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Citation	Elkies, Noam D. 2006. Shimura curves for level-3 subgroups of the (2,3,7) triangle group and some other examples. Lecture Notes in Computer Science (4076): 302-316.
Published Version	doi:10.1007/978-3-540-79456-1_13
Accessed	February 17, 2015 5:03:14 PM EST
Citable Link	http://nrs.harvard.edu/urn-3:HUL.InstRepos:2797445
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Shimura curve computations via K3 surfaces of Néron–Severi rank at least 19

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1   Introduction

In [E1] we introduced several computational challenges concerning Shimura curves, and some techniques to partly address them. The challenges are: obtain explicit equations for Shimura curves and natural maps between them; determine a Schwarzian equation on each curve (a.k.a. Picard–Fuchs equation, a linear second-order differential equation with a basis of solutions whose ratio inverts the quotient map from the upper half-plane to the curve); and locate CM (complex multiplication) points on the curves. We identified some curves, maps, and Schwarzian equations using the maps' ramification behavior; located some CM points as images of fixed points of involutions; and conjecturally computed others by numerically solving the Schwarzian equations.

But these approaches are limited in several ways: we must start with a Shimura curve with very few elliptic points (not many more than the minimum of three); maps of high degree are hard to recover from their ramification behavior, limiting the range of provable CM coordinates; and these methods give no access to the abelian varieties with quaternionic multiplication (QM) parametrized by Shimura curves. Other approaches somewhat extend the range where our challenges can be met. Detailed theoretical knowledge of the arithmetic of Shimura curves makes it possible to identify some such curves of genus at most 2 far beyond the range of [E1] (see e.g. [Rob,GR]), though not their Schwarzian equations or CM points. Roberts [Rob] showed in principle how to find CM coordinates using product formulas analogous to those of [GZ] for differences between CM j-invariants, but such formulas have yet to be used to verify and extend the tables of [E1]. Errthum [Er] recently used Borcherds products to verify all the conjectural rational coordinates for CM points tabulated in [E1] for the curves associated to the quaternion algebras over $\mathbf{Q}$ ramified at $\{2,3\}$ and $\{2,5\}$; it is not yet clear how readily this technique might extend to more complicated Shimura curves. The p-adic numerical techniques of [E3] give access to further maps and CM points. Finally, in the $\{2,3\}$ and $\{2,5\}$ cases Hashimoto and Murabayashi had already parametrized the relevant QM abelian surfaces in 1995 [HM], but apparently such computations have not been pushed further since then.

In this paper we introduce a new approach, which exploits the fact that some Shimura curves also parametrize K3 surfaces of Néron–Severi rank at least 19.

"Singular" K3 surfaces, those whose Néron–Severi rank attains the characteristic-zero maximum of 20, then correspond to CM points on the curve. We first encountered such parametrizations while searching for elliptic K3 surfaces of maximal Mordell–Weil rank over $\mathbf{Q}(t)$ (see [E4]), for which we used the K3 surface corresponding to a rational non-CM point on the Shimura curve $\mathrm{X}(6, 79)/\langle w_{6 \cdot 79} \rangle$ of genus 2. The feasibility of this computation suggested that such parametrizations might be used systematically in Shimura curve computations.

This approach is limited to Shimura curves associated to quaternion algebras over $\mathbf{Q}$. Within that important special case, though, we can compute curves and CM points that were previously far beyond reach. The periods of the K3 surfaces should also allow the computation of Schwarzian equations as in [LY], though we have not attempted this yet. We do, however, find the corresponding QM surfaces using Kumar's recent formulas [Ku] that make explicit Dolgachev's correspondence [Do] between Jacobians of genus-2 curves and certain K3 surfaces of rank at least 17. The parametrizations do get harder as the level of the Shimura curve grows, but it is still much easier to parametrize the K3 surfaces than to work directly with the QM abelian varieties — apparently because the level, reflected in the discriminant of the Néron–Severi group, is spread over 19 Néron–Severi generators rather than the handful of generators of the endomorphism ring.[1] In this paper we illustrate this with several examples of such computations for the curves $\mathrm{X}(N, 1)$ and their quotients. As the example of $\mathrm{X}(6, 79)/\langle w_{6 \cdot 79} \rangle$ shows, the technique also applies to Shimura curves not covered by $\mathrm{X}(N, 1)$, but already for $\mathrm{X}(N, 1)$ there is so much new data that we can only offer a small sample here: the full set of results can be made available online but is much too large for conventional publication. Since we shall not work with $\mathrm{X}(N, M)$ for $M > 1$, we abbreviate the usual notation $\mathrm{X}(N, 1)$ to $\mathrm{X}(N)$ here.

The rest of this paper is organized as follows. In the next section, we review the necessary background, drawn mostly from [Vi,Rot2,BHPV], concerning Shimura curves, the abelian and K3 surfaces that they parametrize, and the structure of elliptic K3 surfaces in characteristic zero; then give A. Kumar's explicit formulas for Dolgachev's correspondence, which we use to recover Clebsch–Igusa coordinates for QM Jacobians from our K3 parametrizations; and finally describe some of our techniques for computing such parametrizations. In the remaining sections we illustrate these techniques in the four cases $N = 6$, $N = 14$, $N = 57$, and $N = 206$. For $N = 6$ we find explicit elliptic models for our family of K3 surfaces S parametrized by $\mathrm{X}(6)/\langle w_6 \rangle$, locate a few CM points to find the double cover $\mathrm{X}(6) \rightarrow \mathrm{X}(6)/\langle w_6 \rangle$, transform S to find an elliptic model with essential lattice $N_{\mathrm{ess}} \supset E_7 \oplus E_8$ to which we can apply Kumar's formulas, and verify that our results are consistent with previous computations of CM points [E1] and Clebsch–Igusa coordinates [HM]. For $N = 14$ we exhibit S and verify the location of a CM point that we computed numerically in [E1] but could not prove using the techniques of [E1,E3]. For $N = 57$, the first case for

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[1]   It would be interesting to quantify the computational complexity of such computations in terms of the level and the CM discriminant; we have not attempted such an analysis.

which $X(N)/\langle w_N\rangle$ has positive genus, we exhibit the K3 surfaces parametrized by this curve, and locate all its rational CM points. For $N = 206$, the last case for which $X(N)/\langle w_N\rangle$ has genus zero, we exhibit the corresponding family of K3 surfaces and the hyperelliptic curves $X(206)$ and $X(206)/\langle w_2\rangle$, $X(206)/\langle w_{103}\rangle$ covering the rational curves $X(206)/\langle w_{206}\rangle$ and $X(206)/\langle w_2, w_{103}\rangle$.

2   Definitions and techniques

Quaternion algebras over Q, Shimura curves, and QM abelian surfaces. Fix a squarefree integer $N > 0$ with an even number of prime factors. There is then a unique indefinite quaternion algebra $\mathsf{A}/\mathbf{Q}$ whose finite ramified primes are precisely the factors of N. Let $\mathcal{O}$ be a maximal order in A. Since A is indefinite, all maximal orders are conjugate in A, and conjugate orders will be equivalent for our purposes. Let $\mathcal{O}_1^*$ be the group of units of reduced norm 1 in $\mathcal{O}$; let Γ be the arithmetic subgroup $\mathcal{O}_1^*/\{\pm 1\}$ of $\mathsf{A}^*/\mathbf{Q}^*$; and let Γ^* be the normalizer of Γ in the positive-norm subgroup of $\mathsf{A}^*/\mathbf{Q}^*$. If $N = 1$ then $\Gamma^* = \Gamma$; otherwise Γ^*/Γ is an abelian group of exponent 2, and for each factor $d|N$ there is a unique element $w_d \in \Gamma^*/\Gamma$ whose lifts to A^* have reduced norms in $d \cdot \mathbf{Q}^{*2}$.

Because A is indefinite, $\mathsf{A} \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic with the matrix algebra $M_2(\mathbf{R})$, so the positive-norm subgroup of $\mathsf{A}^*/\mathbf{Q}^*$ is contained in $\mathrm{PSL}_2(\mathbf{R})$ and acts on the upper half-plane $\mathcal{H}$. The quotient $\mathcal{H}/\Gamma$ is then a complex model of the Shimura curve associated to Γ, usually called $X(N, 1)$. In [E1] we called this curve $\mathcal{X}(1)$ in analogy with the classical modular curve $X(1)$ (see below), since N was fixed and we studied Shimura curves that we called $\mathcal{X}_0(p)$, $\mathcal{X}_1(p)$, etc., associated with various congruence subgroups of $\mathsf{A}^*/\mathbf{Q}^*$. In this paper we restrict attention to $\mathcal{H}/\Gamma$ and its quotients by subgroups of Γ^*/Γ; thus we return to the usual notation, but simplify it to $X(N)$ because we do not need $X(N, M)$ for $M > 1$. If $N = 1$ then $\mathsf{A} \cong M_2(\mathbf{Q})$, and we may take $\mathcal{O} = M_2(\mathbf{Z})$, when $\Gamma = \Gamma^* = \mathrm{PSL}_2(\mathbf{Z})$ and $\mathcal{H}$ must be extended by its rational cusps before we can identify $\mathcal{H}/\Gamma$ with $X(1)$. Here we study curves $X(N)$ and their quotients only for $N > 1$, and these curves have no cusps.

The Shimura curve $X(N)$ associated to a quaternion algebra over $\mathbf{Q}$ has a reasonably simple moduli description. Fix a positive anti-involution ϱ of A of the form $\varrho(\beta) = \mu^{-1}\bar{\beta}\mu$ for some $\mu \in \mathcal{O}$ with $\mu^2 + N = 0$. Then $X(N)$ parametrizes pairs (A, ι) where A is a principally polarized abelian surface and ι is an embedding of $\mathcal{O}$ into the ring $\mathrm{End}(A)$ of endomorphisms of A, such that the Rosati involution is given by ϱ. See [Rot2, §2 and Prop. 4.1]. This gives $X(N)$ the structure of an algebraic curve over $\mathbf{Q}$.

An abelian surface with an action of a (not necessarily maximal) order in a quaternion algebra is said to have "quaternionic multiplication" (QM). A *complex multiplication* (CM) point of $X(N)$ is a point, necessarily defined over $\overline{\mathbf{Q}}$, for which A has complex multiplication, i.e. is isogenous with the square of a CM elliptic curve. We shall use the QM abelian surfaces A to find models for the Shimura curves $X(N)$ and locate some of their CM points.

When $N = 1$, an abelian surface together with an action of $\mathcal{O} \cong M_2(\mathbf{Z})$ is just the square of an elliptic curve, so we recover the classical modular curve X(1). We henceforth fix $N > 1$. Then the group Γ^*/Γ, acting on X(N) by involutions that we also call w_d, is nontrivial. These involutions are again defined over $\mathbf{Q}$, taking (A, ι) to (A_d, ι_d) for some A_d isogenous with A. Specifically, A_d is the quotient of A by the subgroup of the d-torsion group $A[d]$ annihilated by the two-sided ideal of $\mathcal{O}$ consisting of elements whose norm is divisible by d, and the principal polarization on A_d is $1/d$ times the pull-back of the principal polarization on A. In particular A_d is CM if and only if A is. Hence the notion of a CM point makes sense on the quotient of X(N) by Γ^*/Γ or by any subgroup of Γ^*/Γ. If a CM point of discriminant $-D$ on X(N)/(Γ^*/Γ) is rational then the class group of $\mathbf{Q}(\sqrt{-D})$ must be generated by the classes of primes lying over factors $p|D$ that also divide N. Thus the class group has exponent 1 or 2 and bounded size; in particular, only finitely many D can arise. In each of the cases $N = 6$, 14, 57, and 206 that we treat in this paper, N has two prime factors, so the class number is at most 4 and we can cite Arno [Ar] to prove that a list of discriminants of rational CM points is complete. When N has 4 or 6 prime factors we can use Watkins' solution of the class number problem up to 100 [Wa].

We have $A_N \cong A$ as principally polarized abelian surfaces, but for $N > 1$ the embeddings ι, ι_N are not equivalent for generic QM surfaces A. When we pass from A to its Kummer surface we shall lose the distinction between ι and ι_N, and so will at first obtain only the quotient curve X(N)/$\langle w_N \rangle$. We shall determine its double cover X(N) by locating the branch points, which are the CM points on X(N)/$\langle w_N \rangle$ for which A is isomorphic to the product of two elliptic curves with CM by the quadratic imaginary order of discriminant $-N$ or $-4N$; the arithmetic behavior of other CM points will then pin down the cover, including the right quadratic twist over $\mathbf{Q}$.

An abelian surface with QM by $\mathcal{O}$ has at least one principal polarization, and the number of principal polarizations of a generic surface with QM by $\mathcal{O}$ was computed in [Rot1, Theorem 1.4 and §6] in terms of the class number of $\mathbf{Q}(\sqrt{-N})$. Each of these yields a map from X(N)/$\langle w_N \rangle$ to $\mathcal{A}_2$, the moduli three-fold of principally polarized abelian surfaces. This map is either generically 1 : 1 or generically 2 : 1, and in the 2 : 1 case it factors through an involution $w_d = w_{d'}$ on X(N)/$\langle w_N \rangle$ where $d, d' > 1$ are integers such that $N = dd'$ and

$$\mathsf{A} \cong \left(\frac{-N, d}{\mathbf{Q}}\right) \left[= \left(\frac{d, d'}{\mathbf{Q}}\right)\right]. \tag{1}$$

(See the last paragraph of [Rot2, §4], which also notes that a 2 : 1 map occurs for $N = 6$ and $N = 10$, each of which has a unique choice of polarization. In the other cases $N = 14$, 57, 206 that we study in this paper, only 1 : 1 maps arise, because the criterion (1) is not satisfied.) We aim to determine at least one of the maps X(N)/$\langle w_N \rangle {\to} \mathcal{A}_2$ in terms of the Clebsch–Igusa coordinates on $\mathcal{A}_2$,

and thus to find the moduli of the generic abelian surface with endomorphisms by $\mathcal{O}$.[2]

K3 surfaces, elliptic K3 surfaces, and the Dolgachev–Kumar correspondence. Let F be a field of characteristic zero. Recall that a *K3 surface* over F is a smooth, complete, simply connected algebraic surface S/F with trivial canonical class. The *Néron–Severi group* $\mathrm{NS}(S) = \mathrm{NS}_{\overline{F}}(S)$ is the group of divisors on S defined over the algebraic closure $\overline{F}$, modulo algebraic equivalence. For a K3 surface this is a free abelian group whose rank, the *Picard number* $\rho = \rho(S)$, is in $\{1, 2, 3, \ldots, 20\}$. The intersection pairing gives $\mathrm{NS}(S)$ the structure of an integral lattice; by the index theorem for surfaces, this lattice has signature $(1, \rho - 1)$, and for a K3 surface the lattice is *even*: $v \cdot v \equiv 0 \bmod 2$ for all $v \in \mathrm{NS}(S)$. Over $\mathbf{C}$, the cycle class map embeds $\mathrm{NS}(S)$ into the "K3 lattice" $H^2(S, \mathbf{Z}) \cong \mathrm{II}_{3,19} \cong U^3 \oplus E_8\langle -1\rangle^2$, where $U = \mathrm{II}_{1,1}$ is the "hyperbolic plane" (the indefinite rank-2 lattice with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$), and $E_8\langle -1\rangle$ is the E_8 root lattice made negative-definite by multiplying the inner product by -1. The Torelli theorem of Piateckii-Shapiro and Šafarevič [PSS] describes the moduli of K3 surfaces, at least over $\mathbf{C}$: the embedding of $\mathrm{NS}(S)$ into $\mathrm{II}_{3,19}$ is primitive, that is, realizes $\mathrm{NS}(S)$ as the intersection of $\mathrm{II}_{3,19}$ with a $\mathbf{Q}$-vector subspace of $\mathrm{II}_{3,19} \otimes \mathbf{Q}$; for every such lattice L of signature $(1, \rho - 1)$, there is a nonempty (coarse) moduli space of pairs (S, ι), where $\iota : L \to \mathrm{NS}(S)$ is a primitive embedding consistent with the intersection pairing; and each component of the moduli space has dimension $20 - \rho$. Moreover, for $\rho = 20, 19, 18, 17$ these moduli spaces repeat some more familiar ones: isogenous pairs of CM elliptic curves for $\rho = 20$, elliptic and Shimura modular curves for $\rho = 19$, moduli of abelian surfaces with real multiplication or isogenous to products of two elliptic curves for $\rho = 18$, and moduli of abelian surfaces for certain cases of $\rho = 17$. Note the consequence that an algebraic family of K3 surfaces in characteristic zero with $\rho \geq 19$ whose members are not all $\overline{F}$-isomorphic must have $\rho = 19$ generically, else there would be a positive-dimensional family of K3 surfaces with $\rho \geq 20$.

An *elliptic* K3 surface S/F is a K3 surface together with a rational map $t : S \to \mathbf{P}^1$, defined over F, whose generic fiber is an elliptic curve. The classes of the zero-section s_0 and fiber f in $\mathrm{NS}(S)$ then satisfy $s_0 \cdot s_0 = -2$, $s_0 \cdot f = 1$, and $f \cdot f = 0$, and thus generate a copy of U in $\mathrm{NS}(S)$ defined over F. Conversely, *any* copy of U in $\mathrm{NS}(S)$ defined over F yields a model of S as an elliptic surface: one of the standard isotropic generators or its negative is effective, and has 2 independent sections, whose ratio gives the desired map to $\mathbf{P}^1$. We often use this construction to transform one elliptic model of S to another that would be harder to compute directly. (Warning: in general one might have to subtract some base locus from the effective generator to recover the fiber class f.)

Since $\mathrm{disc}(U) = -1$ is invertible, we have $\mathrm{NS}(S) = \langle s_0, f\rangle \oplus \langle s_0, f\rangle^{\perp}$, with the orthogonal complement $\langle s_0, f\rangle^{\perp}$ having signature $(0, \rho - 2)$; we thus write $\langle s_0, f\rangle^{\perp} = N_{\mathrm{ess}}\langle -1\rangle$ for some positive-definite even lattice N_{ess}, the "essential

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[2] Alas we cannot say simply "find the generic abelian surface with endomorphisms by $\mathcal{O}$", even up to quadratic twist, because there are abelian surfaces with rational moduli but no model over $\mathbf{Q}$.

lattice" of the elliptic K3 surface. A vector $v \in N_{\mathrm{ess}}$ of norm 2, corresponding to $v \in \langle s_0, f \rangle^{\perp}$ with $v \cdot v = -2$, is called a "root" of N_{ess}; let $R \subseteq N_{\mathrm{ess}}$ be the sublattice generated by the roots. This root sublattice decomposes uniquely as a direct sum of simple root lattices A_n $(n \geq 1)$, D_n $(n \geq 4)$, or E_n $(6 \leq n \leq 8)$. These simple factors biject with reducible fibers, each factor being the sublattice of N_{ess} generated by the components of its reducible fiber that do not meet s_0. The graph whose vertices are these components, and whose edges are their intersections, is then the A_n, D_n, or E_n root diagram; if the identity component and its intersection(s) are included in the graph then the extended root diagram $\tilde{A}_n$, $\tilde{D}_n$, or $\tilde{E}_n$ results. The quotient group N_{ess}/R is isomorphic with the Mordell–Weil group of the surface over $\overline{F}(t)$; the isomorphism takes a point P to the projection of the corresponding section s_P to $\langle s_0, f \rangle^{\perp}$, and the quadratic form on the Mordell–Weil group induced from the pairing on N_{ess} is the canonical height. Thus the Mordell–Weil regulator is $\tau^2 \operatorname{disc}(N_{\mathrm{ess}})/\operatorname{disc}(R) = \tau^2 |\operatorname{disc}(\mathrm{NS}(S))|/\operatorname{disc}(R)$, where τ is the size of the torsion subgroup of the Mordell–Weil group.

An elliptic surface has Weierstrass equation $Y^2 = X^3 + A(t)X + B(t)$ for polynomials A, B of degrees at most 8, 12 with no common factor of multiplicity at least 4 and 6 respectively, and such that either $\deg(A) > 4$ or $\deg(B) > 6$ (i.e., such that the condition on common factors holds also at $t = \infty$ when A, B are considered as bivariate homogeneous polynomials of degrees 8, 12). The reducible fibers then occur at multiple roots of the discriminant $\Delta = -16(4A^3 + 27B^2)$ where B does not vanish to order exactly 1 (and at $t = \infty$ if $\deg \Delta \leq 22$ and $\deg B \neq 11$). To obtain a smooth model for S we may start from the surface $Y^2 = X^3 + A(t)X + B(t)$ in the $\mathbf{P}^2$ bundle $\mathbf{P}(O(0) \oplus O(2) \oplus O(3))$ over $\mathbf{P}^1$ with coordinates $(1 : X : Y)$, and resolve the reducible fibers, as exhibited in Tate's algorithm [Ta], which also gives the corresponding Kodaira types and simple root lattices. This information can then be used to calculate the canonical height on the Mordell–Weil group, as in [Si].

The *Kummer surface* $\mathrm{Km}(A)$ of an abelian surface A is obtained by blowing up the $16 = 2^4$ double points of $A/\{\pm 1\}$, and is a K3 surface with Picard number $\rho(\mathrm{Km}(A)) = \rho(A) + 16 \geq 17$. In general $\mathrm{NS}(\mathrm{Km}(A))$ need not consist of divisors defined over F, even when $\mathrm{NS}(A)$ does, because each 2-torsion point of A yields a double point of $A/\{\pm 1\}$ whose blow-up contributes to $\mathrm{NS}(\mathrm{Km}(A))$, and typically $\mathrm{Gal}(\overline{F}/F)$ acts nontrivially on $A[2]$. But when A is principally polarized Dolgachev [Do] constructs another K3 surface S_A/F, related with $\mathrm{Km}(A)$ by degree-2 maps defined over $\overline{F}$, together with a rank-17 sublattice of $\mathrm{NS}(S_A)$ that is isomorphic with $U \oplus E_7 \oplus E_8$ and consists of divisor classes defined over F. It is these surfaces that we parametrize to get at the Shimura curves $X(N)$.

If A has QM then $\rho(A) \geq 3$, with equality for non-CM surfaces, so $\rho(S_A) = \rho(\mathrm{Km}(A)) \geq 19$. When A has endomorphisms by $\mathcal{O}$, we obtain a sublattice $L_N \subseteq \mathrm{NS}(S_A)$ of signature $(1, 18)$ and discriminant $2N$. This even lattice L_N is characterized by its signature and discriminant together with the following condition: for each odd $p|N$ the dual lattice L_N^* contains a vector of norm c/p for some $c \in \mathbf{Z}$ such that $\chi_p(c) = -\chi_p(-2N/p)$, where χ_p is the Legendre symbol $(\cdot/p)$; equivalently, N_{ess}^* contains a vector of norm c/p with $\chi_p(c) = -\chi_p(+2N/p)$.

There is a corresponding local condition at 2, but it holds automatically once the conditions at all odd $p|N$ are satisfied; likewise when N is odd it is enough to check all but one $p|N$. The Shimura curve $\mathrm{X}(N)/\langle w_N \rangle$ parametrizes pairs (S, ι) where S is a K3 surface with $\rho(S) \geq 19$ and ι is an embedding $L_N \hookrightarrow \mathrm{NS}(S)$. If $\rho(S) = 20$ then (S, ι) corresponds to a CM point on $\mathrm{X}(N)/\langle w_N \rangle$ whose discriminant equals $\mathrm{disc}(\mathrm{NS}(S))$. The CM points of discriminant $-N$ or $-4N$ are the branch points of the double cover $\mathrm{X}(N)$ of $\mathrm{X}(N)/\langle w_N \rangle$. The arithmetic of other CM points then determines the cover; for instance, if $\mathrm{X}(N)/\langle w_N \rangle$ is rational, we know $\mathrm{X}(N)$ up to quadratic twist, and then a rational CM point of discriminant $D \neq -N, -4N$ lifts to a pair conjugate over $\mathbf{Q}(\sqrt{-D})$.

The correspondence between A and S_A was made explicit by Kumar [Ku, Theorem 5.2]. Let A be the Jacobian of a genus-2 curve C, and let I_2, I_4, I_6, I_{10} be the Clebsch–Igusa invariants of C. (If a principally polarized abelian surface A is not a Jacobian then it is the product of two elliptic curves, and thus cannot have QM unless it is a CM surface.) We give an elliptic model of S_A with $N_{\mathrm{ess}} = R = E_7 \oplus E_8$, using a coordinate t on $\mathbf{P}^1$ that puts the E_7 and E_8 fibers at $t = 0$ and $t = \infty$. Any such surface has the formula

$$Y^2 = X^3 + (at^4 + a't^3)X + (b''t^7 + bt^6 + b't^5) \tag{2}$$

for some a, a', b, b', b'' with $a', b'' \neq 0$. (There are five parameters, but the moduli space has dimension only $5 - 2 = 3$ as expected, because multiplying t by a nonzero scalar yields an isomorphic surface, and multiplying a, a' by λ^2 and b, b' by λ^3 for some $\lambda \neq 0$ yields a quadratic twist with the same moduli.) Kumar shows that setting

$$(a, a', b, b', b'') = \left(-I_4/12, -1, (I_2 I_4 - 3I_6)/108, I_2/24, I_{10}/4\right) \tag{3}$$

in (2) yields the surface $S_{J(C)}$. Starting from any surface (2) we may scale (t, X, Y) to $(-a't, a'^2 X, a'^3 Y)$ and divide through by a'^6 to obtain an equation of the same form with $a' = -1$; doing this and solving (3) for the Clebsch–Igusa invariants I_i, we find

$$(I_2, I_4, I_6, I_{10}) = (-24b'/a', \ -12a, \ 96ab'/a' - 36b, \ -4a'b''). \tag{4}$$

If A has QM by $\mathcal{O}$, but is not CM, then the elliptic surface (2) has a Mordell–Weil group of rank 2 and regulator N, with each choice of polarization of A corresponding to a different Mordell–Weil lattice. The polarizations for which the map $\mathrm{X}(N)/\langle w_N \rangle \to \mathcal{A}_2$ factors through some w_d are those for which the lattice has an involution other than -1. When this happens, two points on $\mathrm{X}(N)/\langle w_N \rangle$ related by w_d yield the same surface (2) but a different choice of Mordell–Weil generators. For example, when $N = 6$ and $N = 10$ these lattices have Gram matrices $\frac{1}{2}\left(\begin{smallmatrix} 5 & 1 \\ 1 & 5 \end{smallmatrix}\right)$ and $\frac{1}{2}\left(\begin{smallmatrix} 8 & 0 \\ 0 & 5 \end{smallmatrix}\right)$ respectively.

Some computational tricks. Often we need elliptic surfaces with an A_n fiber for moderately large n, that is, for which $4A^3 + 27B^2$ vanishes to moderately large order $n + 1$ at some $t = t_0$ at which neither A nor B vanishes. Thus we

have approximately $(A, B) = (-3a^2, 2a^3)$ near $t = t_0$. Usually one lets a be a polynomial that locally approximates $(-A/3)^{1/2}$ at $t = t_0$, and writes

$$(A, B) = (-3(a^2 + 2b), 2(a^3 + 3ab) + c) \tag{5}$$

for some b, c of valuations $v(b) = \nu$, $v(c) = 2\nu$ at t_0. Then $v(\Delta) \geq 2\nu$ always, and $v(\Delta) \geq 3\nu$ if and only if $v(3b^2 - ac) \geq 3\nu$; also if $\mu < \nu$ then $v(\Delta) = 2\nu + \mu$ if and only if $v(3b^2 - ac) = 2\nu + \mu$. See [Ha]; this was also the starting point of our analysis in [E2]. For our purposes it is more convenient to allow extended Weierstrass form and write the surface as

$$Y^2 = X^3 + a(t)X^2 + 2b(t)X + c(t) \tag{6}$$

with polynomials a, b, c of degrees at most $4, 8, 12$ such that $(v(b), v(c)) = (\nu, 2\nu)$. Translating X by $-a/3$ shows that this is equivalent to (5), with a, b divided by 3 (so $\mu = v(b^2 - ac)$ in (6)). But (6) tends to produce simpler formulas, both for the surface itself and for the components of the fiber, which are rational if and only if a is a square. For instance, the Shioda–Hall surface with an A_{18} fiber [Sh,Ha] can be written simply as

$$Y^2 = X^3 + (t^4 + 3t^3 + 6t^2 + 7t + 4)X^2 - 2(t^3 + 2t^2 + 3t + 2)X + (t^2 + t + 1)$$

with the A_{18} fiber at infinity, and this is the quadratic twist that makes all of $\mathrm{NS}(S)$ defined over $\mathbf{Q}$. The same applies to D_n, when $A' := A/t^2$ and $B' := B/t^3$ are polynomials such that $4A'^3 + 27B'^2$ has valuation $n - 4$. See for instance (19) below. When we want singular fibers at several t values we use an extended Weierstrass form (6) for which $(v(b), v(c)) = (\nu, 2\nu)$ holds (possibly with different ν) at each of these t.

Having parametrized our elliptic surface S with $L_N \hookrightarrow \mathrm{NS}(S)$, we seek specializations of rank 20 to locate CM points. In all but finitely many cases S has an extra Mordell–Weil generator. In the exceptional cases, either some of the reducible fibers merge, or one of those fibers becomes more singular, or there is an extra A_1 fiber. Such CM points are easy to locate, though some mergers require renormalization to obtain a smooth model and find the CM discriminant D, as we shall see. When there is an extra Mordell–Weil generator, its height is at least $|D|/2N$, but usually not much larger. (Equality holds if and only if the extra generator is orthogonal to the generic Mordell–Weil lattice; in particular this happens if S has generic Mordell–Weil rank zero.) The larger the height of the extra generator, the harder it typically is to find the surface. This has the curious consequence that while the difficulty of parametrizing S increases with N, the CM points actually become easier to find. In some cases we cannot solve for the coefficients directly. We thus adapt the methods of [E3], exhaustively searching for a solution modulo a small prime p and then lifting it to a p-adic solution to enough accuracy to recognize the underlying rational numbers. We choose the smallest p such that $\chi_p(-D) = +1$, so that reduction mod p does not raise the Picard number, and we can save a factor of p in the exhaustive search by first counting points mod p on each candidate S to identify the one with the correct CM.

For large N we use the following variation of the p-adic lifting method to find the Shimura curve $X(N)/\langle w_N \rangle$ and the surfaces S parametrized by it. First choose some indefinite primitive sublattice $L' \subset L_N$ and parametrize all S with $NS(S) \supseteq L'$. Search in that family modulo a small prime p to find a surface S_0 with the desired L_N. Let f_1, f_2 be simple rational functions on the (S, L') moduli space. We hope that the degrees, call them d_i, of the restriction of f_i to $X(N)/\langle w_N \rangle$ are positive but small; that f_1 is locally $1 : 1$ on the point of $X(N)/\langle w_N \rangle$ parametrizing S_0; and that the map $(f_1, f_2) : X(N)/\langle w_N \rangle \rightarrow \mathbf{A}^2$ is generically $1 : 1$ to its image in the affine plane. For various small lifts $\tilde{f}_1$ of $f_1(S_0)$ to $\mathbf{Q}$, lift S_0 to a surface $S/\mathbf{Q}_p$ with $f_1(S) = \tilde{f}_1$, compute $f_2(S)$ to high p-adic precision, and use lattice reduction to recognize $f_2(S)$ as the solution of a polynomial equation $F(f_2) = 0$ of degree (at most) d_1. Discard the few cases where the degree is not maximal, and solve simultaneous linear equations to guess the coefficients of F as polynomials of degree at most d_2 in $\tilde{f}_1$. At this point we have a birational model $F(f_1, f_2) = 0$ for $X(N)/\langle w_N \rangle$. Then recover a smooth model of the curve (using Magma if necessary), recognize the remaining coefficients of S as rational functions by solving a few more linear equations, and verify that the surface has the desired embedding $L_N \hookrightarrow NS(S)$.

3 $N = 6$: The first Shimura curve

The K3 surfaces. We take $N_{\mathrm{ess}} = R = A_2 \oplus D_7 \oplus E_8$, which has discriminant $3 \cdot 4 \cdot 1 = 12 = 2N$, and the correct behavior at 3 because A_2^* contains vectors of norm $2/3$ with $\chi_3(2) = -\chi_3(2 \cdot 6/3)[= -1]$. We choose the rational coordinate t on $\mathbf{P}^1$ such that the A_2, D_7, and E_8 fibers are at $t = 1$, 0, and ∞ respectively. If we relax the condition at $t = 1$ by asking only that the discriminant vanish to order at least 2 rather than 3 then the general such surface can be written as

$$Y^2 = X^3 + (a_0 + a_1 t)tX^2 + 2a_0 bt^3(t - 1)X + a_0 b^2 t^5 (t - 1)^2 \qquad (7)$$

for some a_0, a_1, b, with $a_1 b \neq 0$ lest the surface be too singular at $t = 0$. The discriminant is then $t^9(t-1)^2 \Delta_1(t)$ with Δ_1 a cubic polynomial such that $\Delta(1) = -64a_0 a_1 (a_0 + a_1)^2 b^2$. Thus $\Delta_1(1) = 0$ if and only if $a_1 = 0$ or $a_0 + a_1 = 0$. In the latter case the surface has additive reduction at $t = 1$. Hence we must have $a_1 = 0$. The non-identity components of the resulting A_2 fiber at $t = 1$ then have $X = O(t - 1)$; we calculate that $X = x_1(t - 1) + O((t - 1)^2)$ makes $Y^2 = (x_1 + b)^2 a_0 (t - 1)^2 + O(t - 1)^3$. Therefore these components are rational if and only if a_0 is a square. We can then replace (X, Y, b) by $(a_0 X, a_0^{3/2} Y, a_0 b)$ in (7) to obtain the formula

$$Y^2 = X^3 + tX^2 + 2bt^3(t - 1)X + b^2 t^5 (t - 1)^2 \qquad (8)$$

for the general elliptic K3 surface with $N_{\mathrm{ess}} = R = A_2 D_7 E_8$ and rational A_2 components. The two components of the D_7 fiber farthest from the identity component then have $X = bt^2 + O(t^3)$, so $Y^2 = b^3 t^6 + O(t^7)$; thus these components are both rational as well if and only b is a square, say $b = r^2$. Then b and r

are rational coordinates on the Shimura curves $X(6)/\langle w_2, w_3 \rangle$ and $X(6)/\langle w_6 \rangle$ respectively, with the involution $w_2 = w_3$ on $X(6)/\langle w_6 \rangle$ taking r to $-r$.

The elliptic surface (8) has discriminant $\Delta = 16b^3 t^9 (t-1)^3 (27b(t^2 - t) - 4)$. Thus the formula (8) fails at $b = 0$, and also of course at $b = \infty$. Near each of these two points we change variables to obtain a formula that extends smoothly to $b = 0$ or $b = \infty$ as well. These formulas require extracting respectively a fourth and third root of β, presumably because $b = 0$ and $b = \infty$ are elliptic points of the Shimura curve. For small b, we take $b = \beta^4$ and replace (t, X, Y) by $(t/\beta^2, X/\beta^2, Y/\beta^3)$ to obtain

$$Y^2 = X^3 + tX^2 + 2t^3(t - \beta^2)X + t^5(t - \beta^2)^2, \qquad (9)$$

with the A_2 fiber at $t = \beta^2$ rather than $t = 1$. When $\beta = 0$, this fiber merges with the D_7 fiber at $t = 0$ to form a D_{10} fiber, but we still have a K3 surface, namely $Y^2 = X^3 + tX^2 + 2t^4 X + t^7$, with $L = R = D_{10} \oplus E_8$. This is the CM point of discriminant -4. For large b, we write $b = 1/\beta^3$ and replace (X, Y) by $(X/\beta^2, Y/\beta^3)$ to obtain

$$Y^2 = X^3 + \beta^2 tX^2 + 2\beta t^3(t - 1)X + t^5(t - 1)^2; \qquad (10)$$

then taking $\beta \to 0$ yields the surface $Y^2 = X^3 + t^5(t - 1)^2$ with $N_{\text{ess}} = R_0 = A_2 \oplus E_8 \oplus E_8$: the $t = 0$ fiber changes from D_7 to E_8, and the $t = 1$ fiber becomes additive but still contributes A_2 to R (Kodaira type IV rather than I_3). This is the CM point of discriminant -3.

Two more CM points. The factor $27b(t^2 - t) - 4$ of Δ is a quadratic polynomial in t of discriminant $27b(27b + 16)$. Hence at $b = -16/27$ we have $N_{\text{ess}} = R = A_1 \oplus A_2 \oplus D_7 \oplus E_8$, and we have located the CM point of discriminant -24. Three points fix a rational coordinate on $\mathbf{P}^1$, so we can compare with the coordinate used in [E1, Table 1], which puts the CM points of discriminant -3, -4, and -24 at ∞, 1, and 0 respectively; thus that coordinate is $1 + 27b/16$. This also confirms that $X(6)$ is obtained by extracting a square root of $-(27r^2 + 16)$.

We next locate a CM point of discriminant -19 by finding b for which the surface (8) has a section s_P of canonical height $19/12$. This is the smallest possible canonical height for a surface with $R = A_2 \oplus D_7 \oplus E_8$, because the naïve height is at least 4 and the height corrections at the A_2 and D_7 fibers can reduce it by at most $2/3$ and $7/4$ respectively, reaching $4 - 2/3 - 7/4 = 19/12$. Let $(X(t), Y(t))$ be the coordinates of a point P of height $19/12$. Then $X(t)$ and $Y(t)$ are polynomials of degree at most 4 and 6 respectively (else s_P intersects s_0 and the naïve height exceeds 4), and X vanishes at $t = 1$ (so s_P passes through a non-identity component of the A_2 fiber) and has the form $bt^2 + O(t^3)$ at $t = 0$ (so s_P meets one of the components of the D_7 fiber farthest from the identity component). That is, $X = b(t^2 - t^3)(1 + t_1 t)$ for some t_1. Substituting this into (8) and dividing by the known square factor $(t^4 - t^3)^2$ yields b^3 times

$$-t_1^3 t^4 + (t_1^3 - 3t_1^2)t^3 + 3(t_1^2 - t_1)t^2 + ((3t_1 - 1) + b^{-1}t_1^2)t + 1, \qquad (11)$$

so we seek b, t_1 such that the quartic (11) is a square. We expand its square root in a Taylor expansion about $t = 0$ and set the t^3 and t^4 coefficients equal to

zero. This gives a pair of polynomial equations in b and t_1, which we solve by taking a resultant with respect to t_1. Eliminating a spurious multiple solution at $b = 0$, we finally obtain $(b, t_1) = (81/64, -9)$, and confirm that this makes (11) a square, namely $(27t^2 - 18t - 1)^2$. Therefore $81/64$ is the b-coordinate of a CM point of discriminant -19. Then $1 + 27b/16 = 3211/2^{10}$, same as the value obtained in [E1].

Clebsch–Igusa coordinates. The next diagram shows the graph whose vertices are the zero-section (circled) and components of reducible fibers of an elliptic K3 surface S with $N_{\text{ess}} = A_2 \oplus D_7 \oplus E_8$, and whose edges are intersections between pairs of these rational curves on the surface. Eight of the vertices form an extended root diagram of type $\tilde{E}_7$, and are marked with their multiplicities in a reducible fiber of type E_7 of an alternative elliptic model for S. We may take either of the unmarked vertices of the $\tilde{D}_7$ subgraph as the zero-section. Then the essential lattice of the new model includes an E_8 root diagram as well as the forced E_7. We can thus apply Kumar's formulas to this model once we compute its coefficients.
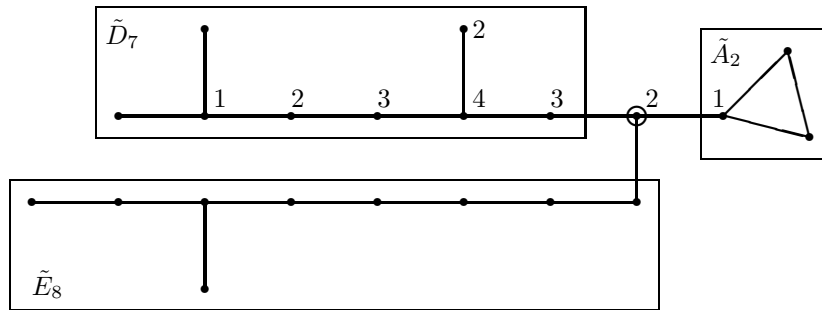

Figure 1: An $\tilde{E}_7$ divisor supported on the zero-section and
fiber components of an $A_2 D_7 E_8$ surface

The sections of the $\tilde{E}_7$ divisor are generated by 1 and $u := X/(t^4 - t^3) + b/t$. Thus $u : S \to \mathbf{P}^1$ gives the new elliptic fibration. Taking $X = (t^3 - t^2)(tu - b)$ in (8) and dividing by $(t^4 - t^3)^2$ yields $Y_1^2 = Q(t)$ for some quartic Q. Using standard formulas for the Jacobian of such a curve, and bringing the resulting surface into Weierstrass form, we obtain a formula (2) with (a, a', b, b', b'') replaced by $(-3b, 1, -2b^2, -(b+1), -b^3)$. As expected this surface has Mordell–Weil rank 2 with generators of height $5/2$, namely

$$\left(r^6 t^4 + 2(r^4 + r^3)t^3 + (r^2 + 1)t^2,\ r^9 t^6 + 3(r^7 + r^6)t^5 + 3(r^5 + r^4 + r^3)t^4 + (r^3 + 1)t^3\right)$$

and the image of this section under $r \leftrightarrow -r$ (recall that $b = r^2$). The formula (4) yields the Clebsch–Igusa coordinates

$$(I_2, I_4, I_6, I_{10}) = ((24b + 1), 36b, 72b(5b + 4), 4b^3). \tag{12}$$

4 $N = 14$: The CM point of discriminant -67

The K3 surfaces. Here we take $N_{\text{ess}} = R = A_3 \oplus A_6 \oplus E_8$, which has discriminant $4 \cdot 7 \cdot 1 = 28 = 2N$, and the correct behavior at 7 because A_6^* contains vectors of norm $6/7$ with $\chi_7(6) = -\chi_7(2 \cdot 14/7)[= -1]$. We put the A_3, A_6, and E_8 fibers at $t = 1$, 0, and ∞ respectively. We then seek an extended Weierstrass form (6) with a, b, c of degrees $2, 4, 7$ such that $t^3 - t^2 | b$, $(t^3 - t^2)^2 | c$, and $(t^3 - t^2)^6 | b^2 - ac$. This gives at least A_3, A_5, E_8. It is then easy to impose the extra condition $t^7 | \Delta$, and we obtain $a = \lambda((s+1)t^2 + (3s^2 + 2s)t + s^3)$, $b = \lambda^2(s+1)((4s+2)t + 2s^2)(t^3 - t^2)$, $c = \lambda^3(s+1)^2(t+s)(t^3 - t^2)^2$ for some s, λ. The twist λ must be chosen so that $a(0)$ and $a(1)$ are both squares; this is possible if and only if $s^2 + s$ is a square, so $s = r^2/(2r+1)$ for some r. Thus r and s are rational coordinates on $X(14)/\langle w_{14} \rangle$ and $X(14)/\langle w_2, w_7 \rangle$ respectively, with the involution $w_2 = w_7$ on $X(14)/\langle w_{14} \rangle$ taking r to $-r/(2r+1)$. The formula in terms of r is cleaner if we let the A_3 fiber move from $t = 1$; putting it at $t = 2r + 1$ yields

$$
\begin{aligned}
a &= ((r+1)^2)t^2 + (3r^4 + 4r^3 + 2r^2)t + r^6), \\
b &= 2(r+1)^2((2r^2 + 2r + 1)t + r^4)(t - (2r+1))t^2, \\
c &= (r+1)^4(t - (2r+1))^2(t + r^2)t^4.
\end{aligned}
\tag{13}
$$

Easy CM points. At $r = 0$, the A_6 fiber becomes E_7, so we have a CM point with $D = -8$; at $r = -1/2$, the A_3 and A_6 fibers merge to A_{10}, giving a CM point with $D = -11$. These have $s = 0$, $s = \infty$ respectively. There is an extra A_1 fiber when $11s^2 + 3s + 8 = 0$; the roots of this irreducible quadratic give the CM points with $D = -56$ (and their lifts to $X(14)/\langle w_{14} \rangle$ are the branch points of the double cover $X(14)$). In [E1] we gave a rational coordinate t on $X(14)/\langle w_2, w_7 \rangle$ for which the CM points of discriminants -8, -11, and -56 had $t = 0$, $t = -1$, and $16t^2 + 13t + 8 = 0$ respectively. Therefore that t is our $-s/(s+1)$.

A harder CM point. At the CM point of discriminant -67 our surface has a section of height $67/28 = 4 - (3/4) - (6/7)$. Thus $Y^2 = X^3 + aX^2 + bX + c$ has a solution in polynomials X, Y of degrees $4, 6$ with $X(0) = X(2h+1) = 0$ and Y having valuation exactly 1 at $t = 0$ and $t = 2h + 1$. An exhaustive search mod 17 quickly finds an example, whose lift to $\mathbf{Q}_{17}$ then yields $r = -35/44$ with

$$
X = \frac{3^4}{5^2 \, 22^5} t \, (22t + 13) \, (527076t^2 + 760364t + 275625).
$$

Thus $s = -1225/1144$, and $-s/(s+1)$ confirms the entry $-1225/81$ in the $|D| = 67$ row of [E1, Table 5].

5 $N = 57$: The first curve $X(N)/w_N$ of positive genus

The K3 surfaces. We cannot have $N_{\text{ess}} = R$ here because there is no root lattice of rank 17 and discriminant $6 \cdot 19$. Instead we take for R the rank-16 lattice

$A_5 \oplus A_{11}$ of discriminant $6 \cdot 12 = 72$, and require an infinite cyclic Mordell–Weil group N_{ess}/R with a generator corresponding to a section that meets the A_5 and A_{11} fibers in non-identity components farthest from the A_5 identity and nearest the A_{11} identity respectively, and does not meet the zero-section (i.e., for which X is a polynomial of degree at most 4 in t). Such a point has canonical height

$$4 - \frac{3 \cdot 3}{6} - \frac{1 \cdot 11}{12} = \frac{19}{12} = \frac{2N}{\mathrm{disc}\, R}. \tag{14}$$

Thus $\mathrm{disc}(N_{\mathrm{ess}})$ has the desired discriminant $2N$. We may check the local conditions by noting that A_5^* contains a vector of norm $4/3$ that remains in N_{ess}^*, and $\chi_3(4) = -\chi_3(2 \cdot 57/3)[= +1]$. We put the A_5 fiber at $t = 0$ and the A_{11} fiber at $t = \infty$. We eventually obtain the following parametrization in terms of a coordinate r on the rational curve $X(57)/\langle w_3, w_{19} \rangle$: let

$$\begin{aligned}
p(r) &= 4(r-1)(r^2-2) + 1, \\
d &= (r^2-1)^2(9t + (2r-1)p(r)), \\
c &= 9t^2 - (2r-1)(8r^2 + 4r - 22)t + (2r-1)^2 p(r), \\
b &= (t - (r^2 - 2r))c + d, \\
a &= (t - (r^2 - 2r))^2 c + 2(t - (r^2 - 2r))d + (r^2-1)^4((4r+4)t + p(r));
\end{aligned} \tag{15}$$

Then the surface is

$$Y^2 = X^3 + aX^2 + 8(r-1)^4(r+1)^5 bt^2 X + 16(r-1)^8(r+1)^{10} ct^4, \tag{16}$$

with a section of height $19/12$ at

$$X = -\frac{4(r-1)^4(r+1)^5(2r-1)t^2}{(r^2-r+1)^2} + \frac{4(r-2)(r+1)^4 t^3}{r^2-r+1}. \tag{17}$$

The components of the A_{11} fiber are rational because the leading coefficient of a is 9, a square; the constant coefficient is $(r^2-r+1)^4 p(r)$, so $X(57)/\langle w_{57} \rangle$ is obtained by extracting a square root of $p(r)$. This gives the elliptic curve with coefficients $[a_1, a_2, a_3, a_4, a_6] = [0, -1, 1, -2, 2]$, whose conductor is 57 as expected (see e.g. Cremona's tables [Cr] where this curve appears as 57-A1(E)).

This curve has rank 1, with generator $P = (2, 1)$. The point at infinity is the CM point of discriminant -19; this may be seen by substituting $1/s$ for r and $(t/s^3, X/s^{12}, Y/s^{18})$ for (t, X, Y), then letting $s \to 0$ to obtain the surface

$$Y^2 = X^3 + (9t^4 - 16t^3 + 4t)X^2 + (72t^5 - 128t^4)X + (144t^6 - 256t^5) \tag{18}$$

with a D_6 fiber at $t = 0$ rather than an A_5. Then we still have a section $(X, Y) = (4t^3 - 8t^2, (3t-5)(t^4 - t^3))$ of height $19/12$, but there is a 2-torsion point $(X, Y) = (-4t, 0)$ so $\mathrm{disc}(N_{\mathrm{ess}}) = -\mathrm{disc}(\mathrm{NS}(S))$ drops to $4 \cdot 12 \cdot (19/12)/2^2 = 19$. The remaining rational CM points on $X(57)/\langle w_{57} \rangle$ come in six pairs $\pm nP$:

n	1	2	3	4	5	8
r	2	1	-1	0	5/4	13/9
$-D$	7	4	16	28	43	163

The last three of these have extra sections $X = -4t$, $X = 0$, and

$$X = -28 \cdot 11^3 \left((t^2/3^6) + (415454t/3^{18})\right)$$

respectively. At $r = 2$, the A_{11} fiber becomes an A_{12} and our generic Mordell–Weil generator becomes divisible by 3; the new generator $(-972t, 26244t^2)$ has height $4 - (5/6) - (40/13) = 7/78$, so disc $N_{\mathrm{ess}} = 6 \cdot 13 \cdot (7/78) = 7$. At $r = 1$, the A_5 and A_{11} fibers together with the section all merge to form a D_{18} fiber: let $r = 1 + s$ and change (t, X) to $(st - 1, -8s^3 X)$, divide by $(-2s)^9$, and let $s \to 0$ to obtain the second Shioda–Hall surface

$$X^3 + (t^3 + 8t)X^2 - (32t^2 + 128)X + 256t \tag{19}$$

with a D_{18} fiber at $t = \infty$ [Sh,Ha]. At $t = -1$, the reducible fibers again merge, this time forming an A_{17} while the Mordell–Weil generator's height drops to $4 - (4 \cdot 14/18) = 8/9$, whence disc$(N_{\mathrm{ess}}) = 16$.

We find four more rational CM values of r that do not lift to rational points on $X(57)/\langle w_{57} \rangle$, namely $r = 5, 1/2, 17/16, -7/4$, for discriminants $-123, -24$, -267, and $-627 = -11 \cdot 57$ respectively. The first of these again has an A_{12} fiber, this time with the section of height $4 - (9/6) - (12/13) = 41/26$; the second has a rational section at $X = 0$; in the remaining two cases we find the extra section by p-adic search:

$$X = -\frac{11^3 3^2}{2^{21} 91^2} t^2 (7840t^2 - 2037t + 3267) \tag{20}$$

for $r = 17/16$, and

$$X = \frac{3^5 11^4 t^2 q(t)}{2^{12}(81920t^3 + 9216t^2 + 23868t + 39339)^2} \tag{21}$$

for $r = -7/4$, where $q(t)$ is the quintic

$$419430400t^5 + 2846883840t^4 + 17148174336t^3$$
$$+ 78784560576t^2 + 175272616341t - 12882888.$$

Using [Ar] we can show that there are no further rational CM values.

6 $N = 206$: The last curve $X(N)/w_N$ of genus zero

Summary of results. Again we take N_{ess} of rank 16 and an infinite cyclic Mordell–Weil group, here $R = A_2 \oplus A_4 \oplus A_{10}$ with a Mordell–Weil generator of height $412/165 = 6 - (1 \cdot 2/3) - (2 \cdot 3)/5 - (2 \cdot 9)/11$. With the reducible fibers placed at $1, 0, \infty$ as usual, the choice of R means $\Delta = t^5(t - 1)^3 \Delta_1$ with Δ_1 of degree $24 - (3 + 5 + 11) = 5$ and $\Delta_1(0), \Delta_1(1) \neq 0$; the generator must then have $X(t) = X_1(t)/(t - t_0)^2$ for some sextic X_1 and some $t_0 \neq 0, 1$, with the corresponding section passing through a non-identity component of the A_2 fiber and

components at distance 2 from the identity of A_4 and A_{10}. We eventually succeed in parametrizing such surfaces, finding a rational coordinate on the modular curve $X(206)/\langle w_{206} \rangle$. These elliptic models do not readily exhibit the involution $w_2 = w_{103}$ on this curve, so we recover this involution from the fact that it must permute the branch points of the double cover $X(206)$ of $X(206)/\langle w_{206} \rangle$. We locate these branch points as simple zeros of the discriminant of Δ_1. As expected, there are 20 (this is the class number of $\mathbf{Q}(\sqrt{-206})$), forming a single Galois orbit. We find a unique involution of the projective line $X(206)/\langle w_{206} \rangle$ that permutes these zeros. This involution has two fixed points, so we switch to a rational coordinate r on $X(206)/\langle w_{206} \rangle$ that makes the involution $r \leftrightarrow -r$. Then $r_0 := r^2$ is a rational coordinate on $X(206)/\langle w_2, w_{103} \rangle$, and the 20 branch points are the roots of $P_{10}(r^2)$ where P_{10} is the degree-10 polynomial

$$P_{10}(r_0) = 8r_0^{10} - 13r_0^9 - 42r_0^8 - 331r_0^7 - 220r_0^6 + 733r_0^5 \qquad (22)$$
$$+ 6646r_0^4 + 19883r_0^3 + 28840r_0^2 + 18224r_0 + 4096.$$

As a further check on the computation, P_{10} has dihedral Galois group, discriminant $-2^{138}103^7$, and field discriminant $-2^{12}103^5$, while $P_{10}(r^2)$ has discriminant $2^{311}103^{14}$ and field discriminant $2^{27}103^{10}$. We find that $r = 0, \pm 1, \pm 2, \infty$ give CM points of discriminants $D = -4, -19, -163, -8$ respectively; evaluating $P_{10}(r^2)$ at any of these points gives $-D$ times a square, showing that the Shimura curve $X(206)$ has the equation $s^2 = -P_{10}(r^2)$ over $\mathbf{Q}$. The curves $X(206)/\langle w_2 \rangle$, $X(206)/\langle w_{103} \rangle$ are then the double covers $s_0^2 = -P_{10}(r_0)$, $s_0'^2 = -r_0 P_{10}(r_0)$ of the r_0-line $X(206)/\langle w_2, w_{103} \rangle$ (in that order, because w_{103} cannot fix a CM point of discriminant -4 or -8).

Acknowledgements

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