Pathologies IV

Citation	Mumford, David B. 1975. Pathologies IV. American Journal of Mathematics 97(3): 847-849.
Published Version	doi:10.2307/2373780
Accessed	February 18, 2015 5:43:40 AM EST
Citable Link	http://nrs.harvard.edu/urn-3:HUL.InstRepos:3612775
Terms of Use	This article was downloaded from Harvard University's DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA

(Article begins on next page)

PATHOLOGIES IV.

By David Mumford.

In this note I would like to use the beautifully simple method introduced by Tony Iarrobino [1]—when he proved that there are 0-dimensional subschemes of $\mathbf{P}^3$ which are not specializations of reduced subschemes—to prove here that there are also reduced and irreducible complete curves which are not specializations of non-singular curves. Since there are no global obstructions in deforming reduced curves, this also shows that there are complete reduced 1-dimensional local rings with no flat deformation which is generically smooth.

Start with a complete non-singular curve C of genus g with no automorphisms over an algebraically closed ground field k. Choose a point $x \in C$ and a large even integer ν. Note that if V is any k-vector space where

$$m_{x,C}^{2\nu} \subset V \subset m_{x,C}^{\nu}$$

then $k+V$ is a subring of $\mathcal{O}_{x,C}$. For each such V, define a new curve:

$$\pi: C \to C(V)$$

by:

(a) π is a bijection, and an isomorphism

$$\operatorname{res}\pi: C-\{x\} \xrightarrow{\approx} C(V)-\{\pi x\}.$$

(b) $\mathcal{O}_{\pi x, C(V)} = k+V$.

Note that if V_1, V_2 are two such vector spaces, then

$$C(V_1) \approx C(V_2) \Rightarrow V_1 = V_2.$$

(In fact, C is the normalization of each $C(V)$; hence any $o: C(V_1) \xrightarrow{\approx} C(V_2)$ lifts to $o': C \to C$ which must be the identity, hence $k+V_1 = \mathcal{O}_{\pi(x), C(V_1)} = \mathcal{O}_{\pi(x), C(V_2)} = k+V_2$.) Moreover, the curves $C(V)$ can all be fitted together into

Manuscript received December 14, 1973.

American Journal of Mathematics, Vol. 97, No. 3, pp. 847–849

a family if we fix the integer $\dim_k V/m_{x,C}^{2\nu}$: for all k, $0 \leqslant k \leqslant \nu$,

let $G=$ Grassmanian of k-dimensional subspaces of $m_{x,C}^{\nu}/m_{x,C}^{2\nu}$,

let $\mathcal{V} \subset (m_{x,C}^{\nu}/m_{x,C}^{2\nu}) \otimes_k \mathcal{O}_G$ be the universal family,

let $C(\mathcal{V})$ be the scheme equal to $C \times G$ as topological space, with structure sheaf defined by:

$$\begin{array}{ccccc}
\mathcal{O}_{C\times G} & \xrightarrow{\alpha} & (\mathcal{O}_{x,C}/m_{x,C}^{2\nu}) \otimes_k \mathcal{O}_G & \longrightarrow 0 \\
 & & \cup & \\
\cup & & [k+(m_{x,C}^{\nu}/m_{x,C}^{2\nu})] \otimes_k \mathcal{O}_G & \\
 & & \cup & \\
\alpha^{-1}(\mathcal{V}+\mathcal{O}_G) & - - - \rightarrow & \mathcal{V}+\mathcal{O}_G & \\
\|\text{def.} & & & \\
\mathcal{O}_{C(\mathcal{V})} & & &
\end{array}$$

Since $[(\mathcal{O}_{x,C}/m_{x,C}^{2\nu}) \otimes_k \mathcal{O}_G]/\mathcal{V}$ is a locally free $\mathcal{O}_G$-sheaf, $\mathcal{O}_{C(\mathcal{V})}$ is flat over $\mathcal{O}_G$, i.e., $C(\mathcal{V})$ is flat over G.

Now choose $k=\nu/2$ and calculate:

(i) $\dim G = k(\nu-k) = \nu^2/4$

(ii) $p_a(C(V)) = g + \dim_k [\mathcal{O}_{x,C}/\mathcal{O}_{\pi(x),C(V)}]$

$$= g + \left(\frac{3\nu}{2} - 1\right)$$

Therefore if $\nu \gg 0, \dim G \geqslant 3p_a(C(V))-3$! I claim that this implies that almost all the curves $C(V)$ are not specializations of non-singular curves, because of:

LEMMA. *Let $p: \mathcal{C} \to S$ be a flat and proper family of reduced and irreducible singular curves $C_s = p^{-1}(s)$ such that*

(a) $\forall s \in S$, $\{s' | C_{s'} \approx C_s\}$ *is finite*

(b) $p_a(C_s) \geqslant 2$, *S is irreducible and* $\dim S \geqslant 3p_a(C_s)-3$,

then almost all curves C_s are not specializations of non-singular curves.

Proof. If the conclusion is false, then after replacing S by a Zariski open subset we can extend the family $\mathcal{C}/S$ like this:

$$\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{C}^* \\
\downarrow & & \downarrow \\
S & \longrightarrow & S^*
\end{array} \quad ; \quad \begin{array}{l} S^* \text{ irreducible,} \\ \dim S^* = \dim S + 1 \end{array}$$

so that $\mathcal{C}^*$ is generically smooth over S^*. In fact $\mathcal{C}$ will carry a relatively ample L, so we may use $p_* L^{\otimes n}$ $(n \gg 0)$ to embed $\mathcal{C}$ in some $\mathbf{P}^N$-bundle $\mathcal{P}$ over S. Moreover, if a C_S $(s \in S_0)$ is abstractly a specialization of a non-singular curve, so is the embedded curve $C_S \subset \mathbf{P}^N$. So take S^* to be a suitable subvariety of the Hilbert scheme of $\mathcal{P}$ over S. Once we have $\mathcal{C}^*/S^*$, consider the two induced families:

$$\mathcal{C}_i^* = \mathcal{C}^* \times_{S^*} (S^* \times S^*) \qquad \text{(formed via } p_i : S^* \times S^* \to S^*, i = 1,2)$$

and the scheme

$$I = \mathbf{Isom}_{S^* \times S^*}(\mathcal{C}_1^*, \mathcal{C}_2^*)$$

whose points over $(s_1, s_2) \in S^* \times S^*$ are isomorphisms $o : C_{s_1} \to C_{s_2}$. Look at the morphisms:

$$q \;\; \underset{S^*}{\overset{I}{\downarrow}}\!\!\big)\, \delta \qquad \begin{aligned} q(o : C_{s_1} \to C_{s_2}) &= s_1 \\ \delta(s) &= [\text{id.} : C_s \to C_s] \end{aligned}$$

Since $\dim S^* = \dim S + 1 > 3p_a(C_s) - 3$, whenever C_s is non-singular, the same non-singular curve must occur in the family $\mathcal{C}^*/S^*$ infinitely often; thus when C_s is non-singular, some component of $q^{-1}(s)$ through $\delta(s)$ is positive-dimensional. Now by upper semi-continuity of dimensions of fibres of a morphism, it follows that for every s, $q^{-1}(s)$ has a positive-dimensional component through $\delta(s)$. Now let $D_1 = \text{Im}(S \to S^*)$, $D_2 = \{s | C_s \text{ is singular}\}$; then $\bar{D}_1$ is a component of D_2 and let $D_1^0 = D_1$-(closure of $D_2 - D_1$). Choose $s \in D_1^0$ and consider how $q^{-1}(s)$ can have a positive-dimensional component γ through $\delta(s)$. By (b), $\text{Aut}(C_s)$ is finite; by (a), there are only finitely many $s' \in D_1$ with $C_s \approx C_{s'}$; certainly $C_s \not\approx C_{s'}$ if $s' \in S^* - D_2$ because C_s is singular while $C_{s'}$ is non-singular; and since $s \not\in$ (closure $D_2 - D_1$), γ cannot lie over $(s) \times \overline{D_2 - D_1}$ in $S^* \times S^*$. Thus there is nowhere for γ to go! Contradiction.

REFERENCES.

[1] A. Iarrobino, "Reducibility of the families of 0-dimensional schemes on a variety," *Inv. Math.* **15** (1972), pp. 72–77.