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THE IRREDUCIBILITY OF THE SPACE OF CURVES OF GIVEN GENUS

by P. DELIGNE and D. MUMFORD [1]

Fix an algebraically closed field k. Let M_g be the moduli space of curves of genus g over k. The main result of this note is that M_g is irreducible for every k. Of course, whether or not M_g is irreducible depends only on the characteristic of k. When the characteristic is 0, we can assume that $k=\mathbf{C}$, and then the result is classical. A simple proof appears in Enriques-Chisini [E, vol. 3, chap. 3], based on analyzing the totality of coverings of $\mathbf{P}^1$ of degree n, with a fixed number d of ordinary branch points. This method has been extended to char. p by William Fulton [F], using specializations from char. 0 to char. p *provided that* $p>2g+1$. Unfortunately, attempts to extend this method to all p seem to get stuck on difficult questions of wild ramification. Nowadays, the Teichmüller theory gives a thoroughly analytic but very profound insight into this irreducibility when $k=\mathbf{C}$. Our approach however is closest to Severi's incomplete proof ([Se], Anhang F; the error is on pp. 344-345 and seems to be quite basic) and follows a suggestion of Grothendieck for using the result in char. 0 to deduce the result in char. p. The basis of both Severi's and Grothendieck's ideas is to construct families of curves X, some *singular*, with $p_a(X)=g$, over *non-singular* parameter spaces, which in some sense contain enough singular curves to link together any two components that M_g might have.

The essential thing that makes this method work now is a recent " stable reduction theorem " for abelian varieties. This result was first proved independently in char. 0 by Grothendieck, using methods of etale cohomology (private correspondence with J. Tate), and by Mumford, applying the easy half of Theorem (2.5), to go from curves to abelian varieties (cf. [M_2]). Grothendieck has recently strengthened his method so that it applies in all characteristics (SGA 7, 1968). Mumford has also given a proof using theta functions in char. $\neq 2$. The result is this:

Stable Reduction Theorem. — *Let* R *be a discrete valuation ring with quotient field* K. *Let* A *be an abelian variety over* K. *Then there exists a finite algebraic extension* L *of* K *such*

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that, if R_L = *integral closure of* R *in* L, *and if* $\mathscr{A}_L$ *is the Néron model of* $A \times_K L$ *over* R_L, *then the closed fibre* $A_{L,s}$ *of* $\mathscr{A}_L$ *has no unipotent radical.*

We shall give two related proofs of our main result. One of these is quite elementary, and follows by quite standard techniques once the Stable Reduction Theorem for abelian varieties is applied, in § 2, to deduce an analogous stable reduction theorem for curves. The other proof is more powerful, and is based on the use of a larger category than the category of schemes, and on proving for the objects of this category many of the standard theorems for schemes, especially the Enriques-Zariski connectedness theorem (EGA 3, (4.3)). Unfortunately, this larger category is not quite a category — it is a simple type of *2-category*; in fact, if X, Y are objects, then Hom(X, Y) is itself a category, but one in which all morphisms are isomorphisms. The objects of this 2-category we call *algebraic stacks* (¹). The moduli space M_g is just the " underlying coarse variety " of a more fundamental object, the *moduli stack* $\mathscr{M}_g$ studied in [M_3]. Full details on the basic properties and theorems for algebraic stacks will be given elsewhere. In this paper, we will only give definitions and state without proof the general theorems which we apply. Using the method of algebraic stacks, we can prove not only the irreducibility of M_g itself, but of all *higher level* moduli spaces of curves too (cf. § 5 below).

§ 1. Stable curves and their moduli.

The key definition of the whole paper is this:

Definition (**1.1**). — *Let* S *be any scheme. Let* $g \geq 2$. *A* stable curve of genus g *over* S *is a proper flat morphism* $\pi : C \to S$ *whose geometric fibres are reduced, connected, 1-dimensional schemes* C_s *such that*:

(i) C_s *has only ordinary double points*;

(ii) *if* E *is a non-singular rational component of* C_s, *then* E *meets the other components of* C_s *in more than* 2 *points*;

(iii) $\dim H^1(\mathcal{O}_{C_s}) = g$.

We will study in this section three aspects of the theory of stable curves: their pluri-canonical linear systems, their deformations, and their automorphisms.

Suppose $\pi : C \to S$ is a stable curve. Since π is flat and its geometric fibres are local complete intersections, the morphism π is locally a complete intersection (i.e., locally, C is isomorphic as S-scheme to $V(f_1, \ldots, f_{n-1}) \subset \mathbf{A}^n \times U$, where $U \subset S$ is open, and $f_1, \ldots, f_{n-1} \in \Gamma(\mathcal{O}_{\mathbf{A}^n \times U})$ are a regular sequence). Therefore, by the theory of duality of coherent sheaves [H], there is a canonical invertible sheaf $\omega_{C/S}$ on C — the unique non-zero cohomology group of the complex of sheaves $f^!(\mathcal{O}_S)$. We need to know the following facts about $\omega_{C/S}$:

a) for all morphisms $f : T \to S$, $\omega_{C \times_S T/T}$ is canonically isomorphic to $f^*(\omega_{C/S})$;

(¹) A slightly less general category of objects, called *algebraic spaces*, has been introduced and studied very deeply by M. Artin [A_1] and D. Knutson [K]. The idea of enlarging the category of varieties for the study of moduli spaces is due originally, we believe, to A. Weil.

b) if $S = \mathrm{Spec}(k)$, k algebraically closed, let $f: C' \to C$ be the normalization of C, $x_1, \ldots, x_n, y_1, \ldots, y_n$ the points of C' such that the $z_i = f(x_i) = f(y_i)$, $1 \leq i \leq n$, are the double points of C. Then $\omega_{C/S}$ is the sheaf of 1-forms η on C' regular except for simple poles at the x's and y's and with $\mathrm{Res}_{x_i}(\eta) + \mathrm{Res}_{y_i}(\eta) = 0$;

c) if $S = \mathrm{Spec}(k)$, and $\mathscr{F}$ is a coherent sheaf on C, then

$$\mathrm{Hom}(H^1(C, \mathscr{F}), k) \cong \mathrm{Hom}_{\mathcal{O}_C}(\mathscr{F}, \omega_{C/S}).$$

Theorem **(1.2)**. — *If* $g \geq 2$ *and* C *is a stable curve of genus g over an algebraically closed field k, then* $H^1(C, \omega_{C/k}^{\otimes n}) = (0)$ *if* $n \geq 2$, *and* $\omega_{C/k}^{\otimes n}$ *is very ample if* $n \geq 3$.

Proof. — Since C is stable, of genus $g \geq 2$, every irreducible component E of C either 1) has (arithmetic) genus ≥ 2 itself, 2) has genus 1, but meets other components of C in at least one point, or 3) is non-singular, rational and meets other components of C in at least three points. But by *b)* above, $\omega_{C/k} \otimes \mathcal{O}_E$ is isomorphic to $\omega_{E/k}(\sum_i Q_i)$, where $\{Q_i\}$ are the points where E meets the rest of C. Since the degree of $\omega_{E/k}$ is $2g_E - 2$, it follows that in any of the cases 1, 2 or 3, $\omega_{C/k} \otimes \mathcal{O}_E$ has positive degree. This shows immediately that $\omega_{C/k}$ is ample on each component E of C, hence is ample.

Next, by *c)* above, $H^1(\omega_{C/k}^{\otimes n})$ is dual to $H^0(\omega_{C/k}^{\otimes 1-n})$. Since $\omega_{C/k} \otimes \mathcal{O}_E$ has positive degree, $\omega_{C/k}^{\otimes 1-n} \otimes \mathcal{O}_E$ has no sections for any E, any $n \geq 2$; therefore $H^0(\omega_{C/k}^{\otimes 1-n}) = (0)$ if $n \geq 2$, and so $H^1(\omega_{C/k}^{\otimes n}) = (0)$ if $n \geq 2$.

To prove that an invertible sheaf $\mathscr{L}$ on any scheme C, proper over k, is very ample, it suffices to show

a) for all closed points $x \neq y$

$$H^0(C, \mathscr{L}) \to H^0(C, (\mathscr{L} \otimes k(x)) \oplus (\mathscr{L} \otimes k(y)))$$

is surjective,

b) for all closed points x,

$$H^0(C, \mathscr{L}) \to H^0(C, \mathscr{L} \otimes \mathcal{O}_x / \mathfrak{m}_x^2)$$

is surjective.

Using the exact sequence of cohomology, these both follow if $H^1(C, \mathfrak{m}_x \cdot \mathfrak{m}_y \cdot \mathscr{L}) = (0)$ for all closed points $x, y \in C$. In our case, $\mathscr{L} = \omega_C^{\otimes n}$, $n \geq 3$, so if we use duality, we must show:

$$(*) \qquad \mathrm{Hom}(\mathfrak{m}_x \cdot \mathfrak{m}_y, \omega_X^{\otimes -n}) = (0), \quad \text{if } n \geq 2.$$

If x is a non-singular point, $\mathfrak{m}_x$ is an invertible sheaf. If x is a double point, let $\pi: C' \to C$ be the result of blowing up x, and let $x_1, x_2 \in C'$ be the two points in $\pi^{-1}(x)$. Then it is easy to check that for any invertible sheaf $\mathscr{L}$ on C:

$$\mathrm{Hom}(\mathfrak{m}_x, \mathscr{L}) \cong H^0(C', \pi^* \mathscr{L})$$
$$\mathrm{Hom}(\mathfrak{m}_x^2, \mathscr{L}) \cong H^0(C', \pi^* \mathscr{L}(x_1 + x_2)).$$

Therefore, we have 3 cases of $(*)$ to check:

Case 1. — x, y non-singular points of C, then $H^0(\omega_C^{-n}(x+y)) = (0)$, if $n \geq 2$.

Case 2. — x double point of C, $\pi: C' \to C$ blowing up x, $\{x_1, x_2\} = \pi^{-1}(x)$, and y a non-singular point of C. Then

$$H^0(\pi^*(\omega_C)^{-n}(y)) = (0), \quad H^0(\pi^*(\omega_C)^{-1}(x_1+x_2)) = (0), \quad \text{if } n \geq 2.$$

Case 3. — x, y double points of C, $\pi: C' \to C$ blowing up x *and* y. Then

$$H^0(\pi^*(\omega_C)^{-n}) = (0) \quad \text{if } n \geq 2.$$

Now since the degree of ω_C^{-n} $(n \geq 2)$ on all components E of C is less than or equal to -2, all of this is clear, except in those cases where $n=2$, the degree of ω_C on some E is 1, and in which two poles are allowed on E. This occurs if:

(i) case 1, $p_a(E) = 1$, E meets $C - E$ at only one point, $x, y \in E$.

(ii) case 1, $p_a(E) = 0$, E meets $C - E$ at only three points, $x, y \in E$.

(iii) case 2, E a rational curve with one double point meeting $C - E$ at one point, $x =$ double point of E.

But in all these cases, C has components besides E and a section in the H^0 in question must definitely vanish on all these other components. So at the points where E meets $C - E$, the section has extra zeroes. Since the sheaf in question has degree 0 on E, the section is zero on E too. Q.E.D.

Corollary. — *Let* $\pi: C \to S$ *be any stable curve of genus* $g \geq 2$. *Then* $\omega_{C/S}^{\otimes n}$ *is relatively very ample if* $n \geq 3$ *and* $\pi_*(\omega_{C/S}^{\otimes n})$ *is a locally free sheaf on* S *of rank* $(2n-1)(g-1)$.

Proof. — In fact, since for all $s \in S$, $H^1(\omega_{C/S}^{\otimes n} \otimes \mathcal{O}_{C_s}) = (0)$, it follows from [EGA, chap. 3, § 7], that $\pi_*(\omega_{C/S}^{\otimes n})$ is locally free and that $\pi_*(\omega_{C/S}^{\otimes n}) \otimes k(s) \cong H^0(\omega_{C/S}^{\otimes n} \otimes \mathcal{O}_{C_s})$. Therefore the corollary follows. Q.E.D.

Taking $n=3$, it follows that every stable curve C/S can be realized as a family of curves in $\mathbf{P}^{5g-6}$ with Hilbert polynomial:

$$P_g(n) = (6n-1)(g-1).$$

Following standard arguments ([M_1], p. 99), it is easy to prove that there is a subscheme

$$H_g \subset \mathbf{Hilb}^{P_g}_{\mathbf{P}^{5g-6}}$$

of " all " tri-canonically embedded stable curves. (**Hilb** is the Hilbert scheme *over* **Z**.) To be precise, there is an isomorphism of functors:

$$\mathrm{Hom}(S, H_g) \cong \left\{ \begin{array}{l} \text{set of stable curves } \pi: C \to S, \text{ plus isomorphisms:} \\ \qquad \mathbf{P}(\pi_*(\omega_{C/S}^{\otimes 3})) \cong \mathbf{P}^{5g-6} \times S \\ \text{(modulo isomorphism)} \end{array} \right\}$$

We will denote by $Z_g \subset H_g \times \mathbf{P}^{5g-6}$ the universal tri-canonically embedded stable curve. The functor of stable curves itself is the sheafification of the quotient of functors: $\underline{H}_g/\underline{PGL}(5g-6)$.

We now consider the deformation theory of stable curves. Let k be any ground field. The deformation theory of X's smooth over k can be found in [SGA, 60-61]; for singular S's, the theory has been worked out in [Sc]. We shall indicate here the results of this theory for a scheme X which is

(i) one-dimensional;
(ii) generically smooth over k;
(iii) locally a complete intersection.

The advantages of this case are two-fold: first, the " cotangent complex " of Grothendieck, Lichtenbaum and Schlessinger reduces, in view of (ii) and (iii), to the single coherent sheaf $\Omega_{X/k}$, the Kähler differentials. Secondly, we have:

Lemma **(1.3)**. — $\mathrm{Ext}^2(\Omega_{X/k}, \mathcal{O}_X)=(0)$.

Proof. — Use the spectral sequence:

$$H^p(X, \mathcal{E}xt^q(\Omega_{X/k}, \mathcal{O}_X)) \Rightarrow \mathrm{Ext}^{p+q}(\Omega_{X/k}, \mathcal{O}_X).$$

Then (i) $H^2(X, \mathcal{E}xt^0)=(0)$ since $\dim X=1$. (ii) Since $\Omega_{X/k}$ is locally free except at a finite number of points, $\mathcal{E}xt^1(\Omega_{X/k}, \mathcal{O}_X)$ has 0-dimensional support, hence $H^1(X, \mathcal{E}xt^1)=(0)$. (iii) Locally, if we embed $X \subset \mathbf{A}^n$, then $\Omega_{X/k}$ has a free resolution of length 2:

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbf{A}^n} \otimes \mathcal{O}_X \to \Omega_{X/k} \to 0$$

where $\mathcal{I}$ = sheaf of ideals defining X. Therefore $\mathcal{E}xt^2=(0)$. Q.E.D.

In Schlessinger's theory, the significance of Lemma (1.3) is that all obstructions vanish, i.e., deformations of X over base schemes $\mathrm{Spec}(A/\mathfrak{I})$ (A = local Artin ring with residue field k) can always be embedded in deformations over Spec(A). Moreover, the theory says that there is a canonical one-one correspondence between $\mathrm{Ext}^1(\Omega_{X/k}, \mathcal{O}_X)$ and the first order deformations of X, i.e., proper, flat morphisms p, and isomorphisms α as follows:

$$\begin{array}{ccccc} X_1 & \supset & X_1 \times_{\mathrm{Spec}\, k[\varepsilon]} \mathrm{Spec}\, k & \xleftarrow[\alpha]{\sim} & X \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow \\ \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2 & \supset & \mathrm{Spec}\, k & = & \mathrm{Spec}\, k. \end{array}$$

Since the obstructions vanish, there is a *versal formal deformation* $\mathcal{X}$ of X over the base scheme

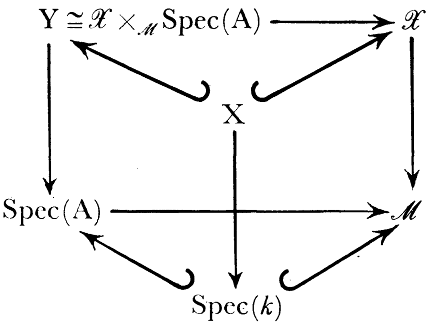
$$\mathcal{M} = \mathrm{Spec}\, \mathfrak{o}_k[[t_1, \ldots, t_N]],$$

where $\mathfrak{o}_k = k$ if the char. is 0, or the complete regular local ring, max. ideal $p.\mathfrak{o}_k$, residue field k, if $\mathrm{char.}(k)=p$, unique (by Cohen's structure theorem)

and $N=\dim_k \mathrm{Ext}^1(\Omega_X, \mathcal{O}_C)$.

This means that $\mathscr{X}$ is a formal scheme, proper and flat over $\mathscr{M}$, with fibre X over Spec(k) and the two properties:

a) Every deformation of X *is induced from* $\mathscr{X}$, i.e., if A is a local Artin $\mathfrak{o}_k$-algebra with residue field k, and $p : Y \to \operatorname{Spec}(A)$ is proper and flat with fibre X over Spec(k), then there is a commutative diagram:

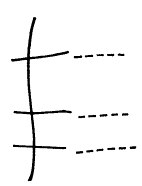


b) If $A = k[\varepsilon]/(\varepsilon^2)$, the above morphism f is uniquely determined by the diagram. This implies that the tangent space to $\mathscr{M} \times_{\mathfrak{o}_k} k$ at its closed point is canonically isomorphic to $\operatorname{Ext}^1(\Omega_{X/k}, \mathscr{O}_X)$.

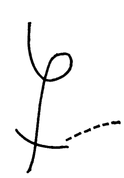
In case $\operatorname{Ext}^0(\Omega_{X/k}, \mathscr{O}_X) = (0)$, $\mathscr{X}/\mathscr{M}$ is, in fact, *universal*: i.e., in property *a)*, f is *always* unique, which means that the functor represented by $\mathscr{M}$ in the category of artin, local $\mathfrak{o}_k$-algebras is isomorphic to the functor of deformations Y/A of X. This fortunately holds for stable curves:

Lemma (**1.4**). — *If* X *is a stable curve*, $\operatorname{Ext}^0(\Omega_{X/k}, \mathscr{O}_X) = (0)$.

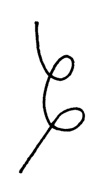
Proof. — We may assume that k is algebraically closed. Now a homomorphism from Ω_X to $\mathscr{O}_X$ is given by an everywhere regular vector field D on X. Such a vector field is given, in turn, by a regular vector field D′ on the normalization X′ of X which vanishes at all points of X′ lying over the double points of X. In particular, D′ and hence D vanishes identically on all components E of X whose normalization E′ has genus ≥ 2. There remain the following possibilities for E:



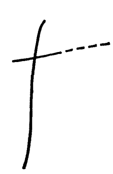
E non-singular rational



E′ rational, E one double pt.



E′ rational, E ≥ 2 double pts.



E non-singular elliptic

E′ elliptic, E ≥ 1 double points

In all cases where E′ is rational, note that D′ has to have at least 3 zeroes; and where E′ is elliptic, D′ has to have at least one zero. So D′ vanishes on all components E′. This proves that $\mathrm{Ext}^0(\Omega_{X/k}, \mathcal{O}_X)=(0)$. Q.E.D.

Schlessinger's theory also allows us to trace what happens to the singularities of X in this deformation $\mathcal{M}$. For each closed point $x \in X$, he studies deformations of the complete local ring $\hat{\mathcal{O}}_{x,X}$ alone, i.e., flat A-algebras $\mathcal{O}$ plus isomorphisms:

$$\mathcal{O}\otimes_A k \cong \hat{\mathcal{O}}_{x,X}.$$

This is a functor of A exactly as before. Since $\mathrm{Ext}^2_{\hat{\mathcal{O}}_x}(\Omega_{\hat{\mathcal{O}}_x/k}, \hat{\mathcal{O}}_x)=(0)$, there are no obstructions to extending deformations. First order deformations with $A=k[\varepsilon]/(\varepsilon^2)$ are classified by $\mathrm{Ext}^1_{\hat{\mathcal{O}}_x}(\Omega_{\hat{\mathcal{O}}_x/k}, \mathcal{O}_x)$. And there is a versal formal deformation $\hat{\mathcal{O}}$, which is a complete local ring and a flat $\mathfrak{o}_k[[t_1, \ldots, t_N]]$-algebra, $N=\dim_k \mathrm{Ext}^1_{\hat{\mathcal{O}}_x}(\Omega_{\hat{\mathcal{O}}_x/k}, \hat{\mathcal{O}}_x)$, such that

$$\hat{\mathcal{O}}/(p, t_1, \ldots, t_N)=\hat{\mathcal{O}}_{x,X}.$$

Whenever X is smooth over k at x, $N=0$, and the theory is uninteresting. The first non-trivial example is a k-rational ordinary double point with k-rational tangent lines:

$$\hat{\mathcal{O}}_{x,X}\cong k[[u, v]]/(u.v).$$

Then $N=1$, and

$$\hat{\mathcal{O}}\cong \mathfrak{o}_k[[u, v, t_1]]/(uv-t_1).$$

In other words, if $\mathcal{O}$ is any deformation of $\hat{\mathcal{O}}_{x,X}$ and u, v are lifted suitably into $\mathcal{O}$, then $u.v=w\in A$ and $\mathcal{O}$ is induced from $\hat{\mathcal{O}}$ via the homomorphism $\mathfrak{o}_k[[t_1]]\to A$ taking t_1 to w. This is easy to prove.

Finally, Schlessinger's theory connects global deformations to local ones. Let $\mathcal{X}/\mathcal{M}_{gl}$ with $\mathcal{M}_{gl}=\mathrm{Spec}(A)$ be the versal global deformation, let $x_1, \ldots, x_k \in X$ be the points where X is not smooth over k, and let $\hat{\mathcal{O}}_i$ as A_i-algebra be the versal deformation of the local ring $\hat{\mathcal{O}}_{x_i,X}$. Let $\mathcal{M}_{lo}=\mathrm{Spec}(A_1\hat{\otimes}_{\mathfrak{o}_k}\ldots\hat{\otimes}_{\mathfrak{o}_k}A_k)$. Then we may consider the local rings

$$\hat{\mathcal{O}}_{x_i,\mathcal{X}}$$

of $\mathcal{X}$ at x_i. There are $\mathfrak{o}_k$-homomorphisms $\varphi_i : A_i\to A$ such that $\hat{\mathcal{O}}_{x_i,\mathcal{X}}\cong\hat{\mathcal{O}}_i\otimes_{A_i}A$. Dualizing, we obtain a morphism

$$\Phi=\prod_i \mathrm{Spec}(\varphi_i) : \mathcal{M}_{gl}\to\mathcal{M}_{lo}$$

which describes exactly how the various singularities of X behave in the versal deformation $\mathcal{X}$. The final fact that we need is:

Proposition **(1.5).** — $\Phi : \mathcal{M}_{gl}\to\mathcal{M}_{lo}$ *is formally smooth*, i.e., *there are isomorphisms*

$$\mathcal{M}_{gl}\cong \mathrm{Spec}\ \mathfrak{o}_k[[t_1, \ldots, t_{N+M}]]$$
$$\mathcal{M}_{lo}\cong \mathrm{Spec}\ \mathfrak{o}_k[[t_1, \ldots, t_N]]$$

such that $\Phi^*(t_i)=t_i$, $1\leq i\leq N$.

Proof. — In view of the functorial significance of $\mathcal{M}_{gl}$ and $\mathcal{M}_{l0}$, this follows if we prove that the natural map:

$$(*) \qquad \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) \to \prod_{i=1}^{k} \mathrm{Ext}^1_{\hat{\mathcal{O}}_{x_i}}(\hat{\Omega}_{x_i}, \hat{\mathcal{O}}_{x_i})$$

is surjective; $(*)$ is the induced map $d\Phi$ on the tangent spaces to $\mathcal{M}_{gl}$ and $\mathcal{M}_{l0}$. Since Ω_X is an invertible $\mathcal{O}_X$-Module outside the x_i's, it follows that:

$$\prod_{i=1}^{k} \mathrm{Ext}^1_{\hat{\mathcal{O}}_{x_i}}(\hat{\Omega}_{x_i}, \hat{\mathcal{O}}_{x_i}) \cong \prod_{i=1}^{k} \mathrm{Ext}_{\mathcal{O}_{x_i}}(\Omega_{x_i}, \mathcal{O}_{x_i})$$
$$\cong \mathrm{H}^0(\mathrm{X}, \mathit{Ext}^1_{\mathcal{O}_x}(\Omega_X, \mathcal{O}_X)).$$

Therefore, $(*)$ is surjective by the spectral sequence used in Lemma (1.3) and the fact that $\mathrm{H}^2(\mathrm{X}, \mathit{Ext}^0(\Omega_X, \mathcal{O}_X)) = (0)$. Q.E.D.

In particular, suppose C is a stable curve over an algebraically closed ground field k, and let $x_1, \ldots, x_k$ be the double points of C. Let $\mathrm{N} = \dim \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$. Then C has a universal formal deformation $\mathscr{C}/\mathcal{M}$ where $\mathcal{M} = \mathrm{Spec}\, \mathfrak{o}_k[[t_1, \ldots, t_N]]$. Note that since in this case, the invertible sheaf $\omega_{\mathscr{C}/\mathcal{M}}$ is relatively ample, $\mathscr{C}$ is not only a formal scheme over $\mathcal{M}$, but also the formal completion of a unique scheme proper and flat over $\mathcal{M}$, which we will also denote by $\mathscr{C}$. $\mathscr{C}$ is clearly a stable curve over $\mathcal{M}$. Now each double point x_i has *one* modulus (cf. our example above) so the versal deformation space of the rings $\mathcal{O}_{x_i, \mathrm{C}}$ is $\mathfrak{o}_k[[t'_1, \ldots, t'_k]]$. By Proposition (1.5), we may identify t_i with t'_i, and we conclude that for suitable u_i, v_i:

$$\hat{\mathcal{O}}_{x_i, \mathscr{C}} \cong \mathfrak{o}_k[[u_i, v_i, t_1, \ldots, t_N]]/(u_i v_i - t_i).$$

In particular, $t_i = 0$ is the locus in $\mathcal{M}$ where " x_i remains a double point ".

The relation between the formal moduli space $\mathcal{M}$ of C and the local structure of H_g at a point x with $\varkappa(x) = k$ corresponding to some tri-canonical model of C is exactly the same as in the case of non-singular curves ([M_1], chap. 5, § 2). Let $\hat{\mathcal{O}}_x$ be the completion of the local ring $\mathcal{O}_{x, \mathrm{H}_g}$, and let $\mathrm{T} = \mathrm{Spec}(\hat{\mathcal{O}}_x)$. Let x denote the closed point of T too. The universal family of stable curves $\mathrm{Z}_g \subset \mathrm{H}_g \times \mathbf{P}^{5g-6}$ induces a family $\mathrm{Z}' \subset \mathrm{T} \times \mathbf{P}^{5g-6}$, whose fibre Z'_x over x is C. Then there is a unique morphism $f : \mathrm{T} \to \mathcal{M}$ such that $\mathrm{Z}' \cong \mathscr{C} \times_{\mathcal{M}} \mathrm{T}$, with this isomorphism restricting to the identity on the fibres over x, both of which are C. I claim that via f, T is formally smooth over $\mathcal{M}$, i.e., $\hat{\mathcal{O}}_x \cong \mathfrak{o}_k[[t_1, \ldots, t_N, t_{N+1}, \ldots, t_M]]$. In fact, by choosing an isomorphism $\mathbf{P}(\pi_*(\omega^{\otimes 3}_{\mathscr{C}/\mathcal{M}})) \cong \mathbf{P}^{5g-6} \times \mathcal{M}$, we obtain a tri-canonical embedding $\mathscr{C} \subset \mathbf{P}^{5g-6} \times \mathcal{M}$ of $\mathscr{C}$, hence a morphism $s : \mathcal{M} \to \mathrm{H}_g$ such that $\mathscr{C}$, with this embedding, is the pull-back of Z_g. Then s factors through T and $s : \mathcal{M} \to \mathrm{T}$ is a section of f. On the other hand, consider the action of $\mathbf{PGL}(5g-6)$ on H_g. Let S_x be the stabilizer of the k-valued point x. Then S_x is finite and reduced. Because if it were not, S_x would have a non-trivial tangent space at the origin, i.e., there would be a $k[\varepsilon]/(\varepsilon^2)$-valued point of $\mathbf{PGL}(5g-6)$ centered at the identity, which maps the embedded stable curve $\mathrm{C} \subset \mathbf{P}^{5g-6}$ corresponding to x into itself. But this action is given by an everywhere regular derivation on C, and we have seen that all such vanish. This means that this $k[\varepsilon]/(\varepsilon^2)$-valued automorphism is the identity at all points of C and, since C is connected and spans $\mathbf{P}^{5g-6}$, the automorphism is the identity everywhere. Thus S_x is finite and reduced. It follows that the action of $\mathbf{PGL}(5g-6)$ on T is *formally free*, and hence that T is formally a principal fibre bundle over $\mathcal{M}$ with group $\mathbf{PGL}(5g-6)$. Therefore T is formally smooth over $\mathcal{M}$ as required.

Putting this together with what we know about $\mathcal{M}$, we conclude the following:
Let k be any algebraically closed field,
let $H'_g = H_g \times \operatorname{Spec}(\mathfrak{o}_k)$, $Z'_g = Z_g \times \operatorname{Spec}(\mathfrak{o}_k)$,
let $x \in H'_g$ be a closed point,
let $C \subset Z'_g$ be the stable curve over x,
let $x_1, \ldots, x_k \in C$ be its double points.

Then

Theorem (**1.6**). — *There are isomorphisms*

$$\hat{\mathcal{O}}_{x, H'_g} \cong \mathfrak{o}_k[[t_1, \ldots, t_N]]$$
$$\hat{\mathcal{O}}_{x_i, Z'_g} \cong \mathfrak{o}_k[[u_i, v_i, t_1, \ldots, t_N]]/(u_i v_i - t_i).$$

Corollary (**1.7**). — H_g *is smooth over* $\mathbf{Z}$. *In particular, for all algebraically closed fields* k, $H_g \times \operatorname{Spec}(k)$ *is a disjoint union of a finite number of non-singular algebraic varieties over* k.

Let

$$H_g^0 = \{x \in H_g \mid \text{the corresponding stable curve } (Z_g)_x \text{ is non-singular}\}.$$
$$S = \{x \in Z_g \mid \text{the projection } \pi : Z_g \to H_g \text{ is } not \text{ smooth at } x\}.$$

Definition (**1.8**). — *Let* $p : X \to Y$ *be a smooth morphism of finite type, with* Y *a noetherian scheme, and let* $D \subset X$ *be a relative Cartier divisor. Then* D has normal crossings relative to Y *if for all* $x \in D$, *the local equation* $d = 0$ *of* D *decomposes in the strict completion* [1] $\tilde{\mathcal{O}}_{x,X}$ *of* $\mathcal{O}_{x,X}$ *as* $d = d_1 \ldots d_k$, *where* $d_1, \ldots, d_k$ *are linearly independent in* $\tilde{\mathfrak{m}}_{x,X}/\tilde{\mathfrak{m}}^2_{x,X} + \mathfrak{m}_{y,Y} \cdot \tilde{\mathcal{O}}_{x,X}$, *with* $y = p(x)$.

Corollary (**1.9**). — $H_g^0 = H_g - S^*$, *where* S^* *is a divisor with normal crossings relative to* $\mathbf{Z}$. Z_g *and* S *are smooth over* $\mathbf{Z}$, *and the projection* $p : S \to S^*$ *is finite and an isomorphism at all points where* S^* *is smooth over* $\mathbf{Z}$, i.e., S *is the normalization of* S^*.

Proof. — In the notation of Theorem (1.6), S^* is defined in $\hat{\mathcal{O}}_{x, H'_g}$ by the local equation $t_1 \ldots t_k = 0$. And

$$\hat{\mathcal{O}}_{x_i, Z'_g} \cong \mathfrak{o}_k[[u_i, v_i, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_N]]$$
$$\hat{\mathcal{O}}_{x_i, S} \cong \hat{\mathcal{O}}_{x_i, Z'_g}/(u_i, v_i)$$
$$\cong \mathfrak{o}_k[[t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_N]].$$

Q.E.D.

(1) The complete local ring, formally etale over $\mathcal{O}_{x,X}$ with residue field the separable closure of $\mathcal{O}_{x,X}/\mathfrak{m}_{x,X}$.

Next we take up the isomorphisms and automorphisms of stable curves. Suppose $p: X \to S$, $q: Y \to S$ are two stable curves:

Definition (**1.10**). — $\underline{\mathrm{Isom}}_S(X, Y)$ *is the functor on* (Sch/S) *associating to each* S-*scheme* S′ *the set of* S′-*isomorphisms between* $X \times_S S'$ *and* $Y \times_S S'$. *If* $X = Y$, *we denote* $\underline{\mathrm{Isom}}_S(X, X)$ *by* $\underline{\mathrm{Aut}}_S(X)$.

Since both X and Y have the canonical polarizations $\omega_{X/S}$, $\omega_{Y/S}$ respectively, any isomorphism $f: X \to Y$ must satisfy $f^*(\omega_{Y/S}) \cong \omega_{X/S}$. Therefore, by Grothendieck's results on the representability of the Hilbert scheme and related functors [Gr_1], we conclude that $\underline{\mathrm{Isom}}_S(X, Y)$ is represented by a scheme $\mathbf{Isom}_S(X, Y)$, quasi-projective over S. Concerning this scheme, we have:

Theorem (**1.11**). — $\mathbf{Isom}_S(X, Y)$ *is finite and unramified over* S.

Proof. — To check that $\mathbf{Isom}_S(X, Y)$ is unramified, we may take S to be the spectrum of an algebraically closed field k, in which case $\mathbf{Isom}_S(X, Y)$ is either empty or isomorphic to $\mathbf{Aut}_k(X)$. A point of $\mathbf{Aut}_k(X)$ with values in $k[\varepsilon]/(\varepsilon^2)$ with image the identity may be identified with a vector field on X. By Lemma (1.4), stable curves have no non-zero vector fields. This proves that $\mathbf{Isom}_S(X, Y)$ is unramified over S, and since it is also of finite type over S, it is quasi-finite over S. It remains to check that $\mathbf{Isom}_S(X, Y)$ is proper over S.

Locally over S, X and Y are the pull-backs of the universal tri-canonically embedded stable curve by some morphisms from S to H_g, so that it suffices to prove the properness of $\mathbf{Isom}_S(X, Y)$ in the " universal " case where $S = H_g \times H_g$, X and Y being the two inverse images of the universal curve on H_g. In that case, the open subset of $\mathbf{Isom}_S(X, Y)$ corresponding to smooth curves is dense, so that the Theorem follows from the valuative criterion of properness which holds by:

Lemma (**1.12**). — *Let* X *and* Y *be two stable curves over a discrete valuation ring* R *with algebraically closed residue field. Denote by* η *and* s *the generic and closed points of* Spec(R), *and assume that the generic fibres* X_η *and* Y_η *of* X *and* Y *are smooth. Then any isomorphism* φ_η *between* X_η *and* Y_η *extends to an isomorphism* φ *between* X *and* Y.

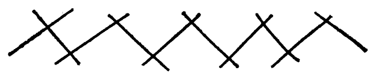
(*A posteriori*, it follows from Theorem (1.11) that the lemma holds for any valuation ring R and without assuming X_η or Y_η smooth.)

Proof. — Another way to put the lemma is that if we start with a smooth curve X_η of genus $g \geq 2$ over the quotient field K of R, there is, up to canonical isomorphism, at most one stable curve X over R with X_η as its generic fibre. We shall deduce this from the analogous uniqueness assertion for minimal models ([L] and [Š]): given a smooth curve X_η of genus $g \geq 1$ over K, there is, up to canonical isomorphism, at most one regular 2-dimensional scheme X, proper and flat over R, with X_η as its generic fibre, without exceptional curves of the first kind in X_s.

Let z denote a generator of the maximal ideal of R and consider the affine plane curve C_n over R given by:

$$xy = z^n.$$

Let $\widetilde{C}_n$ denote the scheme obtained by: 1) blowing up the maximal ideal at the unique singularity of C_n; 2) blowing up the maximal ideal at the unique singularity of this scheme..., and so on $\left[\frac{n}{2}\right]$ times. It is easy to check that $\widetilde{C}_n$ is a regular scheme whose special fibre is the same as that of C_n except that the singular point is replaced by a sequence of $n-1$ projective lines as follows:



Now suppose x is a singular point of the stable curve X over R. At x, X is formally isomorphic as scheme over R to one of the schemes C_n, so we may blow up X the same way we blew up C_n. If we do this for all singular points of X, we get a regular scheme $\widetilde{X}$ with generic fibre X_η. In addition, any non-singular rational component of $\widetilde{X}_s$ is linked to the other irreducible components by at least two points, hence it is not exceptional of first kind. Therefore $\widetilde{X}$ is *the* minimal model of X_η. Note finally that C_n is a normal scheme, hence so is X; therefore X is the unique normal scheme obtained from $\widetilde{X}$ by contracting all non-singular rational components of $\widetilde{X}_s$ linked to the other irreducible components by exactly two points. This proves that X is essentially unique. Q.E.D.

Another important fact about the automorphisms of stable curves is:

Theorem **(1.13)**. — *Let k be an algebraically closed field and* X *a stable curve over k. Let* $\mathbf{Pic}^0(X)$ *denote the group of invertible sheaves on* X *of degree* 0 *on each component. Then the map (of ordinary groups):*

$$\mathrm{Aut}_k(X) \to \mathrm{Aut}_k(\mathbf{Pic}^0(X))$$

is injective.

Proof. — Let φ be an automorphism of X inducing the identity on $\mathbf{Pic}^0 X$.

Lemma **(1.14)**. — *If* X *is smooth, then* φ *is the identity.*

Proof. — If not, by the Lefschetz-Weil fixed point formula, the number n of fixed points of φ, counted with their multiplicities, is

$$n = 1 - \mathrm{Tr}(\varphi, T_l(\mathbf{Pic}^0(X))) + 1 = 2 - 2g < 0$$

which is absurd. Q.E.D.

Lemma **(1.15)**. — *If* X *is irreducible, then* φ *is the identity.*

Proof. — Let φ' be the action of φ on the normalization X′ of X. Each singular point of X, together with an ordering of its 2 inverse images in X′, defines a distinct morphism from $\mathbf{G}_m$ to $\mathbf{Pic}^0(X)$, so that the inverse image S of the singular locus of X is pointwise fixed by φ'. One has either

a) genus$(X') \geq 2$: then conclude by Lemma (1.14);

b) genus$(X') = 1$, $|S| \geq 2$, then φ' is a translation on X′ leaving a point fixed, and so φ' is the identity;

c) X′ is the projective line, $|S| \geq 4$ and φ' is a projectivity leaving more than three points fixed, so is the identity. Q.E.D.

Let Γ be the following (unoriented) graph:

(i) The set of vertices of Γ is the set Γ^0 of irreducible components of X,

(ii) the set of edges of Γ is the set Γ^1 of the singular points of X which lie on two distinct irreducible components,

(iii) an edge $x \in \Gamma^1$ has for extremities the irreducible components on which x lies.

Lemma **(1.16)**. — *If φ induces the identity on Γ, then φ is the identity.*

Proof. — If X_1 is an irreducible component of X, then $\varphi(X_1) = X_1$ and φ leaves fixed the points of intersection of X_1 with the other components. In addition, $\mathbf{Pic}^0(X)$ maps onto $\mathbf{Pic}^0(X_1)$ so that φ acts trivially on $\mathbf{Pic}^0(X_1)$. Either:

a) genus $(X_1) \geq 2$ and $\varphi | X_1$ is the identity by Lemma (1.15);

b) genus $(X_1) = 1$, φ acts by a translation and leaves a point fixed, so is the identity on X_1;

c) X_1 is the projective line and φ leaves fixed at least three points, so is the identity on X_1. Q.E.D.

Lemma **(1.17)**. — (i) *Any edge in Γ has distinct extremities.*

(ii) *Any vertex which is the extremity of* 0, 1 *or* 2 *edges is fixed by* φ.

(iii) φ *acts trivially on* $H^1(\Gamma, \mathbf{Z})$.

Proof. — It is easy to check that the subgroup of $\mathbf{Pic}^0(X)$ corresponding to invertible sheaves whose restriction to each irreducible component of X is trivial is canonically isomorphic to

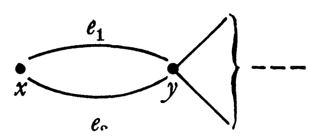
$$H^1(\Gamma, \mathbf{Z}) \otimes \mathbf{G}_m.$$

This implies (iii), and (i) is trivial.

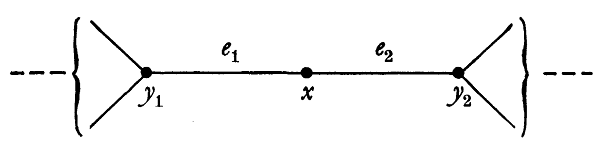
The morphism from $\mathbf{Pic}^0(X)$ to the product $\prod_i \mathbf{Pic}^0(X_i)$, extended over the irreducible components of X, is surjective, so that if $\mathbf{Pic}^0(X_i) \neq \{e\}$, then $\varphi(X_i) = X_i$. This is the case, unless X_i is a projective line, linked to the other components in at least three points. Q.E.D.

We prove now that if an automorphism φ of any finite graph Γ has the properties stated in Lemma (1.17), it is the identity. Make induction on the sum of the number of vertices and edges of Γ. If Γ has an isolated point x, then $\varphi(x) = x$ so let $\Gamma^* = \Gamma - \{x\}$. Then $\varphi =$ identity on Γ^* by induction, so $\varphi =$ identity on Γ too. If Γ has an extremity x, then $\varphi(x) = x$, and again let Γ^* be Γ minus x and the edge abutting at x. Then Γ^* has all the properties Γ has, so $\varphi =$ identity on Γ^*, hence $\varphi =$ identity on Γ. If Γ has a vertex x on which only 2 edges abut, we have one of the two cases:

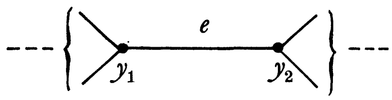
(1)



(2)



In the first case, $\varphi(y)=y$, and let Γ^* be Γ minus x, e_1 and e_2. Then $\varphi=$identity on Γ^* and $\varphi(e_i)=e_i$ too, since if φ reverses the e_i's, this contradicts (iii). In the second case, let Γ^* be Γ minus x, and with e_1 and e_2 identified:



Then $\varphi=$identity on Γ^*, so $\varphi=$identity on Γ. Next, say Γ has an edge e with extremities x and y such that $\varphi(x)=x$, $\varphi(y)=y$, $\varphi(e)=e$. Let Γ^* be Γ minus e. Then $\varphi=$identity on Γ^*, so $\varphi=$identity on Γ. If none of these reductions are possible, we must be in a situation where *a)* every vertex is the abutment of at least three edges and *b)* no edge is left fixed. It is easily seen that the first Betti number b_1 of any of the connected components of Γ is at least 2. Let

$$n_0=\text{number of fixed vertices}$$
$$n_1=\text{number of edges reversed by } \varphi.$$

Then, unless $\Gamma=\emptyset$, the Lefschetz fixed point formula reads:

$$n_0+n_1=b_0-b_1<0,$$

which is impossible. Q.E.D.

§ 2. Degenerations of curves and their jacobians.

We consider the situation:

K $=$discretely-valued field;
R $=$integers in K, $k=\mathrm{R}/\mathfrak{M}=$residue field (assumed algebraically closed);
S $=$Spec(R), η and s its generic and closed points respectively;
C $=$a curve, smooth, geometrically irreducible and proper over K, of genus $g\geq 2$;
J $=$the jacobian variety of C;
$\mathscr{J}=$the Néron model of J over R (cf. [N]);
$\mathscr{J}^0\subset\mathscr{J}$ the open subgroup scheme with $\mathscr{J}^0_s=$identity component of $\mathscr{J}_s$;
$\mathscr{C}=$the minimal model of C over R.

A word about the existence and uniqueness of $\mathscr{C}$ is needed. We recall that $\mathscr{C}$ is to be a regular scheme, flat and proper over R, with generic fibre $\mathscr{C}_\eta=\mathrm{C}$ such that for any other regular scheme $\mathscr{C}'$, flat over R, with generic fibre $\mathscr{C}'_\eta=\mathrm{C}$, the birational map $\mathscr{C}'\to\mathscr{C}$ is a morphism. Šafarevich in [Š] and Lichtenbaum [L] have proven that such a $\mathscr{C}$ (which is obviously unique) exists, provided that there is some regular $\mathscr{C}'$, proper and flat over R, with generic fibre C. And, in fact, that $\mathscr{C}$ is projective over R. To construct such a $\mathscr{C}'$, proceed as follows: first let $\mathscr{C}''$ be any scheme, projective and flat over R with generic fibre C. Let $\hat{\mathrm{R}}$ be the completion of R, and let $\hat{\mathscr{C}}''=\mathscr{C}''\times_{\mathrm{Spec\,R}}\mathrm{Spec}\,\hat{\mathrm{R}}$. Then $\hat{\mathscr{C}}''$ is an excellent surface, so by [Ab] and by unpublished results of Hironaka, there is a sheaf of ideals $\hat{\mathscr{I}}$ with support in the singular locus of $\hat{\mathscr{C}}''$ such that blowing up $\hat{\mathscr{I}}$ leads to a regular surface $\hat{\mathscr{C}}'$. But then

$$\hat{\mathscr{I}} \supset \mathfrak{M}^n . \mathcal{O}_{\hat{\mathscr{C}}''}$$

for some n, so $\hat{\mathscr{I}}$ is induced by a unique sheaf of ideals $\mathscr{I} \subset \mathcal{O}_{\mathscr{C}''}$. Let $\mathscr{C}'$ be obtained by blowing up $\mathscr{I}$. Then

$$\hat{\mathscr{C}}' \cong \mathscr{C}' \times_{\text{Spec R}} \text{Spec } \hat{\text{R}}$$

so $\mathscr{C}'$ is regular. $\mathscr{C}'$ is also projective over R, hence a projective $\mathscr{C}$ exists.

Definition **(2.1)**. — J *has* stable reduction *if* $\mathscr{J}_s$ *has no unipotent radical.*

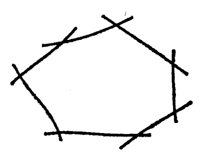
Definition **(2.2)**. — C *has* stable reduction in sense 1 *if* $\mathscr{C}_s$ *is reduced and has only ordinary double points.* C *has* stable reduction in sense 2 *if there is a stable curve* $\mathscr{C}'$ *over* R *with generic fibre* $\mathscr{C}'_\eta = \text{C}$.

Note that if a stable $\mathscr{C}$ exists, then by Theorem (1.11) it is unique.

Proposition **(2.3)**. — *The two senses of stable reduction for* C *are equivalent.*

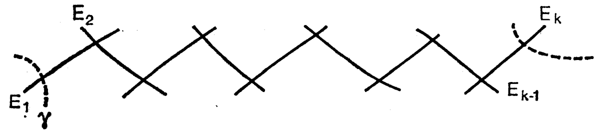
Proof. — Say a stable $\mathscr{C}'$ exists. Blowing up the singularities of $\mathscr{C}'$ as in Lemma (1.12), we obtain the minimal model $\mathscr{C}$ of C and it is seen that $\mathscr{C}_s$ is reduced with only ordinary double points. Conversely, suppose the minimal model $\mathscr{C}$ has this property. Let $\text{E}_1, \ldots, \text{E}_n$ be the non-singular rational components of $\mathscr{C}_s$ which meet the other components in only two points. Then the E_i's divide into several chains of the type:

unless the entire fibre consisted of E_i's and has the type:

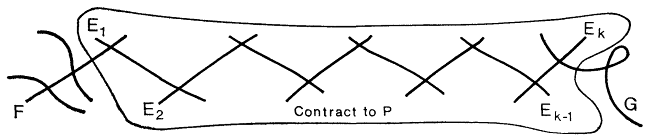


$n \geq 2$
$\mathscr{C}_s = \text{loop of E}_i\text{'s}$
$(\text{E}_i^2)_{\text{on}\,\mathscr{C}} = -2.$

But in this case $\text{genus(C)} = \text{genus}(\mathscr{C}_s) = 1$, which contradicts our assumption. Now, according to Theorem (27.1) of Lipman [Li] (generalizing a result of Artin [A_2], which works for surfaces of finite type over a field) any set of k non-singular rational curves connected in a chain as above on a regular surface X, with self-intersection 2 on X, can be blown down to a rational double point P of type A_k on a normal surface X_0. Reversing the process and blowing up a rational double point of type A_k, it is easy to see that non-singular branches γ on X, crossing transversally only the first or the last rational curve in the chain:



are still non-singular branches when mapped to X_0; and if γ_1, γ_2 intersect E_1 or E_k in distinct points, then they cross transversally on X_0. Therefore, suppose we blow down all the chains of E_i's on $\mathscr{C}$. Let $\mathscr{C}'$ be the normal surface so obtained. Then if one of these chains fits into $\mathscr{C}_s$ like this:



then on $\mathscr{C}'_s$, the images of F and G still have only ordinary double points, each has one non-singular branch through the singular point P, and these branches cross transversally. Therefore $\mathscr{C}'$ is a stable curve over R. Q.E.D.

We are now ready to prove the key result on which our proof of irreducibility depends:

Theorem (**2.4**). — J *has stable reduction if and only if* C *has stable reduction.*

Proof. — The connection between $\mathscr{C}$ and $\mathscr{J}$ is based on the following result of Raynaud [R]:

Theorem (**2.5**). — *If $\mathscr{C}$ and $\mathscr{J}$ are as above, and the greatest common denominator d of the multiplicities of the components of $\mathscr{C}_s$ is* 1, *then $\mathscr{J}^0$ represents the functor* $\underline{\mathrm{Pic}}^0(\mathscr{C}/S)$.

(This result is not stated as such in [R]. It comes out like this, in the terminology of is paper:

a) Condition (N) is verified and $p_*(\mathcal{O}_{\mathscr{C}})=\mathcal{O}_S$, so

b) $\mathscr{C}$ is cohomologically flat over S in dim. 0 by Theorem 4;

c) therefore $\underline{\mathrm{Pic}}^0$ is representable and separated over S by Theorem 3;

d) since $E=\overline{\{0\}}$ and $Q=P/E$, we find $P^0=Q^0$, and since $\mathscr{C}$ is 1-dimensional over S, P^0 and Q^0 are smooth over S; therefore $R^0=Q^0$ and $\widetilde{R}=R$.

e) Then by Theorem 5, $\mathbf{Pic}^0$ is the identity component of the Néron model of $\mathbf{Pic}^0(\mathscr{C}_\eta)=J$.)

Now assume that C has stable reduction. Then $\mathscr{C}_s$ is reduced so $d=1$. By Theorem (2.5), $\mathscr{J}^0=\mathbf{Pic}^0(\mathscr{C}/S)$. Therefore $\mathscr{J}^0_s=\mathbf{Pic}^0(\mathscr{C}_s/k)$. Since $\mathscr{C}_s$ is reduced with only ordinary double points, its generalized jacobian $\mathbf{Pic}^0(\mathscr{C}_s/k)$ is an extension of an abelian variety by a torus, i.e., has no unipotent radical. Therefore J has stable reduction.

The converse is more difficult. Assume J has stable reduction. We first prove that C has stable reduction under the additional hypothesis that C has a K-rational point [1]. In this case, $\mathscr{C}$ has an R-rational point, and since $\mathscr{C}$ is regular, sections of $\mathscr{C}$ over R pass through components of $\mathscr{C}_s$ of multiplicity one. Therefore $d=1$, and

(1) This is in fact the only case which will be needed in our application to questions of irreducibility.

Theorem (2.5) applies. In particular, $\mathbf{Pic}^0(\mathscr{C}_s/k)=\mathscr{I}_s^0$ so $\mathbf{Pic}^0(\mathscr{C}_s/k)$ has no unipotent radical. We apply:

Lemma **(2.6)**. — *Let* D *be a complete* 1*-dimensional scheme over* k *such that* $H^0(\mathscr{O}_D)\simeq k$, *and such that the generalized jacobian of* D *has no unipotent radical. Then*

(i) $\chi(\mathscr{O}_D)=\chi(\mathscr{O}_{D_{red}})$;

(ii) *the singularities of* D_{red} *are all transversal crossings of a set of non-singular branches* (i.e., *analytically isomorphic to the union of the coordinate axes in* $\mathbf{A}^n$).

Proof. — Let $\mathscr{I}\subset\mathscr{O}_D$ be the ideal of nilpotent elements. Filtering $\mathscr{I}$ by a chain of ideals $\mathscr{I}_k$ such that $\mathscr{I}.\mathscr{I}_k\subset\mathscr{I}_{k+1}$, and using the exact sequence:

$$0\to\mathscr{I}_k/\mathscr{I}_{k+1}\xrightarrow[a\mapsto 1+a]{}(\mathscr{O}_D/\mathscr{I}_{k+1})^*\to(\mathscr{O}_D/\mathscr{I}_k)^*\to 0$$

it is easy to deduce that $\mathbf{Pic}^0(D/k)$ is an extension of $\mathbf{Pic}^0(D_{red}/k)$ by a unipotent group. Therefore, since by assumption $\mathbf{Pic}^0(D/k)$ has no unipotent subgroups,

$$\mathbf{Pic}^0(D/k)\cong\mathbf{Pic}^0(D_{red}/k).$$

Since $H^1(\mathscr{O}_D)$, resp. $H^1(\mathscr{O}_{D_{red}})$, is naturally isomorphic to the Zariski tangent space to $\mathbf{Pic}^0(D/k)$, resp. $\mathbf{Pic}^0(D_{red}/k)$, it follows that

$$H^1(\mathscr{O}_D)\cong H^1(\mathscr{O}_{D_{red}})$$

hence (i) is proven. Let $\pi: C\to D_{red}$ be the normalization of D_{red} and let D^* be the local ringed space which, as topological space is D, and whose structure sheaf is given by:

$$\Gamma(U,\mathscr{O}_{D^*})=\{f\in\Gamma(U,\pi_*(\mathscr{O}_C))\mid x_1,x_2\in\pi^{-1}(U), f(x_1)=f(x_2)\text{ if }\pi(x_1)=\pi(x_2)\}$$

It is easy to check that D^* is a 1-dimensional scheme whose singularities are all transversal crossings of a set of non-singular branches and that π factors:

$$C\overset{\pi'}{\to}D^*\overset{\pi''}{\to}D_{red}.$$

I claim that $\pi'': D^*\to D_{red}$ is an isomorphism. Filter $\mathscr{O}_{D^*}/\mathscr{O}_{D_{red}}$ so as to obtain a chain of coherent $\mathscr{O}_{D_{red}}$-algebras:

$$\mathscr{O}_{D^*}=\mathscr{O}^{(0)}\supset\mathscr{O}^{(1)}\supset\ldots\supset\mathscr{O}^{(k)}=\mathscr{O}_{D_{red}}$$

such that $l(\mathscr{O}^{(n)}/\mathscr{O}^{(n+1)})=1$. Equivalently, this factors π'':

$$D^*=D_0\to D_1\to\ldots\to D_k=D_{red}$$

where $\mathscr{O}^{(h)}\cong\mathscr{O}_{D_n}$, and all arrows are homeomorphisms. If $\{x_n\}=\mathrm{Supp}(\mathscr{O}^{(n)}/\mathscr{O}^{(n+1)})$, then we get an exact sequence:

$$0\to\mathscr{O}^*_{D_{n+1}}\to\mathscr{O}^*_{D_n}\to k_{x_n}\to 0$$

(k_y denotes the residue field at y, as sheaf on D), and hence:

$$0\to k\to H^1(\mathscr{O}^*_{D_{n+1}})\to H^1(\mathscr{O}^*_{D_n})\to 0.$$

It follows easily that $\mathbf{Pic}^0(D_{n+1}/k)$ is an extension of $\mathbf{Pic}^0(D_n/k)$ by $\mathbf{G}_a$. But $\mathbf{Pic}^0(D_{red}/k)$ has no unipotent subgroups, and this can only happen if $D_{red}=D^*$. Q.E.D. for lemma.

We apply the lemma to $\mathscr{C}_s$. According to Lichtenbaum [L] and Šafarevich [Š], there is a divisor K on $\mathscr{C}$ such that for all positive divisors D lying over the closed point of Spec(R), we have:

$$\chi(\mathcal{O}_D) = -\frac{(D.(D+kK))}{2}.$$

Let $E_1, \ldots, E_n$ be the components of $\mathscr{C}_s$, $d_1, \ldots, d_n$ their multiplicities. Then conclusion (i) of the lemma implies that:

$$(\sum_i d_i E_i . (\sum_i d_i E_i + K)) = (\sum_i E_i . (\sum_i E_i + K)).$$

But $((\sum_i d_i E_i).E_k) = 0$, all k, since $\sum_i d_i E_i$ is the divisor of a function $\pi \in R$ if (π) = maximal ideal of R. Therefore:

$$(*) \qquad ((\sum_i (d_i - 1).E_i).K) = (\sum_i E_i . \sum_i E_i).$$

Note that at least one d_i equals 1 since $\mathscr{C}$ has a section over Spec(R) and every section must pass through a component of $\mathscr{C}_s$ of multiplicity 1. Moreover the intersection matrix $(E_i.E_j)$ is negative indefinite, with one-dimensional degenerate subspace generated by $\sum_i d_i E_i$, hence *if some* $d_i > 1$, it follows that $(\sum_i E_i . \sum_i E_i) < 0$. Therefore, by $(*)$, $(E_{i_0}.K) < 0$ for some i_0. Then we have:

a) $(E_{i_0}.K) < 0$;

b) $(E_{i_0}.E_{i_0}) < 0$;

c) $(E_{i_0}.(E_{i_0}+K)) = -2\chi(\mathcal{O}_{E_{i_0}}) \geq -2$;

hence in fact $(E_{i_0}.E_{i_0}) = (E_{i_0}.K) = -1$ so E_{i_0} is an exceptional curve of the first kind. This contradicts our assumption that on $\mathscr{C}$ all possible curves have been blown down. Therefore $d_i = 1$, *all* i.

This proves that $\mathscr{C}_s$ is reduced. By conclusion (ii) of the lemma, plus the fact that the dimension of its Zariski tangent-space is everywhere one or two (since $\mathscr{C}_s$ lies on a regular surface $\mathscr{C}$), we deduce that $\mathscr{C}_s$ has only double points. This proves that C has stable reduction in the case that $C(K) \neq \emptyset$.

In the general case, C will acquire a rational point in a finite extension K′ of K.

Let S′ be the spectrum of the localisation at some maximal ideal of the integral closure of R in K′; S′ is the spectrum of a valuation ring and is faithfully flat over S.

We put $S'' = S' \times_S S'$ and denote by C′ and C″ the inverse images of C on $S'_\eta = \mathrm{Spec}(K')$ and S''_η respectively.

$$\begin{array}{ccccc} C'' & \rightrightarrows & C' & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} & S' & \overset{p}{\longrightarrow} & S \end{array}$$

Let $\overline{C}'$ be the stable curve on S′ having C′ as generic fibre. The restriction, C′, of $\overline{C}'$ to S'_η carries a descent datum with respect to p, i.e. an isomorphism, φ_η, between the restrictions of $pr_1^*(\overline{C}')$ and $pr_2^*(\overline{C}')$ to S''_η. It remains only to extend the isomorphism φ_η to an isomorphism, φ, between the $pr_i^*(\overline{C}')$. Then φ will be a descent datum for $\overline{C}'$ with respect to p. As $\overline{C}'$ is canonically polarized (1.2), this descent datum will be effective, and so define a stable curve, $\overline{C}$, over S with generic fibre C.

Because $\mathscr{J}_s^0$ has no unipotent radical, the inverse image, $p^*\mathscr{J}^0$, of $\mathscr{J}^0$ on S′ is the identity component of the Néron Model of the jacobian of C′, so that, defining $q = p \circ pr_1 = p \circ pr_2$, one has

$$\mathbf{Pic}^0(\overline{C}'/S') = p^*\mathscr{J}^0$$
$$\mathbf{Pic}^0(pr_1^*\overline{C}') = \mathbf{Pic}^0(pr_2^*\overline{C}') = q^*\mathscr{J}^0$$

We denote by T the closed subscheme of **Isom**(S″, $pr_1^*(C')$, $pr_2^*(\overline{C}')$) corresponding to those isomorphisms which induce (via the preceeding identifications) the identity on the inverse image of $\mathscr{J}^0$.

By (1.11), T is finite and unramified over S″, and by (1.13) T is radicial over S″. We conclude that the morphism from T to S″ identifies T with a closed subscheme X of S″. As X contains S''_η, which is schematically dense in S″, we have that X is S″ and T "is" the desired section, φ, of **Isom**(S″, $pr_1^*(\overline{C}')$, $pr_2^*(\overline{C}')$) over S″ which extends φ_η. Q.E.D.

Combining Proposition (2.3), Theorem (2.4), and the stable reduction theorem for abelian varieties quoted in the introduction, we obtain the most important consequence:

Corollary **(2.7).** — *Let* R *be a discrete valuation ring with quotient field* K. *Let* C *be a smooth geometrically irreducible curve over* K *of genus* $g \geq 2$. *Then there exists a finite algebraic extension* L *of* K *and a stable curve* $\mathscr{C}_L$ *over* R_L, *the integral closure of* R *in* L, *with generic fibre* $\mathscr{C}_{L,\eta} \cong C \times_K L$.

§ 3. Elementary derivation of the theorem.

Let k be an algebraically closed field of char. $p \neq 0$. We use the notation of § 1, except that we will now denote by H_g the product $H_g \times \mathrm{Spec}(k)$ of the previous H_g with Spec(k): it is a disjoint union of non-singular varieties $H_{g,1}, \ldots, H_{g,n}$ over k and is the subscheme of $\mathbf{Hilb}^{P_g}_{\mathbf{P}^{5g-6}/k}$ of tri-canonical stable curves. Similarly, H_g^0 is the open dense subset of H_g of tri-canonical non-singular curves. By the results of [M_1], we know that a coarse geometric quotient

$$M_g^0 = H_g^0/\mathbf{PGL}(5g-6)$$

exists, that it is a disjoint union of normal varieties over k and is the coarse moduli space for non-singular curves of genus g. Let $S^* = H_g - H_g^0$. Then everything decomposes into the same set of components.

Let

$$H_{g,i} = \text{components of } H_g$$

then $H^0_{g,i} = H_{g,i} \cap H^0_g = \text{components of } H^0_g$

and $M^0_{g,i} = H^0_{g,i}/\mathbf{PGL}(5g-6) = \text{components of } M^0_g$

and $S^* = \text{disjoint union of } S^*_1, \ldots, S^*_n, \quad S^*_i = H_{g,i} \cap S_i.$

We want to prove that M^0_g, or equivalently H^0_g, or equivalently H_g is irreducible. We shall use: (i) the fact that these statements are true in char. 0; (ii) the inductive assumption that these statements are true for smaller genus.

Step I. — No component of M^0_g is complete (i.e., proper over k).

Proof. — Here we use the char. 0 result. By [M_1], there is a scheme X, quasi-projective over Spec(W(k)), W(k) the Witt vectors, whose closed fibre is M^0_g, and whose generic fibre X_η is the char. 0 coarse moduli space over the quotient field of W(k). In particular, X_η is known to be connected. Since X is quasi-projective over W(k), we can embed X as an open dense subset of a scheme $\overline{X}$ projective over W(k). $\overline{X}_\eta$ is still connected, hence by the connectedness theorem of Enriques-Zariski [EGA 3], the closed fibre $\overline{X}_0$ of $\overline{X}$ is connected. But if Y were a complete variety which is a component of M^0_g, then: *a)* Y is an open subset of M^0_g, which is an open subset of $\overline{X}_0$, and: *b)* since Y is proper/k, Y would be a closed subset of $\overline{X}_0$ too. Therefore, $Y = \overline{X}_0$, hence M^0_g is itself irreducible and complete. On the other hand, if A_g is the coarse moduli space of principally pclarized g-dimensional abelian varieties, then the map associating to each curve its jacobian defines a morphism:

$$\theta : M^0_g \rightarrow A_g.$$

If M^0_g were complete, the image of θ would be closed. But it is well known that the closure of the image of θ contains all products of lower dimensional jacobians too, so it is not closed. Q.E.D.

Step II. — No component of H_g consists entirely of non-singular curves, i.e., $S^*_i \neq \emptyset$ for all i.

Proof. — Here we combine Step I with the result of § 2. Take any i. Let $T = \text{Spec } k[[t]]$. Since $M^0_{g,i}$ is not complete, there is a morphism φ of the generic point T_η of T into $M^0_{g,i}$ which does *not* extend to a morphism of T into $M^0_{g,i}$. Now we replace T by its normalization T′ in a finite algebraic extension and let $\varphi' : T'_\eta \rightarrow M^0_{g,i}$ be the induced morphism, which still does not extend to a morphism from T′ to $M^0_{g,i}$. By the results of § 2, if T′ is chosen suitably there exists a stable curve $\pi : C' \rightarrow T'$ over T′ whose generic fibre C'_η is a non-singular curve corresponding to the morphism $\varphi' : T'_\eta \rightarrow M^0_g$ via the functorial properties of the moduli space. Since T′ is the spectrum of a local ring, we can choose an isomorphism

$$\mathbf{P}(\pi_*(\omega^{\otimes 3}_{C'/T'})) \cong \mathbf{P}^{5g-6} \times T'$$

and get a tri-canonical embedding $C' \subset \mathbf{P}^{5g-6} \times T'$. C′, with this embedding, is then induced from the universal tri-canonically embedded stable curve by a morphism $\psi : T' \to H_g$. Since the generic fibre of C' is C'_η, we get a commutative diagram:

$$\begin{array}{ccc} T' & \xrightarrow{\ \psi\ } & H_g \\ \cup & & \cup \\ T'_\eta & \xrightarrow{\ \mathrm{res}(\psi)\ } & H^0_g \\ & \searrow^{\varphi'} & \downarrow \\ & M^0_{g,i} \subset M^0_g & \end{array}$$

If $\mathrm{Im}(\psi) \subset H^0_g$, then φ' would extend to a morphism from T' to M^0_g, hence from T' to $M^0_{g,i}$, and this is a contradiction. Moreover, $\psi(T_\eta)$ must be a point of $H^0_{g,i}$ since its image in M^0_g is in $M^0_{g,i}$. Therefore the image x of the closed point of T' by ψ is in the closure of $H^0_{g,i}$, i.e., in $H_{g,i}$, but not in $H^0_{g,i}$ itself. Q.E.D.

Step III. — S^* is connected.

Proof. — This will follow using only the induction assumption of irreducibility for lower genera. Let $Z \subset H_g \times \mathbf{P}^{5g-6}$ be the universal tri-canonically embedded stable curve. Let $S \subset Z$ be the set of points where Z is not smooth over H_g. As we proved in § 1, S is non-singular and is the normalization of S^*. In particular, this shows that if $x \in S^*$, then the corresponding curve Z_x has exactly one double point if and only if x is a non-singular point of S^*. Stable curves C of genus g with exactly one double point belong to one of the following types:

type 0:  C irreducible
normalization C' of C has genus $g-1$.

type k:  C has two non-singular components C_1, C_2

$1 \leq k \leq \left[\frac{g}{2}\right]$ genus$(C_1) = k$
genus$(C_2) = g-k$

If $0 \leq k \leq \left[\frac{g}{2}\right]$ let $S^*(k) = \{x \in S^* \mid Z_x$ has one double point and is of type $k\}$. Then the open dense subset of S^* of non-singular points is the disjoint union of open subsets $S^*(0), \ldots, S^*\left(\left[\frac{g}{2}\right]\right)$. We first check:

($*$) Each set $S^*(k)$ is irreducible.

Proof of ($*$). — Take the case $k=0$: the cases $1 \leq k \leq \left[\frac{g}{2}\right]$ are similar. Let

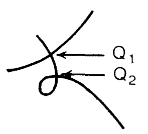
$$T = \{(x_1, x_2) \mid x_1 \neq x_2 \text{ and } \pi(x_1) = \pi(x_2) \in H^0_{g-1}\} \subset Z_{g-1} \times_{H_{g-1}} Z_{g-1}.$$

T is smooth with irreducible fibres over H^0_{g-1}, hence T is irreducible. Consider the correspondence relating $S^*(0)$ and T:

$$\mathrm{W}=\left\{\begin{array}{l}\text{set of pairs } x\in \mathrm{S}^*(o),\ \{x_1, x_2\}\in \mathrm{T} \text{ such that if } y=\pi(x_1)=\pi(x_2),\\ \text{then there exists a birational morphism}\\ \qquad\qquad f: (\mathrm{Z}_{g-1})_y \to (\mathrm{Z}_g)_x\\ \text{such that } f(x_1)=f(x_2).\end{array}\right\}$$

It is easy to check that W is Zariski-closed. Moreover, for any $\{x_1, x_2\}\in \mathrm{T}$, $\mathrm{W}\cap(\mathrm{S}^*(o)\times\{x_1, x_2\})$ is an orbit in $\mathrm{S}^*(o)$ under $\mathbf{PGL}(5g-6)$: so these are non-empty irreducible subsets all of the same dimension. It follows that W itself is irreducible. But the projection from W to $\mathrm{S}^*(o)$ is surjective, so $\mathrm{S}^*(o)$ is irreducible. Q.E.D. *for* $(*)$.

Now for any k, $1\leq k\leq \left[\frac{g}{2}\right]$, choose any stable curve $\mathrm{C}(k)$ of the type:



one non-singular component $\mathrm{C}(k)'$ of genus k.
one component $\mathrm{C}(k)''$ with one double point, normalization of genus $g-k-1$.

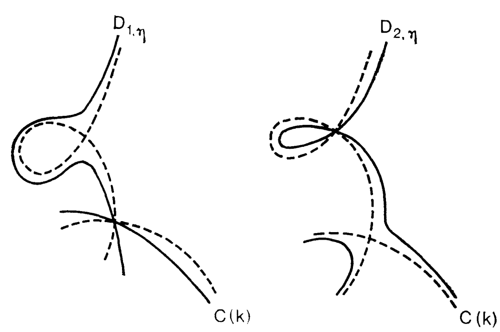
Let $\mathrm{P}(k)$ be a point of H_g such that $\mathrm{Z}_{\mathrm{P}(k)}\cong \mathrm{C}(k)$. Step III will be completed if we prove:

$(**)$ $\mathrm{P}(k)$ is in the closure of $\mathrm{S}^*(o)$ and of $\mathrm{S}^*(k)$, hence $\mathrm{S}^*(o)$, $\mathrm{S}^*(k)$ both lie in the same topological component of S^*.

Proof of $(**)$. — Let $\mathrm{T}=\mathrm{Spec}\, k[[t]]$. Using the fact that S^* has two branches through $\mathrm{P}(k)$, one for each of the double points of $\mathrm{C}(k)$, we see that there exist two morphisms

$$f_1, f_2 : \mathrm{T}\to \mathrm{S}^*$$

$$f_1(\mathrm{T}_s)=f_2(\mathrm{T}_s)=\mathrm{P}(k), \qquad \mathrm{T}_s=\text{closed point of } \mathrm{T}$$



such that if $\pi_1 : \mathrm{D}_1\to\mathrm{T}$, $\pi_2 : \mathrm{D}_2\to\mathrm{T}$ are the two stable curves over T induced by f_1 and f_2, then: *a)* the closed fibres $\mathrm{D}_{1,s}$, $\mathrm{D}_{2,s}$ are $\mathrm{C}(k)$; *b)* there are sections $s_1 : \mathrm{T}\to\mathrm{D}_1$, $s_2 : \mathrm{T}\to\mathrm{D}_2$ whose images are non-smooth points of π_1, π_2 and such that $s_1(\mathrm{T}_s)$ and $s_2(\mathrm{T}_s)$ are the two double points Q_1 and Q_2 of $\mathrm{C}(k)$ respectively, and: *c)* the generic fibres $\mathrm{D}_{1,\eta}$, $\mathrm{D}_{2,\eta}$ have only one double point (cf. figure). I claim:

A) $\mathrm{D}_{1,\eta}$ is of type (k);

B) $\mathrm{D}_{2,\eta}$ is of type (o).

To prove A), let D'_1 be the result of blowing up the subscheme $s_1(T)$ of D_1. Then D'_1 is still flat and proper over T and its special fibre is the special fibre of D_1 with Q_1 blown up (use the fact that formally at Q_1, D_1 is isomorphic to $k[[t, x, y]]/(x.y)$ with the section given by $x=y=0$). Therefore the special fibre of D'_1 is the disjoint union of $C(k)'$, $C(k)''$, so the general fibre of D'_1 is the disjoint union of two irreducible curves which specialize to $C(k)'$, $C(k)''$ respectively. Since $D_{1,\eta}$ has only one double point, $D'_{1,\eta}$ is non-singular, so $D'_{1,\eta}$ is the disjoint union of two non-singular irreducible curves which must then have the same genera as $C(k)'$ and $C(k)''$, i.e., k, $g-k$. Thus $D_{1,\eta}$ has type (k).

To prove B), it suffices to check that $D_{2,\eta}$ is geometrically irreducible. If not, $D_{2,\eta}$ would have two components meeting at the single point $s_2(T_\eta)$. Since D_2 is smooth over T at each generic point of its special fibre, distinct geometric components of $D_{2,\eta}$ have to have specializations which are distinct components of $(D_2)_s$. Then $(D_2)_s$ would have two components meeting at the point $Q_2=s_2(T_s)$. This is false, so B) is proven.

Now because of A), $f_1(T_\eta)\in S^*(k)$, hence $P(k)=f_1(T_s)$ is in the closure of $S^*(k)$. And because of B), $f_2(T_\eta)\in S^*(o)$, hence $P(k)=f_2(T_s)$ is in the closure of $S^*(o)$ too. Q.E.D. *for* (**).

This completes the proof of Step III since we now see that all irreducible components of S^* are part of the same topological component. Finally, from Steps II and III, we see that

a) S^*, being connected, is part of a single component of H_g, while
b) each component of H_g contains part of S^*. Thus H_g is irreducible, as was to be proven.

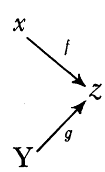
§ 4. Some results on algebraic stacks.

The proofs of the results stated in this section will be given elsewhere.

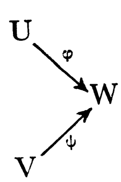
Let C be a category and let $p:\mathscr{S}\to C$ be a category over C. For each $U\in \mathrm{Ob}\, C$, we denote by $\mathscr{S}_U$ the fibre $p^{-1}(U)$. The category $\mathscr{S}$ is *fibered in groupoids* over C if the following two conditions are verified:

a) For all $\varphi: U\to V$ in C and $y\in \mathrm{Ob}\,\mathscr{S}_V$ there is a map $f: x\to y$ in $\mathscr{S}$ with $p(f)=\varphi$.

b) Given a diagram



in $\mathscr{S}$, let



be its image in C. Then for all $\chi : U \to V$ such that $\varphi = \psi\chi$, there is a unique $h : x \to y$ such that $f = g.h$ and $p(h) = \chi$.

Condition *b)* implies that the $f : x \to y$ whose existence is asserted in *a)* is unique up to canonical isomorphism.

Assume that for each $\varphi : U \to V$ in C and each $Y \in \mathrm{Ob}\, \mathscr{S}_V$, such an $f : x \to y$ has been chosen. This x will be written as $\varphi^* y$. Then, φ^* " is " a functor from $\mathscr{S}_V$ to $\mathscr{S}_U$ and if $\varphi\psi$ is a composite morphism in C, the functors $(\varphi\psi)^*$ and $\psi^*\varphi^*$ are canonically isomorphic.

We propose the terminology " stack " for the French word " champ " of non-abelian cohomology (Giraud [G]).

Definition **(4.1)**. — *Let* C *be a category with a Grothendieck topology. We assume products and fibre products exist in* C. *A* stack in groupoids *over* C *is a category over* C, $p : \mathscr{S} \to C$ *such that*:

(i) $\mathscr{S}$ *is fibered in groupoids over* C.

(ii) *For any* $U \in \mathrm{Ob}\, C$ *and any objects* x, y *in* $\mathscr{S}_U$ *the functor from* C/U *to (sets) which to any* $\varphi : V \to U$ *associates* $\mathrm{Hom}_{\mathscr{S}_V}(\varphi^* x, \varphi^* y)$ *is a sheaf.*

(iii) *If* $\varphi_i : V_i \to U$ *is a covering family in* C, *any descent datum relative to the* φ_i, *for objects in* $\mathscr{S}$, *is effective.*

For each $x \in \mathrm{Ob}\, \mathscr{S}_U$, there are given isomorphisms between the inverse images of $x_i = \varphi_i^* x$ and $x_j = \varphi_j^* x$ over $V_{ij} = V_i \times_U V_j$, and the pull-backs of these isomorphisms on $V_{ijk} = V_i \times_U V_j \times_U V_k$ satisfy a " cocycle " condition. In (iii) it is required that reciprocally, any such " descent datum " be defined by some $x \in \mathscr{S}_U$.

In what follows, for the sake of brevity, we will use " stack " to mean " stack in groupoids ".

If $U \in \mathrm{Ob}\, C$ and if $\mathscr{S}$ is a stack over C, the fibre $\mathscr{S}_U$ will be called the category of sections of $\mathscr{S}$ over U.

Let C be as in (4.1). The stacks over C are the objects of a 2-category [B] (stacks/C): 1-morphisms are functors from one stack to another, compatible with the projection into C; and 2-morphisms are morphisms of functors. In this 2-category every 2-morphism is an isomorphism. Products and 2-fibre products exist in this 2-category.

To each $X \in \mathrm{Ob}\, C$ is associated the " representable " stack over C whose category of sections over U is the discrete category whose objects are the morphisms from U to X. This stack will be denoted simply X. For any stack $\mathscr{S}$, the category

$$\mathrm{Hom}(X, \mathscr{S})$$

is canonically equivalent to the category of sections of $\mathscr{S}$ over X. Because of this, $\mathscr{S}$ is sometimes said to " classify " its sections over variable $X \in \mathrm{Ob}\, C$. In the category $\mathrm{Hom}(\mathscr{S}, X)$, all morphisms are identities, i.e., $\mathrm{Hom}(\mathscr{S}, X)$ is just a set.

Let us denote also by C the 2-category having the same objects and morphisms as C, and in which the identities are the only 2-morphisms. The above construction then identifies C with a full sub-2-category of (stacks/C).

For each $S \in \text{Ob } C$, the category C/S satisfies the assumptions of (4.1). Any stack $\mathcal{S}_0$ over C/S (an " S-*stack* ") gives rise to a stack $\mathcal{S}$ over C; a section (φ, η) of $\mathcal{S}$ over $U \in \text{Ob } C$ consists of

(i) a morphism $\varphi : U \to S$;

(ii) a section η of $\mathcal{S}_0$ over (U, φ).

Definition **(4.2)**. — *A* 1*-morphism of stacks over* C, $F : \mathcal{S}_1 \to \mathcal{S}_2$, *will be called* representable *if for any* X *in* C *and any* 1*-morphism* $x : X \to \mathcal{S}_2$, *the fibre product* $X \times_{\mathcal{S}_2} \mathcal{S}_1$ *is a representable stack.*

In down to earth terms, this means the following:

(i) for any $f : Y \to X$ in C, the category whose objects are pairs

$$\{\text{a section, } y, \text{ of } \mathcal{S}_1 \text{ over } Y; \text{ an isomorphism } F(y) \xrightarrow{\sim} f^*(x)\}$$

is equivalent to a category $S(f)$ in which all morphisms are identities;

(ii) the functor $f \mapsto \text{Ob } S(f)$ is representable by some $g : Z \to X$. Such a Z represents the fibre product $X \times_{\mathcal{S}_2} \mathcal{S}_1$.

Let $\boldsymbol{P}$ be a property of morphisms in C, stable by change of base and of a local nature on the target.

Definition **(4.3)**. — *A representable morphism* $F : \mathcal{S}_1 \to \mathcal{S}_2$ *of stacks over* C *has property* $\boldsymbol{P}$ *if for any* 1*-morphism* $x : X \to \mathcal{S}_2$ *the morphism in* C *deduced by base change*: $F' : X \times_{\mathcal{S}_2} S_1 \to X$ *has that property.*

Proposition **(4.4)**. — *Let* $\mathcal{S}$ *be a stack. The diagonal map*

$$\mathcal{S} \to \mathcal{S} \times \mathcal{S}$$

is representable if and only if for all $X, Y \in \text{Ob } C$ *and* 1*-morphisms* $x : X \to \mathcal{S}$, $y : Y \to \mathcal{S}$, *the fibre product* $X \times_{\mathcal{S}} Y$ *is representable.*

If $X \in \text{Ob } C$ and x, y are sections of $\mathcal{S}$ over X, we denote by $\underline{\text{Isom}}(X, x, y)$ the sheaf on C/X which to every Z over X associates the set of isomorphisms between the inverse images of x and y over Z. Then the object representing $\mathcal{S} \times_{(\mathcal{S} \times \mathcal{S})} X$ (the product taken with the map $(x, y) : X \to \mathcal{S} \times \mathcal{S})$ is just $\text{Isom}(X, x, y)$. If $x : X \to \mathcal{S}$, $y : Y \to \mathcal{S}$ are 1-morphisms, then the object representing $X \times_{\mathcal{S}} Y$ is just $\text{Isom}(X \times Y, p_1^* x, p_2^* y)$.

Henceforth, C *will be the category of schemes with the etale topology* (SGAD, IV, (6.3)).

Definition **(4.5)**. — *A stack* $\mathcal{S}$ *is* quasi-separated *if the diagonal morphism from to* $\mathcal{S}$ *to* $\mathcal{S} \times \mathcal{S}$ *is representable, quasi-compact and separated.*

Definition **(4.6)**. — *A stack* $\mathcal{S}$ *is an* algebraic stack [1] *if*

(i) $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ *is representable*;

(ii) *there exists a* 1*-morphism* $x : X \to \mathcal{S}$ *such that for all* $y : Y \to \mathcal{S}$, *the projection morphism* $X \times_{\mathcal{S}} Y \to Y$ *is surjective and etale* (i.e., x *is etale and surjective*).

[1] This definition is the " right " one only for quasi-separated stacks. It will however be sufficient for our purposes.

The 2-category of algebraic stacks contains the representable stacks and is stable under products and fibre products. If $\mathcal{S}$ is a quasi-separated algebraic stack, the diagonal map is unramified and quasi-affine.

Definition **(4.7)**. — *An algebraic stack* $\mathcal{S}$ *is* separated *if* $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ *is proper (or, equivalently, finite). A* 1*-morphism* $f : \mathcal{S}_1 \to \mathcal{S}_2$ *is* separated (resp. quasi-separated) *if for any morphism* $x : X \to \mathcal{S}_2$ *from a separated scheme* X *to* $\mathcal{S}_2$, *the fibre product* $\mathcal{S}_1 \times_{\mathcal{S}_2} X$ *is separated* (resp. *quasi-separated*).

Example **(4.8)**. — Let X be a scheme over S. Let G be a group scheme over S, etale, separated and of finite type over S, which operates on X. We will denote by [X/G] the S-stack whose category of sections over an S-scheme T is the category of principal homogeneous spaces (p.h.s.) E over T, with structural group G (i.e., a p.h.s. under G_T), provided with a G-morphism $\varphi : E \to X$. The principal homogeneous space $G \times X$ over X (G acting only on the first factor) plus the G-morphism $G \times X \to X$ (given by the action of G on X) is a section of [X/G] over X. The corresponding morphism $q : X \to [X/G]$ is etale and surjective, so that [X/G] is an algebraic stack. In addition, X is a principal homogeneous space over [X/G]; the stack [X/G] is representable if and only if X is a principal homogeneous space over a scheme Y, in which case

$$[X/G] \sim Y.$$

If $X = S$, then $[X/G] = [S/G]$ might be called the " classifying stack " of G over S.

Example **(4.9)**. — Suppose a stack $\mathcal{S}$ has the property that in each category $\mathcal{S}_X$ the only morphisms are the identity morphisms. Then $\mathcal{S}_X$ is just a set $\mathcal{F}(X) = \text{Ob}\, \mathcal{S}_X$, and this set, under pull-back, is a contravariant functor $\mathcal{F}$ in X. Conditions (ii) and (iii) of (4.1) assert that the functor $\mathcal{F}$ is a sheaf on C. Artin and Knutson [K] have defined an *algebraic space* to be a sheaf $\mathcal{F}$ such that:

(i) for any morphisms $X \to \mathcal{F}$, $Y \to \mathcal{F}$ of representable functors to $\mathcal{F}$, the fibre product $X \times_{\mathcal{F}} Y$ is representable;

(ii) there exists a morphism $X \to \mathcal{F}$, represented by surjective, etale morphisms of schemes.

This is exactly what we have called an algebraic stack in this case.

Definition **(4.10)**. — *Let* $\mathcal{S}$ *be an algebraic stack. The* etale site $\mathcal{S}_{et}$ *of* $\mathcal{S}$ *is the category with objects the etale morphisms*

$$x : X \to \mathcal{S}$$

and where a morphism from (X, x) *to* (Y, y) *is a morphism of schemes* $f : X \to Y$ *plus a* 2*-morphism between the* 1*-morphism* $x : X \to \mathcal{S}$ *and* $y \cdot f : X \to \mathcal{S}$. *A collection of morphisms* $f_i : (X_i, x_i) \to (X, x)$ *is a covering family if the underlying family of morphisms of schemes is surjective.*

The site $\mathcal{S}_{et}$ is in a natural way ringed. When we speak of sheaves on $\mathcal{S}$ we mean sheaves on $\mathcal{S}_{et}$.

We now explain how many concepts from the theory of schemes may be applied to algebraic stacks.

Let $\boldsymbol{P}$ be a property of morphisms of schemes, stable by etale change of base, and of a local nature (for the etale topology) on the target.

For instance: being an open immersion with dense image, being dominant, birational...

A representable morphism of algebraic stacks $f: T_1 \to T_2$, is said to have property $\boldsymbol{P}$ if for one (and hence for every) surjective etale morphism $x: X \to T_2$, the morphism of schemes deduced by base change $f': X \times_{T_2} T \to X$ has that property.

Let $\boldsymbol{P}$ be a property of morphisms of schemes, which, at source and target, is of a local nature for the etale topology. This means that, for any family of commutative squares

$$\begin{array}{ccc} X_i & \xrightarrow{g_i} & X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ Y_i & \xrightarrow{h_i} & Y \end{array}$$

where the g_i (resp. h_i) are etale and cover X (resp. Y):

$$\boldsymbol{P}(f) \Leftrightarrow \forall_i \boldsymbol{P}(f_i).$$

For instance: f flat, smooth, etale, unramified, normal, locally of finite type, locally of finite presentation.

If $f: T_1 \to T_2$ is a morphism of algebraic stacks, we say f has property $\boldsymbol{P}$ if for one, then necessarily for every, commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{x} & T_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{y} & T_2 \end{array}$$

where X and Y are schemes and x, y are etale and surjective, f' has property $\boldsymbol{P}$.

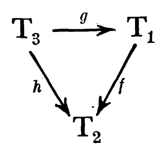
Similarly, if $\boldsymbol{P}$ is a property of schemes, of a local nature for the etale topology, an algebraic stack T will be said to have property $\boldsymbol{P}$ if for one (and hence for every) surjective etale morphism $x: X \to T$, X has property $\boldsymbol{P}$. This applies to, for instance, the properties of being regular, normal, locally noetherian, of characteristic p, reduced, Cohen-Macaulay...

An algebraic stack T will be called *quasi-compact* if there exists a surjective etale morphism $x: X \to T$ with X quasi-compact. A morphism $f: T_1 \to T_2$ of algebraic stacks will be called *quasi-compact* if for any quasi-compact scheme X over T_2, the fiber product $T_1 \times_{T_2} X$ is quasi-compact. It is enough to test the condition for a surjective family $f_i: X_i \to T_2$. We define a morphism $f: T_1 \to T_2$ to be of *finite type*, if it is quasi-compact, and locally of finite type; of *finite presentation*, if it is quasi-compact, quasi-separated, and locally of finite presentation. An algebraic stack is *noetherian*, if it is quasi-compact, quasi-separated, and locally noetherian.

The key point in what follows will be the definition of a " proper morphism " and the analogue of Chow's lemma.

A morphism $f: T_1 \to T_2$ is said to have a property ***P***, locally on T_2, if there exists a surjective etale morphism $x: X \to T_2$ such that the morphism f' deduced from f by the change of base by x has property ***P***.

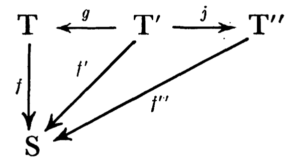
Definition (**4.11**). — *A morphism* $f: T_1 \to T_2$ *is* proper *if it is separated, of finite type and if, locally over* T_2, *there exists commutative diagrams*



with g surjective and h representable and proper.

The following form of Chow's Lemma will be sufficient for our purposes.

Theorem (**4.12**). — *Let* S *be a noetherian scheme and f be a morphism from an etale site* T *to* S. *We assume f to be separated and of finite type. Then, there exists a commutative diagram*



in which T′ *and* T″ *are schemes and such that*

(i) *g is proper, surjective and generically finite*;

(ii) *j is an open immersion*;

(iii) *f″ is projective.*

Using (4.12), it is easy to extend the cohomological theory of coherent sheaves to the present situation. In fact if $f: T_1 \to T_2$ is a proper morphism of noetherian algebraic stacks and if $\mathscr{F}$ is a coherent sheaf on T_1, then the $R^i f_*(\mathscr{F})$ are coherent sheaves on T_2.

However, the $R^i f_*(\mathscr{F})$ don't need to be zero for i large enough. Let S be a scheme and G a finite group. We denote by p the projection $p: [S/G] \to S$. Quasi-coherent (resp. coherent) sheaves of modules on [S/G] may (and will) be identified with quasi-coherent (resp. coherent) sheaves of modules on S, on which G acts. One has

$$R^i p_*(\mathscr{F}) \cong H^i(G, \mathscr{F}).$$

In general, the (quasi-coherent sheaf) cohomology of algebraic stacks appears as a mixture of finite group cohomology and of scheme cohomology.

The *disjoint sum* T of a family $(T_i)_{i \in I}$ of stacks is the stack a section of which over a scheme X consists of

(i) a decomposition $X = \coprod_i X_i$ of X;

(ii) a section of T_i over X_i for each i.

The *void stack* $\emptyset$ is the one represented by the void scheme.

A stack is *connected* if it is non-void and is not the disjoint sum of two non-void stacks.

Proposition **(4.13)**. — *A locally noetherian algebraic stack is in one and only one way the disjoint sum of a family of connected algebraic stacks* (*called its* connected components).

We denote by $\pi_0(T)$ the set of connected components of locally noetherian algebraic stack T. If $x: X \to T$ is surjective and etale, $\pi_0(T)$ is the cokernel of the two maps

$$\pi_0(X \times_T X) \rightrightarrows \pi_0(X) \to \pi_0(T).$$

Proposition **(4.14)**. — *Let* T *be an algebraic stack of finite type over a field k. Then,* T *is connected if and only if there exists a connected scheme* X, *of finite type over k, and a surjective morphism from* X *to* T.

An *open subset* U of an algebraic stack T is a full subcategory $U \subset T$ which is an algebraic stack, which contains together with any $t \in \mathrm{Ob}(T)$ all isomorphic t' and such that the inclusion $j: U \to T$ is representable by open immersions. The open subsets of T corresponds bijectively to the open subsets of its etale site.

For each open subset U of T, there exists one and only one full subcategory $T-U$ of T, which is an algebraic stack, which contains together with any $t \in \mathrm{Ob}(T)$ all isomorphic t' and such that

(i) $T-U$ is reduced;

(ii) the inclusion map $i: T-U \to T$ is representable by closed immersions;

(iii) for any etale surjective morphism $x: X \to T$, the inverse image of $T-U$ on X is the complement of the inverse image of U.

An algebraic stack F in T satisfying (i) and (ii) is a *closed subset* of T, and the functor $U \mapsto T-U$ is an isomorphism of the set of open and the set of closed subsets of T. If F satisfies only (ii), F_{red} satisfies (i) and (ii) so that F defines a closed subset of T.

An algebraic stack T is *irreducible* if it is not the union of two closed subsets, non void and distinct from T.

Proposition **(4.15)**. — *A noetherian algebraic stack* T *is in one and only one way the union of irreducible closed subsets, none of which contains any other. They are called the* irreducible components *of* T. *If* U *is an open dense subset of* T, *the irreducible components of* U *are the non-void intersections of* U *with the irreducible components of* T.

Each irreducible component of T is contained in a connected component of T. Conversely:

Proposition **(4.16)**. — *The connected components of a normal noetherian algebraic stack are irreducible.*

Theorem **(4.17)**. — *Let f be a morphism of finite type from an algebraic stack* T *to a noetherian scheme* S. *For* $s \in S$, *let* $n(s)$ *be the number of connected components of the geometric fibre of* T *at s. Then*

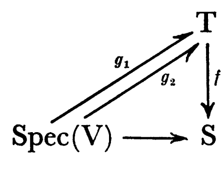
(i) $n(s)$ *is a constructible function of s*;

(ii) *if f is proper and* flat, *then $n(s)$ is lower-semi-continuous;*

(iii) *if f is proper flat, and has geometrically normal fibres, then $n(s)$ is constant.*

Let $f: \mathrm{T} \to \mathrm{S}$ be a morphism of finite type from an algebraic stack T to a noetherian scheme S. Assume that the diagonal map $\mathrm{T} \to \mathrm{T} \times_{\mathrm{S}} \mathrm{T}$ is separated and quasi-compact.

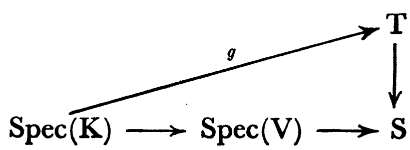
Theorem **(4.18)** (Valuative criterion for separation.) — *The morphism f is separated if and only if, for any complete discrete valuation ring with algebraically closed residue field and any commutative diagram*



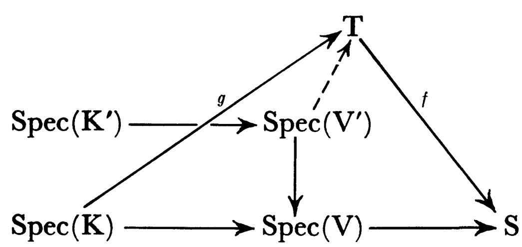
any isomorphism between the restrictions of g_1 and g_2 to the generic point of Spec(V) *can be extended to an isomorphism between g_1 and g_2.*

This criterion is nothing other but the valuative criterion of properness (EGA, II, 7.3.8) applied to the (representable) diagonal morphism.

Theorem **(4.19)** (Valuative criterion for properness.) — *If f is separated, then f is proper if and only if, for any discrete valuation ring* V *with field of fractions* K *and any commutative diagram*



there exists a finite extension K′ *of* K *such that g extends to* Spec(V′), *where* V′ *is the integral closure of* V *in* K′



To prove a given f is proper, it suffices to verify the above criterion under the additional hypothesis that V is complete and has an algebraically closed residue field. Further, given a dense open subset U of T, it is enough to test only g's which factor through U.

Proposition **(4.20)**. — *Let $\mathscr{S}$ be an algebraic stack. The functor which, to any algebraic stack over $\mathscr{S}$, $f: \mathscr{E} \to \mathscr{S}$, associates the $\mathcal{O}_{\mathscr{S}}$ sheaf of algebras $f_* \mathcal{O}_{\mathscr{E}}$ induces an equivalence of categories between:*

(i) *the category of algebraic stacks representable and affine over $\mathscr{S}$;*

(ii) *the dual of the category of quasi-coherent $\mathcal{O}_{\mathscr{S}}$-algebras.*

Let $\mathscr{A}$ be a quasi-coherent sheaf of $\mathcal{O}_{\mathscr{S}}$-algebras on an algebraic stack $\mathscr{S}$. For each etale morphism $x : \mathrm{U} \to \mathscr{S}$, with U affine, let $\mathscr{A}'(\mathrm{U})$ be the integral closure of $\Gamma(\mathrm{U}, \mathcal{O}_{\mathscr{S}})$ in $\mathscr{A}(\mathrm{U})$. By (EGA, II, 6.3.4), the $\mathscr{A}'(\mathrm{U})$ for variable U are the sections over U of a quasi-coherent sheaf $\mathscr{A}'$ on $\mathscr{S}$, which will be called the *integral closure* of $\mathcal{O}_{\mathscr{S}}$ in $\mathscr{A}$.

Let $f : \mathscr{E} \to \mathscr{S}$ be representable and affine. The algebraic stack which is associated by (4.20) to the integral closure of $\mathcal{O}_{\mathscr{S}}$ in $f_* \mathcal{O}_{\mathscr{E}}$ will be called the *normalization of* $\mathscr{S}$ *with respect to* $\mathscr{E}$. Its formation is compatible with any etale change of basis.

Theorem **(4.21)**. — *Let* $\mathscr{S}$ *be a quasi-separated stack over a noetherian scheme* S. *Assume that*

(i) *the diagonal map* $\mathscr{S} \to \mathscr{S} \times_{\mathrm{S}} \mathscr{S}$ *is representable and unramified;*

(ii) *there exists a scheme* X *of finite type over* S *and a smooth and surjective* S*-morphism from* X *to* $\mathscr{S}$.

Then, $\mathscr{S}$ *is an algebraic stack of finite type over* S.

M. Artin has developed powerful methods to relate pro-representability of a stack to the existence of etale surjective maps $x : \mathrm{X} \to \mathscr{S}$.

§ 5. Second proof of the irreducibility theorem.

Let $\mathscr{M}_g$ $(g \geq 2)$ be the stack whose category of sections over a scheme S is the category of stable curves of genus g over S, the morphisms being the isomorphisms of schemes over S. By (1.11), the diagonal morphism $\Delta : \mathscr{M}_g \to \mathscr{M}_g \times \mathscr{M}_g$ is representable, finite and unramified.

We saw in § 1 that the stack classifying the tricanonically embedded stable curves of genus g is represented by a scheme H_g, smooth and of finite type over Spec($\mathbf{Z}$). The " forgetful " morphism

$$\mathrm{H}_g \to \mathscr{M}_g$$

is representable, smooth and surjective. Indeed, if $p : \mathrm{C} \to \mathrm{S}$ is a stable curve over S, defining a morphism

$$\gamma : \mathrm{S} \to \mathscr{M}_g,$$

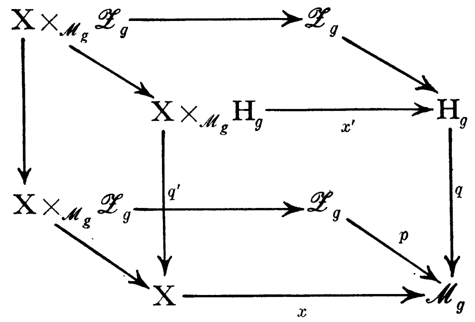
then the fibre product $\mathrm{H}_g \times_{\mathscr{M}_g} \mathrm{S}$ is the scheme, smooth over S, of isomorphisms between the standard projective space of dimension $5g-6$ over S and $\mathbf{P}(p_*(\omega_{\mathrm{C/S}}^{\otimes 3}))$. We deduce from this and (4.21) that:

Proposition **(5.1)**. — $\mathscr{M}_g$ *is a separated algebraic stack of finite type over* Spec($\mathbf{Z}$).

Let us denote by $\mathscr{M}_g^0$ the open subset of $\mathscr{M}_g$ which " consists of " smooth curves, and by $\mathscr{Z}_g$ the " universal curve " over $\mathscr{M}_g$, the algebraic stack classifying pointed stable curves.

Theorem **(5.2)**. — *The algebraic stacks* $\mathscr{M}_g$ *and* $\mathscr{Z}_g$ *are proper and smooth over* Spec($\mathbf{Z}$) *and the complement of* $\mathscr{M}_g^0$ *in* $\mathscr{M}_g$ *is a divisor with normal crossings relative to* Spec($\mathbf{Z}$).

Proof. — Let $x : \mathrm{X} \to \mathcal{M}_g$ be etale and consider the following commutative diagram of algebraic stacks of finite type over Spec($\mathbf{Z}$):



In this diagram, the four horizontal arrows are etale and the four vertical arrows are smooth and surjective. As H_g and Z_g (p. 78). are smooth over Spec($\mathbf{Z}$), so are X and $\mathrm{X} \times_{\mathcal{M}_g} \mathcal{Z}_g$. In addition, $q'^{-1} x^{-1}(\mathcal{M}_g^0) = x'^{-1}(\mathrm{H}_g^0)$ is the complement of a divisor with normal crossings relative to Spec($\mathbf{Z}$), so that the inclusion of $x^{-1}(\mathcal{M}_g)$ in X has the same property. This being true for any x, $\mathcal{M}_g$ and $\mathcal{Z}_g$ are smooth over Spec($\mathbf{Z}$) and $\mathcal{M}_g^0$ is the complement of a divisor with normal crossings relative to Spec($\mathbf{Z}$). In particular $\mathcal{M}_g^0$ is dense in $\mathcal{M}_g$.

We may now use the valuative criterion for properness in its modified form (4.19) to deduce the properness of $\mathcal{M}_g$ from the stable reduction theorem. The properness of $\mathcal{Z}_g$ then results from that of p.

(5.3) Let $p : \mathrm{X} \to \mathrm{S}$ be a stable smooth curve of genus $g \geq 2$ over S. If $k \in \mathbf{N}$ is invertible in $\mathcal{O}_\mathrm{S}$, the sheaf $\mathrm{R}^1 p_*(\mathbf{Z}/k\mathbf{Z})$ on S_{et} is locally free over $\mathbf{Z}/k\mathbf{Z}$, of rank $2g$, and the cup-product is a non degenerate alternating form

$$\mathrm{R}^1 p_*(\mathbf{Z}/k\mathbf{Z}) \otimes \mathrm{R}^1 p_*(\mathbf{Z}/k\mathbf{Z}) \to \mathrm{R}^2 p_*(\mathbf{Z}/k\mathbf{Z}) \sim \boldsymbol{\mu}_k^{\otimes -1}.$$

Locally on S_{et}, $\boldsymbol{\mu}_k$ is isomorphic to $\mathbf{Z}/k\mathbf{Z}$, and thus, locally, $\mathrm{R}^1 p_*(\mathbf{Z}/k\mathbf{Z})$ is provided with a non degenerate symplectic form with values in $\mathbf{Z}/k\mathbf{Z}$, which is determined up to an unit in $\mathbf{Z}/k\mathbf{Z}$, or, as we shall say, $\mathrm{R}^1 p_*(\mathbf{Z}/k\mathbf{Z})$ is provided with an homogeneous symplectic structure. The constant sheaf $(\mathbf{Z}/k\mathbf{Z})^{2g}$ will be provided with the homogeneous symplectic structure induced by the standard symplectic structure of $(\mathbf{Z}/k\mathbf{Z})^{2g}$.

Definition **(5.4)**. — *A* Jacobi structure of level *k on* X *is an isomorphism (respecting their homogeneous symplectic structures) between* $\mathrm{R}^1 p_*(\mathbf{Z}/k\mathbf{Z})$ *and* $(\mathbf{Z}/k\mathbf{Z})^{2g}$.

(5.5) Given a section s of p, and a set of prime numbers **P** including all residue characteristics of S, the specialisation theorem for the fundamental group enables one to construct a pro-objet $\pi_1(\mathrm{X}/\mathrm{S}, s)^{(\mathbf{P})}$ of the category (l.c. $\mathrm{gr}^{(\mathbf{P})}$) of locally constant sheaves of finite groups of order prime to **P**, with the properties

(i) $\mathit{Hom}_\mathrm{S}(\pi_1(\mathrm{X}/\mathrm{S}, s)^{(\mathbf{P})}, \mathrm{G}) = \mathrm{R}^1 p_*(\mathrm{X} \bmod s, p^*\mathrm{G}) = \mathrm{R}^1 p_*(\mathrm{Ker}(p^*\mathrm{G} \to s_*\mathrm{G}))$ functorially in $\mathrm{G} \in (\mathrm{l.c.\ gr}^{(\mathbf{P})})$;

(ii) the formation of $\pi_1(\mathrm{X}/\mathrm{S}, s)^{(\mathbf{P})}$ is compatible with any change of base.

If G and H are two sheaves of groups on S_{et}, we define the sheaf of exterior morphisms from H to G, $Hom^{ext}(H, G)$, as the quotient of $Hom(H, G)$ by the action of H induced by its action on itself by inner automorphism.

The sheaf $$Hom_S^{ext}(\pi_1(X/S, s)^{(\mathbf{P})}, G) \sim R^1 p_*(p^* G) \qquad (\text{for } G \in (\text{l.c. gr}^{(\mathbf{P})}))$$

is " independent " of the choice of s.

We shall denote it as

$$Hom_S^{ext}(\pi_1(X/S)^{(\mathbf{P})}, G).$$

As $p : X \rightarrow S$ admits sections locally for the etale topology, this sheaf makes sense without assuming p to have a global section. It is independent of the choice of **P**, so long as **P** is prime to the order of G and includes all residue characteristics of S.

Definition **(5.6)**. — *Let* G *be a finite group of order n prime to* **P**. *A* Teichmüller structure of level G *on* X *is a surjective exterior homomorphism from* $\pi_1(X/S)$ *to* G.

The finite generation of $\pi_1(X/S)$ (SGA, 60/61, exp. 10) implies:

Lemma **(5.7)**. — *The sheaf on* (Sch/S) *of the Teichmüller structures of level* G *on* X *is represented by an etale covering of* S.

We denote by ${}_G\mathcal{M}_g^0$ the stack classifying the stable smooth curves of genus g and characteristic prime to n, with a Teichmüller structure of level G.

For any algebraic stack $\mathcal{M}$, we denote by $\mathcal{M}[1/n]$ its open subset $\mathcal{M} \times \mathrm{Spec}(\mathbf{Z}[1/n])$. Lemma (5.7) may now be rephrased:

Proposition **(5.8)**. — *The " forgetful " morphism*

$${}_G\mathcal{M}_g^0 \rightarrow \mathcal{M}_g^0[1/n]$$

is representable, finite and etale.

The stack ${}_G\mathcal{M}_g^0$ thus in an algebraic stack. Let ${}_G\mathcal{M}_g$ be the normalisation of $\mathcal{M}_g[1/n]$ with respect to ${}_G\mathcal{M}_g^0$. The stack ${}_G\mathcal{M}_g$, being representable and finite over $\mathcal{M}_g[1/n]$, is proper over $\mathrm{Spec}(\mathbf{Z}[1/n])$.

Theorem **(5.9)**. — *The geometric fibres of the projection of* ${}_G\mathcal{M}_g$ *onto* $\mathrm{Spec}(\mathbf{Z}[1/n])$ *are normal, and, fibre by fibre,* ${}_G\mathcal{M}_g^0$ *is dense in* ${}_G\mathcal{M}_g$.

Proof. — We will use Abhyankar-Artin's lemma, in its " absolute " form:

Lemma **(5.10)**. — *Let* D *be a divisor with normal crossings on an excellent regular scheme* X, Y *an etale covering of* X—D *and* $\overline{Y}$ *the normalisation of* X *with respect to* Y. *Assume that the generic points of the irreducible components of* D *are all of characteristic* 0. *Then every geometric point of* X *has an etale neighbourhood* $x : X' \rightarrow X$ *such that, on* X':

(i) D *becomes a union of regular divisors* $(D_i)_{i \in I}$, D_i *of equation* $t_i = 0$.

(ii) *There exists an integer k prime to residue characteristics of* X' *such that* $\overline{Y}$ *becomes isomorphic to a disjoint union of quotients (by subgroups of* μ_k^I*) of the covering of* X' *obtained by extracting the* k^{th}*-roots of the* t_i*'s.*

If $x : X \rightarrow \mathcal{M}_g\left[\frac{1}{n}\right]$ is an etale morphism, then $X^0 = x^{-1}(\mathcal{M}_g)$ is the complement in X of a divisor with normal crossing relative to Spec(**Z**), $X_1^0 = X \times_{\mathcal{M}_g} ({}_G\mathcal{M}_g^0)$ is an etale covering of X^0 and $X_1 = S \times_{\mathcal{M}_g} ({}_G\mathcal{M}_g)$ is the normalisation of X with respect to this covering of X^0.

By the explicit local description (5.10), for any prime number l prime to n, $X_1 \times \mathrm{Spec}(\overline{\mathbf{F}}_l)$ is the normalisation of $X \times \mathrm{Spec}(\overline{\mathbf{F}}_l)$ with respect to $X_1 \times \mathrm{Spec}(\overline{\mathbf{F}}_l)$. As this is true for any modular family, we get (5.9).

Corollary **(5.11)**. — *The geometric fibres of the projection*

$${}_G\mathcal{M}_g^0 \to \mathrm{Spec}\left(\mathbf{Z}\left[\frac{1}{n}\right]\right)$$

all have the same number of connected components.

Proof. — By (4.17), all geometric fibres ${}_G\mathcal{M}_g \times \mathrm{Spec}(\overline{\mathbf{F}}_l)$ of ${}_G\mathcal{M}_g$ over $\mathrm{Spec}(\mathbf{Z}[1/n])$ have the same number of connected components. These connected components are irreducible (4.16). Furthermore ${}_G\mathcal{M}_g^0 \times \mathrm{Spec}(\overline{\mathbf{F}}_l)$ is dense in ${}_G\mathcal{M}_g \times \mathrm{Spec}(\overline{\mathbf{F}}_l)$ and thus has the same number of connected (or irreducible) components as ${}_G\mathcal{M}_g \times \mathrm{Spec}(\overline{\mathbf{F}}_l)$, and this number is independent of l.

(5.12) Let us denote by Π the fundamental group of an oriented closed differentiable surface S_0 of genus g. The group Π may be defined by generators and relations as follows:

(i) generators: x_i for $1 \leq i \leq 2g$;

(ii) relation: $(x_1, x_{g+1}) \ldots (x_i, x_{g+i}) \ldots (x_g, x_{2g}) = e$, where $(a, b) = aba^{-1}b^{-1}$.

Let (e_i) be the standard basis of $\mathbf{Z}^{2g}$. The morphism φ from Π to $\mathbf{Z}^{2g}$ with $\varphi(X_i) = e_i$ identifies $\Pi/(\Pi, \Pi) = H_1(\Pi, Z)$ with $\mathbf{Z}^{2g}$.

The surface S_0 is a $K(\Pi, 1)$ and thus

$$H^2(\Pi, \mathbf{Z}) = H^2(S_0, \mathbf{Z}) = \mathbf{Z},$$

and the cup product defines a symplectic structure on $\Pi/(\Pi, \Pi)$. This structure is identified by φ with the standard symplectic structure of $\mathbf{Z}^{2g}$.

We denote by $\mathrm{Aut}^0(\Pi)$ the subgroup of $\mathrm{Aut}(\Pi)$ which acts trivially on $H^2(\Pi, \mathbf{Z})$. Dehn has proved that each exterior automorphism of Π is induced by a diffeomorphism of S_0 onto itself and that the map induced by φ:

$$\mathrm{Aut}^0(\Pi) \to \mathbf{Sp}_{2g}(\mathbf{Z}),$$

is surjective (see[Ma]).

Theorem **(5.13)**. — *The number of connected components of any geometric fibre of the projection of* ${}_G\mathcal{M}_g^0$ *onto* $\mathrm{Spec}(\mathbf{Z}[1/n])$ *is equal to the number of orbits of* $\mathrm{Aut}^0(\Pi)$ *in the set of exterior epimorphisms from* Π *to* G.

Proof. — By (5.13), it suffices to prove that ${}_G\mathcal{M}_g^0 \times \mathrm{Spec}(\mathbf{C})$ has the said number of connected components. As results from (4.14),

$$\pi_0({}_G\mathcal{M}_g^0 \times \mathrm{Spec}(\mathbf{C})) = \pi_0({}_G M_g)$$

where ${}_G M_g$ is the coarse moduli scheme classifying stable smooth curves of genus g over $\mathbf{C}$ with a Teichmüller structure of level G.

Recall that a *Teichmüller curve* of genus g is a stable smooth curve C of genus g over $\mathbf{C}$ together with an exterior isomorphism φ of the transcendental fundamental group $\pi_1(C)$ with Π, which induces a symplectic isomorphism [1] between

$$H_1(C, \mathbf{Z}) \sim \pi_1(C)/(\pi_1(C), \pi_1(C))$$

and $\Pi/(\Pi, \Pi)$. By Teichmüller's theory [W], the analytic space T_g classifying Teichmüller curves of a given genus $g \geq 2$ is homeomorphic to a ball, and hence connected.

If ψ is a surjective homomorphism from Π to G, the map

$$(C, \varphi) \mapsto (C, \psi\varphi)$$

defines a morphism $t_\psi : T_g \to {}_G M_g$. Two such maps t_ψ and $t_{\psi'}$ have the same image if and only if $\psi = \psi'\sigma$ for $\sigma \in \mathrm{Aut}^0(\Pi)$, and

$${}_G M_g = \coprod_{\psi \bmod \mathrm{Aut}^0(\Pi)} t_\psi(T_g)$$

which implies (5.13).

(5.14) Let us denote by ${}_n\mathcal{M}_g^0$ the algebraic stack classifying stable smooth curves with a Jacobi structure of level n. This algebraic stack " is " a true scheme for $n \geq 3$ (by [S]).

If φ is a Jacobi structure of level n on a stable smooth curve $p : X \to S$, we define the " multiplicator " $\mu(\varphi)$ of φ by the commutative diagram

$$\begin{array}{ccc} \wedge^2(\mathbf{Z}/n\mathbf{Z})^{2g} & \longrightarrow & \mathbf{Z}/n\mathbf{Z} \\ \Big\downarrow {\scriptstyle \wedge^2\varphi} & & \Big\downarrow {\scriptstyle \mu(\varphi)} \\ \wedge^2 R^1 p_*(\mathbf{Z}/n\mathbf{Z}) & \xrightarrow{\ \wedge\ } & \mu_n^{\otimes -1} \sim R^2 p_*(\mathbf{Z}/n\mathbf{Z}) \end{array}$$

The scheme of isomorphisms between $\mathbf{Z}/n\mathbf{Z}$ and $\mu_n^{\otimes -1}$ is $\mathrm{Spec}(\mathbf{Z}[e^{2\pi i/n}, 1/n])$ thus $\varphi \to \mu(\varphi)$ defines a morphism μ from S to $\mathrm{Spec}(\mathbf{Z}[e^{2\pi i/n}, 1/n])$. This being true for any X and S, μ is induced by

$$\mu : {}_n\mathcal{M}_g^0 \to \mathrm{Spec}(\mathbf{Z}[e^{2\pi i/n}, 1/n]).$$

Theorem **(5.15)**. — *The geometric fibres of the morphism μ are connected.*

Proof. — By definition, ${}_n\mathcal{M}_g^0$ is open and closed in ${}_G\mathcal{M}_g^0$ for $G = (\mathbf{Z}/n\mathbf{Z})^{2g}$. The group $\mathbf{GL}_{2g}(\mathbf{Z}/n\mathbf{Z})$ acts on G, and thus on ${}_G\mathcal{M}_g^0$. One has:

(i) the open subset ${}_n\mathcal{M}_g^0$ of ${}_G\mathcal{M}_g^0$ is stable under the subgroup $H = \mathbf{CSp}_{2g}(\mathbf{Z}/n\mathbf{Z})$ of symplectic similitudes;

(ii) ${}_G\mathcal{M}_g^0 = \coprod_{G/H} \sigma({}_n\mathcal{M}_g^0)$.

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[1] An arbitrary isomorphism φ induces an isomorphism betwen $H_1(C, Z)$ and $\Pi/(\Pi, \Pi)$ which is always symplectic up to sign.

It now results from (5.11) that all geometric fibres of μ have the same number of connected components.

Consider the geometric fibre of μ at the standard complex place of $\mathrm{Spec}(\mathbf{Z}[e^{2\pi i/n}, 1/n])$. This fibre is the algebraic stack classifying complex stable smooth curves C provided with a *symplectic* isomorphism

$$\mathrm{H}^1(\mathrm{C}, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} (\mathbf{Z}/n\mathbf{Z})^{2g}.$$

Reasoning as in (5.13), we are reduced to proving

Lemma **(5.16)**. — *The homomorphism*

$$\mathrm{Aut}^0(\Pi) \to \mathbf{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$$

is surjective.

Proof. — This results from Dehn's theory (5.12) and from the fact that the groups $\mathbf{Sp}_n$, being split semi-simple simply connected groups, are generated by their unipotent elements, so that the reduction map

$$\mathbf{Sp}_{2g}(\mathbf{Z}) \to \mathbf{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$$

is surjective.

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