Random Models and Solvable Skolem Classes

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RANDOM MODELS AND THE MASLOV CLASS

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In [GS] Gurevich and Shelah introduce a novel method for proving that every satisfiable formula in the Gödel class has a finite model (the Gödel class is the class of prenex formulas of pure quantification theory with prefixes $\forall\forall\exists\cdots\exists$). They dub their method "random models": it proceeds by delineating, given any F in the Gödel class and any integer p, a set of structures for F with universe $\{1,\ldots,p\}$ that can be treated as a finite probability space S. They then show how to calculate an upper bound on the probability that a structure chosen at random from S makes F false; from this bound they are able to infer that if p is sufficiently large, that probability will be less than one, so that there will exist a structure in S that is a model for F. The Gurevich-Shelah proof is somewhat simpler than those known heretofore. In particular, there is no need for the combinatorial partitionings of finite universes that play a central role in the earlier proofs (see [G] and [DG, p. 86]). To be sure, Gurevich and Shelah obtain a larger bound on the size of the finite models, but this is relatively unimportant, since searching for finite models is not the most efficient method to decide satisfiability.

Gurevich and Shelah note that the random model method can be used to treat the Gödel class extended by initial existential quantifiers, that is, the prefix-class $\exists\cdots\exists\forall\forall\exists\cdots\exists$; but they do not investigate further its range of applicability to syntactically specified classes. In fact, a reasonably straightforward generalization of the method suffices for the Skolem classes of [DG, Chapter 3], which are classes of prenex formulas with prefixes $\forall\cdots\forall\exists\cdots\exists$ and matrices restricted in the sets of variables that may appear together in atomic subformulas.

The aim of this paper is to present a less straightforward extension of the random model method, to the Maslov class. To describe this class, we define a *signed atomic formula* to be an atomic formula or the negation of one, a *binary disjunction* to be a disjunction $(A \vee B)$ of two signed atomic formulas, and a *Krom formula* to be a conjunction of binary disjunctions. The Maslov class is the class of prenex formulas with prefixes $\forall\cdots\forall\exists\cdots\exists$ and matrices that are Krom formulas. The Maslov class was first shown decidable by Maslov [M]; that every satisfiable Maslov formula has a finite model was first shown by Aanderaa and the author [AG]. The latter

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proof, and the somewhat different one in [DG, Chapter 4, §2], are rather complex. Here the random models method provides an impressive gain in perspicuity and brevity.

The extension of Gurevich and Shelah's method to Maslov formulas involves a key new feature. Their probability measure assigns equal probability to each structure in the space S; thus the probability of a set of structures is simply the ratio of its cardinality to the cardinality of S. At bottom, then, their proof is a counting argument: a bound is calculated on the number of structures in S that make the given formula false; if the size of the structures is sufficiently large, this bound is less than the number of structures in S. In the proof below, however, not all structures in the space receive the same probability. In this way, the argument involves a more intrinsic use of probability.

The proof below splits into two parts. In §1 we present the logical features of Maslov formulas that are needed; in §2 we describe the probability space and calculate the relevant bounds on probabilities.

§1. Let $F = \forall x_1 \cdots \forall x_m \exists x_{m+1} \cdots \exists x_{m+n} G(x_1, \ldots, x_{m+n})$ be a satisfiable Maslov formula, $m, n \geq 1$. We assume that the predicate letters in F are at most m-adic; otherwise, dummy universal quantifiers can be added. We allow F to contain 0-adic predicate letters, that is, sentence letters. In what follows, by "atomic formula" we shall mean only those that contain a predicate letter of F. We assume that F has the following projection property:

> For each i, $1 \leq i \leq m$, and each signed atomic formula D containing exactly the variables $x_1, \ldots, x_i$, there exists an atomic formula C containing the variables $x_1, \ldots, x_{i-1}$ such that F logically implies the universal closure of $(C \equiv \exists x_i D)$.

Any Maslov formula can be transformed into one possessing the projection property by appropriate additions to the matrix. A signed atomic formula D containing $x_1, \ldots, x_i$ is treated by introducing a new $(i-1)$-adic predicate letter P_D, conjoining $(\sim D \vee P_D x_1 \cdots x_{i-1}).(\sim P_D x_1 \cdots x_{i-1} \vee \exists x_i D)$ to the matrix, relabelling the existentially quantified x_i as a new variable, and prenexing. Any model for the resulting formula is a model for the original one; conversely, any model for the original becomes a model for the new formula once P_D is interpreted as an appropriate projection of the extension of the predicate letter in D or the complement of that extension. (Cf. [DG, p. 117].)

A *p-structure* is a structure for the language of F whose universe is $\{1, \ldots, p\}$; a *p-atom* is an atomic formula containing a predicate letter of F and arguments from $\{1, \ldots, p\}$ (we ignore the distinction between the elements $1, \ldots, p$ and names for those elements). If J is a truth-functional combination of p-atoms and ν is a mapping defined on all elements occurring in J, then νJ is the result of replacing each argument of each p-atom in J by its image under ν.

Suppose $\mathbf{N}_0$ is a model for F, with universe N_0. Let Φ be the conjunction of binary disjunctions $(C \vee D)$ whose universal closures are true in $\mathbf{N}_0$, where C and D are signed atomic formulas containing variables among $x_1, \ldots, x_{2m}$; and let Φ_p be the conjunction of instances of Φ over the universe $\{1, \ldots, p\}$. Thus Φ_p is the conjunction of all binary disjunctions $(A \vee B)$ of signed p-atoms such that $\mathbf{N}_0 \models v(A \vee B)$ for every mapping v from $\{1,\ldots,p\}$ to N_0. The conjunction Φ_p is the only information we need to extract from $\mathbf{N}_0$ for the construction of a finite model.

(Incidentally, Φ has the following property, which is central to the solvability proofs for the Maslov class of [M] and [DG, Chapter 4, §1]. Let $\mathbf{M}$ be any model for the universal closure of Φ. Then there is an extension $\mathbf{M}'$ of $\mathbf{M}$ in which the universal closure of Φ remains true and such that $\mathbf{M}' \models \exists x_{m+1} \cdots \exists x_n G[a_1 \cdots a_m]$ for all $a_1,\ldots,a_m$ in $\mathbf{M}$. Iteration of the extension process yields a model for F. The proof below exploits a related property, given in Lemma 4.)

Let J be a truth-functional compound of p-atoms. J is satisfiable if vJ is satisfiable for some mapping v. For, given a truth-assignment making vJ true, we can obtain one making J true by assigning "true" to a p-atom A iff the given assignment assigns "true" to vA. In particular, then $J.\Phi_p$ is satisfiable provided there exists a mapping v from the arguments occurring in J to N_0 such that $\mathbf{N}_0 \models vJ$. For, given such a v, we may extend it arbitrarily to all of $\{1,\ldots,p\}$; then $\mathbf{N}_0 \models v\Phi_p$, since $\mathbf{N}_0 \models v\Phi_p$ for every mapping v. Thus $vJ.v\Phi_p$ is satisfiable, since it is true in $\mathbf{N}_0$; whence $J.\Phi_p$ is satisfiable.

We now prove four fairly simple lemmas. The first gives a general property of Krom formulas, and the second gives a consequence of the projection property. The third, a corollary of the first two, is a crucial underpinning to the construction of the desired probability space. The fourth exploits the condition that the matrix of F is Krom.

LEMMA 1. *Let H be a conjunction of signed atomic formulas and let K be a Krom formula. If $H.K$ is unsatisfiable, then there exist signed atomic formulas A and B that are conjuncts of H such that $A.B.K$ is unsatisfiable.*

PROOF. A truth-functional argument, using a simple induction on the number of conjuncts of the Krom formula K, suffices. See [AG, p. 513] or [DG, p. 109]. ∎

LEMMA 2. *Suppose A and B are signed p-atoms such that $A.B.\Phi_p$ is unsatisfiable. Then there is a p-atom A', whose arguments are common to A and B, such that $A. \sim A'.\Phi_p$ and $B.A'.\Phi_p$ are both unsatisfiable.*

PROOF. If all arguments of A are arguments of B, we may take A' to be A itself. Otherwise, there exist i and j, $0 \leq i < j \leq m$, and a signed atomic formula D containing the variables $x_1,\ldots,x_j$ such that $A = D[q_1,\ldots,q_j]$, where $q_1,\ldots,q_j$ are distinct elements of $\{1,\ldots,p\}$, and $q_1,\ldots,q_i$ are arguments of B but $q_{i+1},\ldots,q_j$ are not. By iteration of the projection property, there exists an atomic formula C containing just the variables $x_1,\ldots,x_i$ such that F logically implies the universal closure of $(C \equiv \exists x_{i+1} \cdots \exists x_j D)$. Hence $(\sim D \vee C)$ is in true in $\mathbf{N}_0$ for all values of $x_1,\ldots,x_j$. Let $A' = C[q_1,\ldots,q_i]$; it follows that $(\sim A \vee A')$ is a conjunct of Φ_p, so that $A.\sim A'.\Phi_p$ is unsatisfiable.

Suppose $B.A'.\Phi_p$ satisfiable. Then $(\sim B \vee \sim A')$ cannot be a conjuct of Φ_p. Hence there is a mapping v from the arguments of B (which include $q_1,\ldots,q_i$) to N_0 such that $\mathbf{N}_0 \models vB.vA'$. Note that vA' is $C[vq_1,\ldots,vq_i]$. Since $(C \supset \exists x_{i+1} \ldots \exists x_j D)$ is true in $\mathbf{N}_0$ for all values of $x_1,\ldots,x_i$, there exist $e_{i+1},\ldots,e_j$ in N_0 such that $\mathbf{N}_0 \models D[vq_1,\ldots,vq_i,e_{i+1} \cdots e_j]$. If v is extended by setting $vq_k = e_k$ for $i < k \leq j$, we have $\mathbf{N}_0 \models D[vq_1,\ldots,vq_j]$, that is, $\mathbf{N}_0 \models vA$. Thus $\mathbf{N}_0 \models vA.vB$, which contradicts the hypothesis that $A.B.\Phi_p$ is unsatisfiable. ∎

If $\mathbf{M}$ is a p-structure and $S \subseteq \{1,\dots,p\}$, then let $(\mathbf{M}|S)$ be the conjunction of signed p-atoms A whose arguments are in S such that $\mathbf{M} \models A$. (In essence, $(\mathbf{M}|S)$ is the diagram of the members of S in the structure $\mathbf{M}$.)

LEMMA 3. *Let $\mathbf{M}$ be a p-structure; let $S_1,\dots,S_k$ be subsets of $\{1,\dots,p\}$ such that $(\mathbf{M}|S_i).\Phi_p$ is satisfiable for each i. Then $(\mathbf{M}|S_1).\cdots.(\mathbf{M}|S_k).\Phi_p$ is satisfiable.*

PROOF. Suppose not. By Lemma 1 there exist signed p-atoms A and B such that A is a conjunct of some $(\mathbf{M}|S_i)$, B is a conjunct of some $(\mathbf{M}|S_j)$, and $A.B.\Phi_p$ is unsatisfiable. By Lemma 2 there exists a p-atom A' whose arguments are common to A and B such that $A.{\sim}A'.\Phi_p$ and $B.A'.\Phi_p$ are both unsatisfiable. If $\mathbf{M} \models {\sim}A'$, then A and ${\sim}A'$ are both conjuncts of $(\mathbf{M}|S_i)$, so that $(\mathbf{M}|S_i).\Phi_p$ is unsatisfiable; but if $\mathbf{M} \models A'$ then B and A' are both conjuncts of $(\mathbf{M}|S_j)$, so that $(\mathbf{M}|S_j).\Phi_p$ is unsatisfiable. In either case the hypothesis is contradicted. ■

LEMMA 4. *Suppose $\mathbf{M}$ is a p-structure in which Φ_p is true. Let $q_1,\dots,q_m \in \{1,\dots p\}$, and let $q_{m+1},\dots,q_{m+n}$ be distinct members of $\{1,\dots,p\} - \{q_1,\dots,q_m\}$. Then $(\mathbf{M}|\{q_1,\dots,q_m\}).G[q_1,\dots,q_{m+n}].\Phi_p$ is satisfiable.*

PROOF. By Lemma 1, it suffices to show that for all signed p-atoms A and B that are conjuncts of $(\mathbf{M}|\{q_1,\dots,q_m\})$, $A.B.G[q_1,\dots,q_{m+n}].\Phi_p$ is satisfiable. Since $\mathbf{M}$ makes both Φ_p and $A.B$ true, $({\sim}A \vee {\sim}B)$ cannot be a conjunct of Φ_p. Hence there is a v such that $\mathbf{N}_0 \models v(A.B)$. We may assume v defined only on the q_i that occur in $A.B$. Extend v arbitrarily to the rest of $q_1,\dots,q_m$. Since $\mathbf{N}_0 \models F$, there exist members $e_{m+1},\dots,e_{m+n}$ of N_0 such that $\mathbf{N}_0 \models G[vq_1,\dots,vq_m,e_{m+1},\dots,e_{m+n}]$. Extend v by setting $vq_{m+i} = e_{m+i}$ for $1 \le i \le n$. Then $\mathbf{N}_0 \models v(A.B.G[q_1,\dots,q_{m+n}])$, which yields the desired conclusion. ■

§2. Let $p > m + n$ be fixed, and let T_p be the set of p-structures in which Φ_p is true. Lemma 3 allows us to treat an arbitrary member $\mathbf{M}$ of T_p as generated by successive choices of truth-value assignments to p-atoms, as follows:

Step 0. Arbitrarily pick values for the sentence letters in such a way that $(\mathbf{M}|\varnothing).\Phi_p$ is satisfiable.

Step i, $0 < i \le m$. Suppose truth-values have been chosen for all p-atoms containing at most $i - 1$ distinct arguments in such a way that $(\mathbf{M}|U).\Phi_p$ is satisfiable for each $U \subseteq \{1,\dots,p\}$ of cardinality $i - 1$. For each $S \subseteq \{1,\dots,p\}$ of cardinality i, proceed thus. Let Δ be the set of possible values H of $(\mathbf{M}|S)$ such that

(a) H extends $(\mathbf{M}|U)$ for each $U \subseteq S$ of cardinality $i - 1$, and

(b) $H.\Phi_p$ is satisfiable.

Δ is not empty, for Lemma 3 and the induction hypothesis yield the satisfiability of the conjunction of Φ_p with all $(\mathbf{M}|U)$ for $U \subseteq S$ of cardinality $i - 1$. Now pick at random an $H \in \Delta$ and then fix the truth-values of the p-atoms whose arguments are precisely the i members of S in such a way that $(\mathbf{M}|S) = H$.

This manner of describing the determination of a random element of T_p is but a picturesque manner of stipulating a probability measure on T_p. This measure is not such that $\mathrm{Prob}[\mathbf{M} = \mathbf{M}_1] = \mathrm{Prob}[\mathbf{M} = \mathbf{M}_2]$ for all $\mathbf{M}_1$ and $\mathbf{M}_2$ in T_p; for at various stages in the process of obtaining $\mathbf{M}_1$ and $\mathbf{M}_2$ there may have been different numbers of alternative choices available. To describe the measure precisely, call two members of T_p *i-equivalent*, where $0 \le i \le m$, iff they agree on all p-atoms that contain at most

i distinct arguments. For $\mathbf{M}$ in T_p, let $\#(0,\mathbf{M})$ be the number of 0-equivalence classes, and let $\#(i+1,\mathbf{M})$ be the number of different $(i+1)$-equivalence classes into which the i-equivalence class of $\mathbf{M}$ splits. Finally, for each $\mathbf{M}_1$ in T_p, let $\mathrm{Prob}[\mathbf{M}=\mathbf{M}_1]$ be $(\prod_{0\le i\le m}\#(i,\mathbf{M}_1))^{-1}$. That is, that a structure chosen at random lie in one 0-equivalence class is equiprobable with its lying in another; given that a structure is in a particular i-equivalence class, the probabilities that it lie in the $(i+1)$-equivalence classes into which the particular i-equivalence class splits are all equal.

We now seek to estimate the probability of the event $\mathbf{M}\models F$.

ESTIMATION LEMMA. *There exists a* δ, $0<\delta<1$, *that does not depend on* p *such that, for any* $q_1,\ldots,q_m$ *in* $\{1,\ldots,p\}$,

$$\mathrm{Prob}[\mathbf{M}\models \sim\exists x_{m+1}\cdots\exists x_{m+n}G[q_1,\ldots,q_m]]\le(1-\delta)^s$$

where $s=[(p-m)/n]$.

PROOF. Let $q_1,\ldots,q_m\in\{1,\ldots,p\}$. Let $q_{m+1},\ldots,q_{m+n}$ be the earliest distinct members of $\{1,\ldots,p\}-\{q_1,\ldots,q_m\}$. For each $\mathbf{M}$ in T_p let $\psi\mathbf{M}$ be a member of T_p such that $\psi\mathbf{M}\models G[q_1,\ldots,q_{m+n}]$ and $(\mathbf{M}|\{q_1,\ldots,q_m\})=(\psi\mathbf{M}|\{q_1,\ldots,q_m\})$. Such a $\psi\mathbf{M}$ always exists, since, by Lemma 4, $(\mathbf{M}|\{q_1,\ldots,q_m\}).G[q_1,\ldots,q_{m+n}].\Phi_p$ is satisfiable. Now let $\mathbf{r}=\langle r_1,\ldots,r_n\rangle$ be any n-tuple of distinct members of $\{1,\ldots,p\}-\{q_1,\ldots,q_m\}$; let ν map $q_1,\ldots,q_m$ to themselves and map each r_i to q_{m+i}, $1\le i\le n$. For any $S\subseteq\{q_1,\ldots,q_m,r_1,\ldots,r_n\}$ let $\nu S=\{\nu e|e\in S\}$. We wish first to estimate the probability of the event

(*) for every p-atom A with arguments among $\{q_1,\ldots,q_m,r_1,\ldots,r_n\}$,

$$\mathbf{M}\models A \quad\text{iff}\quad \psi\mathbf{M}\models\nu A.$$

Note that, since $\psi\mathbf{M}\models G[q_1,\ldots,q_{m+n}]$, (*) implies $\mathbf{M}\models G[q_1,\ldots,q_m,r_1,\ldots,r_n]$, so that $\mathbf{M}\models\exists x_{m+1}\cdots\exists x_{m+n}G[q_1,\ldots,q_m]$. Now (*) is equivalent to the event that, for every $S\subseteq\{q_1,\ldots,q_m,r_1,\ldots,r_k\}$ and every signed p-atom A with arguments in S, A is a conjunct of $(\mathbf{M}|S)$ iff νA is a conjunct of $(\psi\mathbf{M}|\nu S)$; in other words, for every such S, $(\mathbf{M}|S)=\nu^{-1}(\psi\mathbf{M}|\nu S)$. To estimate the probability, we break the event down: let $E_0(\mathbf{M},\mathbf{r})$ be the event $(\mathbf{M}|\varnothing)=(\psi\mathbf{M}|\varnothing)$ and for $1\le i\le m$ let $E_i(\mathbf{M},\mathbf{r})$ be the event: $(\mathbf{M}|S)=\nu^{-1}(\psi\mathbf{M}|\nu S)$ for every $S\subseteq\{q_1,\ldots,q_m,r_1,\ldots,r_n\}$ of cardinality $\le i$. By the choice of $\psi\mathbf{M}$, $E_0(\mathbf{M},\mathbf{r})$ has probability 1. For each i, $1\le i\le m$, let β_i be the number of different atomic formulas containing the variables $x_1,\ldots,x_i$, and let $\alpha_i=2^{\beta_i}$. Then for $\mathbf{M}$ a p-structure and $S\subseteq\{1,\ldots,p\}$ of cardinality i, α_i is an upper bound on the number of possible values H of $(\mathbf{M}|S)$ that extend given values of $(\mathbf{M}|U)$ for all $U\subseteq S$ of cardinality $i-1$. Finally, for $1\le i\le m$ let $c_i=\binom{m+n}{i}-\binom{m}{i}$; c_i is an upper bound on the number of subsets of $\{q_1,\ldots,q_m,r_1,\ldots,r_n\}$ that have cardinality i and that contain at least one r_k (indeed, c_i is exactly that number, unless $i\ge 2$ and $q_1,\ldots,q_m$ are not all distinct).

Now suppose $1\le i\le m$ and $E_{i-1}(\mathbf{M},\mathbf{r})$ occurs. Since $(\mathbf{M}|\{q_1,\ldots,q_m\})=(\psi\mathbf{M}|\{q_1,\ldots,q_m\})$, $(\mathbf{M}|S)=\nu^{-1}(\psi\mathbf{M}|\nu S)$ for each $S\subseteq\{q_1,\ldots,q_m\}$ of cardinality i. Hence $E_i(\mathbf{M},\mathbf{r})$ occurs if correct choices are made for all $(\mathbf{M}|S)$ with $S\subseteq\{q_1,\ldots,q_m,r_1,\ldots,r_n\}$ of cardinality i that contain at least one r_k. Let S be any such set. Let Δ be the set of available alternatives for $(\mathbf{M}|S)$, that is, the set of possible values H for $(\mathbf{M}|S)$ such that H extends $(\mathbf{M}|U)$ for each $U \subseteq S$ of cardinality $i-1$ and $H.\Phi_p$ is satisfiable. Then $|\Delta| \leq \alpha_i$. Moreover, $v^{-1}(\psi\mathbf{M}|vS)$ is in Δ. First, since $E_{i-1}(\mathbf{M},\mathbf{r})$ occurs, $v^{-1}(\psi\mathbf{M}|vS)$ extends $(\mathbf{M}|U)$ for each $U \subseteq S$ of cardinality $i-1$. Second, $v^{-1}(\psi\mathbf{M}|vS).\Phi_p$ is satisfiable: for $(\psi\mathbf{M}|vS).\Phi_p$ is satisfiable, and, if v is extended arbitrarily to all of $\{1,\ldots,p\}$,

$$v(v^{-1}(\psi\mathbf{M}|vS).\Phi_p) = (\psi\mathbf{M}|vS).\Phi_p$$

(Φ_p is invariant under any mapping from $\{1,\ldots,p\}$ to $\{1,\ldots,p\}$).

Thus, $\text{Prob}[(\mathbf{M}|S) = v^{-1}(\psi\mathbf{M}|vS)/E_{i-1}(\mathbf{M},\mathbf{r})] \geq \alpha_i^{-1}$; whence

$$\text{Prob}[E_i(\mathbf{M},\mathbf{r})/E_{i-1}(\mathbf{M},\mathbf{r})] \geq \alpha_i^{-c_i}.$$

It follows that $\text{Prob}[E_m(\mathbf{M},\mathbf{r})] \geq \prod_{1\leq i\leq m}(\alpha_i^{-c_i})$. Let δ be this product; obviously δ does not depend on p, and $\text{Prob}[\text{not } E_m(\mathbf{M},\mathbf{r})] \leq (1-\delta)$.

Now since $s = [\![(p-m)/n]\!]$, there exists for each j, $1 \leq j \leq s$, an n-tuple $\mathbf{r}^j = \langle r_1^j,\ldots,r_n^j\rangle$ of distinct members of $\{1,\ldots,p\} - \{q_1,\ldots,q_m\}$, such that $\mathbf{r}^j$ and $\mathbf{r}^k$ have no members in common when $j \neq k$. As we have just seen, for each j, $1 \leq j \leq s$, $\text{Prob}[\text{not } E_m(\mathbf{M},\mathbf{r}^j)] \leq (1-\delta)$. Moreover, the events $[\text{not } E_m(\mathbf{M},\mathbf{r}^j)]$ and $[\text{not } E_m(\mathbf{M},\mathbf{r}^k)]$ are independent if $j \neq k$, since then $\mathbf{r}^j$ and $\mathbf{r}^k$ are disjoint and whether or not $E_m(\mathbf{M},\mathbf{r}^j)$ occurs depends only on choices made for $(\mathbf{M}|S)$ where S is a subset of $\{q_1,\ldots,q_m,r_1^j,\ldots,r_n^j\}$ that contains at least one of $r_1^j,\ldots,r_n^j$. Hence

$$\text{Prob}[E_m(\mathbf{M},\mathbf{r}^j) \text{ for no } j,\ 1 \leq j \leq s] \leq (1-\delta)^s.$$

Now, as noted above, if $E(\mathbf{M},\mathbf{r}^j)$ then $\mathbf{M} \models \exists x_{m+1}\cdots\exists x_{m+n}G[q_1,\ldots,q_m]$. Hence, if $\mathbf{M} \models {\sim}\exists x_{m+1}\cdots\exists x_{m+n}G[q_1,\ldots,q_m]$ then $E_m(\mathbf{M},\mathbf{r}^j)$ for no j, $1 \leq j \leq s$. It follows that

$$\text{Prob}[\mathbf{M} \models {\sim}\exists x_{m+1}\cdots\exists x_{m+n}G[q_1,\ldots,q_m]] \leq (1-\delta)^s$$

as desired. ∎

COROLLARY. *For large enough p, there exists* $\mathbf{M}$ *in* T_p *such that* $\mathbf{M} \models F$.

PROOF. $\mathbf{M} \models {\sim}F$ iff there exists $q_1,\ldots,q_m$ in $\{1,\ldots,p\}$ such that $\mathbf{M} \models {\sim}\exists x_1\cdots\exists x_{m+n}G[q_1,\ldots,q_m]$. By the estimation lemma, for each m-tuple $\langle q_1,\ldots,q_m\rangle$

$$\text{Prob}[\mathbf{M} \models {\sim}\exists x_1\cdots\exists x_{m+n}G(q_1,\ldots,q_m)] \leq (1-\delta)^s.$$

Since there are p^m m-tuples $\langle q_1,\ldots,q_m\rangle$ of elements of $\{1,\ldots,p\}$,

$$\text{Prob}[\mathbf{M} \models {\sim}F] \leq p^m(1-\delta)^s.$$

Since $s = [\![(p-m)/n]\!]$, for large enough p, $\text{Prob}[\mathbf{M} \models {\sim}F] < 1$. Hence, for large enough p there exists a member of T_p that makes F true. ∎

Let $\alpha = \prod_{1\leq i\leq m}(\alpha_i^{c_i})$, that is, δ^{-1}. Since α is much bigger than m and n, if $p \geq \alpha(\log\alpha)^2$ then p is "large enough" that $\text{Prob}[\mathbf{M} \models {\sim}F] < 1$. This bound on the size of the smallest finite model of F is somewhat better than those obtained from the proofs in [AG] or [DG].

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