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Multi-Item Vickrey-Dutch Auctions*

Debasis Mishra [†] David C. Parkes [‡]

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Abstract

Descending price auctions are adopted for goods that must be sold quickly and in private values environments, for instance in flower, fish, and tobacco auctions. In this paper, we introduce ex post efficient descending auctions for two environments: multiple non-identical items and buyers with unit-demand valuations; and multiple identical items and buyers with non-increasing marginal values. Our auctions are designed using the notion of universal competitive equilibrium (UCE) prices and they terminate with UCE prices, from which the Vickrey payments can be determined. For the unit-demand setting, our auction maintains linear and anonymous prices. For the homogeneous items setting, our auction maintains a single price and adopts Ausubel's notion of "clinching" to compute the final payments dynamically. The auctions support truthful bidding in an ex post Nash equilibrium and terminate with an ex post efficient allocation. In simulation, we illustrate the speed and elicitation advantages of these auctions over their ascending price counterparts.

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1 Introduction

An iterative auction can be described as a monotonic price adjustment procedure that takes bids from buyers in each iteration. Iterative auctions are often preferred over sealed-bid auctions, even in private value environments. The most important reasons are those of transparency, speed, and cost of participation (by avoiding the revelation of unnecessary valuation information through dynamic price discovery) (Cramton, 1998; Perry and Reny, 2005; Compte and Jehiel, 2005). Most of the iterative auction design literature has focused on ascending price auctions (Demange et al., 1986; Gul and Stacchetti, 2000; Parkes and Ungar, 2000; Bikhchandani and Ostroy, 2006; Ausubel, 2004; Ausubel and Milgrom, 2002; de Vries et al., 2007; Mishra and Parkes, 2007). In comparison, there are few results on the design of descending price auctions.¹

Traditionally, in the descending price auction for a single item (called a *Dutch auction*), the seller starts the auction from a very high price and iteratively lowers the price. The first buyer to accept the price wins the auction at that price. The use of such auctions is popular because of its speed. Dutch auctions are used in selling flowers in Netherlands (thus the name Dutch auction) (van den Berg et al., 2001), fish in Israel, and tobacco in Canada. This type of descending price auction is strategically equivalent to a first-price sealed-bid auction and we can expect demand reduction and inefficiency for asymmetric settings (Krishna, 2002).

Contrast this with the simple equilibrium bidding strategies and efficiency in ascending price auctions such as the English auction, and its generalizations for multiple items (Demange et al., 1986; Parkes and Ungar, 2000; Ausubel, 2004; Ausubel and Milgrom, 2002; Mishra and Parkes, 2007). In these auctions, under appropriate assumptions on valuations, it is an *ex post* Nash equilibrium for a buyer to report his true demand set in every iteration; i.e., straightforward bidding is an equilibrium strategy whatever the private valuations of agents. The auctions terminate with the Vickrey-Clarke-Groves (VCG) (Vickrey, 1961; Clarke, 1971; Groves, 1973) outcome. Not only is this bidding strategy simple and robust to incorrect agent beliefs, but it is an *ex post* efficient equilibrium. One might ask whether there exists a descending price auction counterpart of the English auction with such simple bidding as an equilibrium strategy. The answer is “yes”, as noted by Vickrey in his seminal paper (Vickrey, 1961). For the single item setting, Vickrey points out that the Dutch auction can be modified to run until a second buyer accepts an offer and the first buyer to accept an offer wins but pays the price at which the second offer is accepted. Quoting Vickrey (1961):

“On the other hand, the Dutch auction scheme is capable of being modified with advantage to a second-bid price basis, making it logically equivalent to the second-price sealed-bid procedure . . .”

¹One exception is in the work of Mishra and Garg (2006), who propose a generalized Dutch auction for one-to-one assignment setting, the setting in Demange et al. (1986), which terminates at the maximum competitive equilibrium price (approximately) if buyers bid honestly. But, this work provides no game theoretic equilibrium analysis.

Then, he goes on to describe an apparatus that is commonly used to implement the Dutch auction and how the same apparatus can be modified to implement this second-price auction. Quoting [Vickrey \(1961\)](#) again:

“There would be no particular difficulty in modifying the apparatus so that the first button pushed would merely preselect the signal to be flashed, but there would be no overt indication until the second button is pushed, whereupon the register would stop, indicating the price, and the signal would flash, indicating the purchaser.”

The appropriate method to extend Vickrey’s idea to more general environments appears to be a puzzle in the current literature. For instance, in their work on the design of an ascending price auction for the homogeneous items case, [Bikhchandani and Ostroy \(2006\)](#) observe the following while interpreting their auction as a primal-dual algorithm:

“The primal-dual algorithm we describe starts at a low price where there is excess demand. One could start the primal-dual algorithm at a high price at which there would be excess supply, but it is unlikely that this would converge to a marginal pricing equilibrium.”²

In this paper we present generalized “Vickrey-Dutch” auctions for multi-unit and multi-item environments that retain the speed and elicitation advantages that descending price auctions can enjoy over ascending price auctions, while inheriting the robust and simple equilibrium properties that come from termination at the Vickrey prices. The design of these auctions follows the methodology of *universal competitive equilibrium* (UCE) prices ([Mishra and Parkes, 2007](#)) to achieve the VCG outcome. Here we demonstrate, for the first time, the role of UCE prices in the design of auctions with simple prices.

Our Vickrey-Dutch auctions maintain a single price trajectory which can pass “through” a part of the competitive equilibrium price space, before terminating at UCE prices. This dynamics provides for enough demand revelation to realize the VCG outcome. Truthful bidding is an ex post Nash equilibrium strategy, providing robustness to the particular distribution of agent valuations. Our auctions reduce to Vickrey’s descending price auction for the single-item environment, while generalizing Vickrey’s apparatus to the non-identical item and multiple identical items environments.³

For the unit demand environment, the auction maintains a single set of item prices and can be considered to provide the descending analog to the ascending price auctions of

²A marginal pricing equilibrium is a competitive equilibrium price where all buyers get their respective payoffs in the VCG mechanism.

³After the first version of this work, these Vickrey-Dutch auctions have been generalized to the case of multiple heterogeneous items with buyers having combinatorial values in [Mishra and Veeramani \(2006\)](#). But this Vickrey-Dutch auction maintains non-linear and non-anonymous prices and its price dynamics is more complex than the Vickrey-Dutch auctions in the current paper. A possible future research direction is to identify more valuation domains where simple price dynamics can be maintained in Vickrey-Dutch auctions.

Demange et al. (1986).⁴ For multiple identical items and non-increasing marginal valuations (NIMV) we design a “clinging” auction, which provides a descending price analog to the ascending price clinging auction of Ausubel (2004). Just as in Ausubel (2004), our auction maintains a single price in each iteration, with the allocation and payments determined dynamically across iteration. The analysis of the auction establishes that the price in any iteration when coupled with the history of clinging decisions up to that iteration actually provides a concise representation of a non-linear and non-anonymous price vector that terminates at UCE prices.

In completing this section, we will discuss the importance of descending price auctions in some practical auction environments. The rest of the paper is then organized as follows. In Section 2, we introduce our model and introduce the principles behind the design of our Vickrey-Dutch auctions. The specific auctions, for the unit demand setting and the identical items setting are presented in Sections 3 and 4. In Section 5 we present simulation results to illustrate the speed and elicitation-cost advantages that descending price auctions can enjoy over ascending price auctions. We conclude with some future research directions.

1.1 Making the Case for Descending Price Auctions

It is commonly held that an ascending price format is important, in comparison with a sealed-bid format, because it does not reveal the winner’s willingness to pay. The winning bidder may prefer to keep this private when engaged in a long-term strategic interaction with the seller, for instance, to avoid low prices in future periods. This can also limit any “political” problems in second-price sealed-bid auctions, for instance when the price paid by the winner is significantly less than the willingness to pay (Rothkopf et al., 1990).

While providing new privacy for losing bidders, descending price auctions lose this advantage for winning bidders over sealed bid auctions. However, we agree with Perry and Reny (2005), Compte and Jehiel (2005), and Parkes (2005) that there is another more general advantage that ascending price auctions often enjoy over sealed-bid auctions even in private value environments. When participants face costly valuation problems then iterative auctions can provide a significant advantage, with price discovery guiding bidders in deciding how to invest effort in refining their beliefs about their (private) valuations. Importantly, our results show that descending price auctions can enjoy here a significant further advantage.

The valuation problem faced by participants in an auction is often costly and time-consuming. Participants in a flower auction must determine their value for different types and quantities of flowers. Participants in an FCC wireless auction must determine a new business plan to determine the economic value of a particular spectrum allocation. Both activities can require costly information acquisition as well as the cognitive attention of participants. Price discovery in iterative auctions can guide a bidder to determine how accurate

⁴To be precise, our auction is a descending price analog of the variation on the auction in Demange et al. (1986) presented in Sankaran (1994).

he must determine his value for items and on which items to focus attention. In comparison, sealed-bid auctions can require bidders to submit, and consequently determine, significant amounts of unnecessary information about their own valuations for different allocations.⁵ Thus, iterative auctions can enjoy practical economic advantages over sealed-bid auctions even in private-value settings.

In some environments descending price auctions can promote more efficient preference elicitation than ascending price auctions by completely avoiding unnecessary elicitation from losing bidders. The cost is typically a small amount of additional elicitation from winning bidders. As already noted in the introduction, descending price auctions can also have a *speed advantage* over ascending price auctions and terminate after fewer rounds. This can be important when auctioning time-sensitive goods. Descending price auctions also provide less opportunity for collusion, an oft-cited reason for the failure of electronic auctions in the supply chain (Elmaghraby, 2004), since there are less bids submitted and thus less opportunity for bidders to communicate via bids.

We return now to the issue of providing privacy for winning bidders, which can be important both in markets with long-term strategic competition between participants and also for political reasons in second-price auctions. Privacy goals can be addressed through orthogonal approaches to the design of the auction process, both technology and business related. For instance, privacy can be provided in electronic auctions by using a trusted third party to host the auction and private communication channels and also through cryptographic methods. Trusted third parties are abundant in e-commerce, for instance eBay for consumer-to-consumer auctions and companies such as Ariba, Emptoris and CombineNet for business-to-business procurement auctions.⁶ Cryptographic technology, that uses computational hardness to also prove the correctness of an auction to participants while retaining privacy (or even to securely implement an auction without even a trusted third party) can also be adopted (Elkind and Lipmaa, 2004; Parkes et al., 2007).⁷

The simulation results that we present in Section 5 confirm that there are reasonable environments in which the Vickrey-Dutch auctions that we design enjoy both speed and preference-elicitation properties that dominate those of ascending-price auctions. A simple observation that emerges is that when the average clearing price on an item is above the median value on that item the descending auctions have better speed and elicitation properties

⁵The preference elicitation advantage is not necessarily true in the *worst case*. For instance, in the context of iterative combinatorial auctions, Nisan and Segal (2006) construct valuations for which the worst-case communication efficiency of ascending price auctions is equal to that of sealed-bid auctions.

⁶Furthermore, as supply chain relationships become more collaborative it is increasingly common for the winning firms in reverse auctions to share information about their cost base in order to work collaboratively in achieving further cost reductions and process improvements. From this perspective, it seems more desirable that the costs of winners be revealed than the costs of losing bidders.

⁷There remains an advantage for ascending-price auctions over descending-price auctions in *interdependent valuation* environments in which the private information of buyers can influence the valuation of other buyers (Perry and Reny, 2005). We emphasize that we study fully private-value environments and accept this restriction in return for the improved speed and elicitation properties of descending price auctions.

than the ascending auctions.

2 The Model and Preliminaries

To begin we introduce a general model with n heterogeneous indivisible items. In a later section we specialize this model and consider n identical items. We define competitive equilibrium and universal competitive equilibrium prices in this model and provide a general framework for the design of iterative VCG auctions.

The set of items is denoted by $A = \{1, \dots, n\}$. There are m (≥ 2) buyers, denoted by $B = \{1, \dots, m\}$. The set of all bundles of items is denoted by $\Omega = \{S \subseteq A\}$. Naturally, $\emptyset \in \Omega$. For every buyer $i \in B$ and every bundle $S \in \Omega$, the valuation of i on bundle S is denoted by $v_i(S) \geq 0$, assumed to be a non-negative integer. If S is a singleton, we write $v_i(j)$ instead of $v_i(\{j\})$ for simplicity.

We assume a private values setting where each buyer knows his own valuation function and it does not depend on the valuations or allocations of other buyers. The payoff of any buyer $i \in B$ on any bundle $S \in \Omega$ is given by $v_i(S) - p$, where p is the price paid by buyer i on bundle S . Also, if a buyer gets nothing and pays nothing, then his utility is zero: $v_i(\emptyset) := 0 \forall i \in B$. We also assume that $v_i(S) \leq v_i(T) \forall i \in B, \forall S, T \in \Omega$ with $S \subseteq T$. The seller values the items at zero. His payoff (or revenue) is the total payment he receives from buyers.

Let $B_{-i} = B \setminus \{i\}$ be the set of buyers without buyer i . Let $\mathbb{B} = \{B, B_{-1}, \dots, B_{-m}\}$. We will denote the economy with buyers only from set $M \subseteq B$ as $E(M)$. Whenever, $M \neq B$ and $M \in \mathbb{B}$, we call economy $E(M)$ a *marginal economy*. $E(B)$ is called the *main economy*.

Let x denote a feasible allocation in economy $E(M)$ ($M \in \mathbb{B}$). Allocation x is both a partitioning of the set of items and an assignment of the elements of the partition to buyers in M . Allocation x assigns bundle x_i to buyer i for every $i \in M$ and for every $i \neq j$, $x_i \cap x_j = \emptyset$. The possibility of $x_i = \emptyset$ is allowed. We will denote the set of all *feasible* allocations of economy $E(M)$ as $\mathbb{F}(M)$.

An allocation X is **efficient** in economy $E(M)$ if there does not exist another allocation $y \in \mathbb{F}(M)$ such that $\sum_{i \in M} v_i(y_i) > \sum_{i \in M} v_i(x_i)$.

Consider general prices, that can be both non-linear and non-anonymous, and define the **demand set** of buyer $i \in M$ (for some $M \in \mathbb{B}$) at price vector $p \in \mathbb{R}_+^{|M| \times |\Omega|}$ as

$$D_i(p) := \{S \in \Omega : v_i(S) - p_i(S) \geq v_i(T) - p_i(T) \forall T \in \Omega\}$$

and the **supply set** of the seller at price vector $p \in \mathbb{R}_+^{|M| \times |\Omega|}$ in economy $E(M)$ as

$$L(p) := \{x \in \mathbb{F}(M) : \sum_{i \in M} p_i(x_i) \geq \sum_{i \in M} p_i(y_i) \forall y \in \mathbb{F}(M)\}.$$

Define $\pi^s(p) := \sum_{i \in M} p_i(x_i)$, where $x \in L(p)$, as the revenue of the seller at price vector $p \in \mathbb{R}_+^{|M| \times |\Omega|}$ in economy $E(M)$.

Definition 1 Price vector $p \in \mathbb{R}_+^{|M| \times |\Omega|}$ and allocation x are a **competitive equilibrium (CE)** of economy $E(M)$ for some $M \subseteq B$ if $x \in L(p)$, and $x_i \in D_i(p)$ for every buyer $i \in M$. Price p is called a CE price vector of economy $E(M)$.

If $p \in \mathbb{R}_+^{|B| \times |\Omega|}$, then the components of p corresponding to a set of buyers $M \subset B$ will be denoted as p_M (or, p_{-i} if $M = B_{-i}$). A component of p_M will still be denoted as $p_i(\cdot)$ for every $i \in M$.

Definition 2 A price vector p is a **universal competitive equilibrium (UCE)** price vector if p_M is a CE price vector of economy $E(M)$ for **every** $M \in \mathbb{B}$.

A UCE price vector always exists since $p := v$ is a (trivial) UCE price vector. [Mishra and Parkes \(2007\)](#) showed that UCE prices are powerful tools for designing ascending price Vickrey auctions. Specifically, UCE prices are necessary and sufficient to realize the VCG outcome from a CE of the main economy, as shown in the following theorem:

Theorem 1 ([\(Mishra and Parkes, 2007\)](#)) Let (p, x) be a CE of the main economy with $p \in \mathbb{R}_+^{|B| \times |\Omega|}$. The VCG payments of every buyer can be calculated from (p, x) if and only if p is a UCE price vector. Moreover, if p is a UCE price vector, then for every buyer $i \in B$, the VCG payment of every buyer $i \in B$ is $p_i^{vcg} = p_i(x_i) - [\pi^s(p) - \pi^s(p_{-i})]$.

The above result will be sufficient for our purposes because our auctions terminate with UCE prices. [Lahaie et al. \(2005\)](#) provide a more general result, that establishes that UCE prices are necessarily determined in *any* iterative mechanism (whether price-based or otherwise) that also determines the VCG outcome.

2.1 The Design of Vickrey-Dutch Auctions

The underlying idea behind the design of most ascending price auctions is that prices are increased in each iteration in response to demand sets collected from bidders, until the auction terminates with CE prices.

It is typical to stop at the first such price vector and design the auction such that this price vector is buyer optimal across all CE prices. This allows one to design ex post efficient ascending price auctions for restricted classes of valuations, those in which these buyer-optimal CE prices equal VCG payments ([Demange et al., 1986](#); [Bikhchandani and Ostroy, 2006](#); [Ausubel and Milgrom, 2002](#); [de Vries et al., 2007](#)).

Contrast this with a descending price auction for a single item, the so-called Dutch auction. It stops as soon as a buyer agrees to buy the item. If buyers bid truthfully, this terminating condition can be interpreted as “stop when a CE price is reached”. But this CE price, the maximum possible CE price in this setting, does not correspond to payments in the VCG outcome. A similar situation arises in more general settings. For example, [Mishra and Garg \(2006\)](#) show that a generalization of the Dutch auction for the unit demand setting terminates at the unique *maximum* CE price vector under truthful bidding.

In fact, a similar difficulty exists in ascending price auctions for general valuations when no CE price vector corresponds to the VCG payments of buyers (de Vries et al., 2007). Mishra and Parkes (2007) overcome this difficulty by searching for a UCE price vector from which final VCG payments are determined as an adjustment upon termination. This adjustment implements discount $(\pi^s(p) - \pi^s(p_{-i}))$ to every winner. The same method will be used for the design of Vickrey-Dutch auctions.

We state a general theorem about the equilibrium properties of iterative Vickrey auctions. Similar results appear in earlier work (Mishra and Parkes, 2007; de Vries et al., 2007; Bikhchandani and Ostroy, 2006). The proof is omitted because it follows immediately from the dominant-strategy incentive compatibility properties of the VCG mechanism. A buyer bids *truthfully* if he submits true demands sets in each iteration, and follows a *straightforward strategy* with respect to (possibly untruthful) valuation \hat{v} if the buyer submits demand sets that are consistent with some valuation \hat{v} :

Theorem 2 *Consider an iterative (ascending or descending) auction that satisfies the following conditions, for all valuations v ,*

- a) *If every buyer bids truthfully then the auction terminates at a UCE price vector and achieves the VCG outcome.*
- b) *If every buyer except i follows a straightforward strategy, then every feasible strategy available to buyer i is equivalent to some straightforward strategy with respect to some valuation \hat{v}_i , perhaps not his true valuation.*

Such an iterative auction has truthful bidding in an ex post Nash equilibrium.

Condition (b) in Theorem 2 can typically be met in an iterative auction by imposing activity rules (Mishra and Parkes, 2007, e.g.).

The search for a UCE price vector in a descending price auction is very different than in an ascending price auction, and it is not a trivial exercise to design a Vickrey-Dutch auction. One starts from high prices where demand is less than supply and lowers prices until supply and demand balance in all economies. This is in contrast to an ascending price auction where prices are initially low, creating higher demand than supply, and prices are adjusted upwards to match supply and demand in all economies. In many settings, ascending price auctions can achieve UCE prices by adjusting prices until supply equals demand in the main economy because this will often times also balance supply and demand in every marginal economy Demange et al. (1986); Ausubel (2004). This is not true for descending price auctions, even for the single item case.

3 The Unit Demand Environment

In this section we introduce a Vickrey-Dutch auction for the environment with heterogeneous, indivisible items and unit-demand valuations so that each buyer is interested in buying

at most one item. This is the standard *assignment problem*. Our auction maintains an individual price on each item and decreases prices until supply balances demand in the main economy and also in all marginal economies.

For convenience, we will assume that there is a *dummy* item, indexed 0, available such that the value of the dummy item is zero for all buyers, and the dummy item can be allocated to any number of buyers. A feasible allocation x assigns to every buyer $i \in B$ either an item $j \in A$ or the dummy item. No item is assigned more than once (but an item may be unassigned). Let x_i denote the item assigned to buyer i in allocation x .

Let $v_i(j)$ denote buyer i 's value for item $j \in A$. A price vector p denotes linear and anonymous prices with $p \in \mathbb{R}_+^{n+1}$ and $p(0) = 0$. The definition of demand set is modified to restrict to include only singleton bundles, with $D_i(p) = \{j \in A \cup \{0\} : v_i(j) - p(j) \geq \max_{j' \in A \cup \{0\}} [v_i(j') - p(j')]\}$, and the definition of CE is specialized as follows:

Definition 3 *Price vector $p \in \mathbb{R}_+^{n+1}$ is a **competitive equilibrium price vector** if there exists an allocation x such that $x_i \in D_i(p)$ for every $i \in B$ and $p(j) = 0$ for every item j that is not assigned in x .*

If (p, x) is a CE in the assignment problem, then x is an efficient allocation (Gul and Stacchetti, 1999). The set of CE price vectors in this unit demand setting form a complete lattice (Shapley and Shubik, 1972), that is there is a unique minimum and a unique maximum CE price vector. Moreover, the minimum CE price vector corresponds to the VCG payments (Leonard, 1983). We obtain the following simple but useful observation:

Proposition 1 *There is a unique linear and anonymous UCE price vector in the unit-demand environment, and this is equal to the minimum CE price vector and defines the VCG payments.*

Proof: If the UCE price vector p is anonymous then the revenue of the seller in the main economy, $\pi^s(p)$ and in any marginal economy $\pi^s(p_{-i})$, is equal because $p = p_{-i}$. This means that the discounts to buyers in Theorem 1 are all zero and that the CE prices on the efficient allocation already correspond to the VCG payments. Leonard (1983) shows that the minimum CE price vector is one such CE price vector and from the lattice result in (Shapley and Shubik, 1972) this is unique. ■

This observation means that searching for UCE prices will directly (without any discount) give the VCG outcome. To illustrate Proposition 1, consider the following example with two buyers $\{1, 2\}$ and two items $\{1, 2\}$. Valuations are: $v_{11} = 8$, $v_{12} = 4$, $v_{21} = 6$, $v_{22} = 3$. It is easy to verify that the minimum CE price vector is $(3, 0)$, which also gives every buyer his VCG payoff (5 for buyer 1 and 3 for buyer 2). It can be easily verified that $(3, 0)$ is a UCE price vector (item 1 is allocated to the remaining buyer in both marginal economies.) Any other CE price vector in this example will reduce the payoff of at least one buyer below his VCG payoff, and it is not a UCE price vector because $p(2) > 0$ and the seller will still want to sell both items in each marginal economy.

3.1 The Vickrey-Dutch Auction (Unit-Demand Environment)

In every iteration, the auctioneer reports prices $p^t \in \mathbb{R}_+^{n+1}$ and receives demands from each buyer. Let $D(p^t) = \{D_i(p^t)\}_{i \in B}$ and $D_{-i}(p^t) = \{D_k(p^t)\}_{k \in B_{-i}}$ denote the vector of demand sets received in iteration t from buyers in B and in B_{-i} respectively.

Given an allocation x , the *revenue* is the sum of prices of all the items allocated. A buyer i is *satisfied* in an allocation x at price vector p if $x_i \in D_i(p)$. An *admissible allocation* at a price vector is an allocation that allocates to every buyer either the dummy item or exactly one item from his demand set. A *provisional allocation* is an admissible allocation that generates the maximum revenue across all admissible allocations, breaking ties in favor of satisfying the maximum number of buyers and then at random. Let $X(D(p^t))$ denote the set of provisional allocations at price vector p^t and let $X(D_{-i}(p^t))$ denote the set of provisional allocations for economy $E(B_{-i})$. Given an allocation x , let $S(x) \subseteq A$ denote the *set of allocated items with positive prices*.

The following concept plays a central role in defining the auction:

Definition 4 *Item $j \in A$ is **universally allocated**, written $j \in U(p, D(p), x)$ given a price vector p , demand sets $D(p)$, and provisional allocation $x \in X(D(p^t))$, if $p(j) = 0$ or item j is provisionally allocated to some buyer i with $S(x) = S(y)$ for some $y \in X(D_{-i}(p))$.*

A universally allocated item j should either have a price of zero or be provisionally allocated to some buyer i such that all items with positive prices (including item j) that are allocated can also be allocated in the marginal economy without buyer i given the current demand. We give two examples in Section 3.2 to illustrate the idea.

The Vickrey-Dutch auction in this environment seeks prices for which all items are universally allocated and reduces prices on items that are not universally allocated until this is achieved. The final prices are UCE prices by definition of *universal allocation*. We refer to the auction as the *linear-price Vickrey-Dutch* (LVD) auction:

Definition 5 *The **linear-price Vickrey-Dutch** (LVD) auction for the unit demand environment is defined as follows:*

(S0) *Start from a high price p^0 where no buyer demands any item from A . Set $t := 0$.*

(S1) *In iteration t of the auction, with price vector p^t :*

(S1.1) *Collect the demand sets $D(p^t)$ of all the buyers at p^t .*

(S1.2) *Based on the demand sets of buyers at p^t , calculate a provisional allocation $x^t \in X(D(p^t))$.*

(S1.3) *Find the universally allocated set of items, $U(p^t, D(p^t), x^t)$.*

(S1.4) *If $U(p^t, D(p^t), x^t) = A$ (the set of all items), go to Step (S2). Else, set $p^{t+1}(j) := p^t(j) - 1 \forall j \in (A \setminus U(p^t, D(p^t), x^t))$. Set $t := t + 1$ and repeat from Step (S1).*

(S2) The auction terminates in current iteration T with price vector p^T and the provisional allocation x^T . If $x_i^T = j$ for buyer i , then he pays an amount $p^T(j)$ and gets item j .

The problem of finding a provisional allocation $x^t \in X(D(p^t))$ is a variant on the standard assignment problem.⁸ We will now provide a computationally efficient procedure to determine the set of universally allocated items. This is used both to adjust prices and also to check for termination.

3.2 Identifying Universally Allocated Items

Consider the examples in Figures 1(a) and 1(b), which illustrate demand sets and an allocation at two different price vectors. Suppose all prices are positive. A solid line between a buyer and an item means that the buyer is provisionally allocated to the item. A dashed line between a buyer and an item means that the buyer has an item in his demand set but is not provisionally allocated to the item. Each figure represents all the information required to determine the set of universally allocated items.

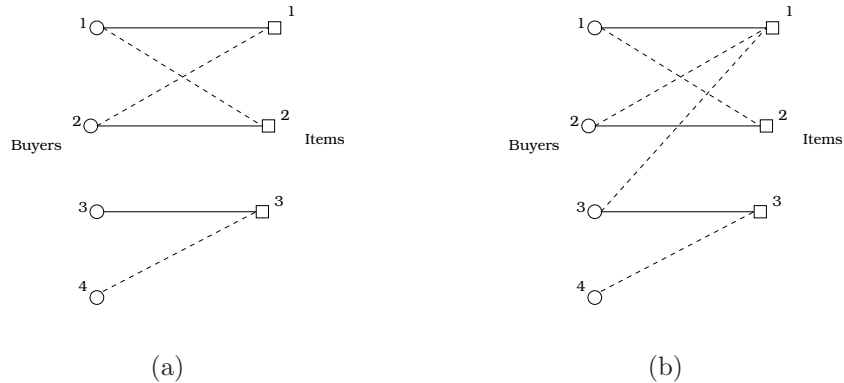


Figure 1: Identifying universally allocated items. Dashed line: item in demand set but not provisionally allocated. Solid line: provisional allocation. All prices are positive.

In Figure 1(a), item 3 is universally allocated: remove buyer 3 (provisionally allocated to item 3), then allocate buyer 4 to item 3 without changing the total set of allocated items. But, no other items are universally allocated. In case of item 1, if we remove buyer 1, the only buyer that demands item 1 is buyer 2 and thus we cannot allocate item 1 without reducing the total set of provisionally allocated items. Item 2 is also not universally allocated, by symmetry. In Figure 1(b), all 3 items are universally allocated. Buyer 4 will take buyer 3's item. Without buyer 1 (allocated to item 1), we can allocate buyer 4 to item 3 and buyer 3

⁸It can be solved with two linear programs (LPs). The first LP computes an admissible allocation that maximizes total revenue given demand sets $D(p^t)$. A second LP is then formulated to break ties in favor of maximizing the number of satisfied buyers, with the objective defined as such and a constraint included to ensure that the revenue from the allocation is equal to that obtained in solving the first LP.

to item 1 without changing the total set of allocated items. Without buyer 2, we can allocate buyer 4 to item 3, buyer 3 to item 1 and buyer 1 to item 2.

It is instructive that item 3, which is demanded by an unallocated buyer (4), is a universally allocated item and the starting point for finding other universally allocated items. We use this idea to develop a procedure to determine the set of universally allocated items.

We first handle items with zero price. Given $(p, D(p), x)$, let $(p', D'(p'), x')$ denote the restriction to items with positive price, with $p' = (p(j) : j \in A, p(j) > 0)$, $D'_i(p') = \{j : j \in D_i(p), p(j) > 0\}$, and $x'_i = x_i$ when $p(x_i) > 0$ and $x'_i = 0$ otherwise. Let $A_0(p) = \{j \in A, p(j) = 0\}$ denote the items with zero price.

Lemma 1 $U(p, D(p), x) = U(p', D'(p'), x') \cup A_0(p)$ where $(p', D'(p'), x')$ is the restriction of $(p, D(p), x)$ to items with positive price and $A_0(p)$ is the set of items with zero price.

Proof: To show $U(p, D(p), x) \supseteq U(p', D'(p'), x') \cup A_0(p)$, if $j \in A_0(p)$ then $j \in U(p, D(p), x)$ by definition. If $j \in U(p', D'(p'), x')$ then j is allocated in x' and thus also in x , and moreover there is some $y' \in X(D'_{-i}(p'))$ (where j is allocated to i in x) with $S(y') = S(x')$. Now, consider $z \in X(D_{-i}(p))$. Clearly, $S(z) = S(y')$ because z must allocate as many items with positive price as y' for it to be a provisional allocation. Since $S(y') = S(x') = S(x)$, we have $S(z) = S(x)$.

To show $U(p', D'(p'), x') \cup A_0(p) \supseteq U(p, D(p), x)$, consider $j \in U(p, D(p), x)$. If $p(j) = 0$ then $j \in A_0(p)$. If $p(j) \neq 0$ then j is allocated in x , to say i . Then, there is some $y \in X(D_{-i}(p))$ with $S(y) = S(x)$. Construct $y' \in X(D'_{-i}(p'))$ with $S(y') = S(x') = S(x)$ by assigning the dummy item to any agent $k \neq i$ allocated to an item with zero price in allocation y . Thus, item $j \in U(p', D_{-i}(p'), x')$. ■

We now formalize the intuition in the example, in which we identified a sequence of reassignments of items, starting with a currently unallocated buyer. Given allocation x , let a *well-defined chain* with respect to buyer i , allocated to item j in x , be $z_{-i}(x, D(p)) = j_0 i_1 j_1 \dots i_c j_c$, with $c \geq 1$ (this is a sequence of alternating buyers and items) and with the property that:

- (i) item $j_0 = 0$ and item $j_c = j$
- (ii) buyer i_r , for $1 \leq r \leq c$ is assigned item j_{r-1} in allocation x
- (iii) buyer $i \notin \{i_1, \dots, i_c\}$
- (iv) item $j_r \in D_{i_r}(p)$ for all $1 \leq r \leq c$

Such a chain defines a reassignment of items, with a modified allocation x' defined with $x'_{i_r} = j_r$ for all $r \in \{1, \dots, c\}$, $x'_k = x_k$ for all $k \notin \{i_1, \dots, i_c\} \cup \{i\}$, and $x'_i = 0$.

Lemma 2 Given a price vector p with non-zero prices for all the items, item $j \in U(p, D(p), x)$ if and only if there is a well-defined chain $z_{-i}(x, D(p)) = j_0 i_1 j_1 i_2 j_2 \dots i_c j$ where buyer i is allocated j in allocation x .

Proof: Given such a chain $z_{-i}(x, D(p))$, the modified allocation x' induced by the chain when applied to x satisfies $x' \in X(D_{-i}(p))$ and $S(x') = S(x)$ by definition. Now, consider

some $j \in U(p, D(p), x)$. We construct a well-defined chain $z_{-i}(x, D(p))$, where i is the buyer assigned j in x . Consider allocation $y \in X(D_{-i}(p))$ with $S(y) = S(x)$. Let $i^{(1)}$ denote the buyer assigned to j in y (such a buyer must exist). Let $j^{(1)}$ denote the item assigned to buyer $i^{(1)}$ in x . If $j^{(1)} = 0$ then the chain is $0i^{(1)}j$. Otherwise, let $i^{(2)}$ denote the buyer assigned to $j^{(1)}$ in y and let $j^{(2)}$ denote the item assigned to this buyer in x . Now, if $j^{(2)} = 0$ then the chain is $0i^{(2)}j^{(1)}i^{(1)}j$. Eventually, for some $q \leq n$ there must be a buyer $i^{(q)}$ assigned item $j^{(q-1)}$ in y but unallocated in x and this completes the chain $0i^{(q)}j^{(q-1)} \dots i^{(1)}j$. Such a buyer must exist because y must allocate to at least one buyer not allocated in x since $S(y) = S(x)$. ■

In Figure 1(a) the well-defined chain that explains item 3 is $j_0i_1j_1 = 043$. In Figure 1(b) the well-defined chain that explains item 1 is 04331 and the chain that explains item 2 is 0433112.

Let $\hat{U}(p, D(p), x) = \{j' \in A : j' \in D'_i(p')$ for some $i \in B$ with $x'_i = 0\}$, where $(p', D'(p'), x')$ is the restriction of $(p, D(p), x)$ to positive prices. We can determine the set of universally allocated items $U(p, D(p), x)$ as follows.

Procedure:	UAI
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Step 0:	Initialize $U^{(0)}(p, D(p), x) = \hat{U}(p, D(p), x)$. Set $r := 1$.
Step 1:	Let $T^{(r)}$ denote the set of buyers allocated to items in $U^{(r-1)}(p, D(p), x)$
Step 2:	Let $W^{(r)} = \bigcup_{i \in T^{(r)}} D_i(p) \cap S(x)$ denote the set of provisionally allocated items with positive price demanded by buyers in $T^{(r)}$.
Step 3:	If $W^{(r)} \subseteq U^{(r-1)}(p, D(p), x)$, output $U^{(r-1)}(p, D(p), x) \cup A_0(p)$ and STOP . Else, $U^{(r)}(p, D(p), x) := U^{(r-1)}(p, D(p), x) \cup W^{(r)}$. $r := r + 1$. Repeat from Step 1 .

As an illustration, we apply the UAI algorithm to the example in Figure 1(b). In the first round of the UAI algorithm, set $U(p, D(p), x) = \{3\}$. This gives $T = \{3\}$ and $W = \{1, 3\}$. We update $U(p, D(p), x) = \{1, 3\}$. Now, $T = \{1, 3\}$ and $W = \{1, 2, 3\}$. We update $U(p, D(p), x) = \{1, 2, 3\}$. Now, $T = \{1, 2, 3\}$ and $W = \{1, 2, 3\}$ and we stop because $W = U(p, D(p), x)$.

Proposition 2 *The UAI procedure determines all universally allocated items.*

Proof: Each repetition of steps 1–3 in procedure **UAI** is referred to as a round, indexed $r = 1, 2, \dots$. By Lemma 1 we can reduce the problem of determining the set of universally allocated items to that of finding the set $U(p', D'(p'), x')$ where $(p', D'(p'), x')$ is restricted to items with positive prices. For these, we know from Lemma 2 that an item j is universally allocated if and only if there is a well-defined chain, starting from an unallocated agent in x' and terminating with the item j . The UAI procedure completes the closure of all

items reachable by any well-defined chain, first initializing $U(p, D(p), x)$ to the set of items demanded by an agent unallocated in x' (and thus reachable by a chain of length 1), and then in each round $r = 1, 2, \dots$ identifying all allocated items W reachable by some chain of length $r + 1$. The items with zero price $A_0(p)$ are finally added to the set of universally allocated items. ■

Every round of the UAI procedure either finds new universally allocated items or stops, and thus the maximum number of rounds of the UAI algorithm is n , the total number of items. The computations in each round can be done in polynomial time, and therefore the UAI algorithm runs in polynomial time.⁹

3.3 Theoretical Analysis

When designing ascending price auctions it is common to construct a price trajectory such that the seller is satisfied in every iteration and the buyers are all satisfied upon termination. Our analysis of the LVD auction will establish the reverse for descending price auctions: every buyer is satisfied with the provisional allocation in every iteration and the seller is satisfied upon termination.

Proposition 3 *In the provisional allocation in every iteration of the LVD auction, every buyer is satisfied under truthful bidding.*

Proof: The proof is by induction on the iteration $t \geq 0$ of the auction. The base case is easy because every buyer demands only the dummy item. Now, suppose the claim holds in iteration $t - 1$ and consider iteration $t > 0$. Let $x^{t-1} \in X(D(p^{t-1}))$. By assumption all buyers are satisfied in x^{t-1} .

We will first show that x^{t-1} remains *admissible*, so that every buyer is still satisfied with provisional allocation x^{t-1} in iteration t .

Case 1: Buyer i is unallocated in iteration $t - 1$. Since i is satisfied, $0 \in D_i(p^{t-1})$ and for all $j \in D_i(p^{t-1}) \setminus \{0\}$ then j is universally allocated in iteration $t - 1$ by Lemma 2. Thus, the price is unchanged in iteration t on all items in the demand set of i , while the price falls by at most 1 on all other items. This implies $D_i(p^t) \supseteq D_i(p^{t-1})$.

Case 2: Buyer i is allocated item $j \neq 0$, but j is not universally allocated in iteration $t - 1$. Thus, the price on this item falls in iteration t and since the price on no other item falls by more than this, $j \in D_i(p^t)$.

Case 3: Buyer i is allocated item $j \neq 0$ that is universally allocated. We show that for every $j' \in D_i(p^{t-1})$, we have $j' \in U(p^{t-1}, D(p^{t-1}), x^{t-1})$. Consider the interesting case that

⁹Contrast this with finding a minimal set of over-demanded items that is needed to be computed in each iteration of the ascending-price auction in Demange et al. (1986). A naive algorithm will require verifying the condition for over-demanded items for exponential number of bundles of items in the worst case. Sankaran (1994) observed this and introduced a modified price adjustment that is computationally feasible and provides an auction with the same theoretical properties.

$p^{t-1}(j') > 0$ and suppose that j' is not allocated in x^{t-1} . But, the auction could have then achieved more revenue in iteration $t-1$ with allocation $y \in D_{-i}(p^{t-1})$ such that $S(y) = S(x)$ (which exists since j is universally allocated) and then augmenting allocation y to assign item j' to agent i . Thus, this would contradict with x^{t-1} being a provisional allocation. Now consider the chain $z_{-i}(x^{t-1}, D(p^{t-1})) = 0i_1j_1 \dots i_cj$ that establishes that item j is universally allocated. If buyer i' , allocated to item j' in x^{t-1} , is not represented in the chain then the well-defined chain $0i_1j_1 \dots i_cjij'$ establishes that j' is also universally allocated. On the other hand, if i' is in the chain, for instance $z_{-i}(x^{t-1}, D(p^{t-1})) = 0i_1j_1i_2j'i'j_3 \dots i_cj$ then consider truncated chain $0i_1j_1i_2j'$ which is well-defined for $z_{-i'}(x^{t-1}, D(p^{t-1}))$ and establishes that j' is universally allocated. Since all items in $D_i(p^{t-1})$ are universally allocated, the price of every item remains unchanged in p^t while the price on other items falls by at most 1 on any other item. This implies $D_i(p^t) \supseteq D_i(p^{t-1})$.

Second, we show that a provisional allocation x^t exists in iteration t that satisfies every buyer. We provide a sequence of transformations to construct such a provisional allocation from any provisional allocation \hat{x}^t in iteration t (i.e. \hat{x}^t may assign the dummy item to some agents that do not have it in their demand sets). Initialize $B^{(0)}$ to the set of buyers not satisfied in \hat{x}^t and initialize $A^{(0)} = \emptyset$, to denote the set of items whose assignment is fixed. Pick any $i \in B^{(0)}$ and consider item j allocated to i in x^{t-1} . By the above reasoning we know that $j \in D_i(p^t)$, and thus we construct admissible allocation $x^{(1)}$ by assigning item j to buyer i and assigning the buyer assigned to j in \hat{x}^t to the dummy item. Set $A^{(1)} = \{j\}$ to indicate that the allocation of item j is now fixed. The revenue achieved by the seller in $x^{(1)}$ is the same as in $x^{(0)}$. Define $B^{(1)}$ as the buyers not satisfied in allocation $x^{(1)}$. Continue, picking a buyer $i \in B^{(1)}$, fixing the allocation of another item, and so on. Since this process assigns buyers to their assignments in x^{t-1} , eventually an admissible allocation is achieved with the same revenue as \hat{x}^t that satisfies every buyer. ■

Next, we show that the LVD auction terminates at the unique UCE price vector and consequently the minimum CE price vector, and therefore collects the VCG payment.

Proposition 4 *The LVD auction terminates at the minimum CE price vector when every buyer follows the truthful bidding strategy.*

Proof: The price is reduced on at least one item with positive price in each iteration and thus the auction must terminate since any item with zero price is universally allocated. Upon termination, every item is universally allocated and thus every item with positive price is allocated and the allocation is in the supply set of the seller. Taken together with every buyer being satisfied (Proposition 3) we see that the final prices and the final provisional allocation are a CE. We will next show that the final price vector, p^T , is a UCE price vector. Consider a buyer i . Let $x_i^T = j$. If $j = 0$, then clearly (p^T, x_{-i}^T) is a CE of the marginal economy without buyer i . If $j \neq 0$, then because this item is universally allocated there exists an allocation $y^T \in X(D_{-i}(p^T))$ such that all items with positive price are allocated (and so it is in the supply set), and all the buyers except i are satisfied. This is because of Lemma 2, wherein

the well-defined chain shows that upon moving from allocation x^T to allocation y^T , that any agent $\neq i$ allocated a non-dummy item in x^T is still allocated a non-dummy item in y^T . One additional buyer, unallocated (but satisfied) in x^T is now also allocated a non-dummy item in y^T . Hence, (p^T, y^T) is a CE in the marginal economy without buyer i . This shows that p^T is a UCE price vector, and thus the minimum CE price vector by Proposition 1. ■

For truthful bidding to be an ex post Nash equilibrium it is sufficient to ensure that the only feasible strategy available to a bidder is to bid straightforwardly, for some perhaps untruthful valuation. For this we introduce an activity rule that ensures that there is some valuation function that is consistent with the revealed preference information implied by the demand sets bid by a buyer in each iteration of the auction (we refer to such an activity rule as *revealed preference activity rule*).

Let $\mathbb{T} = \{1, \dots, T\}$ be the set of iterations of the LVD auction till iteration T . Consider a buyer i in the LVD auction. Let \hat{v}_i be a (integer valued) valuation function that satisfies the following inequalities:

$$\hat{v}_i(j) - p^{t'}(j) = \hat{v}_i(k) - p^{t'}(k) \quad \forall j, k \in D_i(p^{t'}), \forall t' \in \mathbb{T} \quad (\mathbf{C-I})$$

$$\hat{v}_i(j) - p^{t'}(j) \geq \hat{v}_i(k) - p^{t'}(k) + 1 \quad \forall j \in D_i(p^{t'}), \forall k \notin D_i(p^{t'}) \forall t' \in \mathbb{T}. \quad (\mathbf{C-II})$$

A bidding strategy of buyer i is *consistent* until iteration T , if the system of equations (C-I) and (C-II) has a feasible solution. The activity rule is defined to ensure feasibility and can be checked by validating these constraints in each iteration.¹⁰

Our main theorem for the LVD auction then follows from Theorem 2, along with the equivalence between the minimum CE price vector and the VCG payments in the unit-demand environment.

Theorem 3 *Truthful bidding is an ex post Nash equilibrium in the LVD auction with the revealed-preference activity rule and the auction is ex post efficient.*

The LVD auction may achieve a CE price vector before termination and has a price trajectory that traverses *through* the CE price vector space to reach the minimum CE price vector. This is illustrated in the next section. Losing buyers must continue to bid even after the first CE price vector has been identified, and thus after it is known to the auction that the buyer is not a winner. We keep buyers ignorant of this by disclosing only the price information in each iteration and withholding information about the provisional allocation or demand sets. Vickrey’s remarks on implementing the Vickrey-Dutch auction for a single item using the “flash button” apparatus point to similar obfuscation.

¹⁰This check can be implemented with a linear program with $O(Tn^2)$ constraints. A simple, non-computational activity rule does not appear to be possible in this auction because of the simple linear prices adopted in the auction. Nevertheless, this activity rule can be made accessible to buyers by providing a bid interface that explicitly restricts the demand sets that a buyer can submit in any iteration to those consistent with previous reports.

3.4 An Example: Unit Demand

There are two buyers $\{1, 2\}$ and two items $\{1, 2\}$. Valuations are: $v_{11} = 8$, $v_{12} = 4$, $v_{21} = 6$, $v_{22} = 3$. The starting price of the LVD auction is $(9, 9)$.

Iteration	Price	$D_1()$	$D_2()$	Provisional Allocations	Universally Allocated Items
0	(9, 9)	{0}	{0}	-	\emptyset
1	(8, 8)	{0, 1}	{0}	$1 \rightarrow 1$	\emptyset
2	(7, 7)	{1}	{0}	$1 \rightarrow 1$	\emptyset
3	(6, 6)	{1}	{0, 1}	$1 \rightarrow 1$	{1}
4	(6, 5)	{1}	{0, 1}	$1 \rightarrow 1$	{1}
5	(6, 4)	{1}	{0, 1}	$1 \rightarrow 1$	{1}
6	(6, 3)	{1}	{0, 1, 2}	$1 \rightarrow 1, 2 \rightarrow 2$	\emptyset
7	(5, 2)	{1}	{1, 2}	$1 \rightarrow 1, 2 \rightarrow 2$	\emptyset
8	(4, 1)	{1}	{1, 2}	$1 \rightarrow 1, 2 \rightarrow 2$	\emptyset
9	(3, 0)	{1}	{1, 2}	$1 \rightarrow 1, 2 \rightarrow 2$	{1, 2}

A CE of the main economy is achieved in iteration 6 but this is not a UCE price vector. In the final iteration, item 2 is universally allocated since its price is zero and item 1 is universally allocated since it can be allocated to buyer 2 in the absence of buyer 1. The final price vector $(3, 0)$ is the UCE price vector identified in Section 3. One can also observe that both buyers are satisfied in every iteration and that the demand set of buyer 2 monotonically increases while he is not allocated. Note, though, that the set of universally allocated items is not monotonically increasing.

We plot the CE price vector space, price trajectory of the LVD auction, and price trajectory of the [Demange et al. \(1986\)](#) (DGS) auction in Figure 2. It is interesting to note that the price trajectory never touches the maximum CE price vector ($p^{max} = (7, 3)$). We can also see that the LVD auction price path travels through a significant portion of the entire CE price vector space before reaching the minimum CE price vector, whereas the first CE price vector in the DGS auction is the minimum CE price vector.

3.5 Remark: The Single-Item Special Case

When there is exactly one item for sale it is universally allocated when the number of buyers demanding the item becomes more than one. Hence, the auction will allocate the item to the first buyer who demands it but terminates at a price for which the item is also demanded by the buyer with the second-highest value. Clearly, this translates to the flash-button apparatus implementation described by Vickrey.

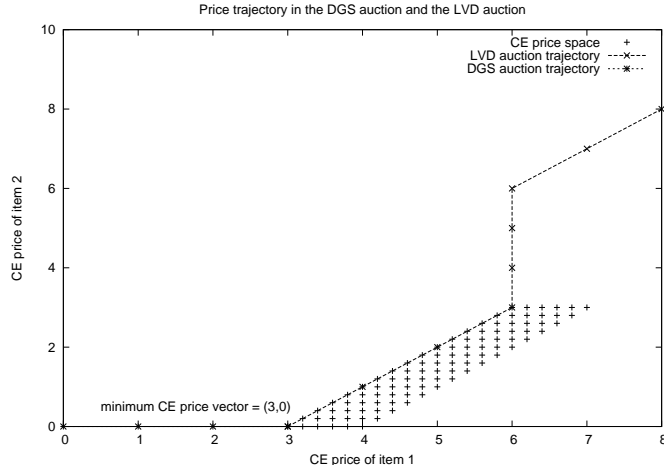


Figure 2: Plot of price trajectory of the LVD auction and the DGS auction

4 The Homogeneous Items Environment with Decreasing Marginal Values

In this section we introduce a Vickrey-Dutch auction for selling multiple units of a homogeneous item for buyers with marginal-decreasing values for each additional unit.

The auction maintains a single price rather than a vector of prices and provides a descending-price analog to the ascending price “clinch” auction of Ausubel (2004): buyers also clinch items in our auction and the final payments are determined from demand information revealed during the auction. The underlying philosophy of the design of this auction remains the UCE price concept: all buyers are satisfied in the provisional allocation in every iteration, and the auction eventually terminates with supply equal to demand in the main economy as well as in every marginal economy.

In introducing the auction we first provide a *stylized clinching auction* that is defined using a non-linear and non-anonymous price vector. This stylized version of the auction makes the UCE framework clear and facilitates our analysis. Ultimately, we will show that the auction can be implemented with the *simple clinching auction*, which maintains a price on a single unit of the item in each iteration. This price is best thought of as defining the current ask price for a marginal unit over and above the units already clinched by any buyer.

4.1 A Stylized Clinching Auction

By a slight abuse of notation, let n denote the number of units of the item for sale. Let $v_i(j)$ denote the value of buyer i for j units of the item, assumed to be a non-negative integer. Assume that $v_i(0) = 0$ for every buyer and consider only *non-increasing marginal values* (NIMV), so that $v_i(j) - v_i(j - 1) \geq v_i(j + 1) - v_i(j)$ for every buyer i and every unit

$1 \leq j < n$.

For the stylized clinching auction, which is used for analysis only, let $p \in \mathbb{R}_+^{m \times (n+1)}$ denote the non-anonymous and non-linear price maintained in each iteration. Let $p_i(j)$ denote the price of buyer i for j units where $0 \leq j \leq n$. Let $D_i(p) = \{j \in \{0, 1, \dots, n\} : v_i(j) - p_i(j) \geq \max_{0 \leq j' \leq n} v_i(j') - p_i(j')\}$. Let the *maximal demand* of buyer i at price vector p be $\overline{D}_i(p)$, defined as the maximum number of units demanded. Allocation x , where x_i denotes the number of units allocated to buyer i , is a *feasible allocation* of economy $E(M)$ for $M \in \mathbb{B}$ if $\sum_{i \in M} x_i \leq n$, and *admissible* if it is feasible and allocation $x_i \in D_i(p) \cup \{0\}$ for every $i \in M$. A *provisional allocation* is an admissible allocation that maximizes the revenue to the seller over all admissible allocations, breaking ties in favor of satisfying as many buyers as possible.

Given prices p , define $\alpha(M, p) := \max(0, n - \sum_{i \in M} \overline{D}_i(p))$ and $\alpha_i(p) := \max_{M \in \mathbb{B}: i \in M} \alpha(M, p)$. Quantity $\alpha(M, p)$ denotes the *under-demand* in economy $E(M)$ while $\alpha_i(p)$ denotes the *under-demand* for buyer i , i.e. the minimal number of additional units that buyer i needs to demand for demand to meet supply in every economy in which he is present.

Definition 6 *The (stylized) clinching Vickrey-Dutch (CVD) auction is defined as:*

(S0) Initialize prices as $p_i^0(j) = q^0 j$ for all $i \in B$ and for all $j \leq n$, where q^0 is a high integer. Set $t := 0$.

(S1) In iteration t of the auction with price vector p^t :

(S1.1) Collect the demand sets $D(p^t)$ of all the buyers at p^t . Impose $\overline{D}_i(p^t) \geq \overline{D}_i(p^{t-1})$ for every buyer $i \in B$ if $t > 0$.

(S1.2) Based on the demand sets of buyers at p^t , calculate the under-demand $\alpha_i(p^t)$ for every buyer.

(S1.3) If $\alpha_i(p^t) = 0$ for every buyer or $q^t = 0$ then go to Step (S2). Else, $q^{t+1} := q^t - 1$ and for every i with $\alpha_i(p^t) > 0$, set

$$p_i^{t+1}(j) := \begin{cases} p_i^t(j) & \text{for all } j \leq \overline{D}_i(p^t) \\ p_i^t(\overline{D}_i(p^t)) + q^{t+1}(j - \overline{D}_i(p^t)) & \text{, otherwise.} \end{cases}$$

Set $t := t + 1$ and repeat from Step (S1).

(S2) The auction terminates in current iteration T with price vector p^T . Final allocation is $x^T \in X(D(p^T))$, and for each buyer $i \in B$ with $x_i^T > 0$, his payment is $p_i^T(x_i^T) - [\pi^s(p^T) - \pi^s(p_{-i}^T)]$.

The auction includes a simple activity rule in Step S1.1, which requires that maximal demand should not decrease across iterations (we also require that no buyer has maximal demand more than zero in the first iteration). The only feedback provided to each buyer in each iteration is the current price vector p_i^t . While there is under-demand for buyer i ,

then parameter q^t represents the ask price in round t on each marginal unit over and above the current number of items demanded by the buyer. The auction terminates when this marginal price is zero, or when there is no under-demand for any buyer. The price vector faced by buyer i in round t is defined recursively in terms of the price in the previous round but adjusted downwards on numbers of units greater than its current demand. The payment is finally determined as a discount from the prices at the end of the auction.

It is useful to define the provisional allocation that is associated with each iteration of the auction, even though this is not explicit in the auction rules. For this purpose, let $x_M^t \in X(D_M(p^t))$ denote the provisional allocation for economy $E(M)$. We will show that provisional allocation, x_M^t , satisfies every buyer in every economy $E(M)$ in every iteration, and that upon termination these provisional allocations are also in the supply set for the seller in every economy. Thus we have UCE prices. This resembles our analysis for the LVD auction.

We first make the following observations. These are made for a variant on CVD without the activity rule in Step (S1.1) of the auction (and serve in part to show that the activity rule does not constrain a truthful bidding strategy):

Lemma 3 *Suppose every buyer follows truthful bidding strategy, and consider the stylized CVD auction without the activity rule in Step (S1.1). Then, in every iteration $t \geq 0$ of this auction in the homogeneous items NIMV environment, we have (a) $\overline{D}_i(p^{t+1}) \geq \overline{D}_i(p^t)$, (b) $D_i(p^t) = \{0, 1, \dots, \overline{D}_i(p^t)\}$, and (c) $v_i(j) - v_i(j-1) \leq p_i^t(j) - p_i^t(j-1)$ for all $0 < j \leq n$, for all buyers $i \in B$.*

Proof: Consider any buyer $i \in B$. The base case for $t = 0$ is easy because the prices are initially high and $\overline{D}_i(p^0) = 0$ (and thus $\overline{D}_i(p^1) \geq \overline{D}_i(p^0)$), $D_i(p^0) = \{0, \dots, \overline{D}_i(p^0)\}$ and $v_i(j) - v_i(j-1) \leq q^0$ for all $0 < j \leq n$. For iteration $t > 0$, consider the interesting case that $\overline{D}_i(p^{t-1}) < n$. Otherwise the prices are unchanged. By the inductive hypothesis (b) and the price change rule, we have $v_i(j) - v_i(j-1) = p_i^{t-1}(j) - p_i^{t-1}(j-1) = p_i^t(j) - p_i^t(j-1)$ for all $0 < j \leq \overline{D}_i(p^{t-1})$, and $v_i(\overline{D}_i(p^{t-1}) + 1) - v_i(\overline{D}_i(p^{t-1})) < p_i^{t-1}(\overline{D}_i(p^{t-1}) + 1) - p_i^{t-1}(\overline{D}_i(p^{t-1})) = q^{t-1}$. By NIMV, $v_i(j) - v_i(j-1) \leq v_i(\overline{D}_i(p^{t-1}) + 1) - v_i(\overline{D}_i(p^{t-1})) < p^{t-1}(j) - p^{t-1}(j-1) = q^{t-1} = q^t + 1$ for all $\overline{D}_i(p^{t-1}) < j \leq n$ by NIMV. This implies that $v_i(j) - v_i(j-1) \leq q^t = p_i^t(j) - p_i^t(j)$ for all $\overline{D}_i(p^{t-1}) < j \leq n$, where the last equality follows from the price change rule. This establishes inductive hypothesis (c) for iteration t . Property (a) is true since $\overline{D}_i(p^t) \geq \overline{D}_i(p^{t-1})$, and the price on marginal unit $\overline{D}_i(p^{t-1}) + 1$ falls while the price on all units $\leq \overline{D}_i(p^{t-1})$ is unchanged. Finally, for inductive hypothesis (b), assume $j \in D_i(p^t)$. If $j > 0$, then we will show that $j-1 \in D_i(p^t)$. Since $j \in D_i(p^t)$, $v_i(j) - v_i(j-1) \geq p_i^t(j) - p_i^t(j-1)$. From (c), which was proved earlier, we have $v_i(j) - v_i(j-1) \leq p_i^t(j) - p_i^t(j-1)$. Together, it gives $v_i(j) - v_i(j-1) = p_i^t(j) - p_i^t(j-1)$. This implies $j-1 \in D_i(p^t)$. Hence (b) holds. ■

A consequence of Lemma 3 is that when every buyer bids truthfully in the auction, then $q^t = 0$ in some iteration t implies that $\alpha_i(p^t) = 0$ for all $i \in B$ (this is because of non-negative marginal values).

Define the *marginal price* of the $(j + 1)^{\text{st}}$ unit ($0 \leq j < n$) for buyer i , given price p^t in iteration t , as $p_i^t(j + 1) - p_i^t(j)$. The role of q^t in the auction definition is to capture this marginal price, and the marginal price on the $(j + 1)^{\text{st}}$ unit for $j \geq \overline{D}_i(p^t)$ is exactly q^t , for every buyer and in every iteration.

Lemma 4 *In every iteration of the auction and for every buyer, the marginal price of j^{th} unit is less than or equal to the marginal price of $(j + 1)^{\text{st}}$ unit for $0 \leq j \leq n - 1$. Moreover, all buyers face the same marginal price on each unit for all units j greater than the maximal demand of the buyer in that iteration.*

Proof: By definition of the auction, for j greater than or equal to the maximal demand of a buyer, the marginal price of $(j + 1)^{\text{st}}$ unit for that buyer is q^t in iteration t . The prices of units less than maximal demand are not changed. By the activity rule in Step (S1.1), the maximal demand of every buyer is non-decreasing from iteration to iteration (starting at zero in iteration 0). Hence, the marginal price of j^{th} unit is less than or equal to the marginal price of $(j + 1)^{\text{st}}$ unit for $0 \leq j \leq n - 1$. ■

Proposition 5 *Suppose every buyer follows the truthful bidding strategy and $\alpha(M, p^t) = 0$ for economy $E(M)$ for some $M \in \mathbb{B}$ in iteration t of the stylized CVD auction in the homogeneous items NIMV environment. Then the provisional allocation x of economy $E(M)$ is in the supply set of the seller in that economy in iteration t .*

Proof: Assume for contradiction that x is not in the supply set of the seller in economy $E(M)$ in iteration t . By definition, x maximizes revenue of the seller across all *admissible* allocations of economy $E(M)$ in iteration t . This means no allocation in the supply set of the seller is an admissible allocation in economy $E(M)$ (if there are allocations in the supply set that are admissible then one of them must be selected, because they satisfy more buyers than an allocation that is not admissible). Consider some allocation y in the supply set of the seller in economy $E(M)$ in iteration t . Since y is not admissible, and by Lemma 3 (which says that zero is in demand set of every buyer throughout the auction), there is some buyer $i \in M$ such that $y_i > \overline{D}_i(p^t)$. Since $\alpha(M, p^t) = 0$, there exists $k \neq i$ such that $y_k < \overline{D}_k(p^t)$. Now, construct a new allocation z of economy $E(M)$ such that $z_i = y_i - 1$, $z_k = y_k + 1$, and $z_{i'} = y_{i'}$ for all $i' \in M \setminus \{i, k\}$. From Lemma 4, the revenue from z is greater than or equal to the revenue from y because because the marginal price of a unit less than the maximal demand of a buyer is greater than the marginal price of a unit greater than maximal demand. This process of constructing a new allocation can be repeated till we get an allocation z such that $z_i \leq \overline{D}_i(p^t)$ for all $i \in M$. Thus, we get an admissible allocation of economy $E(M)$ with revenue greater than or equal to the revenue from y . This is a contradiction. ■

These results lead to the main result of this section.

Proposition 6 *Suppose every buyer follows the truthful bidding strategy. Then the stylized CVD auction terminates at a UCE price vector in the homogeneous items NIMV environment.*

Proof: Since buyers are truthful, the maximal demand of all the buyers will be n when the marginal price reaches zero. Thus, the zero marginal price provides a lower bound for the prices in the auction and it will terminate after a finite number of iterations. There exists a provisional allocation of economy $E(M)$, in every $M \in \mathbb{B}$, for which every buyer is satisfied in every iteration t . This is easy to see: every provisional allocation allocates every buyer either zero units or the number of units requested in his demand set. By Lemma 3, receiving zero units is in the demand set of every buyer in every iteration. Then, upon termination we will have $\sum_{i \in M} \bar{D}_i(p^T) \geq n$ for all $M \in \mathbb{B}$, and thus the provisional allocation in every economy $E(M)$ is in the supply set of the seller by Proposition 5. Thus, p^T is a UCE price vector. ■

4.2 A Simpler Representation of the CVD Auction: Clinching

We now present a simpler representation of the stylized CVD auction that only maintains the marginal price, q^t , in each iteration and makes the clinching behavior of the CVD auction explicit. We call it the *simple-CVD auction*. The analysis re-parameterizes the demand sets and under-demand in terms of this marginal price. The simplified auction design follows from the following observations:

- By Lemma 3, a buyer can report his entire demand set by revealing only his maximal demand in each iteration. Since q^t is the marginal price of a unit, the maximal demand given q^t is:

$$\bar{D}_i(q^t) = \begin{cases} 0 & , \text{ if } v_i(1) - v_i(0) < q^t \\ \max_{j \in \{0, 1, \dots, n\}} j \text{ s.t. } v_i(j) - v_i(j-1) \geq q^t & , \text{ otherwise.} \end{cases}$$

- The maximal demand reported by buyers across all rounds can be used to determine the provisional allocation in each economy $E(M)$. This is important because it is not possible to find the provisional allocation from just the maximal demands reported by buyers in the current iteration. Consider economy $E(M)$ and iteration t . If $\alpha(M, q^t) > 0$ then define the provisional allocation by assigning $\bar{D}_i(q^t)$ to each buyer $i \in M$. In the first period t in which $\alpha(M, q^t) = 0$, allocate $\bar{D}_i(q^{t-1})$ to each buyer $i \in M$ and complete the allocation by assigning additional units (i.e., $n - \sum_{i \in M} \bar{D}_i(q^t)$) to buyers at random such that no buyers gets more than his maximal demand in iteration t . We refer to this allocation as a **sequential allocation**.
- The final payments must also be determined from the demand sets reported by buyers across iterations. Let t^m denote the first iteration in which $\sum_{i \in B} x_i^t \geq n$, and thus

for which $\alpha(B, q^t) = 0$. Define the *residual demand* without buyer i as $r_{-i}(q^t) = \min(x_i, \sum_{j \neq i} [\bar{D}_j(q^t) - x_j])$ for all iterations $t \geq t^m$. For iterations $t < t^m$, define $r_{-i}(q^t) = 0$ for all $i \in B$. For iterations $t \geq t^m$, the residual demand without i defines the amount of the supply that is allocated to buyer i in the main economy that is also demanded (in aggregate) by other buyers. The change in residual demand from iteration to iteration, from t^m till termination also provides enough information to define the final payments of each buyer (see Lemma 5).

- By definition, in iterations $t \geq t^m$, $\alpha(B_{-i}, q^t) = 0$ if and only if $r_{-i}(q^t) = x_i^t$, for all buyers $i \in B$. Thus, the residual demand information also provides a *termination condition*: the CVD auction terminates if and only if $t \geq t^m$ and either $r_i(q^t) = x^t$ for all i , or $q^t = 0$ (under truthful bidding, we do not need to check for $q^t = 0$).

Lemma 5 *Suppose every buyer follows a truthful bidding strategy in the homogeneous items NIMV environment, then the final payment of buyer i is equal to $\sum_{t \geq t^m} [q^t * (r_{-i}(q^t) - r_{-i}(q^{t-1}))]$.*

Proof: Let p^T be the final price vector in the stylized CVD auction. Let x and y be efficient allocations of the main economy and the marginal economy corresponding to buyer i respectively. The VCG payment of buyer i when all buyers are truthful is:

$$\begin{aligned} p_i^{vcg} &= v_i(x_i) - \sum_{k \in B} v_k(x_k) + \sum_{k \in B_{-i}} v_k(y_k) \\ &= \sum_{k \in B_{-i}} [v_k(y_k) - v_k(x_k)] = \sum_{k \in B_{-i}} [p_k^T(y_k) - p_k^T(x_k)] \quad (\text{From Lemma 3}) \end{aligned}$$

The above expression for the VCG payment of buyer i can be rewritten as

$$p_i^{vcg} = \sum_{k \neq i} [p_k^T(y_k) - p_k^T(y_k - 1)] + \dots + [p_k^T(x_k + 1) - p_k^T(x_k)] \quad (1)$$

For any $k \in B_{-i}$ and for any $j \in \{x_k, \dots, y_k - 1\}$, the marginal price of $(j + 1)$ st unit in the final iteration, $p_k^T(j + 1) - p_k^T(j)$, is the marginal price of $(j + 1)$ st unit in the iteration in which the maximal demand of k increased from j to $j + 1$. Since every buyer sees the same marginal price in any iteration of our auction, we can monitor the change in maximal demand of all buyers in B_{-i} simultaneously in every iteration after iteration t^m to compute the VCG payment of buyer i . The change in residual demand without buyer i in an iteration $t \geq t^m$ reflects the total change in maximal demand of buyers in B_{-i} . The price of these units get fixed after this. Hence $q^t * (r_{-i}(q^t) - r_{-i}(q^{t-1}))$ is the price component of buyer i for $r_{-i}(q^t) - r_{-i}(q^{t-1})$ units in Equation 1. Adding over all the iterations after iteration t^m gives the VCG payment of buyer i , i.e., $p_i^{vcg} = \sum_{t \geq t^m} q^t * (r_{-i}(q^t) - r_{-i}(q^{t-1}))$. ■

We can now define the simple clinching auction for the multiple, homogeneous items environment.

Definition 7 *The simple-CVD auction is an iterative procedure with the following steps:*

(S0) *Start from a high price q^0 . Set $t := 0$. Set the total number of units **clinched** by buyers, (c_1, \dots, c_m) , to zero. Set the total payments of buyers, (s_1, \dots, s_m) , to zero.*

(S1) *In iteration t of the auction with price q^t :*

(S1.1) *Collect maximal demand $\bar{D}_i(q^t)$ of every buyer i at price q^t . Impose $\bar{D}_i(q^t) \geq \bar{D}_i(q^{t-1})$ for every buyer for all $t > 0$.*

(S1.2) *If $\sum_{i \in B} \bar{D}_i(q^t) < n$ then $c_i = \bar{D}_i(q^t)$ for all $i \in B$. Set $q^{t+1} := q^t - 1$, $t := t + 1$, and repeat from Step (S1.1).*

(S1.3) *If $\sum_{i \in B} \bar{D}_i(q^t) \geq n$ and $\sum_{i \in B} \bar{D}_i(q^{t-1}) < n$, then set $t^m := t$ and set c to be any sequential allocation.*

(S1.4) *Set $s_i := s_i + q^t * (r_{-i}(q^t) - r_{-i}(q^{t-1}))$ for all $i \in B$.*

(S1.5) *If $r_{-i}(q^t) = c_i$ for all $i \in B$ or $q^t = 0$, then go to Step (S2). Else, set $q^{t+1} := q^t - 1$, $t := t + 1$, and repeat from Step (S1.1).*

(S2) *Final allocation is (c_1, \dots, c_m) and the final payment vector is (s_1, \dots, s_m) .*

The simple-CVD auction maintains the marginal price of an additional unit over and above the clinched units and lowers it in every iteration. Buyers clinch all units they currently demand while the total demand is less than the total number of units. In the first period in which the total demand exceeds supply, then a *sequential allocation* (defined above) is used to determine the clinched units. The clinched units of a buyer are priced by monitoring the residual demand in the marginal economy corresponding to that buyer. The auction stops when all the clinched units of the buyers are priced and occurs exactly when the residual demand without each buyer equals the number of units clinched by that buyer, and thus exactly when supply equals demand in every economy. The computational requirements of this auction are very light: simple linear time operations are performed in every iteration.

The activity rule, which requires that maximal demand weakly increases during the auction is analogous to the activity rule in Ausubel's ascending price auction (Ausubel, 2004) where the demand of each buyer must weakly *decrease* from iteration to iteration. As a consequence of the activity rule, we have the following result.

Lemma 6 *Any strategy available to any buyer in the simple-CVD auction under the activity rule is equivalent to some straightforward bidding strategy in the homogeneous items NIMV environment.*

Proof: Consider a buyer $i \in B$. Now, consider the following valuation function \hat{v}_i for buyer i : $\hat{v}_i(0) = 0$, $\hat{v}_i(j) - \hat{v}_i(j-1) = q^t$, if the maximal demand of buyer i is greater than or equal to j units for the first time in the simple-CVD auction in iteration t , and $\hat{v}_i(j) - \hat{v}_i(j-1) = 0$ if the maximal demand of buyer i is less than j units in every iteration of the simple-CVD

auction. From the activity rule, \hat{v}_i must satisfy NIMV because the marginal price is non-increasing in the simple-CVD auction. So, any strategy of any buyer can be mapped to a straightforward bidding strategy with respect to some NIMV valuation. ■

Using Theorem 2 with Proposition 6 and Lemma 6 gives us our main result for the CVD auction:

Theorem 4 *Truthful bidding is an ex post Nash equilibrium strategy in the simple-CVD auction under the activity rule and the auction is ex post efficient in the homogeneous items NIMV environment.*

4.3 An Example: Homogeneous Items

j	1	2	3	4
$v_1(j)$	7	9	10	10
$v_2(j)$	8	13	15	15
$v_3(j)$	4	8	10	10

Table 1: An example with 4 units of a homogeneous item and 3 buyers, each with non-increasing marginal values. The values in the table provide the total value of each buyer for some number of units of the item. The efficient allocation is depicted in bold.

Consider the simple example in Table 1. The price adjustment in the stylized CVD auction and in the simple CVD auction is shown in Tables 2 and 3 respectively.

The first column in Table 2 indicates the iteration number, the next three columns indicate the personalized prices of each buyer for each number of units, and the last column indicates the excess supply in economy $E(M)$ for every $M \in \mathbb{B}$. Parentheses around prices for a buyer indicate which quantity of units are in the demand set of a buyer in each iteration. The final price vector in the auction is a UCE price vector. Also, $\pi^s(B) = 24$, $\pi^s(B_{-1}) = 21$, $\pi^s(B_{-2}) = 17$, $\pi^s(B_{-3}) = 22$. Using discounts from UCE price as in Eq. (1), the payment of buyer 1 can be calculated as $p_1^{vcg} = p_1(1) - [\pi^s(B) - \pi^s(B_{-1})] = 7 - [24 - 21] = 4$. Similarly, $p_2^{vcg} = 13 - [24 - 17] = 6$, $p_3^{vcg} = 4 - [24 - 22] = 2$. It is a simple matter to check that these are indeed the VCG payments in this example.

We illustrate the steps of the simple-CVD auction in Table 3. Buyers clinch units as soon as they demand them while there is available supply. For instance, in iteration 2, buyer 2 demands a unit and clinches it as there are 4 units of supply available in the main economy (and no other demand). In iteration 6 the total demand in economy $E(B_{-1})$ is 4 units and buyers in B_{-1} have clinched 3 units. This means there is a residual demand of 1 unit in the economy and the single unit clinched by buyer 1 is priced at the current price, which is 4. Similarly, the total demand in economy $E(B_{-2})$ is 3 units and buyers in B_{-2} have clinched 2 units, creating a residual demand of 1 unit. Thus, only one of the two units clinched by

#	q^t	Buyer 1				Buyer 2				Buyer 3				$\alpha(\cdot)$			
		1	2	3	4	1	2	3	4	1	2	3	4	B	B_{-1}	B_{-2}	B_{-3}
1	9	9	18	27	36	9	18	27	36	9	18	27	36	4	4	4	4
2	8	8	16	24	32	(8)	16	24	32	8	16	24	32	3	3	4	3
3	7	(7)	14	21	28	(8)	15	22	29	7	14	21	28	2	3	3	2
4	6	(7)	13	19	25	(8)	14	20	26	6	12	18	24	2	3	3	2
5	5	(7)	12	17	22	(8)	(13)	18	23	5	10	15	20	1	2	3	1
From iteration 6 onwards, we have a CE of the main economy.																	
6	4	(7)	11	15	19	(8)	(13)	17	21	(4)	(8)	12	16	0	0	1	0
7	3	(7)	10	13	16	(8)	(13)	17	20	(4)	(8)	11	14	0	0	1	0
8	2	(7)	(9)	11	13	(8)	(13)	17	19	(4)	(8)	(10)	12	0	0	0	0

Table 2: Non-linear and non-anonymous price trajectory in the stylized CVD auction

buyer 2 can be priced at the current price of 4. The next change in residual demand occurs in iteration 8, when the remaining units are priced, and the auction terminates.

t	Price	Demand			Units clinched			Residual demand			Payments		
	q^t	κ_1^t	κ_2^t	κ_3^t	c_1	c_2	c_3	r_1^t	r_2^t	r_3^t	s_1	s_2	s_3
1	9	0	0	0	0	0	0	0	0	0	0	0	0
2	8	0	1	0	0	1	0	0	0	0	0	0	0
3	7	1	1	0	1	1	0	0	0	0	0	0	0
4	6	1	1	0	1	1	0	0	0	0	0	0	0
5	5	1	2	0	1	2	0	0	0	0	0	0	0
From iteration 6 onwards, we have a CE of the main economy.													
6	4	1	2	2	1	2	1	1	1	0	4	4	0
7	3	1	2	2	1	2	1	1	1	0	4	4	0
8	2	2	3	3	1	2	1	1	2	1	4	4+2	2

Table 3: Marginal price trajectory in the simple-CVD auction

4.4 Remark: Relating to LVD and the Unit-Demand Special Case

Due to the homogeneous nature of items in this setting it is difficult to give a direct analogy to the notion of a universally allocated item, as defined for the unit demand setting. But the idea of pricing a unit when the residual demand of a buyer increases is similar to the idea of pricing a unit when it can still be allocated in the absence of that buyer. Moreover, the prices in our auction are decreased until all units are allocated in all economies, which mirrors the termination condition in the LVD auction.

Consider now the special case of unit-demand preferences with multiple units of a homogeneous item, so that each buyer demands a single unit of the item. This satisfies the NIMV

requirement because the marginal value of k^{th} unit with $k > 1$ is zero. In the unit-demand setting, the maximal demand of all the buyers never exceed one (except for the trivial case when q^t is zero). A buyer is awarded a unit as soon as he demands it. So, the number of clinched units of a buyer does not exceed one (as long as $q^t > 0$). Hence, the residual demand of a buyer cannot exceed one. The residual demand of every buyer who has clinched a unit becomes equal to one when the number of buyers demanding a unit becomes greater than the number of units. In that iteration, all the buyers are priced - the payment of every buyer (who has clinched a unit) is the marginal price in that iteration. Notice that (a) the terminating condition in this case requires all units to be allocated in an economy without a buyer, which is equivalent to the universally allocated item condition of the LVD auction; (b) The CVD auction translates to the well known uniform second-price auction for multiple units, where buyers pay the (same) price of the highest losing buyer. Thus, the CVD auction translates to a simpler representation of the LVD auction in the homogeneous item unit demand setting. When there is exactly one unit for auction (i.e., $n = 1$), the auction stops as soon as a second buyer demands the unit, translating to the Vickrey's flash-button apparatus.

5 An Experimental Analysis of Speed and Preference Elicitation

In this section, we compare the speed and preference elicitation properties of our auctions with their ascending price counterparts. Our analysis is based on computer simulations. For speed, our metric for comparison is the number of iterations in the auction. For preference elicitation, we define a new metric (the *uncertainty index*), which provides a measure of the accuracy with which agent valuations are revealed to the auction.

In our computer simulations the input parameters are the number of buyers, the number of items (or units), and the *density*. In the unit-demand setting the density is a real number between 0 and 1, and reflects the probability with which a buyer will have positive value on an item: a higher density means that more buyers have positive value on items. In the homogeneous item setting, a higher density reflects a higher probability that a buyer has positive marginal value for a unit. All values are integers and the bid decrement in both auctions is set to one throughout. All the results are averaged over 100 trials.

5.1 The LVD Auction: Unit Demand

We simulate the LVD auction and the DGS auction¹¹ using a computer program. Valuations of buyers for items are drawn from a uniform distribution with range $[0, 100]$ and assigned according to the density. The starting price on every item in the LVD auction is 100 (the upper limit of valuation) and zero in the DGS auction (the lower limit of valuation). Thus, this is completely symmetric for the two different auctions. Before presenting the simulation results, we propose a novel way of measuring preference elicitation in iterative auctions.¹²

5.1.1 A Preference Elicitation Measure

We propose a metric that measures the amount of revealed preference information that is implied by the bids placed by buyers in response to increasing or decreasing prices. In the equilibrium of both the DGS and the LVD auctions, the demand set reported by a buyer in a round implies constraints on an agent's valuation. These constraints are those defined in Section 3.3 in the context of the revealed preference activity rule, i.e. constraints (C-I) and (C-II). Following the language introduced there, the set of feasible valuations for a buyer is the set of *consistent* valuations; i.e., those valuations that are consistent with the demand sets submitted across the sequence of iterations. The size of the set of consistent valuations indicates the amount of information revealed: a large set corresponds with considerable uncertainty about valuations and a small set corresponding to little uncertainty.

The consistent valuations in the LVD and DGS auctions form a *lattice*, the structure of which leads to a natural metric.¹³

Theorem 5 *Suppose the valuation of every buyer is drawn from a distribution with finite support. Then the set of consistent valuations of every buyer in the LVD auction and the DGS auction forms a complete lattice.*

Proof: Let v_i^1 and v_i^2 be two consistent valuations of buyer i in the LVD auction. Define $v_i^3 := \min(v_i^1, v_i^2)$ and $v_i^4 := \max(v_i^1, v_i^2)$. Consider an iteration t of the LVD auction. We will show that v_i^3 satisfies Equations (C-I) and (C-II). Consider $j \in D_i(p^t)$. Suppose $v_{ij}^3 = v_{ij}^1$. Consider the case when $k \in D_i(p^t)$. If $v_{ik}^3 = v_{ik}^1$, then $v_{ij}^3 - v_{ik}^3 = v_{ij}^1 - v_{ik}^1 = p_j^t - p_k^t$. Else, $v_{ik}^1 > v_{ik}^2$. This gives, $v_{ij}^1 - v_{ik}^1 < v_{ij}^2 - v_{ik}^2$. But, $v_{ij}^1 - v_{ik}^1 = v_{ij}^2 - v_{ik}^2 = p_j^t - p_k^t$. This gives a

¹¹The ascending price auction in Demange et al. (1986) has been modified in Sankaran (1994). This modification has a computationally faster price adjustment procedure than the original ascending price auction of Demange et al. (1986), and is suitable for practical implementation. We refer to the auction of Demange et al. (1986), as modified by Sankaran (1994), as the **DGS auction**.

¹²See (Parkes and Kalagnanam, 2005) for a related, but different, approach in which the probability mass that is consistent with the revealed information is adopted as a metric of the remaining uncertainty about preferences.

¹³Given two consistent valuations v_i^1 and v_i^2 of buyer i , we say that the set of consistent valuations of buyer i form a *lattice* if $\min(v_i^1, v_i^2)$ and $\max(v_i^1, v_i^2)$ are also consistent valuations of buyer i . Here, for any two vectors x^1 and x^2 , $\min(x^1, x^2)$ is defined as by taking minimum of every component of x^1 and x^2 , and similarly $\max(x^1, x^2)$ is defined.

contradiction, implying that $v_{ik}^3 = v_{ik}^1$ and $v_{ij}^3 - v_{ik}^3 = v_{ij}^1 - v_{ik}^1 = p_j^t - p_k^t$. Now, consider the case when $k \notin D_i(p^t)$. Suppose $v_{ij}^3 = v_{ij}^1$. $v_{ij}^3 - v_{ik}^3 = v_{ij}^1 - v_{ik}^3 \geq v_{ij}^1 - v_{ik}^1 = p_j^t - p_k^t$. Thus, Equations (C-I) and (C-II) are satisfied by valuation function v_i^3 . So, v_i^3 is a consistent valuation. Similar arguments show that v_i^4 is a consistent valuation. This shows that the set of consistent valuations is a lattice in the LVD auction. A similar proof follows for the DGS auction. Since the valuations are drawn from a distribution with finite support, this lattice is complete. ■

The previous theorem says that for every buyer i there exists a unique *greatest consistent valuation*, v_i^{max} , and a unique *least consistent valuation*, v_i^{min} . We define the *uncertainty index* UI_i for buyer i (in both the LVD and the DGS auction) as:

$$UI_i = \sum_{j \in A_i} \frac{[v_{ij}^{max} - v_{ij}^{min}]}{100 \times |A_i|}, \quad (2)$$

where $A_i = \{j \in A : v_{ij} > 0\}$. The uncertainty index for buyer i is the sum of remaining uncertainty (in terms of the difference between the greatest and smallest consistent values) about the buyer's valuation on items for which it has non-zero value. We normalize this by the size of the domain of distribution from which the valuations are drawn (here, it is $100-0=100$) and the number of items on which it has non-zero value. We choose not to consider the residual uncertainty on items for which a buyer's value is zero because we think of the preference information on these items as easy for buyers to report: it is easy to identify items for which a buyer has no value but hard for a buyer to determine its exact value for other items.¹⁴

The uncertainty index (UI) of an auction is the average of uncertainty indices of all the buyers, averaged in turn over a number of instances.¹⁵ Note that the UI index captures the amount of (costly) preference elicitation **not** elicited from the buyers and thus a large UI index is better than a small UI index.

¹⁴The results that we present are qualitatively unchanged if the uncertainty index is defined as the average residual value uncertainty across all items.

¹⁵To illustrate the idea of UI, we give an example of auctioning of a single item. Suppose there are four buyers with values 10,8,6, and 4 respectively, drawn from a uniform distribution with range [0,10]. Suppose the starting price in the Vickrey-Dutch auction is 10 and that in the English auction is 0. Both the Vickrey-Dutch auction and the English auction will terminate at price 8. All buyers except the first buyer will reveal their valuations in the English auction, whereas the final price gives a lower bound of 9 on the valuation of the first buyer. So, the least consistent valuation profile and the greatest consistent valuation profile in the English auction are (9, 8, 6, 4) and (10, 8, 6, 4) respectively, and the UI is the average of UI of all the buyers, i.e., $\frac{1+0+0+0}{4 \times 10} = 0.025$. In the Vickrey-Dutch auction, the first and second buyers reveal their valuations. The final price gives an upper bound of 7 on the valuations of the third and fourth buyers. So, the greatest consistent valuation profile and the least consistent valuation profile in the Vickrey-Dutch auction are (10, 8, 7, 7) and (10, 8, 0, 0) respectively. So, the UI of the Vickrey-Dutch auction is $\frac{0+0+7+7}{4 \times 10} = 0.35$.

5.1.2 Simulation Results

We fix the number of items at 5, the density at 0.75 (i.e. on average, every buyer has non-zero value for 75% of the items), and vary the number of buyers from 5 to 50. A comparison of the speed of the LVD auction and the DGS auction is shown in Figure 3(a), while a comparison of preference elicitation (using the uncertainty index) is shown in Figure 3(b). We also plot the *average market clearing price*, which is defined as the ratio between the total payments made by the bidders and the total number of items in the market. The clearing price is a useful proxy for the overall level of competition in the market as we vary the number of buyers. The clearing price is normalized by dividing it over 100 for Figure 3(b), in order to include this alongside a plot of UI.

In both the figures, we see that when the number of buyers are relatively low (less than 8), the DGS auction has better speed and preference elicitation properties. As the number of buyers increase, the LVD auction does better in terms of both speed and preference elicitation. It is nice to observe that the cross-over point, beyond which the LVD auction dominates the DGS auction, occurs for prices around 50 which is the median of the domain of values on items (0,100) in this environment (note that 0 and 100 also represent the initial prices in the DGS and LVD auctions respectively).¹⁶

In terms of the average uncertainty index, it is notable that the LVD auction is able to achieve approaching 90% UI for a large number of buyers. Roughly, this number indicates the average remaining value uncertainty on items for which agents have non-zero value at the end of the auction. This is a significant improvement over the UI of the DGS auction for a large range of the number of buyers.¹⁷

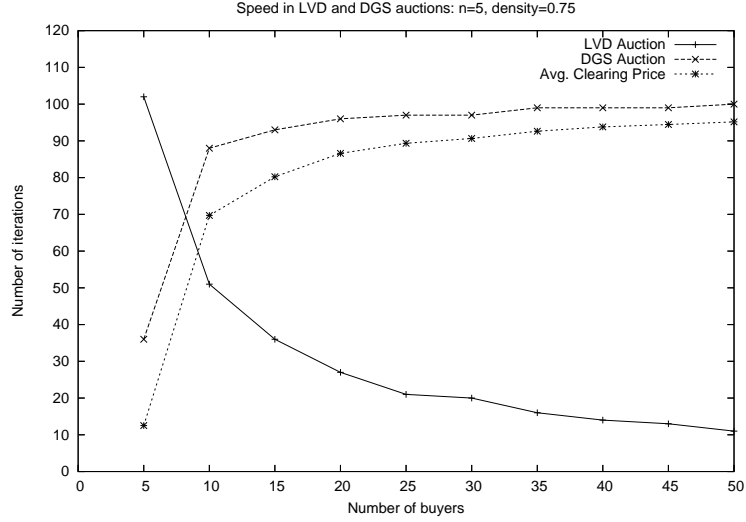
We see these trends in other environments also. We experimented in two other environments: (a) fix the density and the number of buyers, and vary the number of items; (b) fix the number of buyers and the number of items, and vary the density. In both these environments, we see very similar effects: at high average market clearing prices the LVD auction does better than the DGS auction but at low market clearing prices the DGS auction does better than the LVD auction. The cross-over point occurs consistently around an average clearing price of 50.

5.2 The Clinching Vickrey Dutch Auction: Homogeneous Items

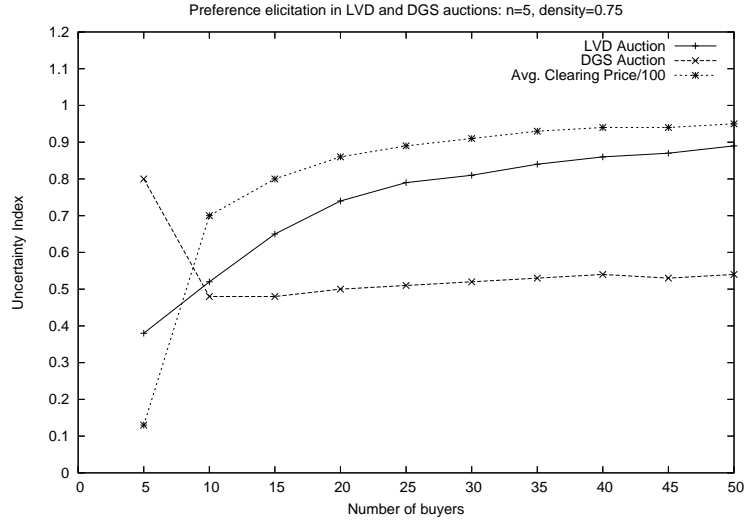
Turning to the environment with multiple units of a homogeneous item we define a distribution on NIMV valuations. The marginal value of the first unit is drawn uniformly from

¹⁶Indeed, the number of iterations in the DGS auction tracks the clearing price while the number of iterations in the LVD auction tracks 100 minus the clearing price.

¹⁷One surprising observation from this simulation is that although the UI for the DGS auction initially decreases with the number of buyers, it is approximately steady (and even slightly improving) as the number of buyers increases from 10 towards 50. We expect that this results from two factors: (a) the number of iterations in DGS is fairly constant (and large) on this range; (b) as the clearing prices become higher on this range more buyers are “priced-out” and stop revealing information.



(a)



(b)

Figure 3: The speed and average uncertainty index in the LVD and the DGS auctions for unit-demand valuations.

the integers on the range $[50, 100]$. For buyer i and $k > 1$, if \hat{v}_i^{k-1} is the marginal value of the $(k-1)^{\text{st}}$ unit, then the marginal value of the k^{th} unit is drawn uniformly from integers in the range $[\frac{\hat{v}_i^{k-1}}{2}, \hat{v}_i^{k-1}]$ with probability equal to density, where density is an input to our simulations. With probability equal to $(1-\text{density})$, we set the marginal value of k^{th} ($k > 1$) unit of buyer i at zero (and zero value on future units as well).

Before presenting the simulation results, we describe how we measure preference elicita-

tion in this setting. The revealed preference information implicit in the demand sets in the equilibrium of the CVD and AUS auctions in this environment provides information about the *marginal value* on each unit but not necessarily the absolute value. This is because the prices in these auctions define the price on a marginal unit relative to the number of units already clinched. For this reason, the uncertainty index (UI) is defined here in terms of information about the *marginal valuations*, but is otherwise unchanged: we measure the difference between the greatest consistent *marginal* valuation and the least consistent *marginal* valuation (normalized by dividing it over 100, and considering only those units whose marginal value is non-zero).¹⁸

5.2.1 Simulation Results

We fix the number of units at 20, the density at 0.75, and vary the number of buyers from 5 to 50. The plots for speed and for preference elicitation are shown in Figures 4(a) and 4(b). We continue to include the average market clearing price, defined here the total payment made by the bidders divided by the number of units, and again normalized by dividing by 100 in the preference elicitation plot.

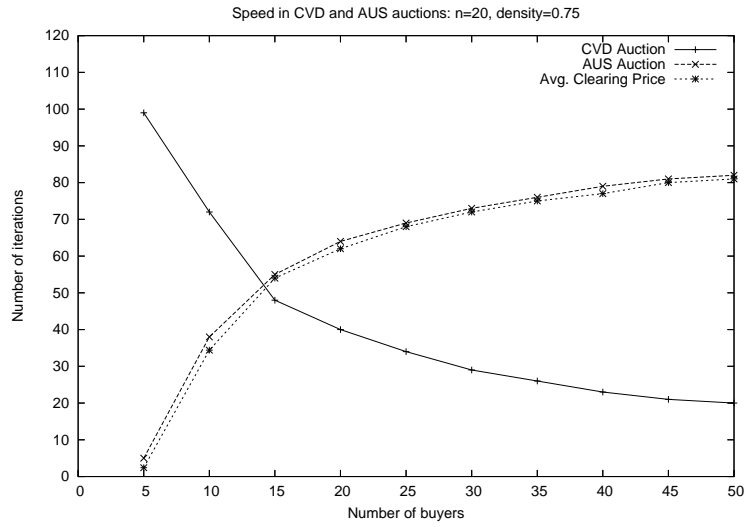
We again see an excellent match between the average clearing price and the number of iterations, with the CVD auction converging more quickly than AUS for around 15 or more buyers, corresponding again to average clearing price of 50. The descending auction also enjoys better preference elicitation properties than the ascending auction for a large number of buyers, with the cross-over again occurring at around 15 buyers and a clearing price of around 50. In this simulation the UI of the AUS auction continues to worsen as the number of buyers increases, apparently trending towards 100% elicitation for a large number of buyers. The preference elicitation properties of the CVD auction are significantly better than the AUS auction in this environment for a moderate to large number of buyers.

We again conducted our simulations for two other environments: (a) fixing the density and the number of buyers, and varying the number of units; (b) fixing the number of items and the number of buyers, and varying the density. In both these environments, we found similar conclusions: at higher market clearing prices the CVD auction performs better than the AUS auction in terms of both speed and preference elicitation, with the cross-over occurring at an average market price of around 50.

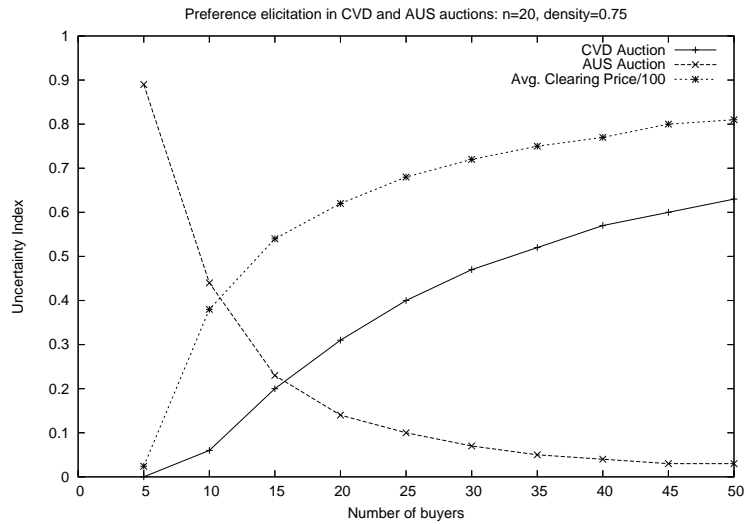
6 Conclusions

We proposed ex post efficient Vickrey-Dutch auctions for environments with unit demand and with multiple homogeneous items and non-decreasing marginal values. These Vickrey-Dutch auctions are designed within the framework of universal competitive equilibrium prices. The

¹⁸The consistent marginal valuations form a lattice in both auctions. This can be verified along the lines of proof of Theorem 5, and is omitted in the interest of space.



(a)



(b)

Figure 4: The average number of iterations to termination and preference elicitation in the CVD and the AUS auctions for NIMV and homogeneous items.

Vickrey-Dutch auctions in both settings maintain simple prices, which along with their preference elicitation properties and speed, especially in environments with high competition, appears to make them very attractive for practical use in private value settings.

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