of other agents.

$$\mathcal{B}^t_{\mathrm{BR},i} = \{(S, p^t_{\mathrm{br},i}(S)) \ : \ v_i(S) - p^t_{\mathrm{br},i}(S) + \epsilon \geq \pi^t_{\mathrm{br},i}\}$$

The following two lemmas hold for both pivotal subproblems, $\mathcal{K}^t_{\mathrm{pivot}} = (\mathcal{I} \setminus pivot^t)$, and for the problem with all agents, $\mathcal{K}^t_{\mathrm{pivot}} = \mathcal{I}$.

Lemma 3 (cs1a). *If agent i follows its MBR bidding strategy in round t, then the provisional allocation $S^t(\mathcal{K}^t_{\mathrm{pivot}})$ for pivotal subproblem, $\mathcal{K}^t_{\mathrm{pivot}}$, satisfies ($\epsilon$-CS1a) for agent i whenever $S^t(\mathcal{K}^t_{\mathrm{pivot}}) \neq \emptyset$.*

Proof. Let $S_i = S_i^t(\mathcal{K}_{\text{pivot}}^t)$, and $p_{\text{br},i}^t$ denote the effective ask price for agent i. By MBR, if $S_i \neq \emptyset$, then $v_i(S_i) - p_{\text{br},i}^t(S_i) + \epsilon \geq \max_{S'} v_i(S') - p_{\text{br},i}^t(S')$, from which $v_i(S_i) - p_{\text{ask},i}^t(S_i) + 2\epsilon \geq \max_{S'} v_i(S') - p_{\text{ask},i}^t(S')$, since $p_{\text{ask},i}^t(S) - \epsilon \leq p_{\text{br},i}^t(S) \leq p_{\text{ask},i}^t(S)$ for all S. Now, if $\max_{S'} v_i(S') - p_{\text{ask},i}^t(S') \geq 0$ then $\pi_i = \max_{S'} v_i(S') - p_{\text{ask},i}^t(S')$ and (ϵ-CS1a) immediately follows. Otherwise, if $\pi_i = 0$, we must have $0 \leq v_i(S_i) - p_{\text{br},i}^t(S_i)$, from which $0 \leq v_i(S_i) - (p_{\text{ask},i}^t(S_i) - \epsilon)$. Since $\pi_i = 0$, we have $v_i(S_i) + \epsilon \geq \pi_i + p_{\text{ask},i}^t(S_i)$, and ($\epsilon$-CS1a). □

Lemma 4 (maintenance). *If all agents follow MBR in every round, then once prices satisfy (ϵ-CS1a) for every agent in a pivotal subproblem, $\mathcal{K}_{\text{pivot}}$, and prices satisfy (CS1b), then (ϵ-CS1a) and (CS1b) are maintained in $\mathcal{K}_{\text{pivot}}$ in all future rounds.*

Proof. Let t' denote the round in which provisional allocation $S^{t'}(\mathcal{K}_{\text{pivot}})$ first satisfies (ϵ-CS1a) and (CS1b). We prove that this allocation continues to solve $\text{WD}^t(\mathcal{K}_{\text{pivot}})$ in all future rounds, and therefore satisfy (CS2). Price increases may still occur because of unhappy agents in other pivotal subproblems, but because (CS1b) holds for all agents in $\mathcal{K}_{\text{pivot}}$, only agents that receive a bundle in $S^*(\mathcal{K}_{\text{pivot}})$ will continue bidding in other pivotal subproblems and cause price increases. Moreover, a single agent can only increase the price on any one allocation by ϵ in any round, since anonymous price increases are due to *safe* bids, and trivially for non-anonymous price increases. In addition, the redundancy check performed in step 4) of SelectDrop (Figure6) ensures that all anonymous price increases are covered by an agent in every pivotal subproblem, even if price increases are due to the pivotal agent for $\mathcal{K}_{\text{pivot}}$ in another pivotal subproblem. Suppose x agents drive up the price in a particular round, then the revenue from the current provisional allocation increases by $x\epsilon$, and it remains maximal for the seller across all feasible allocations. □

Continuing, introduce (ϵ-CS2),

$$y(k) > 0 \Rightarrow \pi^s - \sum_{[i,S]\in k} p_{\text{ask},i}^t(S) \leq \min\{m, n\}\,\epsilon \qquad (\epsilon\text{-CS2})$$

in which $\pi^s = \max_{k\in\mathcal{K}} \sum_{[i,S]\in k} p_{\text{ask},i}^t(S)$, and $\mathcal{K}$ is the set of all feasible allocations. In words, the provisional allocation must approximately maximize the revenue to the seller at the current prices, across all feasible allocations.

Before establishing (ϵ-CS2), we must first introduce some technical lemmas. First, a couple of useful definitions.

Definition 21 (cover). *If bundle S is covered by agent j in round t then the agent will bid for the bundle, or some bundle S' that weakly dominates S, (at least) until the ask price to agent j for the bundle is increased due to bids from another agent.*

Definition 22 (strict positive price). *Bundle S has a strict positive price at prices, p^t, if no bundle weakly dominates S at the prices.*

In addition, we say there is a *strict price increase* on bundle S if $p^{t+1}(S) > p^t(S)$ and bundle S has a strict positive price at p^{t+1}.

Lemma 5 (covering property). *If agent i follows MBR, and effects a strict increase in the anonymous ask price on bundle S in round t, then* both *agent i and at least one other agent will cover bundle S in future rounds.*

Proof. See the appendix. □

Notice that an agent that covers a bundle in the final round that it faces anonymous prices will then cover the bundle *indefinitely* while it faces non-anonymous prices because the individualized price on the bundle can never be increased by bids from another agent. The last-and-final bid ϵ-discount is essential to allow an agent to continue to cover a bundle in all future rounds. This covering property provides a level of redundancy to price-increases, the same anonymous price increases would occur without any one agent. The following lemma is immediate, since strict positive anonymous ask prices can only exist as the result of a price increase.

Lemma 6 (anonymous cover). *If agents follow MBR, all bundles with strict positive anonymous ask price are covered by at least two agents.*

Let us refer to the bundles covered by an agent as a result of strict anonymous price increases the *anonymous cover* of an agent.

Lemma 7 (cover safety). *If agents follow MBR, the set of bundles with strict positive ask price in the anonymous cover of any one agent in a particular round are all mutually non-disjoint.*

Proof. An MBR agent bids for all bundles in its anonymous cover in every round, and whenever additional bundles are introduced to its anonymous cover the agent must have submitted safe bids. □

Lemma 8 (non-anonymous cover). *If agents follow MBR, then for every agent, i, facing non-anonymous prices, and all bundles, S', with a strict positive non-anonymous ask price, $p^t_{\mathrm{ind},i}(S')$, then either (a) the bundle receives a bid from agent i, or (b) the anonymous ask price, $p^t_{\mathrm{anon}}(S') \geq p^t_{\mathrm{ind},i}(S')$, and the smallest bundle that weakly-dominates S' at anonymous prices is covered by some other agent.*

Proof. See the appendix. □

Recall that $\Gamma(\mathcal{B})$ is the set of allocations consistent with agent bids, $\mathcal{B}$. The challenge in proving (ϵ-CS2) is to show that this restriction is unimportant. Given allocation, S, let $\pi^s(p, S) = \sum_i p_i(S_i)$ denote the revenue to the seller given prices p. Ignoring for now the possibility that bid prices may be slightly less than ask prices, the following *transformation* lemma states that it is not restrictive to only consider allocations that are compatible with agent bids.

Lemma 9 (transformation). *If agents follow MBR, then any allocation, S, that is not consistent with bids from agents can be transformed to a bid-consistent allocation, S', with at least as much revenue at the ask prices, in all rounds.*

Proof. See the appendix. $\square$

The transformation is quite straightforward when every agent faces anonymous prices, as the safety of bids in an agent's cover set immediately implies that there exists a reallocation of bundles in an allocation to covering agents. At the other extreme, if every agent faces non-anonymous prices in every round, then the initial allocation is already consistent with bids from agents, by Lemma 8.

The transformation lemma leads to a statement about ϵ-CS2, or the payoff to the seller in each round, both in Phase I and Phase II of the auction.

Lemma 10 (CS2-master). *If agents follow MBR, the provisional allocation that solves the winner-determination, WD($\mathcal{I}$), problem with all agents satisfies (ϵ-CS2), in all rounds.*

Proof. By contradiction. Assume that the solution to $\text{WD}^t(\mathcal{I})$, $S^t(\mathcal{I})$, has value $\sum_i p^t_{\text{bid},i}(S^t_i) < \pi^s - \min\{m,n\}\epsilon$. Let S denote the allocation that maximizes surplus at ask prices, solving π^s. By Lemma 9, a transformed allocation, S', has at least as much surplus at ask prices, and is consistent with agent bids. Now, for any allocation S'', we have $\sum_i p^t_{\text{bid},i}(S''_i) \geq \sum_i p^t_{\text{ask},i}(S''_i) - \min\{m,n\}\epsilon$, because there can be no more bundles allocated than there are items or agents, and $p^t_{\text{bid},i}(S) \geq p^t_{\text{ask},i}(S) - \epsilon$ for all i, all S. Therefore, the value of transformed allocation, S', at bid prices, $\sum_i p^t_{\text{bid},i}(S'_i) \geq \pi^s - \min\{m,n\}\epsilon$, and $S^t(\mathcal{I})$ cannot be an optimal solution to $\text{WD}^t(\mathcal{I})$. $\square$

Proposition 13 (phase I termination). *If agents never bid for bundles with negative surplus, then Phase I terminates.*

Proof. Phase I continues while there is at least one agent bidding unsuccessfully at the ask price on at least one bundle, and the price on that bundle increases by ϵ in the next round. Termination follows from the finiteness of the problem, and the boundedness of agent values. $\square$

Proposition 14 (CE prices). *If agents follow MBR, the ask prices are competitive equilibrium prices for $\text{CAP}(\mathcal{I})$ at the end of Phase I, and throughout Phase II, as the bid increment, $\epsilon \to 0$.*

Proof. By definition, when Phase I terminates, then (CS1b) holds, in addition to (ϵ-CS1a) and (ϵ-CS2), which hold by Lemmas 3 and 10. Together, this establishes that Phase I terminates with CE prices as $\epsilon \to 0$. Then, because of the maintenance of the CS conditions (Lemmas 4 and 10), CE prices are maintained throughout Phase II. $\square$

Theorem 4 (allocative efficiency). *If agents follow MBR, the total value of the allocation computed at the end of Phase I is within $3\min\{m,n\}\epsilon$ of the total value of the efficient allocation, for bid increment ϵ, m goods, and n agents.*

Proof. At the end of Phase II prices continue satisfy (ϵ-CS1a), (CS1b) and (ϵ-CS2) for allocation, S^*, computed at the end of Phase I. Let p^T_{ask} denote the ask prices at the end of Phase II. Summing (ϵ-CS1a) over all agents in the final allocation, and with $\pi_i = 0$ for agents not in the allocation by (CS1b), then $\sum_{i \in I} \pi_i \leq \sum_i v_i(S^*_i) - \sum_i p^T_{\text{ask},i}(S^*_i) + 2\min\{m,n\}\epsilon$, because an allocation can include no more bundles than there are items or agents. Introducing (ϵ-CS2), and with $y(k^*) = 1$ for allocation, k^*, that corresponds with allocation, S^*, then $\pi^s \leq \sum_i p^T_{\text{ask},i}(S^*_i) + \min\{m,n\}\epsilon$. Finally, adding these two equations, we have:

$$\pi^s + \sum_i \pi_i \leq \sum_i v_i(S^*_i) + 3\min\{m,n\}\epsilon \tag{2}$$

Since $\pi + \sum_i \pi_i \geq V(\mathcal{I})$ for all dual solutions by weak duality, then $V(\mathcal{I}) \leq \sum_i v_i(S^*_i) + 3\min\{m,n\}\epsilon$. $\square$

As $\epsilon \to 0$ then *i*BEA terminates with the efficient allocation.

5.2 Vickrey Payments

In this section we prove that the adjusted prices at the end of *i*BEA implement Vickrey payments when agents follow MBR strategies.

The first lemma describes an important robustness property of *i*BEA. The (ϵ-CS2) property holds for the subproblem CAP$(\mathcal{I} \setminus j)$, *whatever* the bidding strategy of agent j. This is despite of the fact that bids from agent j can change the anonymous prices in the auction, and follows from the choice embedded in the proof of the transformation lemma 9, which in turn follows from the robustness provided in the anonymous-covering lemma 6. This robustness property is used to prove that MBR is an *ex post* Nash equilibrium of the auction.

Lemma 11 (CS2-pivot). *For any pivotal subproblem, $\mathcal{K}_{\text{pivot}}$, if the agents in $\mathcal{K}_{\text{pivot}}$ follow MBR then the provisional allocation, $S^t(\mathcal{K}_{\text{pivot}})$, satisfies ($\epsilon$-CS2) for subproblem* CAP$(\mathcal{K}_{\text{pivot}})$ *in all rounds.*

Proof. The same transformation used to prove Lemma 9 can be used to transform any allocation, $S^t \in \Gamma(\mathcal{K}_{\text{pivot}})$, to one consistent with bids from agents in set $\mathcal{K}_{\text{pivot}}$. Each step of the transformation presents a *choice* of agents, because there are always at least two agents that cover any bundle with strict-positive anonymous price (Lemma 6). Simply choose an agent $j \in \mathcal{K}_{\text{pivot}}$ that covers the bundle. $\square$

It is quite a pleasing property of *i*BEA that additional preference elicitation, beyond that required to first adjust to competitive equilibrium, is not necessary in the special case that Vickrey payoffs can be supported in a single competitive equilibrium.

Proposition 15 (agents are substitutes). *If agents follow MBR, and agents are substitutes,* i*BEA terminates immediately at the end of Phase I.*

Proof. Prices are approximately Quasi-CE in all subproblems, $\text{CAP}(\mathcal{I} \setminus j)$, in all rounds by Lemma 11. By Proposition 8), when agents are substitutes, then as soon as prices are also CE (at the end of Phase I), then these Universal Quasi-CE prices are also Universal CE prices, and *i*BEA terminates. $\square$

Proposition 16 (Phase II termination). *If agents never bid for bundles with negative surplus, then Phase II terminates.*

Proof. Straightforward, using a similar argument as that made for termination of Phase I. $\square$

Theorem 5 (Universal CE). *If agents follow MBR, then* i*BEA terminates with prices that approach Universal CE prices as the minimal bid increment, $\epsilon \to 0$.*

Proof. When Phase II terminates, (ϵ-CS1a) and (CS1b) was achieved for each agent in every pivotal subproblem in some earlier round, and continue to hold by Lemma 4. In addition, prices continue to satisfy (ϵ-CS2) in all subproblems by Lemma 11. Finally, prices continue to be in CE for $\text{CAP}(\mathcal{I})$ by Proposition 14, and as $\epsilon \to 0$ these CS conditions imply that prices are Universal CE. $\square$

Lemma 12 (pivotal efficiency). *If agents follow MBR, the total value of the final allocation, $S^*(\mathcal{K}_{\text{pivot}})$, in pivotal subproblem, $\mathcal{K}_{\text{pivot}}$, is within $3\min\{m,n\}\epsilon$ of the total value of the efficient allocation for subproblem $\text{CAP}(\mathcal{K}_{\text{pivot}})$.*

Proof. From the Universal-CE property of prices, using the same error-bounding techniques as in the proof of the error-bound on allocative efficiency for the problem with all agents in Theorem 4. $\square$

At the end of Phase II, the adjusted payment to agent $j \in \mathcal{T}^*$ is computed as $p_{\text{ibea},j} = p_j^*(S_j^*) - \Delta_{\text{ibea},j}$, where $\Delta_{\text{ibea},j} = \pi^s(p^*, S^*) - \pi^s(p^*, S^*(\mathcal{I} \setminus j))$, with $\pi^s(p^*, S^*) = \sum_i p_i^*(S_i^*)$ and $\pi^s(p^*, S^*(\mathcal{I} \setminus j)) = \sum_{i \neq j} p_i^*(S_i^*(\mathcal{I} \setminus j))$, and $p^* = \min\{p_{\text{bid},i}^T, p_{\text{ask},i}^T\}$. As before, let $\pi^s(\mathcal{K})$ denote the maximal seller surplus over all allocations consistent with agents in $\mathcal{K}$, and $\pi^s(k)$ denote the surplus from allocation k, both evaluated at the final ask prices. Shorthand π^s is used to denote $\pi^s(\mathcal{I})$.

Theorem 6 (Vickrey payoffs). *If all agents follow MBR, then* i*BEA terminates with individual agent payoffs $\pi_{\text{ibea},i} \geq \pi_{\text{vick},i} - (2\epsilon + 4\min\{m,n\}\epsilon)$, and total agent payoffs $\sum_i \pi_{\text{ibea},i} \leq \sum_i \pi_{\text{vick},i} + (4n-2)\min\{m,n\}\epsilon$, for bid increment ϵ, m goods, and n agents.*

Proof. First, establish the lower-bound, $\pi_{\text{ibea},i} \geq \pi_{\text{vick},i} - (2\epsilon + 4\min\{m,n\}\epsilon)$, on the final payoff to an individual agent. Substituting terms, payoff $\pi_{\text{ibea},j} = v_j(S_j^*) - p_j^*(S_j^*) + \pi^s(p^*, S^*) - \pi^s(p^*, S^*(\mathcal{I} \setminus j))$. Now, $\pi^s(p^*, S^*) \geq \pi^s - \min\{m,n\}\epsilon$, by ($\epsilon$-CS2), and $\pi^s \geq V(\mathcal{I}) - \sum_i \pi_i$ for all feasible dual solutions. Also, $\pi^s(p^*, S^*(\mathcal{I} \setminus j)) \leq \pi^s(\mathcal{I} \setminus j)$, since $p_i^*(S) \leq p_{\text{ask},i}^T(S)$ for all S, and

$\pi^s(\mathcal{I} \setminus j) \leq V(\mathcal{I} \setminus j) - \sum_{i \neq j} \pi_i + 3\min\{m,n\}\epsilon$ by Lemma 12. Putting this all together, we have:

$$\begin{aligned}\pi_{\text{ibea},j} \geq & v_j(S_j^*) - p_j^*(S_j^*) + V(\mathcal{I}) - \sum_i \pi_i - \min\{m,n\}\epsilon \\ & - V(\mathcal{I} \setminus j) + \sum_{i \neq j} \pi_i - 3\min\{m,n\}\epsilon\end{aligned}$$

Substituting, $v_j(S_j^*) - p_j^*(S_j^*) \geq v_j(S_j^*) - p^T_{\text{ask},j}(S_j^*)$, and $v_j(S_j^*) - p^T_{\text{ask},j}(S_j^*) \geq \pi_j - 2\epsilon$ by (ϵ-CS1a), we have $\pi_{\text{ibea},j} \geq V(\mathcal{I}) - V(\mathcal{I} \setminus j) - 4\min\{m,n\}\epsilon - 2\epsilon$, and the lower-bound with $\pi_{\text{vick},j} = V(\mathcal{I}) - V(\mathcal{I} \setminus j)$.

Second, establish the upper-bound on the total payoff over all agents, $\sum_i \pi_{\text{ibea},i} \leq \sum_i \pi_{\text{vick},i} + (4n-2)\min\{m,n\}\epsilon$. Substituting terms, the total payoff, $\sum_i \pi_{\text{ibea},i} = \sum_i v_i(S_i^*) - \sum_i p_i^*(S_i^*) + \sum_i \pi^s(p^*, S^*) - \sum_i \pi^s(p^*, S^*(\mathcal{I} \setminus i))$. Now, $\sum_i v_i(S_i^*) \leq V(\mathcal{I})$, and $\sum_i p_i^*(S_i^*) \geq \pi^s - \min\{m,n\}\epsilon$ by (ϵ-CS2). Also, $\pi^s(p^*, S^*(\mathcal{I} \setminus i)) \geq \pi^s(\mathcal{I} \setminus i) - \min\{m,n\}\epsilon$ by (ϵ-CS2) in the pivotal subproblems. Finally, $\pi^s(p^*, S^*) \leq \pi^s$ because $p_i^*(S) \leq p^T_{\text{ask},i}(S)$ for all S. Putting this all together, we have:

$$\sum_i \pi_{\text{ibea},i} \leq V(\mathcal{I}) - \pi^s + \min\{m,n\}\epsilon + \sum_i \pi^s - \sum_i \pi^s(\mathcal{I} \setminus i) + \sum_i \min\{m,n\}\epsilon$$

Substituting, $\pi^s \leq V(\mathcal{I}) + 3\min\{m,n\}\epsilon - \sum_i \pi_i$, from Theorem 4, and $\pi^s(\mathcal{I} \setminus i) \geq V(\mathcal{I} \setminus i) - \sum_{j \neq i} \pi_j$, valid for all feasible dual solutions, we have:

$$\begin{aligned}\sum_i \pi_{\text{ibea},i} &\leq V(\mathcal{I}) + (n-1)\left(V(\mathcal{I}) + 3\min\{m,n\}\epsilon - \sum_i \pi_i\right) \\ &\quad - \sum_i V(\mathcal{I} \setminus i) - \sum_i \sum_{j \neq i} \pi_j + (n+1)\min\{m,n\}\epsilon \\ &= \sum_i \pi_{\text{vick},i} + (4n-2)\min\{m,n\}\epsilon\end{aligned}$$

where $\sum_i \sum_{j \neq i} \pi_j$ cancels with $\sum_i \pi_i$. □

As $\epsilon \rightarrow 0$, the adjusted prices in *i*BEA converge to the VCG payments.

5.3 Strategic Analysis

We prove that myopic best-response is an *ex post* Nash equilibrium of *i*BEA. The approach is to show that any feasible strategy from agent i has the effect of selecting a Vickrey outcome for some reported valuation function, $\hat{v}_i$, given that the other agents play MBR, for any values v_{-i}. It follows from the dominance of truth-revelation in the VCG that it is always weakly preferable to select the Vickrey outcome that corresponds with its true valuation function, and follow MBR. A similar proof method is adopted in Gul & Stacchetti [26], to establish the incentive-compatibility of an ascending-price auction for gross-substitutes preferences.

Theorem 7 (incentive-compatibility). *Myopic best-response is an ex-post Nash equilibrium of* iBEA*, as the minimal bid increment* $\epsilon \to 0$*.*

Proof. Without loss of generality, let $v = (v_1, \ldots, v_n)$ denote agent valuations, and suppose that agent 1 follows some strategy, $\hat{\sigma}_1$, while the other agents follow MBR. Let p^* denote the prices at the end of Phase II, from which price adjustments are computed, and let $\hat{S}$ denote the allocation implemented by *i*BEA, which is the allocation computed at the end of Phase I. Agent 1 receives bundle $\hat{S}_1$ and makes payment $p_{\text{ibea},1}(\hat{\sigma}_1) = p_1^*(\hat{S}_1) - \Delta_{\text{ibea},1}$, where $\Delta_{\text{ibea},1} = [\pi^s(p^*, \hat{S}) - \pi^s(p^*, \hat{S}(\mathcal{I} \setminus 1))]^+$, and $\hat{S}(\mathcal{I} \setminus 1)$ is the allocation computed for subproblem $\text{CAP}(\mathcal{I} \setminus 1)$. The payoff to agent 1 is $\pi_1(\hat{\sigma}_1) = v_1(\hat{S}_1) - p_1^*(\hat{S}_1) + [\pi^s(p^*, \hat{S}) - \pi^s(p^*, \hat{S}(\mathcal{I} \setminus 1))]^+$.

First, let us consider the outcome of the auction from the perspective of agents $j \neq 1$. Conditions (ϵ-CS1a) and (CS1b) hold for all agents $j \neq 1$ in allocation $\hat{S}$ at the end of Phase I, trivially by Lemma 3 and the termination properties of Phase I. Notice that (CS2) does not necessarily hold, because of the strategy of agent 1. We must show that conditions (ϵ-CS1a) and (CS1b) are maintained for $\hat{S}$ and agents $j \neq 1$ in Phase II, despite agent 1's strategy. The concern is to make sure that agents $j \neq 1$ facing anonymous prices continue to bid for bundles $\hat{S}_j$. It is sufficient to observe from the redundancy in the anonymous-cover lemma 6 that any anonymous price increases due to agent 1 are redundant with respect to safe bids from some agent, $k \neq 1$ in allocation $\hat{S}$. From this, agent k will continue to bid for its bundle $\hat{S}_k$, even while it faces anonymous prices, and while anonymous prices are increased by agent 1.

Prices are in *full* competitive equilibrium for subproblem $\text{CAP}(\mathcal{I} \setminus 1)$, without agent 1, because of the redundancy in Lemma 11; i.e. it is only necessary that agents $j \neq 1$ follow a MBR strategy for this property to hold.

Let $\pi^s(p)$ denote the maximal surplus to the seller at prices p. We show that the payoff to agent 1, $\pi_1(\hat{\sigma}_1)$, is equivalent to its payoff in the VCG mechanism for a reported valuation function, $\hat{v}_1$, which we construct as:

$$\hat{v}_1(S) = \begin{cases} p_1^*(S) & \text{, for } S = \hat{S}_1 \\ [p_1^*(S) - \delta]^+ & \text{, for } S \neq \hat{S}_1 \end{cases}$$

where $\delta = \pi^s(p^*) - \pi^s(p^*, \hat{S})$, the amount by which allocation $\hat{S}$ violates (CS2) for the seller. We construct prices, $\hat{p}$, that are: CE prices for $\text{CAP}(\mathcal{I})$; and CE prices for $\text{CAP}(\mathcal{I} \setminus 1)$, with agent values $(\hat{v}_1, v_2, \ldots, v_n)$. Let $\hat{p}_1(S) = \hat{v}_1(S)$, with $\hat{p}_j(S) = p_j^*(S)$ for all $j \neq 1$. For CE prices, first we have (CS2) for the seller, because the prices to agent 1 were reduced by δ on all bundles except $\hat{S}_1$ and $\pi^s(\hat{p}) = \pi^s(\hat{p}, \hat{S}) = \pi^s(p^*) - \delta$. Then, ($\epsilon$-CS1a) and (CS1b) trivially hold for agent 1, because $\hat{v}_1 = \hat{p}_1$, and continue to hold for all $j \neq 1$ because nothing has changed for those agents. Similarly, this is easy to show for CE prices in $\text{CAP}(\mathcal{I} \setminus 1)$ because the prices, valuations, and allocation to agents $j \neq 1$ are unchanged.

Now, the prices are CE in subproblem $\text{CAP}(\mathcal{I} \setminus 1)$, and we can compute the payment in the VCG mechanism to agent 1 with reported values $(\hat{v}_1, v_2, \ldots, v_n)$

with the $\Delta_{\text{adjust},1}$ discount, and $p_{\text{vick},1}(\hat{v}_1) = \hat{p}_1(\hat{S}_1) - [\pi^s(\hat{p}, \mathcal{I}) - \pi^s(\hat{p}, \mathcal{I} \setminus 1)] = p_1^*(\hat{S}_1) - \left[\pi^s(p^*, \hat{S}) - \pi^s(p^*, \hat{S}(\mathcal{I} \setminus 1))\right]$, since $\hat{S}$ and $\hat{S}(\mathcal{I} \setminus 1)$ satisfy (CS2). This payment, $p_{\text{vick},1}(\hat{v}_1)$, is equivalent to the payment, $p_{\text{ibea},i}(\hat{\sigma}_1)$, made by agent 1 in *i*BEA with strategy $\hat{\sigma}_1$. Moreover, the bundle allocated to agent 1 is efficient given preferences $(\hat{v}_1, v_2, \ldots, v_n)$, and *i*BEA implements the VCG outcome for agents with preferences $(\hat{v}_1, v_2, \ldots, v_n)$.

We have shown that the effect of any strategy, $\hat{\sigma}_1$, is to select a VCG outcome for some possibly non-truthful, $\hat{v}_1$. By the strategyproofness of the VCG, the payoff from playing MBR and selecting its Vickrey payoff, $\pi_{\text{vick},1} = V(\mathcal{I}) - V(\mathcal{I} \setminus 1)$, always weakly-dominates any other outcome. This is true for *any* values, v_{-1}, of the other agents, and therefore MBR is an *ex post* Nash equilibrium of the auction. □

6 Discussion

*i*BEA stands out as the first ascending combinatorial auction design to terminate with Vickrey payments in all CAP instances, and the first ascending combinatorial auction design to supports Vickrey payments in problem instances in which Vickrey payoffs are not supported simultaneously to every agent in any competitive equilibrium. The main advantage of *i*BEA over other proposed ascending combinatorial auction designs follows from this robust equilibrium solution concept, straightforward bidding is an *ex post* Nash equilibrium. This means that for all preferences of other agents, as long as they follow a straightforward bidding strategy (or myopic best-response), then an agent's best-response is truthful straightforward bidding. *i*BEA achieves this strong property of truth-revelation from implementing the Vickrey outcome.

Earlier ascending Vickrey auctions were proposed for special cases of CAP. Demange *et al.* [DGS86] [19] stands out as the first non-trivial theoretical result that connects ascending-price auctions with Vickrey auctions, proposing an ascending Vickrey auction for the unit-demand problem. Recently, Bikchandani et al. [10] demonstrate that the [DGS86] auction implements a primal-dual algorithm for the LP relaxation of the CAP, which was shown to support Vickrey payments in the minimal dual solution by Leonard [34].[9] Ausubel [5] proposed a clinching-mechanism for a multi-unit homogeneous item auction, that implements Vickrey payments when agents have decreasing-marginal values for items.

A number of ascending auctions for the general combinatorial allocation problem have been proposed, including AUSM [8] and RAD [20], and more recently A*k*BA [55], *i*Bundle [40, 44] and an ascending-price proxy auction [4].

[9] Earlier, Crawford & Knoer [15], for linear-additive preferences, and Kelso& Crawford [29], for gross-substitutes preferences, had proposed ascending auctions that terminated with CE prices, but not necessarily Vickrey payments. The auction of Gul & Stacchetti [GS00] [26] is a generalization of Demange *et al.* to a combinatorial auction, allowing agents to bid for bundles of items while still retaining linear ask prices. The auction terminates with group-minimal linear CE prices for gross-substitutes preferences, but these do not necessarily support Vickrey payoffs.

Parkes & Ungar [44] prove the efficiency of *i*Bundle with straightforward bidding, and recently, and Parkes [42, 43, chapter 6] proved that *i*Bundle(3), the variation of *i*Bundle in which non-anonymous prices are maintained for all agents throughout the auction, implements the Vickrey outcome in the special case of agents are substitutes. *i*BEA is an extension of this earlier *i*Bundle design, that requires more dynamic price-discrimination and an extra phase to elicit additional preference information when agents are not substitutes. Just as in standard one-shot VCG mechanisms, *i*BEA remains vulnerable to collusion. During Phase II of *i*BEA an agent's bids neither affect the final allocation or the agent's final price. Instead, the only effect of continued bidding is to *decrease* the final payment made by other agents in the efficient allocation.

Ausubel & Milgrom [4] introduce an interesting equilibrium analysis for their ascending-price proxy auction design, which is a slight variation on *i*Bundle(3). The analysis provides a formal motivation for the free-riding strategies observed in ascending combinatorial auctions when Vickrey payoffs are not supported in the group-minimal competitive equilibrium. An asymmetric equilibrium is developed in which one agent bids slowly and receives its Vickrey payoff, while the other agents share the cost of implementing an equilibrium outcome. Ausubel & Milgrom claim revenue advantages and robustness to collusion in equilibria, in comparison with the equilibria in the VCG, but selection of the equilibrium across participants remains a significant problem. Essentially participants are bargaining about how to share the cost of the gap in seller-revenue between the VCG outcome and the group-minimal CE outcome. *i*BEA nicely avoids this bargaining problem, defaulting to the uniquely defined payoffs in the VCG outcome.

6.1 Proxy Bidding Agents

Proxy bidding agents may provide a useful method to reduce the strategy space available to agents in iterative mechanisms [45, 42, 43, 4]. Consider a proxy bidding agent as a device that sits between a real participant and the auction, with partial (and perhaps untruthful) information about a participant's preferences. In one reasonable implementation, the proxy agents follow myopic best-response for participants, only submitting bids when there is enough preference information to identify the appropriate best-response strategy, requesting additional preference information otherwise.

Proposition 17. *If an agent must follow MBR for some ex ante fixed reported valuation function, $\hat{v}_i$, then it is a weakly dominant strategy to choose truthful $\hat{v}_i = v_i$.*

Proof. Trivial, since with MBR strategies for fixed $(\hat{v}_1, \ldots, \hat{v}_n)$, *i*BEA has a simple interpretation as an algorithm to compute the VCG outcome with bids $(\hat{v}_1, \ldots, \hat{v}_n)$. □

In practice, the appropriate use of proxy agents represents a tradeoff between the positive effect that proxy agents can have to boost the robustness of the equilibrium solution concept, and the negative effects that less incremental preference revelation can have on preference elicitation costs. At one extreme, if an agent must report its preferences to a proxy agent, then the auction is *completely identical* to the VCG mechanism and loses all the preference elicitation benefits associated with iterative auctions. The proposed *i*BEA design lies somewhere closer to the other extreme, in one sense we already force "click-box" style proxy bidding by making agents bid at the current ask price and forbidding jump bids.

One important role for proxy bidding agents is to restrict opportunities for collusion between agents in *i*BEA. In fairly static environments without common-value learning, it is reasonable for a proxy-agent to enforce consistency of preference information across rounds. An interesting question for future analysis is to understand the extent to which this is successful at curtailing the opportunity to drive up prices and collude towards the end of the auction. Notice that proxy agents also provide an additional level of information-hiding between the current prices and allocation in *i*BEA, and the auction participants, which can in itself be useful to reduce the ability of a participant to determine when *i*BEA is in Phase II.

7 Conclusions

We propose an efficient ascending-price combinatorial auction, in which myopic best-response is an *ex post* Nash equilibrium, preference elicitation is minimized, and final bid prices are adjusted to implement Vickrey payments. Linear programming duality is used to analyze *i*BEA, and primal-dual methods are extended beyond the agents are substitutes case that has previously been considered a significant barrier to progress in iterative auction design [25, 10]. *i*BEA elicits just enough information, beyond that required to compute a single competitive equilibrium, to implement the Vickrey outcome.

In future work it would be interesting to design special-cases of *i*BEA for particular restrictions on agent preferences that violate the agents are substitutes condition, but allow computationally tractable winner-determination and price-adjustment [49, 18]. Empirical tests will be useful to compute average preference elicitation costs in the GVA, with *i*BEA, with and without asynchronous price-updates, and with and without dynamic price-discrimination. Finally, it would be interesting to explore proxy-agent interfaces into *i*BEA, to enhance the preference-elicitation languages supported within the auction, to further accelerate auction progress, and as a method to restrict opportunities for collusion.

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Appendix

Lemma 1 (ntv1).

Proof. To show that CE and NTV prices are sufficient for group-minimal CE prices, we first substitute for $P(\mathcal{I}) = V(\mathcal{I}) - \sum_{i \in \mathcal{T}^*} c_{\mathrm{ce},i}$ and

$$P(\mathcal{I} \setminus \mathcal{L}) = \max_{\mathcal{K} \subseteq \mathcal{I} \setminus \mathcal{L}} \left[V(\mathcal{K}) - \sum_{i \in \mathcal{K}} c_{\mathrm{ce},i} \right], \quad \forall \mathcal{L} \subseteq \mathcal{T}^*, \mathcal{L} \neq \emptyset$$

into [RD-CS'], where this second expression follows because NTV prices satisfy CE conditions in subproblem $\textsc{cap}(\mathcal{I} \setminus \mathcal{L})$ for those agents selected in the surplus-maximizing allocation in that subproblem, but not necessarily for *all* agents.

Now, consider the term $\max\limits_{\mathcal{K} \subseteq \mathcal{I} \setminus \mathcal{L}} \left[V(\mathcal{K}) - \sum_{i \in \mathcal{K}} c_{\mathrm{ce},i} \right]$, and consider a constraint defined on some set $\mathcal{L}'$ for which $\mathcal{K}^* \subset \mathcal{I} \setminus \mathcal{L}'$ is optimal. This constraint is redundant, by comparison with the constraint defined over set $\mathcal{L}'' = (\mathcal{I} \setminus \mathcal{K}^*)$. Notice that $\mathcal{K}^* \supseteq \mathcal{T}^*$, because $c_{\mathrm{ce},i} = 0$ for all $i \notin \mathcal{T}^*$, and that $V(\mathcal{K})$ is (weakly) increasing in $\mathcal{K}$. Also, we have $\mathcal{L}'' \supseteq \mathcal{L}'$, because $\mathcal{K}^* \subset \mathcal{I} \setminus \mathcal{L}'$. We have $\mathcal{L}' \subset \mathcal{L}'' \subseteq \mathcal{T}^*$, and the constraint for $\mathcal{L}''$ dominates that for $\mathcal{L}'$.

We are free to replace the constraint on $\mathcal{L}'$ with the weaker constraint, $\sum_{i \in \mathcal{L}'} \Delta_i \leq V(\mathcal{I}) - \sum_{i \in \mathcal{T}^*} c_{\mathrm{ce},i} - [V(\mathcal{I} \setminus \mathcal{L}') - \sum_{i \in \mathcal{I} \setminus \mathcal{L}'} c_{\mathrm{ce},i}]$. Finally, the constraints in [RD-CS'] can be expressed as $\sum_{i \in \mathcal{L}} \Delta_i \leq V(\mathcal{I}) - V(\mathcal{I} \setminus \mathcal{L}) - \sum_{i \in \mathcal{L}} c_{\mathrm{ce},i}$, for all $\mathcal{L} \subseteq \mathcal{T}^*$, since $-\sum_{i \in \mathcal{T}^*} c_{\mathrm{ce},i} + \sum_{i \in \mathcal{I} \setminus \mathcal{L}} c_{\mathrm{ce},i} = -\sum_{i \in \mathcal{L}} c_{\mathrm{ce},i}$.

A simple variable substitution, with $\Delta'_i = \Delta_i + c_{\mathrm{ce},i}$, changes the objective to $\max - \sum_i c_{\mathrm{ce},i} + \sum_i \Delta'_i$, subject to exactly the constraints in the group-minimal CE price formulation [RD']. Therefore, at the optimal solution $\sum_i \Delta'_i = \sum_i \overline{\pi}_i$, for group-minimal agent payoffs $\overline{\pi}$, and a solution $(\Delta^g_{\mathrm{adjust},1}, \ldots, \Delta^g_{\mathrm{adjust},n})$ to [RD-NTV] satisfies $\sum_i \Delta^g_{\mathrm{adjust},i} = \sum_i \overline{\pi}_i - \sum_i c_{\mathrm{ce},i}$. Since the payoff to agent i at the initial NTV and CE prices is precisely $c_{\mathrm{ce},i}$, the total adjusted payoff is equal to the total agent payoff at some set of group-minimal CE prices. $\square$

Lemma 2 (ntv2).

Proof. Let $P(\mathcal{K})$ denote the maximal surplus to the seller at Quasi-CE prices, p_{qce}, over all allocations restricted to agents in set $\mathcal{K}$. Let $\pi_i = v_i(S^*_i) - p_{\mathrm{qce},i}(S^*_i)$ for all $i \in \mathcal{T}^*$ and $\pi_i = 0$ otherwise. To show that NTV prices can be constructed with the same surplus properties as Quasi-CE prices, the proof is to construct NTV prices:

$$p_{\mathrm{ntv},i}(S) = \begin{cases} [v_i(S) - \pi_i]^+ & \text{, if } i \in \mathcal{T}^* \\ v_i(S) & \text{, otherwise.} \end{cases}$$

and show that the prices on all bundles in surplus-maximizing allocations to the seller in $\textsc{cap}(\mathcal{I})$ and $\textsc{cap}(\mathcal{K})$ for all $\mathcal{K} \in \mathcal{C}$ remain unchanged, while the price on every other bundle (weakly) decreases. First, consider prices to an agent $i \in \mathcal{T}^*$. Initially $p_{\mathrm{qce},i}(S) = v_i(S) - \pi_i$ on any bundle S it is allocated in the solution to $P(\mathcal{I})$ or $P(\mathcal{K})$ for some $\mathcal{K} \in \mathcal{C}$, by Quasi-CE. In addition, the initial price $p_{\mathrm{qce},i}(S) \geq v_i(S) - \pi_i$ on any other bundle, and therefore the adjusted prices are (weakly) smaller than the initial prices on these bundles. Second, for an agent $i \notin \mathcal{T}^*$, because initial prices are CE then $p_i(S) \geq v_i(S)$ for all S, and the adjusted prices are (weakly) smaller. $\square$

Lemma 5.[covering property]

Proof. Consider bundle S' with a strict increase in its anonymous ask price in round t, and show that at least two agents cover this bundle.

There must be some agent, $i \in \mathit{unhappy}^t$, with $i \in \mathit{anon}^{t+1}$ and $i \in \mathit{safe}^t$, with $p^t_{\mathrm{bid}}(S') = p^t_{\mathrm{anon}}(S)$. Without loss of generality, consider only bids on bundles S'' with $p^t_{\mathrm{br},i}(S'') = p^t_{\mathrm{anon}}(S'')$. Proceed by case analysis on the relationship between $\pi^t_{\mathrm{br},i}$ and $\pi^{t+1}_{\mathrm{br},i}$. Case (a), $\pi^{t+1}_{\mathrm{br},i} = \pi^t_{\mathrm{br},i} - \epsilon$. Let $\pi^t_{\mathrm{br},i}(S) = v_i(S) - p^t_{\mathrm{br},i}(S)$. Since $S' \in \mathcal{B}^t_i$, we have $\pi^t_{\mathrm{br},i}(S') + \epsilon \geq \pi^t_{\mathrm{br},i}$, and therefore $\pi^{t+1}_{\mathrm{br},i}(S') + \epsilon = \pi^t_{\mathrm{br},i}(S') \geq \pi^t_{\mathrm{br},i} - \epsilon = \pi^{t+1}_{\mathrm{br},i}$, and bundle S' remains in the agent's MBR set. Case (b), $\pi^t_{\mathrm{br},i} \geq \pi^{t+1}_{\mathrm{br},i} > \pi^t_{\mathrm{br},i} - \epsilon$. Now, no bundle $S'' \notin \mathcal{B}^t_i$ can solve $\pi^{t+1}_{\mathrm{br},i}$, even if the price on every other bundle falls by ϵ. For $\pi^{t+1}_{\mathrm{br},i} > \pi^t_{\mathrm{br},i} - \epsilon$, the new maximal surplus must equal the surplus on some bundle $\hat{S} \in \mathcal{B}^t_i$ with the same bid price across rounds. This requires either $p^t_{\mathrm{br},i}(\hat{S}) < p^t_{\mathrm{ask},i}(\hat{S})$ and $p^t_{\mathrm{ask},i}(\hat{S}) \leq v_i(\hat{S})$, or $v_i(\hat{S}) - p^t_{\mathrm{ask},i}(\hat{S}) \leq \epsilon$; either way we have $v_i(\hat{S}) - p^{t+1}_{\mathrm{br},i}(\hat{S}) \leq \epsilon$, and $\pi^{t+1}_{\mathrm{br},i} \leq \epsilon$. Finally, $\pi^{t+1}_{\mathrm{br},i}(S') \geq 0$, and bundle S' remains in the agents MBR set.

In addition, the test in step 4) of SelectDrop (Figure 6) requires *redundant*$(i, (\mathcal{L}_{\mathrm{anon}} \setminus i) \cup \mathcal{L}_{\mathrm{extra}})$ for all agents $i \in \mathcal{L}_{\mathrm{anon}}$ that cause an increase in anonymous prices. In this expression, the set used to check redundancy includes safe bids from agents other than i that continue to face anonymous prices in $t+1$ and for which anonymous price increases based on their bids would be redundant in the next round. An equivalent analysis to that made above for agent i, then shows that at least one of these agents $j \neq i$ will also cover the anonymous price increase on any bundle S' that is caused by agent i. $\square$

Lemma 8(non-anonymous cover).

Proof. Let t' denote the first round in which agent i faced non-anonymous ask prices. To show that bundle S' with strict-positive non-anonymous ask price, $p^t_{\mathrm{ind},i}(S')$, in some round $t \geq t'$ is covered, we first observe that (a) the bundle had a strict-positive anonymous ask price in round $t'-1$, or (b) the bundle received a bid from agent i in some round $t'' \geq t'$. In case (a), then either agent i covered S' in round $t'-1$ and continues to bid for S' by the covering property, or another agent must cover a bundle that weakly-dominates S' at the current ask prices, which can only have increased from round $t'-1$. In case (b), agent i will continue to bid for S', because bundle S' was in its surplus-maximizing set in round t'' and it it easy to show that once a bundle enters the surplus-maximizing set it never leaves. $\square$

Lemma 9 (transformation).

Proof. To show that an allocation S can be transformed into some allocation S' that is compatible with agent bids, while maintaining seller surplus given the current ask prices, we can first restrict attention to allocations over bundles with strict-positive ask prices. In addition, suppose that agents $1, \ldots, l$ face anonymous prices, while agents $l+1, \ldots, n$ face non-anonymous prices. To transfer S to S', successively implement one of the following steps until *every bundle in the allocation is either associated with an agent that covers the bundle, is allocated to an agent that faces non-anonymous ask prices and has a strict-positive ask price.* Notice that in each step, every bundle reallocated receives a bid from the associated agent, by covering lemmas 6 and 8.

(i) agent $i \in anon^t$ is allocated a bundle S_i that it does not cover, and an agent, j, that covers the bundle faces anonymous prices and is not currently allocated a bundle: *move the bundle to agent j.*

(ii) agent $i \in anon^t$ is allocated a bundle S_i that it does not cover, and an agent, j, that covers the bundle faces anonymous prices and is currently allocated a bundle, S_j: *switch the bundles between the two agents.*

(iii) agent $i \in anon^t$ is allocated a bundle S_i that it does not cover, and an agent, j, that covers the bundle faces non-anonymous prices and is not currently allocated a bundle: *move the smallest bundle that weakly dominates S_i at non-anonymous prices $p^t_{\text{ind},j}(S)$ to agent j.*

(iv) agent $i \in anon^t$ is allocated a bundle S_i that it does not cover, and an agent, j, that covers the bundle faces non-anonymous prices and is currently allocated bundle, S_j: *switch the bundles, agent j receives the smallest bundle that weakly dominates S_i at non-anonymous prices $p^t_{\text{ind},j}(S)$, and agent i receives the smallest bundle that weakly dominates S_j at anonymous prices $p^t_{\text{anon}}(S)$.*

Each step (weakly) increases the revenue to the seller. Steps (i) and (ii) have a neutral effect. Step (iii) has a (weakly) positive effect because the ask price on the bundle can have only increased since the covering agent started facing non-anonymous prices. For step (iv), let t' denote the round in which agent j first faces non-anonymous prices. Define $\delta_j = p^t_{\text{ind},j}(S_i) - p^{t'}_{\text{ind},j}(S_i)$, and note that $p^t_{\text{anon}}(S_i) = p^{t'}_{\text{anon}}(S')$, so that $p^t_{\text{ind},j}(S_i) - p^t_{\text{anon}}(S_i) = \delta$, since $p^{t'}_{\text{ind},j}(S_i) = p^{t'}_{\text{anon}}(S_i)$. Now, we must have $p^t_{\text{ind},j}(S_j) - p^{t'}_{\text{anon}}(S_j) \leq \delta$ because agent j bids for S_i whenever it bids for S_j. Then, with $p^t_{\text{anon}}(S_j) \geq p^{t'}_{\text{anon}}(S_j)$, we have $p^t_{\text{ind},j}(S_j) - p^t_{\text{anon}}(S_j) \leq \delta$, which gives $p^t_{\text{anon}}(S_j) + p^t_{\text{ind},j}(S_i) \geq p^t_{\text{anon}}(S_i) + p^t_{\text{ind},j}(S_j)$. Finally, the transformation procedure terminates because each step successfully allocates at least one more bundle to an agent. Only bundles allocated to agents facing anonymous prices that do not cover a bundle are moved, and every time a bundle is moved at least one new bundle is associated with an agent that covers the bundle, and no agent is ever removed from a bundle that it covers. $\square$

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