Optimal Constants in the Rosenthal Inequality for Random Variables with Zero Odd Moments.

OPTIMAL CONSTANTS IN THE ROSENTHAL INEQUALITY FOR RANDOM VARIABLES WITH ZERO ODD MOMENTS[1]

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Abstract. We obtain estimates for the best constant in the Rosenthal inequality $E\left|\sum_{i=1}^{n}\xi_i\right|^{2m} \le C(2m)max\left(\sum_{i=1}^{n}E\xi_i^{2m},\left(\sum_{i=1}^{n}E\xi_i^2\right)^m\right)$ for independent random variables $\xi_1,...,\xi_n$ with l zero first odd moments, $l \ge 1$. The estimates are sharp in the extremal cases l=1 and l=m, that is, in the cases of random variables with zero mean and random variables with m zero first odd moments.

Rosenthal (1970) proved the following inequality:

$$E\left|\sum_{i=1}^{n}\xi_i\right|^t \le C(t)max\left(\sum_{i=1}^{n}E|\xi_i|^t,\left(\sum_{i=1}^{n}E\xi_i^2\right)^{t/2}\right) \tag{1}$$

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for all positive integers n and all independent random variables (r.v.'s) $\xi_1,\dots,\xi_n$ with $E\xi_i = 0$, $E|\xi_i|^t < \infty$, $i=1,\dots, n$, $t > 2$, where $C(t)$ is a constant depending only on t. A number of papers have focused on refinements and extensions of inequality (1) and related problems (see Prokhorov, 1962; Sazonov, 1974; Nagaev and Pinelis, 1977; Pinelis, 1980, 1994; Pinelis and Utev, 1984; Johnson, Schechtman and Zinn, 1985; Utev, 1985; Hitczenko, 1990, 1994; Bestsennaya and Utev, 1991, Ibragimov and Sharakhmetov, 1995, 1997, 2001a, b; Figiel, Hitczenko, Johnson, Schechtman and Zinn, 1997; Ibragimov, 1997; de la Peña and Giné, 1999; and de la Peña, Ibragimov and Sharakhmetov, 2003). Figiel et al. (1997) and Ibragimov and Sharakhmetov (1995, 1997) derived the following expressions for the best constant $C^*_{sym}(t)$ in inequality (1) for symmetric r.v.'s: $C^*_{sym}(t) = 1 + \dfrac{2^{t/2}\Gamma\left(\frac{t+1}{2}\right)}{\sqrt{\pi}}$, $2<t<4, C^*_{sym}(t) = E|\theta_1 - \theta_2|^t$, $t\geq 4$, where $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$ and θ_1, θ_2 are independent Poisson r.v.'s with parameter 0.5. The proof of the expressions for $C^*_{sym}(t)$ in Ibragimov and Sharakhmetov (1995, 1997) relies on the work by Utev (1985), who obtained, among other results, sharp upper and lower bounds on $E\left|\sum_{i=1}^n \xi_i\right|^t$, $t\geq 4$, where $\xi_1,\dots,\xi_n$ are independent symmetric r.v.'s with finite tth moment, in terms of $\sum_{i=1}^n E|\xi_i|^t$ and $\left(\sum_{i=1}^n E\xi_i^2\right)^{t/2}$. Bestsennaya and Utev (1991) derived a similar upper bound on even moments of sums independent mean-zero r.v.'s $\xi_1,\dots,\xi_n$, from which the best constant in general Rosenthal's inequality (1) in the case t=2m can be deduced. Using a different proof technique, the expression for the best constant in general inequality (1) for even moments t=2m of sums of mean-zero r.v.'s was independently obtained in Ibragimov and Sharakhmetov (2001a). Ibragimov and Sharakhmetov (2001b) obtained the best constant in the analogue of inequality (1) for nonnegative r.v.'s. The results in Ibragimov and Sharakhmetov (1995, 1997, 2001a, b) were also presented in Ibragimov (1997). de la Peña *et al.* derived sharp analogues of the Burkholder–Rosenthal inequalities and related estimates for the expectations of functions of sums of dependent nonnegative r.v.'s and conditionally symmetric martingale differences with bounded conditional moments as well as for sums of multilinear forms.

The present paper deals with estimating the best constants in the Rosenthal's inequality for r.v.'s with l zero first odd moments. Namely, let $C_l^*(t)$ denote the best constants in inequality (1) for all positive integers n and all independent r.v.'s $\xi_1,...,\xi_n$ with $E\xi_i^{2s-1}=0$, $s=1,2,...,l$. Then the following theorem holds.

Theorem 1. If t=2m, $m\in\boldsymbol{N}$, then

$$C_l^*(2m)\le(2m)!\sum_{j=1}^{2m}\sum_{r=1}^{j}\sum\prod_{k=1}^{r}\frac{(m_k!)^{-j_k}}{j_k!}, \tag{2}$$

where the inner sum is taken over all natural $m_1>m_2>...>m_r$ and $j_1,...,j_r$ satisfying the conditions $m_1j_1+...+m_rj_r=2m$, $j_1+...+j_r=j$, $m_i\ne 2s-1$, $i=1,2,...,r$, $s=1,2,...,l$.

Remark 1. The value $(2m)!\sum_{j=1}^{2m}\sum_{r=1}^{j}\sum\prod_{k=1}^{r}\frac{(m_k!)^{-j_k}}{j_k!}$ in inequality (2) has a simple combinatorial sense (e.g., Sachkov, 1996): it equals the number of partitions of a set consisting of $2m$ elements into parts the number of elements in which is not equal to $2s-1$, $s=1,2,...,l$.

Remark 2. As follows from the results in Pinelis and Utev (1984), Bestsennaya and Utev (1991) and Ibragimov and Sharakhmetov (1997, 2001a), bounds (2) are sharp for l=1 and l=m; in addition, when l=m, the right-hand side of (2) equals to the best constant $C^*_{sym}(2m)$ in the Rosenthal's inequality for symmetric r.v.'s. It is also interesting to note that, in the case l=0, the expression on the right-hand side of (2), with the inner sum taken over all natural $m_1>m_2>...>m_r$ and $j_1,...,j_r$ satisfying the conditions $m_1j_1+...+m_rj_r=2m$, $j_1+...+j_r=j$, equals to the best constant the analogue of inequality (1) for nonnegative r.v.'s (see Ibragimov and Sharakhmetov, 2001b). Similar to Remark 1, the latter expression equals to the total number of partitions of a set consisting of $2m$ elements (the $2m$-th Bell number).

Let us formulate some auxiliary results needed for the proof of Theorem 1. The following lemma follows from Corollary 2 in Utev (1985) and the formula representing moments by semi invariants.

Lemma 1. Let $\xi_1,...,\xi_n$ be independent r.v.'s with $E\xi_i^{2s-1}=0$, $s=1,2,...,l$. Set $A_{k,n}=\sum_{i=1}^{n}E\xi_i^k$, k=1, 2,..., 2m, $B_n=A_{2,n}^{1/2}$. The following inequality holds:

$$E\left(\sum_{i=1}^{n}\xi_i\right)^{2m}\le(2m)!\sum_{r=0}^{2m}\sum\prod_{k=1}^{r}\frac{A_{m_k,n}^{j_k}(m_k!)^{-j_k}}{j_k!},\qquad(5)$$

where the inner sum is taken over all natural $m_1>m_2>...>m_r$ and $j_1,...,j_r$, satisfying the conditions $m_1j_1+...+m_rj_r=2m$, $j_1+...+j_r=j$, $m_i \neq 2s-1$, $i=1,2,...,r$, $s=1,2,...,l$.

Let A_{2m}, B, D>0. Denote $M_1^l(m,A_{2m},B)=\sup\limits_{n,\,\xi_k} ES_n^{2m}$, where $\sup$ is taken over positive integers n and all independent r.v.'s $\xi_1,...,\xi_n$ with $E\xi_i^{2s-1}=0$, $s=1,2,...,l$ and fixed $A_{2m,n}=A_{2m}$, $B_n=B$; $M_2^l(m,A_{2m},B)=\sup\limits_{n,\,\xi_k} ES_n^{2m}$, where $\sup$ is taken over positive integers n and all independent r.v.'s $\xi_1,...,\xi_n$ with $E\xi_i^{2s-1}=0$, $s=1,2,...,l$, for which $A_{2m,n}\le A_{2m}$, $B_n\le B$; $M^l(m,D)=\sup\limits_{n,\,\xi_k} ES_n^{2m}$, where $\sup$ is taken over positive integers n and all independent r.v.'s $\xi_1,...,\xi_n$ with $E\xi_i^{2s-1}=0$, $s=1,2,...,l$, and fixed $\max(A_{2m,n}, B_n^{2m})=D$.

The following lemma is well-known (see, e.g., Pinelis and Utev, 1984).

Lemma 2. For $2<s<2m$

$$\left|A_{s,n}\right| \le \left(A_{2m,n}^{s-2} B_n^{2(2m-s)}\right)^{1/(2m-2)}. \tag{6}$$

Relation (5) and Lemma 2 imply the following

Lemma 3. For A_{2m}, B>0

$$M_i^{\,l}(m,A_{2m},B)\le(2m)!\sum_{j=1}^{2m}\left(\sum_{r=1}^{j}\sum\prod_{k=1}^{r}\frac{(m_k\,!)^{-j_k}}{j_k\,!}\right)\left(A_{2m}^{\,m-j}B^{\,2m(j-1)}\right)^{1/(m-1)},$$

i=1,2, where the inner sum is taken over all natural $m_1>m_2>...>m_r$ and $j_1,...,j_r$, satisfying the conditions $m_1j_1+...+m_rj_r=2m$, $j_1+...+j_r=j$, $m_i\ne 2s-1$, $i=1,2,...,r$, $s=1,2,...,l$.

Proof of Theorem 1. From Lemma 3 and the evident inequality

$$M^{\,l}(m,D)\le M_2^{\,l}(m,D,D^{\,1/2m})$$

it follows that

$$M^{\,l}(m,\;D)\le(2m)!\sum_{j=1}^{2m}\sum_{r=1}^{j}\sum\prod_{k=1}^{r}\frac{(m_k\,!)^{-j_k}D}{j_k\,!},$$

where the inner sum is taken over all natural $m_1>m_2>...>m_r$ and $j_1,..., j_r$, satisfying the conditions $m_1j_1+...+m_rj_r=2m$, $j_1+...+j_r=j$, $m_i\ne 2s-1$, $i=1,2,...,r$, $s=1,2,...,l$. Since

$$C_l^{\,*}(2m)=\sup_{D>0}\frac{M^{\,l}(m,D)}{D},\tag{7}$$

this implies (2).

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