Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Bernstein–Walsh type theorems for real analytic functions in several variables

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No. 1390
Berlin 2008

2000 *Mathematics Subject Classification.* 41A17, 41A10, 41A60, 41A63, 41A25, 32U35, 32U05, 32E30, 32D20.

Key words and phrases. Polynomial approximation in higher dimensions, Bernstein-Walsh's type theorems, real-analytic functions in $\mathbb{R}^N$, Maximal convergence, Plurisubharmonicity, Pluricomplex Green functions.

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Abstract

The aim of this paper is to extend the classical maximal convergence theory of Bernstein and Walsh for holomorphic functions in the complex plane to real analytic functions in $\mathbb{R}^N$. In particular, we investigate the polynomial approximation behavior for functions $F : L \to \mathbb{C}$, $L = \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\}$, of the type $F = g\overline{h}$, where g and h are holomorphic in a neighborhood of a compact set $K \subset \mathbb{C}^N$. To this end the maximal convergence number $\rho(S_c, f)$ for continuous functions f defined on a compact set $S_c \subset \mathbb{C}^N$ is connected to a maximal convergence number $\rho(S_r, F)$ for continuous functions F defined on a compact set $S_r \subset \mathbb{R}^N$. We prove that $\rho(L, F) = \min\{\rho(K, h)), \rho(K, g)\}$ for functions $F = g\overline{h}$ if K is either a closed Euclidean ball or a closed polydisc. Furthermore, we show that $\min\{\rho(K, h)), \rho(K, g)\} \leq \rho(L, F)$ if K is regular in the sense of pluripotential theory and equality does not hold in general.
Our results are achieved by methods based on the theory of plurisubharmonic Green's function with pole at infinity and Lundin's formula for the extremal function Φ. Further, an important role plays a properly chosen transformation of Joukowski structure.

1 Introduction and main results

1.1 Maximal convergence

An important field in constructive approximation theory is the investigation of the relation between the smoothness of a function and the speed at which it can be approximated by polynomials. Classical one dimensional results in this context are for instance Jackson theorems and maximal convergence theorems of Bernstein and Walsh. Both kind of theorems have attracted much attention and some endeavor has recently been made to extend them to higher dimensions, e.g. Bernstein–Walsh type theorems for holomorphic functions in $\mathbb{C}^N$ ([Sic62], [Zah76], [Sic81], [Blo89]), squared modulus holomorphic functions in $\mathbb{R}^2$ ([Kra07]), harmonic functions in $\mathbb{R}^N$ ([And93], [BL91], [SZ01]), pluriharmonic functions in $\mathbb{C}^N$ ([Sic96]) and solutions of elliptic equations in $\mathbb{R}^N$ ([BL93], [BL94]).
The main intention of this paper is to extend the existing theory of maximal convergence to real analytic functions in $\mathbb{R}^N$. In particular, we investigate the polynomial approximation behavior for functions of holomorphic–antiholomorphic type, i. e.

$$F(x_1, \ldots, x_N, y_1, \ldots, y_N) = g(x_1 + iy_1, \ldots, x_N + iy_N)\overline{h(x_1 + iy_1, \ldots, x_N + iy_N)},$$

where g and h are holomorphic. In this context we define a real maximal convergence number and connect this number to the corresponding maximal convergence number for holomorphic functions in several complex variables.
To state Bernstein–Walsh type theorems we first need to define some approximation measure. As usual, we choose the n–th polynomial approximation error as follows:

(i) *Let $K \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be compact and let $F : K \to \mathbb{R}$ be a continuous function. Then we define*

$$E_n(K, F) := \inf\{\|F - P_n\|_K : \ P_n : \mathbb{R}^N \to \mathbb{R},\ P_n \text{ a polynomial of degree} \leq n\},$$

where $n \in \mathbb{N}$ and $\|\cdot\|_K$ denotes the supremum norm on K.

(ii) Let $K \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be compact and let $F : K \to \mathbb{C}$ be a continuous function. Then we define

$$E_n^c(K,F) := \inf\{\|F - P_n\|_K : \ P_n : \mathbb{R}^N \to \mathbb{C},\ P_n \text{ a polynomial of degree} \leq n\},$$

where $n \in \mathbb{N}$ and $\|\cdot\|_K$ denotes the supremum norm on K.

(iii) Let $K \subset \mathbb{C}^N$, $N \in \mathbb{N}$, be compact and let $f : K \to \mathbb{C}$ be a continuous function. Then we define

$$e_n(K,f) := \inf\{\|f - p_n\|_K : \ p_n : \mathbb{C}^N \to \mathbb{C},\ p_n \text{ a polynomial of degree} \leq n\},$$

where $n \in \mathbb{N}$ and $\|\cdot\|_K$ denotes the supremum norm on K.

Now let $\rho \in (1,\infty]$ and $f : K \to \mathbb{C}$ be a continuous function on the compact set $K \subset \mathbb{C}^N$, $N \in \mathbb{N}$, such that

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K,f)} = \frac{1}{\rho}. \tag{1.1}$$

Then we say a sequence $\{p_n\}_{n\in\mathbb{N}}$ of polynomials p_n of degree $\leq n$ converges maximally to f, if for every $R \in (1,\rho)$ the estimate

$$\|f - p_n\|_K \leq \frac{M}{R^n}, \qquad n \in \mathbb{N},$$

holds, where $M > 0$ is some constant independent of n.
Theorems which describe the connection between ρ and f as in equation (1.1) are called maximal convergence theorems. Analogously, we use this terminology for functions defined on compact sets in $\mathbb{R}^N$. Since we consider functions f defined on sets in $\mathbb{C}^N$ and functions F defined on sets in $\mathbb{R}^N$ simultaneously, we will distinguish them for more clarity by small and capital letters.
A famous result that marks the beginning of a series of studies on maximal convergence is the Bernstein theorem:

Theorem 1.1 ([Ber52], 1912)
Let $F : [-1,1] \to \mathbb{R}$ be continuous and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1],F)} \leq \frac{1}{\rho}$$

if and only if F has a holomorphic extension to the set

$$\{z \in \mathbb{C} : \ |h(z)| < \rho\},$$

where $h : \mathbb{C} \to \mathbb{C}\backslash\{z \in \mathbb{C} : |z| < 1\}$ is defined by $h(z) = z + \sqrt{z^2 - 1}$[1].

In the year 1934 Walsh (and Russell) discovered an outstanding extension of Theorem 1.1. The interval $[-1,1]$ in Theorem 1.1 can be replaced by compact sets $K \subset \mathbb{C}$ whose complement is connected and regular in the sense that for $\hat{\mathbb{C}}\backslash K$, $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, Green's function g_K with pole at infinity exists[2].
We recall, Green's function g_K is the uniquely determined function which has a logarithmic singularity at infinity, is continuous in $\mathbb{C}$, harmonic in $\mathbb{C}\backslash K$ and identically zero on K.

[1] The branch of the square root is chosen such that $h(x) > 1$ for $x > 1$.

[2] The generalization of Theorem 1.1 is due to Walsh [Wal26] in the case that $\hat{\mathbb{C}}\backslash K$ is simply connected in $\hat{\mathbb{C}}$ and due to Walsh and Russell [WR34] in the case that $\hat{\mathbb{C}}\backslash K$ is connected and regular. However in the literature Theorem 1.1 and Theorem 1.2 are just called the Bernstein-Walsh theorems.

Theorem 1.2 ([Wal35], 1934)
Let K be a compact subset of $\mathbb{C}$ such that $\hat{\mathbb{C}}\backslash K$ is connected and regular. Furthermore, let $f : K \to \mathbb{C}$ be continuous and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K, f)} \leq \frac{1}{\rho}$$

if and only if $f \equiv \tilde{f}|_K$, where $\tilde{f}$ is a holomorphic function in

$$L_\rho = \{z \in \mathbb{C} : \, e^{g_K(z)} < \rho\}.$$

A first step to an extension of the Bernstein–Walsh theorems to higher dimensions was taken by Sapagov in 1956. He stated the following analogous result to the Bernstein theorem.

Theorem 1.3 ([Sap56], 1956)
Let $F : K \subset \mathbb{R}^N \to \mathbb{R}$ be a continuous function, where $K := K_1 \times K_2 \times \cdots \times K_N$, $K_j = [-1, 1]$, $1 \leq j \leq N$, and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(K, F)} \leq \frac{1}{\rho}$$

if and only if F has a holomorphic extension to

$$L_{\rho_1} \times L_{\rho_2} \times \cdots \times L_{\rho_N},$$

where $L_{\rho_j} = \big\{z \in \mathbb{C} : \big|h(z)\big| < \rho\big\}$, $1 \leq j \leq N$, and h is defined as in Theorem 1.1.

The proof of Theorem 1.3 uses concepts of the proof of Bernstein's theorem. The function F is considered on the intervals K_j, $j = 1, 2, \ldots, N$, separately. In a similar way Theorem 1.2 can be generalized if the compact set $K \subset \mathbb{C}^N$ is the Cartesian product of compact subsets in the complex plane. However, for an arbitrary (sufficiently nice) compact set $K \subset \mathbb{C}^N$ the situation is much more involved. Siciak [Sic62] was the first who managed to extend Theorem 1.2 to appropriate compact sets $K \subset \mathbb{C}^N$, see Theorem 1.4. His key to this result was the introduction of an extremal function Φ for compact sets K in $\mathbb{C}^N$, which behaves in many ways like the (generalized) Green's function for $\hat{\mathbb{C}}\backslash K$ with pole at infinity. Later Zaharjuta found a different approach to Theorem 1.4, using the technique of Hilbert scales, compare [Zah76]. A refinement of Siciak's proof of Theorem 1.4 can be found in [Sic81]. We also refer to Bloom [Blo89] for an ingenious modification of Siciak's latter proof.

Theorem 1.4 ([Sic62], 1962)
Let $K \subset \mathbb{C}^N$ be a compact set such that the extremal function $\Phi(z, K)$ is continuous in $\mathbb{C}^N$. Further, let $f : K \to \mathbb{C}$ be continuous and $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K, f)} \leq \frac{1}{\rho}$$

if and only if f has a holomorphic extension to

$$L_{N,\rho} = \{z \in \mathbb{C}^N : \Phi(z, K) < \rho\}.$$

1.2 Results for real analytic functions in $\mathbb{R}^N$

In this work we prove maximal convergence theorems for real analytic functions[3] in $\mathbb{R}^N$, especially for functions of holomorphic–antiholomorphic type.
Let us start with an approximation question raised by Braess which encounters in the numerical treatment of elliptic differential equations. In [Bra01] it was conjectured that functions $F : \overline{B}_2 \to \mathbb{R}$, $\overline{B}_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, defined by

$$F(x, y) = \frac{1}{\left((x - x_0)^2 + (y - y_0)^2\right)^s}, \tag{1.2}$$

where $s \in (0, \infty)$ and $(x_0, y_0) \in \mathbb{R}^2$ such that $\rho_0 := \sqrt{x_0^2 + y_0^2} > 1$, satisfy the relation

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho_0}. \tag{1.3}$$

Clearly, the function F in (1.2) can be expressed as the squared modulus of a holomorphic function in some neighborhood of the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$. If we set $g(z) := 1/(z - z_0)^s$, where $z_0 = x_0 + iy_0$, then F can be written as

$$F(x, y) = \frac{1}{\left((x - x_0)^2 + (y - y_0)^2\right)^s} = g(z)\overline{g(z)}.$$

Further, g is holomorphic in $\mathbb{D}_{\rho_0} := \{z \in \mathbb{C} : |z| < \rho_0\}$ but in no neighborhood containing $\overline{\mathbb{D}}_{\rho_0}$, cf. Theorem 1.2. These facts indicate that the approximation behavior for functions of squared holomorphic type on the closed unit disk in $\mathbb{R}^2$ is determined by the approximation behavior of the corresponding holomorphic function on the closed unit disk in $\mathbb{C}$. Indeed, it was shown in [Kra07]:
Let $F : \overline{B}_2 \to \mathbb{R}$ be given by

$$F(x, y) = |g(x + iy)|^2,$$

where $g \in \mathcal{H}(\overline{\mathbb{D}})$. Further, let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \leq \frac{1}{\rho}$$

if and only if g has a holomorphic extension to $\mathbb{D}_\rho$

On the other hand the function F of (1.2) can be continued analytically to some open neighborhood of $\overline{B}_2$ in $\mathbb{C}^2$. Therefore Theorem 1.4 gives rise to ask if there exists a similar result for real–valued continuous functions defined on compact sets $K \subset \mathbb{R}^N$. For that reason we shed some light on Siciak's machinery which he used to prove Theorem 1.4, especially on Siciak's extremal function. We will show that Theorem 1.4 can be carried over to $\mathbb{R}^N$. In particular, there exists non–empty compact sets K in $\mathbb{R}^N$ such that Siciak's extremal function Φ is continuous.

Theorem 1.5
Let $K \subset \mathbb{R}^N$ be a compact set such that the extremal function $\Phi(z, K)$ is continuous in $\mathbb{C}^N$. Furthermore, let $F : K \to \mathbb{R}$ be continuous and $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(K, F)} \leq \frac{1}{\rho}$$

if and only if F has a holomorphic extension to

$$L_{N,\rho} = \left\{z \in \mathbb{C}^N : \Phi(z, K) < \rho\right\}.$$

[3] A function F defined on an open set $U \subset \mathbb{R}^N$ with range $\mathbb{R}$ or $\mathbb{C}$ is said to be real analytic in U, if for each $x \in U$ the function F may be represented by a convergent power series in some non–empty neighborhood of x in U.

Now, from some "theoretical" point of view the maximal convergence problem in $\mathbb{R}^N$ is solved. We obtain analogous to the complex case a *real* maximal convergence number. However, bearing for example Braess's problem in mind, we also would like to calculate the real maximal convergence number ρ for a given function F defined on a compact set $K \subset \mathbb{R}^N$. Consequently, we need the explicit formula of Φ, which requires even for simple compact sets, e.g. that of a closed unit ball in $\mathbb{R}^N$, much effort. An explicit representation of Φ for compact, convex and symmetric sets S in $\mathbb{R}^N$ with non–empty interior $\mathrm{Int}S$ is due to Lundin [Lun85][4]:
Let S be a compact, convex and symmetric (with respect to 0) subset of $\mathbb{R}^N$ with $\mathrm{Int}S \neq \emptyset$ in $\mathbb{R}^N$. Then

$$\Phi(z, S) = \max_{y \in \partial B_N} \left| h\big(a(y)\langle z, y\rangle\big)\right| \quad \textit{for } z \in \mathbb{C}^N, \tag{1.4}$$

where $h : \mathbb{C} \to \mathbb{C}\backslash\{z \in \mathbb{C} : |z| < 1\}$, $h(z) = z + \sqrt{z^2 - 1}$ and $a(y) := 1/\max_{x\in S}\langle x, y\rangle$ for $y \in \partial B_N := \big\{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \big(\sum_{j=1}^N |x_j|^2\big)^{1/2} = 1\big\}$. The symbol $\langle\cdot,\cdot\rangle$ means the standard scalar product in $\mathbb{R}^N$ and $\mathbb{C}^N$ respectively.

Formula (1.4) was obtained by a representation of Φ in terms of plurisubharmonic functions. It took more than twenty years to verify the identity

$$\log \Phi(z, K) = \sup\{u(z) : u \in \mathcal{L}, u|_K \leq 0\}, \qquad z \in \mathbb{C}^N, \tag{1.5}$$

for compact sets $K \subset \mathbb{C}^N$, where $\mathcal{L}$ denotes the set of all plurisubharmonic functions v in $\mathbb{C}^N$ which satisfy the growth condition $\sup_{z\in\mathbb{C}^N} |v(z) - \log(1 + |z|)| < \infty$.
Zaharjuta [Zah76] showed this identity under the assumption that Φ is continuous. He studied various properties of Hilbert spaces of analytic functions in this context. For the general case Siciak provides two different proofs, see [Sic81] and [Sic82]. His first proof is based on an approximation theorem by means of spectral theory and the latter proof was obtained by deep classical results of several complex variables.

It turns out that in general the real maximal convergence number ρ can't be determined explicitly even if the explicit formula of Φ is known, see Paragraph §2.7. Regarded from this point of view it is even more desirable to establish a link between the real maximal convergence number for functions of squared–modulus holomorphic type in $\mathbb{R}^N$ and the corresponding complex maximal convergence number, since then ρ can often be easily calculated for that kind of functions.
Before we state some results of this type let us introduce some notations.
$\overline{\mathcal{B}}_{N,r}$ and $\overline{B}_{2N,r}$ stand for the closed balls with center r in $\mathbb{C}^N$ and $\mathbb{R}^{2N}$ with respect to the Euclidean norm whereas $\overline{\mathcal{D}}_{N,r}$ and $\overline{D}_{2N,r}$ denote the closed polydiscs with center r in $\mathbb{C}^N$ and $\mathbb{R}^{2N}$ equipped with the maximum norm.

Theorem 1.6
(i) Let $g \in \mathcal{H}(\overline{\mathcal{B}}_{N,r})$ and $F : \overline{B}_{2N,r} \to \mathbb{R}$ be given by

$$F(x, y) = |g(x + iy)|^2, \quad (x, y) \in \overline{B}_{2N,r}, \quad x, y \in \mathbb{R}^N.$$

Further, let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N,r}, F)} \leq \frac{1}{\rho}$$

[4] For a different approach to this formula see [BT86]. A generalization of Lundin's formula for some special classes of compact, convex and symmetric subsets of $\mathbb{C}^N$ was discovered by Baran [Bar88]. It was achieved by considering various properties of a function of Joukowski type and making use of equation (1.5).

if and only if g has a holomorphic extension to $\mathcal{B}_{N,r\rho} = \left\{z \in \mathbb{C}^N : \left(\sum_{j=1}^N |z_j|^2\right)^{1/2} < r\rho\right\}$. (ii) Let $g \in \mathcal{H}(\overline{\mathcal{D}}_{N,r})$ and $F : \overline{D}_{2N,r} \to \mathbb{R}$ be given by

$$F(x,y) = |g(x+iy)|^2, \quad (x,y) \in \overline{D}_{2N,r}, \quad x,y \in \mathbb{R}^N.$$

Further, let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N,r}, F)} \le \frac{1}{\rho}$$

if and only if g has a holomorphic extension to $\mathcal{D}_{N,r\rho} = \left\{z \in \mathbb{C}^N : \max_{j=1,\dots,N} |z_j| < r\rho\right\}$.

The proof of the above theorem requires a lengthy preparation with several rather technical auxiliary results. To this end we fall back on Lundin's formula and the representation of Siciak's extremal function in terms of plurisubharmonic functions. In this context an additional change of variables is of crucial importance.

Further we prove that for functions of holomorphic–antiholomorphic type the real maximal convergence number can be also derived from the corresponding complex maximal convergence numbers.

Theorem 1.7
(i) Let $F : \overline{B}_{2N,r} \to \mathbb{R}$ be given by

$$F(x,y) = g(x+iy)\overline{h(x+iy)}, \quad (x,y) \in \overline{B}_{2N,r}, \quad x,y \in \mathbb{R}^N,$$

where $g, h \in \mathcal{H}(\overline{\mathcal{B}}_{N,r})$ and $g \not\equiv 0, h \not\equiv 0$. Further, let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n^c(\overline{B}_{2N,r}, F)} \le \frac{1}{\rho}$$

if and only if g and h have holomorphic extensions to $\mathcal{B}_{N,r\rho}$.
(ii) Let $F : \overline{D}_{2N,r} \to \mathbb{R}$ be given by

$$F(x,y) = g(x+iy)\overline{h(x+iy)}, \quad (x,y) \in \overline{D}_{2N,r}, \quad x,y \in \mathbb{R}^N,$$

where $g, h \in \mathcal{H}(\overline{\mathcal{D}}_{N,r})$ and $g \not\equiv 0, h \not\equiv 0$. Further, let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n^c(\overline{D}_{2N,r}, F)} \le \frac{1}{\rho}$$

if and only if g and h have holomorphic extensions to $\mathcal{D}_{N,r\rho}$.

The maximal convergence number ρ for F in Theorem 1.7 was determined by the largest Euclidean ball and polydisc in $\mathbb{C}^N$ to which g and h have holomorphic extensions. A different approach is described in Theorem 1.8. Here, the maximal convergence number is received by the holomorphic extension of F itself and its singularities.

Theorem 1.8
(i) Assume $F \in \mathcal{H}(\overline{B}_{2N,r})$ has the representation

$$F(x,y) = g(x+iy)\overline{h(x+iy)}, \quad (x,y) \in \overline{B}_{2N,r}, \quad x,y \in \mathbb{R}^N,$$

where $g, h \in \mathcal{H}(\overline{\mathcal{B}}_{N,r})$ and $g \not\equiv 0, h \not\equiv 0$. Then the following conditions are equivalent:

(a) $\limsup_{n\to\infty} \sqrt[n]{E_n^c(\overline{B}_{2N,r}, F)} \le \frac{1}{\rho}$

(b) F has a holomorphic extension to

$$\Big\{z=(z_1,\dots,z_{2N})\in\mathbb{C}^{2N}:\sum_{j=1}^{2N}\Big|\frac{z_j}{r}\Big|^2+\Big|\sum_{j=1}^{2N}\Big(\frac{z_j}{r}\Big)^2-1\Big|<\frac{1}{2}\Big(\rho^2+\frac{1}{\rho^2}\Big)\Big\}.$$

(c) F has a holomorphic extension to

$$\Big\{z=(z_1,\dots,z_{2N})\in\mathbb{C}^{2N}:\Big(\sum_{j=1}^{N}|z_{2j-1}+iz_{2j}|^2\Big)^{\frac{1}{2}}<r\rho\ \wedge\ \Big(\sum_{j=1}^{N}|z_{2j-1}-iz_{2j}|^2\Big)^{\frac{1}{2}}<r\rho\Big\}.$$

(d) F has no singular points on

$$\Big\{z=(z_1,\dots,z_{2N})\in\mathbb{C}^{2N}: z_{2j-1}=\frac{rR_j}{2R}\Big(Re^{it_j}+\frac{1}{Re^{it_j}}\Big), z_{2j}=\pm\frac{rR_j}{2iR}\Big(Re^{it_j}-\frac{1}{Re^{it_j}}\Big),$$
$$\sum_{j=1}^{N}R_j^2=R^2,\ R\in(1,\rho),\ R_j\in[0,R],\ t_j\in[0,2\pi],\ j=1,\dots,N\Big\}.$$

(ii) Let $F\in\mathcal{H}(\overline{D}_{2N,r})$ be of the form

$$F(x,y)=g(x+iy)\overline{h(x+iy)},\quad (x,y)\in\overline{D}_{2N,r},\quad x,y\in\mathbb{R}^N,$$

where $g,h\in\mathcal{H}(\overline{\mathcal{D}}_{N,r})$ and $g\not\equiv 0, h\not\equiv 0$. Then the following conditions are equivalent:

(a) $\limsup_{n\to\infty}\sqrt[n]{E_n^c(\overline{D}_{2N,r},F)}\le\frac{1}{\rho}$

(b) F has a holomorphic extension to

$$\Big\{z=(z_1,z_2,\dots,z_{2N})\in\mathbb{C}^{2N}:\max_{1\le j\le N}\Big(\Big|\frac{z_{2j-1}}{r}\Big|^2+\Big|\frac{z_{2j}}{r}\Big|^2+\Big|\Big(\frac{z_{2j-1}}{r}\Big)^2+\Big(\frac{z_{2j}}{r}\Big)^2-1\Big|\Big)<$$
$$\frac{1}{2}\Big(\rho^2+\frac{1}{\rho^2}\Big)\Big\}.$$

(c) F has a holomorphic extension to

$$\Big\{z=(z_1,z_2,\dots,z_{2N})\in\mathbb{C}^{2N}:\max_{1\le j\le N}|z_{2j-1}+iz_{2j}|<r\rho\ \wedge\ \max_{1\le j\le N}|z_{2j-1}-iz_{2j}|<r\rho\Big\}.$$

(d) F has a no singular points on

$$\Big\{z=(z_1,\dots,z_{2N})\in\mathbb{C}^{2N}: z_{2j-1}=\frac{rR_j}{2R}\Big(Re^{it_j}+\frac{1}{Re^{it_j}}\Big), z_{2j}=\pm\frac{rR_j}{2iR}\Big(Re^{it_j}-\frac{1}{Re^{it_j}}\Big),$$
$$\max_{1\le j\le N}R_j=R,\ R\in(1,\rho),\ R_j\in[0,R],\ t_j\in[0,2\pi],\ j=1,\dots,N\Big\}.$$

In the next theorem it is shown that the real maximal convergence number is always greater or equal than the corresponding complex maximal convergence numbers.

Theorem 1.9
Let $K \subset \mathbb{C}^N$ be a compact set such that Siciak's extremal function Φ is continuous in $\mathbb{C}^N$ and define $L = \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\}$. Moreover, let $F : L \to \mathbb{R}$ have the representation

$$F(x,y) = g(x+iy)\overline{h(x+iy)},$$

where g, h are holomorphic functions in an open connected neighborhood of K. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n^c(L,F)} \le \frac{1}{\rho}$$

if g and h have holomorphic extensions to $\left\{z \in \mathbb{C}^N : \Phi(z,K) < \rho\right\}$.

The example below illustrates that the opposite direction of Theorem 1.9 is not true in general, cf. [Kra07]. In particular, it is not sufficient to assume that the set K is regular in the sense of pluripotential theory.

Example 1.10
Consider

$$F(x,y) = \frac{1}{\left((x-\rho_0)^2 + y^2\right)^s} = g(z)\overline{g(z)},$$

where $g(z) = \dfrac{1}{(z-\rho_0)^s}$, $s \in (0,\infty)$ and $\rho_0 \in (1,\infty)$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1]\times[-1,1], F)} = \frac{1}{\rho_0}$$

but

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K,g)} = \frac{1}{|\psi(\rho_0)|} > \frac{1}{\rho_0}.$$

Here, ψ maps $\hat{\mathbb{C}}\backslash K$ univalently onto $\hat{\mathbb{C}}\backslash\{z \in \mathbb{C} : |z| \le 1\}$ such that $\psi(\infty) = \infty$[5].

The paper is organized as follows. Section 2 starts with a discussion of maximal convergence theory in $\mathbb{C}^N$ and $\mathbb{R}^N$. We give a short comparison about maximal convergence concepts in $\mathbb{C}$ and $\mathbb{C}^N$ and introduce Siciak's extremal function Φ. In Paragraph §2.4 we focus on necessary and sufficient conditions for the continuity of Φ which are essential for Theorem 1.5. Then we prove Theorem 1.5 and discuss how the maximal convergence number ρ can be computed. Here, some ingredients of plurisubharmonicity are required which are provided in Paragraph §2.6. In Section 3 we set the stage for the main results. We construct some transformations of Joukowski type and prove several upper and lower bounds for the real maximal convergence number. The estimates are based on the characterization of possible singularities of functions of squared modulus holomorphic and holomorphic–antiholomorphic type. Finally, we establish Theorems 1.6, 1.7, 1.8 and 1.9. The proofs are rather technical and quite lengthy.

2 Maximal convergence in $\mathbb{C}^N$ and $\mathbb{R}^N$

2.1 Notations

At first let us become acquainted with some notations and definitions in $\mathbb{R}^N$ and $\mathbb{C}^N$ which we need throughout our work.

[5]The conformal mapping ψ is up to a rotation uniquely determined.

An element of $\mathbb{R}^N$ is denoted by $x = (x_1, x_2, \dots, x_N)$ and an element of $\mathbb{C}^N$ by $z = (z_1, z_2, \dots, z_N)$. We equip the space $\mathbb{C}^N$ with the Euclidean norm

$$\|z\| := \sqrt{z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_N\overline{z_N}}$$

and the maximum norm

$$|z| := \max\{|z_1|, \dots, |z_N|\},$$

where we regard $\mathbb{R}^N$ as a subset of $\mathbb{C}^N$. The *open polydisc* in $\mathbb{C}^N$ with center $a \in \mathbb{C}^N$ and radius $r > 0$ is abbreviated by

$$\mathcal{D}_N(a, r) := \{z \in \mathbb{C}^N : |z - a| < r\}.$$

In particular, we denote for simplification

$$\mathcal{D}_{N,r} := \mathcal{D}_N(0, r) \qquad \text{and} \qquad \mathcal{D}_N := \mathcal{D}_N(0, 1).$$

The *closed polydisc* in $\mathbb{C}^N$ with center $a \in \mathbb{C}^N$ and radius $r > 0$ is defined by

$$\overline{\mathcal{D}}_N(a, r) := \{z \in \mathbb{C}^N : |z - a| \le r\}.$$

Similar as before, we put

$$\overline{\mathcal{D}}_{N,r} := \overline{\mathcal{D}}_N(0, r) \qquad \text{and} \qquad \overline{\mathcal{D}}_N := \overline{\mathcal{D}}_N(0, 1).$$

The symbols $D_{2N}(a, r)$ and $\overline{D}_{2N}(a, r)$ are used for the sets

$$D_{2N}(a, r) = \{x = (x_1, \dots, x_{2N}) \in \mathbb{R}^{2N} : \max_{1 \le j \le N} |x_{2j-1}^2 + x_{2j}^2 - a| < r\}$$

and

$$\overline{D}_{2N}(a, r) = \{x = (x_1, \dots, x_{2N}) \in \mathbb{R}^{2N} : \max_{1 \le j \le N} |x_{2j-1}^2 + x_{2j}^2 - a| \le r\},$$

where $a \in \mathbb{R}^{2N}$ and $r > 0$.
Polydiscs are balls with respect to the maximum norm. Open and closed balls in $\mathbb{C}^N$ with respect to the Euclidean norm are abbreviated by $\mathcal{B}_N(a, r)$ and $\overline{\mathcal{B}}_N(a, r)$, whereas $B_N(a, r)$ and $\overline{B}_N(a, r)$ stand for the open and closed balls in $\mathbb{R}^N$.

2.2 Comparison of maximal convergence in $\mathbb{C}$ and $\mathbb{C}^N$

As a preparation for our further considerations we give in this paragraph a rough outline of the ideas behind the proofs of Theorem 1.2 and Theorem 1.4.

The "only if"–part of Theorem 1.2 is based on the so–called Bernstein-Walsh property:

Let K be a compact subset of $\mathbb{C}$ such that $\hat{\mathbb{C}} \backslash K$ is connected and possesses a Green's function g_K with pole at infinity[6]*. Then g_K has the representation*

$$g_K(z) = \max\left\{0, \sup\left\{\frac{1}{\deg p} \log |p(z)|\right\}\right\}, \quad z \in \mathbb{C}, \quad \deg p : \textit{ degree of } p, \tag{2.1}$$

where the supremum is taken over all non-constant polynomials p satisfying $\|p\|_K \le 1$. Furthermore, if

$$L_\rho := \{z \in \mathbb{C} : e^{g_K(z)} < \rho\},$$

where $\rho > 1$, then

$$|p(z)| \le \|p\|_K \rho^{\deg p} \quad \textit{for } z \in L_\rho.$$

[6] Let G be a domain in $\hat{\mathbb{C}}$. Then there exists a unique Green's function for G if and only if ∂G is non–polar.

As we will see in Paragraph §2.3 there exists an extension of the Bernstein–Walsh property to several complex variables. Using this generalization the "only if"–part of Theorem 1.4 can be proved quite similar to the one dimensional case.
In the complex plane one shows that there exists a sequence of polynomials $p_n : \mathbb{C} \to \mathbb{C}$, $n \in \mathbb{N}$, such that the series $p_0 + \sum_{n=1}^{\infty}(p_n - p_{n-1})$ converges uniformly on compact subsets of L_ρ to a holomorphic function $\tilde{f}$ which agrees with f on K.

The "if"–part of Walsh's theorem can be established by using series expansions for holomorphic functions f in the region L_ρ, which can be approximated by lemniscates. To be more precisely, one can construct a sequence of lemniscates

$$\Omega_n := \{z \in \mathbb{C} : |p_n(z)| < r_n\}, \quad n \in \mathbb{N},$$

where p_n is a polynomial of degree $\leq n$ and r_n is a positive number, such that Ω_n increases up to L_ρ and contains K for n sufficiently large. Within the lemniscates Ω_n, $n \in \mathbb{N}$, the function f can be expanded into a Jacobi–series of the form

$$f(z) = \sum_{j=0}^{\infty} q_j(z)\,[p_n(z)]^j\,,$$

where q_j is a polynomial of degree $\leq n-1$. The Jacobi–series of f converges uniformly on $\overline{\Omega}'_n := \{z \in \mathbb{C} : |p_n(z)| \leq r'_n\}$, $0 < r'_n < r_n$. Truncating the series appropriately we get a suitable polynomial approximant to f. For details see Section III and Section IV of [Wal35].

In several complex variables the lemniscates can be replaced by a sequence of polynomial polyhedra which contains K and increases up to $L_{N,\rho}$. A polynomial polyhedra is defined as follows:

$$\Omega_n := \big\{z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N : |z_i| < r_n,\ |p_j(z)| < r_n, \quad i = 1, \ldots, N,\ j = 1, \ldots, k\big\},$$

where $r_n > 0$, p_j are complex–valued polynomials of degree $\leq n$ and $n, k \in \mathbb{N}$.

Now, a holomorphic function f in $\mathbb{C}^N$ can be expanded into a series of polynomials analogously to the one dimensional case. To this end one has to fall back on a deep theorem in several complex variables, namely the Oka–Weil extension theorem, see [Hoe66].

Sequences of polynomials which converge maximally to the corresponding holomorphic function can also be constructed by interpolation. A proper choice of interpolation points are for instance the extremal points of a compact set $K \subset \mathbb{C}^N$ introduced in [Sic62]. These points coincide with the well–known Fekete points if $K \subset \mathbb{C}$, see e.g. [Gai80].

Note, Green's function which plays the central role in approximating and interpolating holomorphic functions by polynomials in the complex plane is replaced by $\log \Phi$, where Φ is Siciak's extremal function introduced in [Sic62].

2.3 The extremal function Φ

Let $K \subset \mathbb{C}^N$ be compact and define for every $n \in \mathbb{N}$ the function $\Phi_n : \mathbb{C}^N \to \mathbb{R} \cup \{\infty\}$ by

$$\Phi_n(z, K) := \sup\big\{|p(z)| : p \in \mathcal{P}_n^c, |p(z)| \leq 1 \text{ for } z \in K\big\}, \tag{2.2}$$

where $\mathcal{P}_n^c = \{p : \mathbb{C}^N \to \mathbb{C} : p \text{ a polynomial of degree} \leq n\}$. Then the extremal function Φ may be introduced by means of Φ_n.

Definition 2.1
The function $\Phi : \mathbb{C}^N \to \mathbb{R} \cup \{\infty\}$ defined by

$$\Phi(z, K) = \sup_{n \in \mathbb{N}} \sqrt[n]{\Phi_n(z, K)} \tag{2.3}$$

is called the extremal function for the compact set $K \subset \mathbb{C}^N$.

A wide variety of polynomial estimates can be derived from the extremal function Φ. The cause depends upon the different ways to express Φ, cf. [Sic62]. In Paragraph §2.6 we will be acquainted with an important representation of Φ in terms of plurisubharmonic functions.

We also would like to point out that the extremal function Φ may be written in an analogous form to equation (2.1). In fact,

$$\log \Phi(z, K) = \max \left\{ 0, \sup \left\{ \frac{1}{\deg p} \log |p(z)| \right\} \right\}, \quad z \in \mathbb{C}^N,$$

where the supremum is taken over all non-constant polynomials $p : \mathbb{C}^N \to \mathbb{C}$ satisfying $\|p\|_K \le 1$.

To generalize the Bernstein-Walsh inequality in higher dimensions the compact set $K \subset \mathbb{C}^N$ has to satisfy an additional property, i.e. K has to be *unisolvent*:

A set $S \subset \mathbb{C}^N$ is called unisolvent, if every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ that vanishes on S is identical zero on $\mathbb{C}^N$.

A first glimpse about unisolvent sets gives the following examples.

(i) Let $K_j \subset \mathbb{C}$, $j = 1, \dots, N$, be an arbitrary set consisting of at least $n + 1$ different points. Then $K := K_1 \times \dots \times K_N$ is unisolvent of order n.

(ii) A compact set $K \subset \mathbb{R}^N$ with non–empty interior in $\mathbb{R}^N$ is unisolvent.

(iii) If $K \subset \mathbb{C}^N$ is unisolvent, then $\tilde{K} \supset K$ is also unisolvent.

Now let us state the Bernstein–Walsh inequality in higher dimensions.

If $K \subset \mathbb{C}^N$ is a unisolvent compact set and $p_n \in \mathcal{P}_n^c$ then

$$|p_n(z)| \le \|p_n\|_K [\Phi(z, K)]^n \quad \textit{for } z \in \mathbb{C}^N. \tag{2.4}$$

A useful tool for our further work is the preceeding theorem due to Siciak [Sic62], which describes the extremal function Φ for Cartesian products of compact sets. Observe, Theorem 1.3 is then just an application of Theorem 2.2. Moreover, we see that Cartesian products of compact intervals with non–empty interior have a continuous extremal function Φ.

Theorem 2.2 ([Sic62])
Let $K_1 \subset \mathbb{C}^{N_1}$ and $K_2 \subset \mathbb{C}^{N_2}$ be compact sets, $N_1, N_2 \in \mathbb{N}$. Then the extremal function Φ for $K_1 \times K_2$ is given by

$$\Phi((z, w), K_1 \times K_2) = \max\{\Phi(z, K_1), \Phi(w, K_2)\}, \qquad (z, w) \in \mathbb{C}^{N_1 + N_2}.$$

We refer the reader to [Kli91] for a nice proof of Theorem 2.2.

2.4 Necessary and sufficient conditions for the continuity of Φ

Theorem 1.5 is based on the assumption "Φ is continuous in $\mathbb{C}^N$". For that reason we like to discuss some necessary and sufficient conditions for this prerequisite.
At first we show that the extremal function $\Phi(z, K)$ can only be continuous in $\mathbb{C}^N$ if K is unisolvent.

Lemma 2.3
If $K \subset \mathbb{C}^N$ is a compact set such that $\Phi(z, K)$ is bounded in some closed proper neighborhood $\overline{U}$ of K, then K is unisolvent.
In particular, if $\Phi(z, K)$ is continuous in $\mathbb{C}^N$ then K is unisolvent.

Proof:
We assume K is not unisolvent. Then there exists a polynomial $\hat{p}_n \in \mathcal{P}_n^c$ for some $n \in \mathbb{N}$, such that

$$\|\hat{p}_n\|_K = 0 \qquad \text{but} \qquad \|\hat{p}_n\|_{\overline{U}} = t, \quad t > 0.$$

Now, since $\Phi(z, K)$ is bounded in $\overline{U}$, there exists some constant $M > 0$ such that

$$|\Phi(z, K)| < M, \qquad z \in \overline{U}.$$

As $(M^n \cdot \hat{p}_n)/t \in \mathcal{P}_n^c(K)$ we obtain due to the definition of Φ the inequality

$$\left|\frac{M^n}{t}\hat{p}_n(z)\right| \leq [\Phi(z)]^n, \qquad z \in \mathbb{C}^N.$$

In particular,

$$\left|\frac{M^n}{t}\hat{p}_n(z)\right| < M^n \qquad \text{for } z \in \overline{U}.$$

The latter is clearly impossible, since we have $|\hat{p}(\hat{z})| = t$ for some $\hat{z} \in \overline{U}$. Hence K is unisolvent.
■

In the sequel sufficient conditions for the continuity of Φ are described.

Remark 2.4 ([Sic82])
Let K be a compact subset of $\mathbb{C}^N$. Then the following conditions are equivalent:

(i) *Φ is continuous at every point $z \in K$, that is* $\lim\limits_{\substack{z_k \to z \in K, \\ z_k \in \mathbb{C}^N}} \Phi(z_k, K) = \Phi(z, K)$.

(ii) *Φ is continuous in $\mathbb{C}^N$.*

(iii) *To each real number $R > 1$ there exist an open neighborhood U of K and a constant $M > 0$ such that*

$$\|p\|_U \leq M \|p\|_K R^n$$

for every $p \in \mathcal{P}_n^c$, $n \in \mathbb{N}$.

Remark 2.5
Baouendi and Goulaouic [BG74] as well as Siciak and Nguyen Thanh Van [SN74] provided an additional equivalent condition in Remark 2.4 in the case that the compact set K is not too "small". The requirement "K should not be too small" means that any holomorphic function defined on a connected open neighborhood of K with $f|_K \equiv 0$ is identical zero.
Now, let K be such a set. Then (i), (ii) and (iii) of Remark 2.4 are equivalent to the statement:
If f is continuous and $\limsup_{n\to\infty} \sqrt[n]{e_n(f, K)} < 1$, then f extends to a uniquely determined holomorphic function in a neighborhood of K.

In view of Theorem 1.6 and Theorem 1.7 we prove that closed balls in $\mathbb{R}^N$ are not too "small" compact sets.

Lemma 2.6
Let $F : \overline{B}_N \to \mathbb{R}$ be continuous. If $F : \overline{B}_N \to \mathbb{R}$ has a holomorphic extension $\tilde{F}$ to some neighborhood of $\overline{B}_N$ in $\mathbb{C}^N$, then $\tilde{F}$ is uniquely determined.

Proof:
Suppose this were not true. Then there exist two different holomorphic extensions $\tilde{F}_1 : G_1 \to \mathbb{C}$ and $\tilde{F}_2 : G_2 \to \mathbb{C}$, where G_1 and G_2 are appropriate chosen neighborhoods of B_N in $\mathbb{C}^N$. In particular, these extensions are holomorphic in $\mathcal{D}_N(0,\varepsilon) = \{z \in \mathbb{C}^N : |z| < \varepsilon\} \subset G := G_1 \cap G_2$ for $\varepsilon > 0$ sufficiently small. There we may expand $\tilde{F}_1$ and $\tilde{F}_2$ into their power series

$$\tilde{F}_1(z) = \sum_{\alpha \in \mathbb{N}_0^N} a_\alpha z^\alpha \quad \text{and} \quad \tilde{F}_2(z) = \sum_{\alpha \in \mathbb{N}_0^N} b_\alpha z^\alpha, \quad z \in \mathcal{D}_N(0,\varepsilon),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As we have

$$\sum_{\alpha \in \mathbb{N}_0^N} a_\alpha z^\alpha = \sum_{\alpha \in \mathbb{N}_0^N} b_\alpha z^\alpha$$

for $z \in D_N(0,\varepsilon) = \{x \in \mathbb{R}^N : |x| < \varepsilon\}$, we get by the identity principle of power series

$$a_\alpha = b_\alpha, \qquad \alpha \in \mathbb{N}_0^N,$$

and therefore

$$\tilde{F}_1(z) = \tilde{F}_2(z) \quad \text{for } z \in \mathcal{D}_N(0,\varepsilon).$$

Since $\mathcal{D}_N(0,\varepsilon)$ is a non-empty open set of G we can apply the identity principle of holomorphic functions and obtain

$$\tilde{F}_1|_G \equiv \tilde{F}_2|_G.$$

■

A useful geometric criterion to check the continuity of $\Phi(z, K)$ in $\mathbb{C}^N$ goes back to Plesniak [Ple84]. We also refer the reader [Sic97].

Theorem 2.7 ([Ple84])
Let Ω be a bounded open subset of $\mathbb{C}^N$ with C^1–boundary. Then the extremal function Φ for $\overline{\Omega}$ is continuous in $\mathbb{C}^N$.

2.5 Maximal convergence in $\mathbb{R}^N$

Now we are well prepared to prove Theorem 1.5.

Proof of Theorem 1.5:

"$\Leftarrow$": Let us associate to each complex–valued polynomial $p_n : \mathbb{C}^N \to \mathbb{C}$ of degree $\leq n$,

$$p_n(z) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha| \leq n} a_\alpha z^\alpha,$$

the real–valued polynomial $P_n : \mathbb{R}^N \to \mathbb{R}$,

$$P_n(x) = \sum_{\alpha\in\mathbb{N}_0^N,\,|\alpha|\le n} \mathrm{Re}(a_\alpha)x^\alpha \ .$$

Notice, $P_n(x) = \mathrm{Re}\, p_n(x)$ for $x \in \mathbb{R}^N$. Therefore we obtain the inequality

$$\|F - P_n\|_K = \|F - \mathrm{Re}\, p_n\|_K \le \|F - p_n\|_K$$

and in view of Theorem 1.4 we achieve

$$\limsup_{n\to\infty} \sqrt[n]{E_n(K,F)} \le \limsup_{n\to\infty} \sqrt[n]{e_n(K,F)} \le \frac{1}{\rho}.$$

"$\Rightarrow$": For an arbitrary real polynomial $\hat{P}_n : \mathbb{R}^n \to \mathbb{R}$ of degree $\le n$ we define

$$\hat{p}_n(z) := \hat{P}_n(z), \quad z \in \mathbb{C}^N.$$

Since $\hat{p}_n \in P_n^c$ we derive the estimate

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K,F)} \le \limsup_{n\to\infty} \sqrt[n]{E_n(K,F)} \le \frac{1}{\rho}.$$

Theorem 1.4 now implies that F has a holomorphic extension to

$$L_\rho := \left\{z \in \mathbb{C}^N : \Phi(z,K) < \rho\right\}.$$

■

2.6 On some representations of Φ

Explicit representations of Φ are mainly based on the identity (1.5). For that reason we take for the convenience of the reader a short "pluricomplex interlude". We refer [Hoe66], [Kli91] and [Kra01] for a comprehensive discussion about this topic.

2.6.1 Plurisubharmonicity

Let us first recall the definitions of subharmonic and plurisubharmonic functions.

Definition 2.8
Let $\Omega \subset \mathbb{C}$ be an open set. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called subharmonic, if
- *(i) u is upper semicontinuous;*
- *(ii) the local submean inequality holds, i.e. for every $z_0 \in \Omega$ there exists an $\rho > 0$ such that*

$$u(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})dt$$

for any $r \in (0,\rho)$.

Definition 2.9
Let $\Omega \subset \mathbb{C}^N$ be an open set. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called plurisubharmonic, if
- *(i) u is upper semicontinuous;*
- *(ii) to each $z \in \Omega$ and $w \in \mathbb{C}^N$ correspond a neighborhood U of 0 in $\mathbb{C}$ such that the function*

$$\tau \mapsto u(z + \tau w)$$

is subharmonic in U.

The set of all plurisubharmonic functions defined on an open set $\Omega \subset \mathbb{C}^N$ is denoted by $PSH(\Omega)$. Typical examples of plurisubharmonic functions are $\log|f|$ and $|f|^\alpha$ for $\alpha > 0$, if f is holomorphic. A function $u \in PSH(\mathbb{C}^N)$ is said to be of *minimal growth at infinity* if

$$u(z) - \log(1+|z|) \leq O(1) \quad \text{as} \quad |z| \to \infty.$$

The family of all such functions will be denoted by

$$\mathcal{L} := \left\{ u \in PSH(\mathbb{C}^N) : u(z) \leq \beta + \log(1+|z|) \text{ for } z \in \mathbb{C}^N \right\}, \tag{2.5}$$

where $\beta \in \mathbb{R}$ may depend on u.
An attractive feature of plurisubharmonic functions with minimal growth at infinity is the full description by polynomials, see [Sic82].

Further, we put for any set $S \subset \mathbb{C}^N$

$$\mathcal{L}(S) := \{u \in \mathcal{L} : u(z) \leq 0 \quad \text{for } z \in S\},$$

and define for every $z \in \mathbb{C}^N$ the function

$$V(z,S) := \sup\{u(z) : u \in \mathcal{L}(S)\}.$$

The function V is called *the pluricomplex Green's function* to emphasize the analogy to the one-dimensional case.
Now, if S is compact, then the pluricomplex Green's function coincides with Siciak's extremal function Φ.

Theorem 2.10 (cf. [Sic82])
Let $K \subset \mathbb{C}^N$ be compact. Then

$$V(z,K) = \log \Phi(z,K) \quad \textit{for } z \in \mathbb{C}^N.$$

2.6.2 An explicit representation of Φ for compact, convex and symmetric sets in $\mathbb{R}^N$

Theorem 2.10 is the gist of Lundin's formula for the extremal function Φ for compact, convex and symmetric (with respect to 0) subsets S of $\mathbb{R}^N$ whose interior $\mathrm{Int}S$ is not empty. These sets have the nice property that they can be described by a continuous function with range in $[-1,1]$. More precisely:
If $S \subset \mathbb{R}^N$ is a compact, convex and symmetric (with respect to 0) subset of $\mathbb{R}^N$ and $\mathrm{Int}S \neq \emptyset$ in $\mathbb{R}^N$, then S can be described by

$$S = \left\{ x \in \mathbb{R}^N : a(y)\langle\, x, y\,\rangle \in [-1,1] \quad \text{for every } y \in \partial B_N \right\},$$

where $a(y) := 1/\max_{x\in S}\langle x, y\,\rangle$ is a continuous function defined on ∂B_N and $\langle\,\cdot\,,\cdot\,\rangle$ means the standard scalar product in $\mathbb{R}^N$ and $\mathbb{C}^N$ respectively.

Lundin's formula:
Let S be a compact, convex and symmetric (with respect to 0) subset of $\mathbb{R}^N$ with $\mathrm{Int}S \neq \emptyset$ in $\mathbb{R}^N$. Then

$$\Phi(z,S) = \max_{y \in \partial B_N} \left| h\big(a(y)\langle\, z, y\,\rangle\big)\right| \quad \textit{for } z \in \mathbb{C}^N,$$

where $h : \mathbb{C} \to \mathbb{C}\backslash\mathbb{D}$ is defined by $h(\eta) = \eta + \sqrt{\eta^2 - 1}$ and $a(y) := 1/\max_{x\in S}\langle x, y\,\rangle$ for $y \in \partial B_N$.

Now, if S is the closed unit ball in $\mathbb{R}^N$ then the formula for Φ can even be refined, cf. [Bar88].

Corollary 2.11
Let $h : \mathbb{C} \to \mathbb{C}\backslash\mathbb{D}$ be defined by $h(\eta) = \eta + \sqrt{\eta^2 - 1}$. Then

$$\Phi(z, \overline{B}_N) = \sqrt{h(\|z\|^2 + |\langle\, z, \overline{z}\,\rangle - 1|)} \quad \textit{for} \quad z \in \mathbb{C}^N.$$

2.7 Computation of ρ

In this paragraph we like to show how Theorem 1.5 and Lundin's formula can be utilized to get some information on the real maximal convergence number ρ for a given continuous function. For that purpose we first combine Theorem 1.5 and Corollary 2.11 to the following

Lemma 2.12
(i) Let $F : \overline{B}_N \to \mathbb{R}$ be a continuous function and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_N, F)} = \frac{1}{\rho}$$

if and only if F has a holomorphic extension $\tilde{F}$ to

$$L_{N,\rho} := \left\{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \|z\|^2 + \left| \sum_{j=1}^{N} z_j^2 - 1 \right| < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \right\}$$

but to no larger domain containing $\overline{L}_{N,\rho}$.
(ii) Let $F : \overline{D}_{2N} \to \mathbb{C}$ be a continuous function and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N}, F)} = \frac{1}{\rho}$$

if and only if F has a holomorphic extension $\tilde{F}$ to

$$\mathcal{L}_{2N,\rho} := \left\{ z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1\le j\le N} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + \big|z_{2j-1}^2 + z_{2j}^2 - 1\big| \right\}$$

but to no larger domain containing $\overline{\mathcal{L}}_{2N,\rho}$.

Thus in view of Lemma 2.12 the number ρ is the largest root of the equation

$$\gamma = \inf_{z\in P} \left\{ \|z\|^2 + \left| \sum_{j=1}^{N} z_j^2 - 1 \right| \right\} = \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \tag{2.6}$$

and

$$\gamma = \inf_{z\in P} \left\{ \max_{1\le j\le N} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + \big|z_{2j-1}^2 + z_{2j}^2 - 1\big| \right\} = \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \tag{2.7}$$

respectively, where P denotes the set of all non–removable singularities[7] of $\tilde{F}$.

Let us apply this lemma to a class of functions whose approximation behavior is also of special interest in the numerical treatment of elliptic differential equations, cf. [Sau], [Bra01] and [Kra06]. *Let $\hat{F} : \overline{B}_{2N} \to \mathbb{R}$ be given by*

$$F(x, y) = \frac{1}{\big(\sum_{j=1}^{N} \big((x_j - x_{0,j})^2 + (y_j - y_{0,j})^2\big)\big)^s}, \tag{2.8}$$

[7] A point $\tilde{z} \in \mathbb{C}^N$ is called a non–removable singularity of F if the function $\tilde{F}$ has no analytic continuation to a non–empty open neighborhood of $\tilde{z}$.

where $s \in (0,\infty)$ and $(x_0, y_0) \in \mathbb{R}^{2N}$ with $\rho_0 := \sqrt{\sum_{j=1}^N x_{0,j}^2 + y_{0,j}^2} > 1$.
Then

$$\tilde{F}(z_1, z_2, \ldots, z_{2N}) = \frac{1}{\left(\sum_{j=1}^N (z_{2j-1} - x_{0,j})^2 + (z_{2j} - y_{0,j})^2\right)^s}$$

is the uniquely determined holomorphic extension of $\hat{F}$ to $\mathbb{C}^{2N} \backslash P$, where $P = \{(z_1, z_2, \ldots, z_{2N}) \in \mathbb{C}^{2N} : z_{2j-1} = x_{0,j} \pm i(z_{2j} - y_{0,j}), j = 1, 2, \ldots, N\}$, see Remark 2.6 for the argument of uniqueness. Thus γ of equation (2.6) takes on the form

$$\gamma = \inf_{z \in P} \left\{ \sum_{j=1}^N \left(|z_{2j}|^2 + |x_{0,j} \pm i(z_{2j} - y_{0,j})|^2 \right) + \left| \sum_{j=1}^N \left(x_{0,j}^2 - y_{0,j}^2 \pm 2ix_{0,j}(y_{0,j} - z_{2j}) + 2y_{0,j} z_{2j} \right) - 1 \right| \right\}.$$

Obviously, the exact value for γ can't be given straight away and consequently the same holds for ρ. So the way we proposed to determine ρ by means of the maximal convergence number for the corresponding holomorphic function has the advantage that it can be done explicitly, i.e. we have $\rho = \rho_0$. Here, in contrast, we have the possibility to find easily some upper bounds for ρ by evaluating $||\hat{z}||^2 + \left|\sum_{j=1}^{2N} \hat{z}_j^2 - 1\right|$ for a given non-removable singularity $\hat{z}$ of F. We can also calculate ρ numerically. However, we have to bear in mind that the computation of γ gets more involved if N increases.
The following example illustrates how a sharp lower bound for the approximation error of $\hat{F}$ in (2.8) can be derived by a suitable chosen non–removable singularity.

Example 2.13
The function $\hat{F}$ in (2.8) has a non–removable singularity at

$$\begin{aligned}(z_1, z_2, \ldots, z_{2N-1}, z_{2N}) = \\ \Bigg(\frac{1}{2\rho_0}\left((x_{0,1} + iy_{0,1})^2 + 1\right), \frac{1}{2i\rho_0}\left((x_{0,1} + iy_{0,1})^2 - 1\right), \ldots, \\ \frac{1}{2\rho_0}\left((x_{0,N} + iy_{0,N})^2 + 1\right), \frac{1}{2i\rho_0}\left((x_{0,N} + iy_{0,N})^2 - 1\right) \Bigg).\end{aligned}$$

After a lengthy but straight forward computation we receive

$$\begin{aligned}\inf_{z \in P} \left\{ ||z|| + \left| \sum_{j=1}^{2N} z_j^2 - 1 \right| \right\} \leq \\ \sum_{j=1}^N \frac{|x_{0,j} + iy_{0,j}|^2}{2\rho_0^2} \left(|x_{0,j} + iy_{0,j}|^2 + \left| \frac{1}{x_{0,j} + iy_{0,j}} \right|^2 \right) + \left| \sum_{j=1}^N \frac{|x_{0,j} + iy_{0,j}|^2}{\rho_0^2} - 1 \right| \\ = \frac{1}{2}\left(\rho_0^2 + \frac{1}{\rho_0^2} \right),\end{aligned}$$

where P is the set of all non–removable singularities of $\tilde{F}$. This implies

$$\limsup_{n \to \infty} \sqrt[n]{E_n(\overline{B}_{2N}, \hat{F})} \geq \frac{1}{\rho_0}.$$

3 Proofs of Theorems 1.6, 1.7, 1.8 and 1.9

The proof of Theorem 1.6 in one complex variable is based on some special factorizations of holomorphic functions g like $g = \tilde{g}B$, where B is a Blaschke product, see [Kra07]. As inner functions of that form lack in several complex variables we have to choose a different approach in order to establish Theorem 1.6. Here, a useful tool is Lemma 2.12. We will see that

$$F \in \mathcal{H}(L_{2N,\rho}) \quad \text{if and only if} \quad g \in \mathcal{H}(\mathcal{B}_{N,\rho}).$$

In this context an additional change of variables is of crucial importance.
Now, let us set off on proving Theorem 1.6 and Theorem 1.7.

3.1 Auxiliary results

In the preceeding lemma we construct a transformation of Joukowski type for the domains $\mathbb{C}^2\backslash\{(z_1,z_2)\in\mathbb{C}^2 : z_1=0 \vee z_2=0\}$ and $\mathbb{C}^2\backslash\{(z_1,z_2)\in\mathbb{C}^2 : z_1=\pm iz_2\}$.

Lemma 3.1
Consider the map $h : \mathbb{C}^2\backslash\{(\xi,\eta)\in\mathbb{C}^2 : \xi=0 \vee \eta=0\} \to \mathbb{C}^2\backslash\{(z_1,z_2)\in\mathbb{C}^2 : z_1=\pm iz_2\}$,

$$(\xi,\eta)\mapsto\Big(\xi\frac{1}{2}\Big(\eta+\frac{1}{\eta}\Big),\xi\frac{1}{2i}\Big(\eta-\frac{1}{\eta}\Big)\Big).$$

Then h is surjective.
In particular, any point $(z_1,z_2)\in\mathbb{C}^2\backslash\{(z_1,z_2)\in\mathbb{C}^2 : z_1=\pm iz_2\}$ can be expressed as

$$(z_1,z_2)=\Big(\xi\frac{1}{2}\Big(\eta+\frac{1}{\eta}\Big),\xi\frac{1}{2i}\Big(\eta-\frac{1}{\eta}\Big)\Big),$$

if $\xi,\eta\in\mathbb{C}\backslash\{0\}$ are chosen appropriately.

Proof:
Observe, h maps $\mathbb{C}^2\backslash\{(\xi,\eta)\in\mathbb{C}^2 : \xi=0 \vee \eta=0\}$ in $\mathbb{C}^2\backslash\{(z_1,z_2)\in\mathbb{C}^2 : z_1=\pm iz_2\}$.
Now let $(z_1,z_2)\in\mathbb{C}^2\backslash\{(z_1,z_2)\in\mathbb{C}^2 : z_1=\pm iz_2\}$ be an arbitrary point. Then

$$(\xi,\eta)=\left(\sqrt{z_1^2+z_2^2},\frac{z_1+iz_2}{\sqrt{z_1^2+z_2^2}}\right)$$

is an element of $\mathbb{C}^2\backslash\{(\xi,\eta)\in\mathbb{C}^2 : \xi=0 \vee \eta=0\}$ and we calculate

$$\begin{aligned}
h(\xi,\eta) &= \Big(\xi\frac{1}{2}\Big(\frac{\eta^2+1}{\eta}\Big),\xi\frac{1}{2i}\Big(\frac{\eta^2-1}{\eta}\Big)\Big)\\
&= \left(\xi\frac{1}{2}\frac{(z_1+iz_2)^2+z_1^2+z_2^2}{(z_1+iz_2)\sqrt{z_1^2+z_2^2}},\xi\frac{1}{2i}\frac{(z_1+iz_2)^2-z_1^2-z_2^2}{(z_1+iz_2)\sqrt{z_1^2+z_2^2}}\right)\\
&= \left(\xi\frac{1}{2}\frac{2z_1^2+2iz_1z_2}{(z_1+iz_2)\sqrt{z_1^2+z_2^2}},\xi\frac{1}{2i}\frac{2iz_1z_2-2z_2^2}{(z_1+iz_2)\sqrt{z_1^2+z_2^2}}\right)\\
&= \Big(z_1,-\frac{-z_1z_2-iz_2^2}{z_1+iz_2}\Big)=(z_1,z_2).
\end{aligned}$$

■

The next two lemmata may be regarded as the nub for determining non–removable singularities of functions F of squared modulus holomorphic and holomorphic–antiholomorphic type.

Lemma 3.2
Let $\rho_j \in (0,\infty)$, $j=1,2,\dots,N$, *be arbitrary real numbers such that* $\rho := \sqrt{\sum_{j=1}^{N} \rho_j^2} > 1$.
Then the function $h : \mathbb{C}^N \to \mathbb{R}$ *defined by*

$$h(w) = \sum_{j=1}^{N} \frac{|w_j|^4}{2\rho_j^2} + \Big| \sum_{j=1}^{N} w_j^2 - 1 \Big|, \qquad w = (w_1, w_2, \dots, w_N) \in \mathbb{C}^N,$$

attains its minimum at the points $\hat{w} = (\pm\rho_1/\rho, \pm\rho_2/\rho, \dots, \pm\rho_N/\rho)$. *In particular,*

$$h(w) > \frac{1}{2\rho^2} \qquad \text{for} \quad w \in \mathbb{C}^N \backslash \{\hat{w}\}.$$

Proof:
Note, we may consider h as a function of $2N$ real variables. For that reason we define the function $\hat{h} : \mathbb{R}^{2N} \to \mathbb{R}$ by $\hat{h}(x,y) = h(x+iy)$, where $x, y \in \mathbb{R}^N$. Then the function $\hat{h}$ can be written as

$$\begin{aligned}
\hat{h}(x,y) &= \sum_{j=1}^{N} \frac{(x_j^2+y_j^2)^2}{2\rho_j^2} + \Big| \sum_{j=1}^{N} (x_j + iy_j)^2 - 1 \Big| \\
&= \sum_{j=1}^{N} \frac{(x_j^2+y_j^2)^2}{2\rho_j^2} + \sqrt{\Big(\sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 \Big)^2 + \Big(2 \sum_{j=1}^{N} x_j y_j \Big)^2}.
\end{aligned}$$

Observe, for $\hat{x} = (\pm\rho_1/\rho, \pm\rho_2/\rho, \dots, \pm\rho_N/\rho)$ and $\hat{y} = (0,0,\dots,0)$ the function $\hat{h}$ takes the value

$$\hat{h}(\hat{x},\hat{y}) = \sum_{j=1}^{N} \frac{\rho_j^2}{2\rho^4} + \Big| \sum_{j=1}^{N} \frac{\rho_j^2}{\rho^2} - 1 \Big| = \frac{1}{2\rho^2}.$$

Since our intention is to show that $\hat{h}$ assumes its minimum at the points $(\hat{x},\hat{y})$, we compute the first partial derivatives of $\hat{h}$ which necessarily vanish at critical points:

$$\frac{\partial \hat{h}(x,y)}{\partial x_j} = \frac{2(x_j^2+y_j^2)2x_j}{2\rho_j^2} + \frac{2\Big(\sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 \Big) 2x_j + 2\Big(2 \sum_{j=1}^{N} x_j y_j \Big) 2y_j}{2\sqrt{\Big(\sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 \Big)^2 + \Big(2 \sum_{j=1}^{N} x_j y_j \Big)^2}} = 0 \tag{3.1}$$

$$\frac{\partial \hat{h}(x,y)}{\partial y_j} = \frac{2(x_j^2+y_j^2)2y_j}{2\rho_j^2} + \frac{2\Big(\sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 \Big)(-2y_j) + 2\Big(2 \sum_{j=1}^{N} x_j y_j \Big) 2x_j}{2\sqrt{\Big(\sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 \Big)^2 + \Big(2 \sum_{j=1}^{N} x_j y_j \Big)^2}} = 0 \tag{3.2}$$

for $j = 1,2,\dots,N$. Now $(3.1)y_j - (3.2)x_j$ gives

$$\frac{8\Big(\sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 \Big) x_j y_j + 4\Big(\sum_{j=1}^{N} 2x_j y_j \Big)(y_j^2 - x_j^2)}{2\sqrt{\Big(\sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 \Big)^2 + \Big(2 \sum_{j=1}^{N} x_j y_j \Big)^2}} = 0$$

for $j = 1, 2, \ldots, N$.

For simplification we set $A := \sum\limits_{j=1}^{N}(x_j^2 - y_j^2) - 1$ and $B := \sum\limits_{j=1}^{N} 2x_jy_j$. Hence the last equation assumes the form

$$\frac{8Ax_jy_j + 4B(y_j^2 - x_j^2)}{2\sqrt{A^2 + B^2}} = 0$$

for $j = 1, 2, \ldots, N$.

Note, the assumption $A \neq 0$ implies $B = 0$ and $A = 0$ entails $B = 0$ since

$$2x_jy_jA = B(x_j^2 - y_j^2), \quad j = 1, 2, \ldots, N, \tag{3.3}$$

and therefore

$$AB = A\Big(\sum_{j=1}^{N} 2x_jy_j\Big) = B\Big(\sum_{j=1}^{N} x_j^2 - y_j^2\Big) = B(A+1).$$

Thus, we only have to distinguish the two cases:

(i) $A \neq 0$ and $B = 0$

(ii) $A = 0$ and $B = 0$

Case (i): By equation (3.3) we obtain

$$x_j = 0 \ \text{ or } \ y_j = 0 \quad \text{ for } \ j = 1, 2, \ldots, N.$$

If $x_j = y_j = 0$ for $j = 1, 2, \ldots, N$, we have $\hat{h}(0,0) = 1$. Hence $\hat{h}$ can't have an absolute minimum at $(0,0)$. Therefore we may assume that there exists at least one x_k or y_k, $k \in \{1, 2, \ldots, N\}$, which is not zero. Thus equations (3.1) and (3.2) simplify to

$$\frac{x_k^3}{\rho_k^2} + \frac{x_kA}{|A|} = 0 \tag{3.4}$$

if $x_k \neq 0$, and

$$\frac{y_k^3}{\rho_j^2} - \frac{y_kA}{|A|} = 0 \tag{3.5}$$

if $y_k \neq 0$. Equation (3.4) implies $A < 0$ and (3.5) shows $A > 0$. Thus either $x_j = 0$ for all $j = 1, 2, \ldots, N$, or $y_j = 0$ for all $j = 1, 2, \ldots, N$. Since $\hat{h}(0, y) \geq 1$ for $y = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^N$, we only have to study the case $y_j = 0$ for all $j = 1, 2, \ldots, N$. Because of (3.4) we get

$$x_j = 0 \ \text{ or } \quad \text{ or } \ x_j = \pm\rho_j, \qquad j = 1, 2, \ldots, N.$$

Without loss of generality we may assume

$$x_j = \pm\rho_j \qquad \text{for} \quad j = 1, 2, \ldots, m, \quad m \leq N,$$

and

$$x_j = 0 \qquad\quad \text{for} \quad j = m+1, m+2, \ldots, N.$$

Then we obtain for such a point the estimate

$$\hat{h}(\pm\rho_1, \ldots, \pm\rho_m, 0, \ldots, 0) = \frac{1}{2}\sum_{j=1}^{m} \rho_j^2 + \Big|\sum_{j=1}^{m} \rho_j^2 - 1\Big| \geq \frac{1}{2} > \frac{1}{2\rho^2}$$

as

$$\frac{1}{2}\sum_{j=1}^{m}\rho_j^2+\left|\sum_{j=1}^{m}\rho_j^2-1\right|=\begin{cases}\frac{1}{2}\sum\limits_{j=1}^{m}\rho_j^2+\sum\limits_{j=1}^{m}\rho_j^2-1 & \text{for } \sum\limits_{j=1}^{m}\rho_j^2\geq 1\\ \frac{1}{2}\sum\limits_{j=1}^{m}\rho_j^2+1-\sum\limits_{j=1}^{m}\rho_j^2 & \text{for } \sum\limits_{j=1}^{m}\rho_j^2< 1\end{cases}$$

$$\geq\begin{cases}\frac{1}{2} & \text{for } \sum\limits_{j=1}^{m}\rho_j^2\geq 1\\ 1-\frac{1}{2}\sum\limits_{j=1}^{m}\rho_j^2>\frac{1}{2} & \text{for } \sum\limits_{j=1}^{m}\rho_j^2< 1.\end{cases}$$

Consequently, $\hat{h}$ has a chance to take on an absolute minimum only if $A=0$. This fact leads us indispensably to the second case:
Case (ii): Here, our minimum problem consists in the following extremum problem with side conditions:
Find the minimum of

$$\hat{h}(x,y)=\sum_{j=1}^{N}\frac{(x_j^2+y_j^2)^2}{2\rho_j^2}$$

under the conditions

$$g_1(x,y)=\sum_{j=1}^{N}(x_j^2-y_j^2)-1=0 \quad\text{and}\quad g_2(x,y)=\sum_{j=1}^{N}x_jy_j=0.$$

We shall solve this problem by the Lagrange multiplication formalism as the hypotheses for this machinery are fulfilled.
Consequently, we have to determine the minimum of the function

$$\tilde{h}(x,y,\lambda_1,\lambda_2)=\sum_{j=1}^{N}\frac{(x_j^2+y_j^2)^2}{2\rho_j^2}+\lambda_1\left(\sum_{j=1}^{N}(x_j^2-y_j^2)-1\right)+\lambda_2\sum_{j=1}^{N}x_jy_j$$

for $x,y\in\mathbb{R}^N$ and $\lambda_1,\lambda_2\in\mathbb{R}$.
Thus the following conditions must be fulfilled:

$$\frac{\partial\tilde{h}(x,y,\lambda_1,\lambda_2)}{\partial x_j}=\frac{2x_j(x_j^2+y_j^2)}{\rho_j^2}+\lambda_1 2x_j+\lambda_2 y_j=0,\quad j=1,2,\ldots,N, \tag{3.6}$$

$$\frac{\partial\tilde{h}(x,y,\lambda_1,\lambda_2)}{\partial y_j}=\frac{2y_j(x_j^2+y_j^2)}{\rho_j^2}+\lambda_1 2(-y_j)+\lambda_2 x_j=0,\quad j=1,2,\ldots,N, \tag{3.7}$$

$$\frac{\partial\tilde{h}(x,y,\lambda_1,\lambda_2)}{\partial\lambda_1}=\sum_{j=1}^{N}(x_j^2-y_j^2)-1=0, \tag{3.8}$$

$$\frac{\partial\tilde{h}(x,y,\lambda_1,\lambda_2)}{\partial\lambda_2}=\sum_{j=1}^{N}x_jy_j=0. \tag{3.9}$$

In order to determine λ_1 and λ_2, we consider the equations

$$(3.6)x_j-(3.7)y_j=\frac{2(x_j^2-y_j^2)(x_j^2+y_j^2)}{\rho_j^2}+\lambda_1 2(x_j^2+y_j^2)=0,\qquad j=1,2,\ldots,N, \tag{3.10}$$

and

$$(3.6)y_j + (3.7)x_j = \frac{4x_j y_j(x_j^2 + y_j^2)}{\rho_j^2} + \lambda_2(x_j^2 + y_j^2) = 0, \qquad j = 1, 2, \ldots, N. \tag{3.11}$$

Now, equations (3.10) and (3.11) imply for $j = 1, 2, \ldots, N$,

$$(x_j^2 - y_j^2) = -\lambda_1 \rho_j^2 \quad \text{or} \quad x_j = y_j = 0 \tag{3.12}$$

and

$$4x_j y_j = -\lambda_2 \rho_j^2 \quad \text{or} \quad x_j = y_j = 0. \tag{3.13}$$

Without loss of generality we may assume $x_j \neq 0$ or $y_j \neq 0$ for $j = 1, 2, \ldots, m$, $m \leq N$, and $x_j = y_j = 0$ for $j = m + 1, \ldots, N$.
As the case $x_j = y_j = 0$ for $j = 1, 2, \ldots, N$, does not meet the side conditions we can exclude it. Therefore we obtain by (3.12)

$$\sum_{j=1}^{N}(x_j^2 - y_j^2) - 1 = \sum_{j=1}^{m}(-\lambda_1 \rho_j^2) - 1 = 0$$

and by (3.13)

$$\sum_{j=1}^{N} x_j y_j = -\frac{1}{4}\lambda_2 \sum_{j=1}^{m} \rho_j^2 = 0.$$

Thus it follows

$$\lambda_1 = -\frac{1}{\sum_{j=1}^{m} \rho_j^2} \quad \text{and} \quad \lambda_2 = 0.$$

Further, we receive from equations (3.6) and (3.7)

$$x_j = \pm\sqrt{-\lambda_1}\rho_j \quad \text{and} \quad y_j = 0 \quad \text{for} \quad j = 1, 2, \ldots, m.$$

Inserting x_j and y_j, $j = 1, 2, \ldots, N$, into $\hat{h}$ gives

$$\hat{h}(x, y) = \frac{1}{2\Big(\sum_{j=1}^{m} \rho_j^2\Big)^2} \sum_{j=1}^{m} \rho_j^2 = \frac{1}{2\sum_{j=1}^{m} \rho_j^2}.$$

Since

$$\sum_{j=1}^{m} \rho_j^2 < \sum_{j=1}^{N} \rho_j^2$$

we conclude that $\hat{h}$ assumes its minimum if and only if

$$x_j = \pm\frac{\rho_j}{\sqrt{\sum_{j=1}^{N} \rho_j^2}} \quad \text{and} \quad y_j = 0 \quad \text{for} \quad j = 1, 2, \ldots, N,$$

and we are done. ■

Lemma 3.3
Let $\rho > 1$ be arbitrary.
(i) Consider the sets

$$L_{2N,\rho} := \Big\{ z = (z_1, \dots, z_{2N}) \in \mathbb{C}^{2N} : \sum_{j=1}^{2N} |z_j|^2 + \Big| \sum_{j=1}^{2N} z_j^2 - 1 \Big| < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \Big\}$$

and

$$T_{2N,\rho} := \Big\{ z = (z_1, ..., z_{2N}) \in \mathbb{C}^{2N} \colon \Big(\sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2 \Big)^{\frac{1}{2}} < \rho \ \wedge \ \Big(\sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2 \Big)^{\frac{1}{2}} < \rho \Big\}.$$

Then

$$L_{2N,\rho} \subset T_{2N,\rho}.$$

(ii) Analogously, the sets

$$\mathcal{L}_{2N,\rho} := \Big\{ z = (z_1, z_2, \dots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1 \le j \le N} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1| \big) < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \Big\}$$

and

$$\mathcal{T}_{2N,\rho} := \Big\{ z = (z_1, z_2, \dots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1 \le j \le N} |z_{2j-1} + iz_{2j}| < \rho \ \wedge \max_{1 \le j \le N} |z_{2j-1} - iz_{2j}| < \rho \Big\}$$

satisfy the inclusion

$$\mathcal{L}_{2N,\rho} \subset \mathcal{T}_{2N,\rho}.$$

Proof:
To (i): To prove this inclusion let an arbitrary point $z = (z_1, \dots, z_{2N}) \in \mathbb{C}^{2N} \backslash T_{2N,\rho}$ be given. Without loss of generality we may assume that

$$z_{2j-1} = \pm\, iz_{2j} \qquad \text{for} \quad j = 1, \dots, m, \quad m \in \mathbb{N}_0,$$

and

$$z_{2j-1} \ne \pm\, iz_{2j} \qquad \text{for} \quad j = m+1, \dots, N.$$

Now, for $j = m+1, \dots, N$ we choose the representation of Lemma 3.1

$$z_{2j-1} = \xi_j \frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big) \qquad \text{and} \qquad z_{2j} = \xi_j \frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big), \qquad \xi_j, \eta_j \in \mathbb{C}\backslash\{0\}.$$

In addition, we set

$$\tilde{\rho}_j := |\xi_j \eta_j| \qquad \text{as well as} \qquad \hat{\rho}_j := \Big| \xi_j \frac{1}{\eta_j} \Big|, \qquad j = m+1, \dots, N.$$

Thus we obtain

$$\begin{aligned} \sum_{j=1}^{2N} |z_j|^2 + \Big| \sum_{j=1}^{2N} z_j^2 - 1 \Big| &= \sum_{j=1}^{m} 2|z_{2j}|^2 + \sum_{j=m+1}^{N} \Big(\Big| \frac{\xi_j}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big) \Big|^2 + \Big| \frac{\xi_j}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big) \Big|^2 \Big) + \\ &\quad \Big| \sum_{j=m+1}^{N} \Big(\Big(\frac{\xi_j}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big) \Big)^2 + \Big(\frac{\xi_j}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big) \Big)^2 \Big) - 1 \Big| \\ &= \sum_{j=1}^{m} 2|z_{2j}|^2 + \sum_{j=m+1}^{N} |\xi_j|^2 \frac{1}{2}\Big(|\eta_j|^2 + \frac{1}{|\eta_j|^2} \Big) + \Big| \sum_{j=m+1}^{N} \xi_j^2 - 1 \Big| \end{aligned}$$

$$\underset{\left|\frac{1}{\eta_j}\right|=\frac{|\xi_j|}{\tilde\rho_j}}{=} \sum_{j=1}^m 2|z_{2j}|^2 + \frac{1}{2}\sum_{j=m+1}^N \tilde\rho_j^2 + \frac{1}{2}\sum_{j=m+1}^N \frac{|\xi_j|^4}{\tilde\rho_j^2} + \Big|\sum_{j=m+1}^N \xi_j^2 - 1\Big|$$

$$= \sum_{j=1}^m 2|z_{2j}|^2 + \frac{1}{2}\tilde\rho^2 + \frac{1}{2}\sum_{j=m+1}^N \frac{|\xi_j|^4}{\tilde\rho_j^2} + \Big|\sum_{j=m+1}^N \xi_j^2 - 1\Big|$$

as well as

$$\sum_{j=1}^{2N}|z_j|^2 + \Big|\sum_{j=1}^{2N} z_j^2 - 1\Big| = \sum_{j=1}^m 2|z_{2j}|^2 + \sum_{j=m+1}^N |\xi_j|^2\frac{1}{2}\Big(|\eta_j|^2 + \frac{1}{|\eta_j|^2}\Big) + \Big|\sum_{j=m+1}^N \xi_j^2 - 1\Big|$$

$$\underset{|\eta_j|=\frac{|\xi_j|}{\hat\rho_j}}{=} \sum_{j=1}^m 2|z_{2j}|^2 + \frac{1}{2}\sum_{j=m+1}^N \hat\rho_j^2 + \frac{1}{2}\sum_{j=m+1}^N \frac{|\xi_j|^4}{\hat\rho_j^2} + \Big|\sum_{j=m+1}^N \xi_j^2 - 1\Big|$$

$$= \sum_{j=1}^m 2|z_{2j}|^2 + \frac{1}{2}\hat\rho^2 + \frac{1}{2}\sum_{j=m+1}^N \frac{|\xi_j|^4}{\hat\rho_j^2} + \Big|\sum_{j=m+1}^N \xi_j^2 - 1\Big|,$$

where $\tilde\rho = \Big(\sum_{j=m+1}^N \tilde\rho_j^2\Big)^{\frac{1}{2}}$ and $\hat\rho = \Big(\sum_{j=m+1}^N \hat\rho_j^2\Big)^{\frac{1}{2}}$.
By the definition of $T_{2N,\rho}$ we have for the chosen element $z = (z_1, \ldots, z_{2N})$ either the estimate

$$\sum_{j=1}^N |z_{2j-1} + iz_{2j}|^2 = \sum_{j=1}^m |z_{2j-1} + iz_{2j}|^2 + \tilde\rho^2 \geq \rho^2$$

or the estimate

$$\sum_{j=1}^N |z_{2j-1} - iz_{2j}|^2 = \sum_{j=1}^m |z_{2j-1} - iz_{2j}|^2 + \hat\rho^2 \geq \rho^2.$$

Therefore we conclude

$$\sum_{j=1}^m 2|z_{2j}|^2 \geq \begin{cases} \max\{\frac{1}{2}(\rho^2 - \tilde\rho^2), 0\} & \text{if } \sum_{j=1}^m |z_{2j-1} + iz_{2j}|^2 + \tilde\rho^2 \geq \rho^2. \\ \max\{\frac{1}{2}(\rho^2 - \hat\rho^2), 0\} & \text{if } \sum_{j=1}^m |z_{2j-1} - iz_{2j}|^2 + \hat\rho^2 \geq \rho^2. \end{cases}$$

Lemma 3.2 implies now

$$\sum_{j=1}^{2N}|z_j|^2 + \Big|\sum_{j=1}^{2N} z_j^2 - 1\Big|$$

$$\geq \begin{cases} \max\{\frac{1}{2}(\rho^2 - \tilde\rho^2), 0\} + \frac{1}{2}\big(\tilde\rho^2 + \frac{1}{\tilde\rho^2}\big) & \text{if } \sum_{j=1}^m |z_{2j-1} + iz_{2j}|^2 + \tilde\rho^2 \geq \rho^2. \\ \max\{\frac{1}{2}(\rho^2 - \hat\rho^2), 0\} + \frac{1}{2}\big(\hat\rho^2 + \frac{1}{\hat\rho^2}\big) & \text{if } \sum_{j=1}^m |z_{2j-1} - iz_{2j}|^2 + \hat\rho^2 \geq \rho^2. \end{cases}$$

This guarantees

$$\sum_{j=1}^{2N}|z_j|^2 + \Big|\sum_{j=1}^{2N} z_j^2 - 1\Big| \geq \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big).$$

Since $z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N}\backslash T_{2N,\rho}$ was arbitrary we derive

$$L_{2N,\rho} \subset T_{2N,\rho}$$

as required.

To (ii): Let $z = (z_1, \dots, z_{2N}) \in \mathbb{C}^{2N} \backslash \mathcal{T}_{2N,\rho}$ be an arbitrary point. We may assume that

$$z_{2j-1} = \pm\, i z_{2j} \qquad \text{for} \quad j = 1, \dots, m, \quad m \in \mathbb{N}_0,$$

and

$$z_{2j-1} \neq \pm\, i z_{2j} \qquad \text{for} \quad j = m+1, \dots, N.$$

If $j = m+1, \dots, N$, then we take the representation of Lemma 3.1

$$z_{2j-1} = \xi_j \frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big) \qquad \text{and} \qquad z_{2j} = \xi_j \frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big), \qquad \xi_j, \eta_j \in \mathbb{C}\backslash\{0\}$$

and set

$$\tilde{r}_j := |\xi_j \eta_j| \qquad \text{as well as} \qquad \hat{r}_j := \Big|\xi_j \frac{1}{\eta_j}\Big|, \qquad j = m+1, \dots, N.$$

Hence we obtain either the estimate

$$\max_{1 \le j \le m} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1|\big) \ge \max_{1 \le j \le m} \Big(\frac{1}{2}|z_{2j-1} \pm i z_{2j}|^2 + 1\Big) \ge$$

$$\ge \frac{1}{2}\rho^2 + 1 > \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \qquad \text{if} \ \max_{1 \le j \le m} \{|z_{2j-1} + i z_{2j}|, |z_{2j-1} - i z_{2j}|\} \ge \rho$$

or

$$\max_{m+1 \le j \le N} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1|\big) = \max_{m+1 \le j \le N} \Big(|\xi_j|^2 \frac{1}{2}\Big(|\eta_j|^2 + \frac{1}{|\eta_j|^2}\Big) + \Big|\xi_j^2 - 1\Big|\Big)$$

$$\ge \begin{cases} \displaystyle\max_{m+1 \le j \le N} \Big(\tfrac{1}{2}\tilde{r}_j^2 + \tfrac{|\xi_j|^4}{2\tilde{r}_j^2} + \Big|\xi_j^2 - 1\Big|\Big) \ge \tfrac{1}{2}\big(\rho^2 + \tfrac{1}{\rho^2}\big) & \text{if} \ \displaystyle\max_{m+1 \le j \le N} |z_{2j-1} + i z_{2j}| \ge \rho. \\ \displaystyle\max_{m+1 \le j \le N} \Big(\tfrac{1}{2}\hat{r}_j^2 + \tfrac{|\xi_j|^4}{2\hat{r}_j^2} + \Big|\xi_j^2 - 1\Big|\Big) \ge \tfrac{1}{2}\big(\rho^2 + \tfrac{1}{\rho^2}\big) & \text{if} \ \displaystyle\max_{m+1 \le j \le N} |z_{2j-1} - i z_{2j}| \ge \rho. \end{cases}$$

Putting all things together gives

$$\max_{1 \le j \le N} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1|\big) \ge \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big)$$

but this means that

$$\mathcal{L}_{2N,\rho} \subset \mathcal{T}_{2N,\rho}.$$

■

To make the proof of Theorem 1.6 for the reader more convenient we state an additional lemma and definition.

Lemma 3.4
Let $\rho \in (1, \infty)$ and $\hat{\rho}_j \in (0, \infty)$, $j = 1, 2, \dots, N$, be arbitrary numbers such that $\hat{\rho} := \sqrt{\sum_{j=1}^N \hat{\rho}_j^2} \in (1, \rho)$. Furthermore, let $\varepsilon > 0$ be any real number satisfying

$$\varepsilon < \min\left\{\frac{\rho^2 + \frac{1}{\rho^2} - \hat{\rho}^2 - \frac{1}{\hat{\rho}^2}}{28N}, \min_{j=1,2,\dots,N} \frac{\hat{\rho}_j}{\hat{\rho}}\right\}.$$

Then

$$\frac{1}{2}\hat{\rho}^2 + \frac{1}{2}\sum_{j=1}^N \frac{|\xi_j|^4}{\hat{\rho}_j^2} + \Big|\sum_{j=1}^N \xi_j^2 - 1\Big| < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big)$$

for $\xi = (\xi_1, \dots, \xi_N) \in U_{N,\varepsilon} := \Big\{z = (z_1, \dots, z_N) \in \mathbb{C}^N : \max_{1 \le j \le N} \Big|z_j - \frac{\hat{\rho}_j}{\hat{\rho}}\Big| < \varepsilon\Big\}.$

Proof:
We first determine an upper bound for the expression

$$\Big|\sum_{j=1}^{N}\xi_j^2-1\Big|^2=\Big(\sum_{j=1}^{N}\big((\operatorname{Re}\xi_j)^2-(\operatorname{Im}\xi_j)^2\big)-1\Big)^2+4\Big(\sum_{j=1}^{N}\operatorname{Re}\xi_j\operatorname{Im}\xi_j\Big)^2.$$

If $\xi=(\xi_1,\xi_2,\dots,\xi_N)\in U_{N,\varepsilon}$ we may estimate

$$\begin{aligned}\Big|\sum_{j=1}^{N}\xi_j^2-1\Big|^2 &\le \Big(\Big|\sum_{j=1}^{N}(\operatorname{Re}\xi_j)^2-1\Big|+\sum_{j=1}^{N}(\operatorname{Im}\xi_j)^2\Big)^2+4\Big(\sum_{j=1}^{N}(\operatorname{Re}\xi_j)^2\Big)\Big(\sum_{j=1}^{N}(\operatorname{Im}\xi_j)^2\Big)\\
&< \Big(\max\Big\{\sum_{j=1}^{N}\Big(\frac{\hat\rho_j}{\hat\rho}+\varepsilon\Big)^2-1,1-\sum_{j=1}^{N}\Big(\frac{\hat\rho_j}{\hat\rho}-\varepsilon\Big)^2\Big\}+N\varepsilon^2\Big)^2+4\sum_{j=1}^{N}\Big(\frac{\hat\rho_j}{\hat\rho}+\varepsilon\Big)^2N\varepsilon^2\\
&< \Big(\frac{2\varepsilon}{\hat\rho}\sum_{j=1}^{N}\hat\rho_j+2N\varepsilon^2\Big)^2+16N^2\varepsilon^2<(2\varepsilon N+2N\varepsilon^2)^2+16N^2\varepsilon^2\le 32N^2\varepsilon^2.\end{aligned}$$

Thus for $\xi=(\xi_1,\xi_2,\dots,\xi_N)\in U_{N,\varepsilon}$ we obtain

$$\begin{aligned}\frac12\hat\rho^2+\frac12\sum_{j=1}^{N}\frac{|\xi_j|^4}{\hat\rho_j^2}+\Big|\sum_{j=1}^{N}\xi_j^2-1\Big| &< \frac12\hat\rho^2+\frac12\sum_{j=1}^{N}\frac{\Big(\frac{\hat\rho_j}{\hat\rho}+\varepsilon\Big)^4}{\hat\rho_j^2}+6N\varepsilon\\
&= \frac12\hat\rho^2+\frac12\sum_{j=1}^{N}\frac{(\hat\rho_j+\varepsilon\hat\rho)^4}{\hat\rho^4\hat\rho_j^2}+6N\varepsilon=\frac12\hat\rho^2+\frac12\sum_{j=1}^{N}\frac{(\hat\rho_j^2+2\varepsilon\hat\rho\hat\rho_j+\varepsilon^2\hat\rho^2)^2}{\hat\rho^4\hat\rho_j^2}+6N\varepsilon\\
&< \frac12\hat\rho^2+\frac12\sum_{j=1}^{N}\frac{(\hat\rho_j^2+3\varepsilon\hat\rho\hat\rho_j)^2}{\hat\rho^4\hat\rho_j^2}+6N\varepsilon=\frac12\hat\rho^2+\frac12\sum_{j=1}^{N}\frac{\hat\rho_j^4+6\varepsilon\hat\rho\hat\rho_j^3+9\varepsilon^2\hat\rho^2\hat\rho_j^2}{\hat\rho^4\hat\rho_j^2}+6N\varepsilon\\
&< \frac12\hat\rho^2+\frac{1}{2\hat\rho^2}+\frac{15}{2}\varepsilon\sum_{j=1}^{N}\frac{\hat\rho_j}{\hat\rho^3}+6N\varepsilon<\frac12\hat\rho^2+\frac{1}{2\hat\rho^2}+14N\varepsilon\\
&< \frac12\hat\rho^2+\frac{1}{2\hat\rho^2}+\frac12\Big(\rho^2+\frac{1}{\rho^2}-\hat\rho^2-\frac{1}{\hat\rho^2}\Big)=\frac12\Big(\rho^2+\frac{1}{\rho^2}\Big).\end{aligned}$$

■

Definition 3.5
Let $\Omega\subset\mathbb{C}^N$ be a Reinhardt domain, where a Reinhardt domain is characterized by the property that $(z_1,\dots,z_N)\in\Omega$ implies $(e^{i\theta_1}z_1,\dots,e^{i\theta_N}z_N)\in\Omega$ for all $\theta_j\in[0,2\pi]$, $j=1,\dots,N$. Furthermore, let $0\in\Omega$ and a function $f\in\mathcal{H}(\Omega)$ with its homogeneous expansion

$$f(z)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}a_\alpha z^\alpha,\qquad z\in\Omega,\quad \alpha\in\mathbb{Z}_+^N,$$

be given[8]*. Then we define the function*

$$f_-(z)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}\overline{a_\alpha}z^\alpha,\qquad z\in\Omega.$$

[8]The following result in several complex variables is well-known: Let $\Omega\subset\mathbb{C}^N$ be a Reinhardt domain with $0\in\Omega$. If f is a holomorphic function in Ω, then f can be expanded into a series of homogeneous polynomials converging locally uniformly on Ω.

Clearly, we have $f_- \in \mathcal{H}(\Omega)$.

3.2 Proof of Theorem 1.6 and Theorem 1.7

After this lengthy preparation all basic tools are now available to prove the main results.

Proof of Theorem 1.6:

Without loss of generality we may assume that $r = 1$. Otherwise consider the scaled function $\hat{F}(x, y) = F(rx, ry) = |g(r(x + iy))|^2$ for $(x, y) \in \mathbb{R}^{2N}$.
Proof of (i):
Let us begin with some notes which we need for the verification of both directions. We define for $\rho \in (1, \infty)$ the sets

$$S_{2N,\rho} = \Big\{ z = (z_1, z_2, \ldots, z_{2N}) \in \mathbb{C}^{2N} : z_{2j-1} = \xi_j \frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big), z_{2j} = \xi_j \frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big),$$
$$\xi_j, \eta_j \in \mathbb{C}\backslash\{0\},\ j = 1, \ldots, N,\ \Big(\sum_{j=1}^{N} |\xi_j \eta_j|^2\Big)^{\frac{1}{2}} < \rho\ \wedge\ \Big(\sum_{j=1}^{N} \Big|\xi_j \frac{1}{\eta_j}\Big|^2\Big)^{\frac{1}{2}} < \rho \Big\}$$

and

$$T_{2N,\rho} = \Big\{ z = (z_1, ..., z_{2N}) \in \mathbb{C}^{2N} : \Big(\sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2\Big)^{\frac{1}{2}} < \rho\ \wedge\ \Big(\sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2\Big)^{\frac{1}{2}} < \rho \Big\}.$$

Then $S_{2N,\rho} \subset T_{2N,\rho}$. To see this inclusion, let us choose for any element $z = (z_1, z_2, \ldots, z_{2N})$ of $S_{2N,\rho}$ the representation

$$(z_1, z_2, z_3, \ldots, z_{2N}) = \Big(\xi_1 \frac{1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \xi_1 \frac{1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \xi_2 \frac{1}{2}\Big(\eta_2 + \frac{1}{\eta_2}\Big), \ldots, \xi_N \frac{1}{2i}\Big(\eta_N - \frac{1}{\eta_N}\Big)\Big).$$

We receive

$$\sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2 = \sum_{j=1}^{N} |\xi_j \eta_j|^2 < \rho^2$$

and

$$\sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2 = \sum_{j=1}^{N} \Big|\xi_j \frac{1}{\eta_j}\Big|^2 < \rho^2.$$

Consequently, $S_{2N,\rho} \subset T_{2N,\rho}$.
"$\Leftarrow$": By hypothesis we have $g \in \mathcal{H}(\mathcal{B}_{N,\rho})$. We now show that F has a holomorphic extension to $L_{2N,\rho}$, where $L_{2N,\rho}$ is defined as in Lemma 2.12, because then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} \leq \frac{1}{\rho}$$

follows from Lemma 2.12. Therefore we define the function $f_1 : T_{2N,\rho} \to \mathbb{C}$ by

$$f_1(z_1, z_2, ..., z_{2N-1}, z_{2N}) = g(z_1 + iz_2, ..., z_{2N-1} + iz_{2N}) g_-(z_1 - iz_2, ..., z_{2N-1} - iz_{2N}), \tag{3.14}$$

where g_- is specified in Definition 3.5. Since g is holomorphic in $\mathcal{B}_{N,\rho}$ we deduce from the definition of $T_{2N,\rho}$ that f_1 is holomorphic in $T_{2N,\rho}$. Moreover, f_1 is a holomorphic extension of F to $T_{2N,\rho}$

as $f_1 = F|_{\overline{B}_{2N}}$ and $\overline{B}_{2N} \subset T_{2N,\rho}$. Since $L_{2N,\rho} \subset T_{2N,\rho}$ by Lemma 3.3 we are done.
"$\Rightarrow$": This direction is proved by contradiction. We suppose

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} \leq \frac{1}{\rho}$$

but g has no holomorphic extension to $\mathcal{B}_{N,\rho}$.
Then there exists a number $\tilde{\rho} \in (1,\rho)$ such that $g \in \mathcal{H}(\mathcal{B}_{N,\tilde{\rho}}) \backslash \mathcal{H}(\overline{\mathcal{B}}_{N,\tilde{\rho}})$. Hence we can find to an arbitrary $\varepsilon_0 > 0$ a non–removable singularity $\hat{z} = (\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_N)$ of g with

$$\hat{\rho} := \|\hat{z}\| \in [\tilde{\rho}, \rho) \cap [\tilde{\rho}, \tilde{\rho} + \varepsilon_0). \tag{3.15}$$

Further, let us set $\hat{\rho}_j := |\hat{z}_j|$, $j = 1, 2, \ldots, N$.
Now, for more clarity we divide the proof of this direction into two steps. Step 1: $\hat{z}_j \neq 0$ for $j = 1, 2, \ldots, N$ and Step 2: $\hat{z}_k = 0$ for at least one $k \in \{1, \ldots, N\}$.
Step 1: We define the function $f_1 : T_{2N,\tilde{\rho}} \to \mathbb{C}$ as in the "if"–direction. Then f_1 can be expressed by

$$\begin{aligned} f_1(z_1, z_2, z_3, \ldots, z_{2N}) &= f_1\left(\frac{\xi_1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \frac{\xi_1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \frac{\xi_2}{2}\Big(\eta_2 + \frac{1}{\eta_2}\Big), \ldots, \frac{\xi_N}{2i}\Big(\eta_N - \frac{1}{\eta_N}\Big)\right) \\ &= g\Big(\xi_1\eta_1, \ldots, \xi_N\eta_N\Big)\, g_{-}\Big(\xi_1\frac{1}{\eta_1}, \ldots, \xi_N\frac{1}{\eta_N}\Big), \end{aligned}$$

if $(z_1, z_2, \ldots, z_{2N}) \in S_{2N,\tilde{\rho}}$, $z_{2j-1} = \xi_j\frac{1}{2}(\eta_j + \frac{1}{\eta_j}), z_{2j} = \xi_j\frac{1}{2i}(\eta_j - \frac{1}{\eta_j})$, $\xi_j, \eta_j \in \mathbb{C}\backslash\{0\}$, $j = 1, 2, \ldots, N$. From the "if"–direction we know that f_1 is holomorphic in $T_{2N,\tilde{\rho}}$ and $f_1 = F|_{\overline{B}_{2N}}$. In addition, we infer from Lemma 2.12 that F has a holomorphic extension $\tilde{F}$ to $L_{2N,\rho}$. Thus, in view of Remark 2.6 and by the identity principle we obtain that f_1 has a unique holomorphic extension to $L_{2N,\rho}$ and that $f_1|_{L_{2N,\rho}} \equiv \tilde{F}|_{L_{2N,\rho}}$.
Let us now define the set

$$U_{N,\tilde{\varepsilon}} := \left\{ z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N : \max_{1\leq j\leq N} \left| z_j - \frac{|\hat{z}_j|}{\hat{\rho}} \right| < \tilde{\varepsilon} \right\},$$

where $\tilde{\varepsilon} = \min\Big\{(\rho^2 + 1/\rho^2 - \hat{\rho}^2 - 1/\hat{\rho}^2)/(28N), \min\limits_{1\leq j\leq N}(|\hat{z}_j|/\hat{\rho}), (\sqrt{\hat{\rho}} - 1)\min\limits_{1\leq j\leq N}(|\hat{z}_j|/\hat{\rho})\Big\}$.
Then for $\eta_j = \hat{z}_j/\xi_j$, $j = 1, 2, \ldots, N$, and $\xi = (\xi_1, \ldots, \xi_N) \in U_{N,\tilde{\varepsilon}}$ we may express the non–removable singularity $\hat{z}$ in the form

$$\hat{z} = (\hat{z}_1, \ldots, \hat{z}_N) = (\xi_1\eta_1, \ldots, \xi_N\eta_N)$$

and obtain

$$\left(\xi_1\frac{1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \xi_1\frac{1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \xi_2\frac{1}{2}\Big(\eta_2 + \frac{1}{\eta_2}\Big), \ldots, \xi_N\frac{1}{2i}\Big(\eta_N - \frac{1}{\eta_N}\Big)\right) \in L_{2N,\rho},$$

since for $\xi \in U_{N,\tilde{\varepsilon}}$ the inequality

$$\sum_{j=1}^{N} |\xi_j|^2 \left(\left|\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\right|^2 + \left|\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\right|^2 \right) +$$

$$\left| \sum_{j=1}^{N} \xi_j^2 \left(\Big(\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\Big)^2 + \Big(\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\Big)^2 \right) - 1 \right| =$$

$$= \sum_{j=1}^{N} |\xi_j|^2 \frac{1}{2}\Big(|\eta_j|^2 + \frac{1}{|\eta_j|^2}\Big) + \Big| \sum_{j=1}^{N} \xi_j^2 - 1 \Big|$$

$$= \frac{1}{2}\hat{\rho}^2 + \frac{1}{2}\sum_{j=1}^{N} \frac{|\xi_j|^4}{\hat{\rho}_j^2} + \Big| \sum_{j=1}^{N} \xi_j^2 - 1 \Big| < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \tag{3.16}$$

is valid, where the upper bound follows from Lemma 3.4.
In addition, if $\eta_j = \hat{z}_j/\xi_j$, $j = 1, 2, \ldots, N$, and $\xi = (\xi_1, \ldots, \xi_N) \in U_{N,\tilde{\varepsilon}}$, we derive from the estimate

$$\sum_{j=1}^{N} \Big|\xi_j \frac{1}{\eta_j}\Big|^2 = \sum_{j=1}^{N} \Big|\xi_j^2 \frac{1}{\hat{z}_j}\Big|^2 < \sum_{j=1}^{N} \frac{\Big(\frac{|\hat{z}_j|}{\hat{\rho}} + \tilde{\varepsilon}\Big)^4}{|\hat{z}_j|^2} \le \sum_{j=1}^{N} \frac{|\hat{z}_j|^4}{\hat{\rho}^4} \frac{(1+\sqrt{\hat{\rho}}-1)^4}{|\hat{z}_j|^2} = 1, \tag{3.17}$$

that

$$\Big(\xi_1 \frac{1}{\eta_1}, \xi_2 \frac{1}{\eta_2}, \ldots, \xi_N \frac{1}{\eta_N}\Big) \in \mathcal{B}_N.$$

We now claim

$$g_{\text{-}}\Big(\xi_1^2 \frac{1}{\hat{z}_1}, \xi_2^2 \frac{1}{\hat{z}_2}, \ldots, \xi_N^2 \frac{1}{\hat{z}_N}\Big) = 0 \quad \text{for } \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in U_{N,\tilde{\varepsilon}}.$$

Proof of the claim: This is done by contradiction. Therefore we assume there exists some $\hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_N) \in U_{N,\tilde{\varepsilon}}$ such that

$$g_{\text{-}}\Big(\hat{\xi}_1^2 \frac{1}{\hat{z}_1}, \hat{\xi}_2^2 \frac{1}{\hat{z}_2}, \ldots, \hat{\xi}_N^2 \frac{1}{\hat{z}_N}\Big) \neq 0.$$

Then we also have

$$g_{\text{-}}\Big(\hat{\xi}_1^2 \frac{1}{\hat{z}_1} + w_1, \hat{\xi}_2^2 \frac{1}{\hat{z}_2} + w_2, \ldots, \hat{\xi}_N^2 \frac{1}{\hat{z}_N} + w_N\Big) \neq 0$$

for $w = (w_1, w_2, \ldots, w_N) \in \mathcal{D}_N(0, \varepsilon)$ if $\varepsilon > 0$ is sufficiently small. Further, in view of the inequalities (3.16) and (3.17) we may suppose that for $w = (w_1, w_2, \ldots, w_N) \in \mathcal{D}_N(0, \varepsilon)$

$$\left(\frac{\hat{\xi}_1}{2}\Big(\frac{\hat{z}_1}{\hat{\xi}_1} + \frac{\hat{\xi}_1}{\hat{z}_1}\Big) + w_1, \frac{\hat{\xi}_1}{2i}\Big(\frac{\hat{z}_1}{\hat{\xi}_1} - \frac{\hat{\xi}_1}{\hat{z}_1}\Big), \frac{\hat{\xi}_2}{2}\Big(\frac{\hat{z}_2}{\hat{\xi}_2} + \frac{\hat{\xi}_2}{\hat{z}_2}\Big) + w_2, \ldots, \frac{\hat{\xi}_N}{2i}\Big(\frac{\hat{z}_N}{\hat{\xi}_N} - \frac{\hat{\xi}_N}{\hat{z}_N}\Big) \right) \in L_{2N,\rho}$$

and

$$\Big(\hat{\xi}_1^2 \frac{1}{\hat{z}_1} + w_1, \hat{\xi}_2^2 \frac{1}{\hat{z}_2} + w_2, \ldots, \hat{\xi}_N^2 \frac{1}{\hat{z}_N} + w_N\Big) \in \mathcal{B}_N.$$

Next, we consider the functions

$$h(w_1, w_2, \ldots, w_N) :=$$

$$f_1\left(\frac{\hat{\xi}_1}{2}\Big(\frac{\hat{z}_1}{\hat{\xi}_1} + \frac{\hat{\xi}_1}{\hat{z}_1}\Big) + w_1, \frac{\hat{\xi}_1}{2i}\Big(\frac{\hat{z}_1}{\hat{\xi}_1} - \frac{\hat{\xi}_1}{\hat{z}_1}\Big), \frac{\hat{\xi}_2}{2}\Big(\frac{\hat{z}_2}{\hat{\xi}_2} + \frac{\hat{\xi}_2}{\hat{z}_2}\Big) + w_2, \ldots, \frac{\hat{\xi}_N}{2i}\Big(\frac{\hat{z}_N}{\hat{\xi}_N} - \frac{\hat{\xi}_N}{\hat{z}_N}\Big) \right)$$

and

$$\hat{g}_-(w_1, w_2, \ldots, w_N) := g_-\Big(\hat{\xi}_1^2 \frac{1}{\hat{z}_1} + w_1, \hat{\xi}_2^2 \frac{1}{\hat{z}_2} + w_2, \ldots, \hat{\xi}_N^2 \frac{1}{\hat{z}_N} + w_N\Big)$$

for $w = (w_1, w_2, \ldots, w_N) \in \mathcal{D}_N(0, \varepsilon)$. Then h and $\hat{g}_-$ are holomorphic in $\mathcal{D}_N(0, \varepsilon)$. Furthermore, $\hat{g}_-(w_1, w_2, \ldots, w_N) \neq 0$ for $w = (w_1, w_2, \ldots, w_N) \in \mathcal{D}_N(0, \varepsilon)$. Thus the function

$$l(w_1, w_2, \ldots, w_N) := \frac{h(w_1, w_2, \ldots, w_N)}{\hat{g}_-(w_1, w_2, \ldots, w_N)}$$

is holomorphic for $w = (w_1, \ldots, w_N) \in \mathcal{D}_N(0, \varepsilon)$. Since for $\varepsilon_0 > 0$ sufficiently small $\mathcal{D}_N(0, \varepsilon) \cap \big\{w = (w_1, \ldots, w_N) \in \mathbb{C}^N : \big(\sum_{j=1}^N |\hat{z}_j + w_j|^2\big)^{1/2} < \tilde{\rho}\big\}$ is certainly a non–empty open set in $\mathbb{C}^N$ (see (3.15)), we obtain that g has a holomorphic extension $\tilde{g}$ to some non–empty neighborhood of $\hat{z}$. To be more precisely,

$$\tilde{g}(\hat{z}_1 + w_1, \hat{z}_2 + w_2, \ldots, \hat{z}_N + w_N) = l(w_1, w_2, \ldots, w_N)$$

for $w = (w_1, w_2, \ldots, w_N) \in \mathcal{D}_N(0, \varepsilon)$ which contradicts the hypothesis that g has a non–removable singularity at $\hat{z}$. These aspects show

$$g_-\Big(\xi_1^2 \frac{1}{\hat{z}_1}, \xi_2^2 \frac{1}{\hat{z}_2}, \ldots, \xi_N^2 \frac{1}{\hat{z}_N}\Big) = 0 \quad \text{for } \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in U_{N,\tilde{\varepsilon}}$$

and the claim is proved.

By the identity principle we conclude

$$g_-(z) = 0 \qquad \text{for} \quad z \in \mathcal{B}_{N,\tilde{\rho}}$$

and therefore

$$g(z) = 0 \qquad \text{for} \quad z \in \mathcal{B}_{N,\tilde{\rho}},$$

which is clearly a contradiction to the assumption that g has no holomorphic extension to some neighborhood of $\overline{\mathcal{B}}_{N,\tilde{\rho}}$.

Step 2: Now let $\hat{z}_k = 0$ for some $k \in \{1, 2, \ldots, N\}$. Without loss of generality we may assume that

$$\hat{z}_j \neq 0 \qquad \text{for } j = 1, 2, \ldots, m, \quad m < N,$$

and

$$\hat{z}_{m+l} = 0 \qquad \text{for } l = 1, 2, \ldots, N - m.$$

Next, we consider instead of $S_{2N,\tilde{\rho}}$ the set

$$\begin{aligned}\tilde{S}_{2N,\tilde{\rho}} := \Big\{ z = (z_1, \ldots, z_{2m}, w_1, \ldots, w_{2N-2m}) \in \mathbb{C}^{2N} \colon z_{2j-1} = \frac{\xi_j}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big), z_{2j} = \frac{\xi_j}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big), \\ \xi_j, \eta_j \in \mathbb{C}\backslash\{0\}, j = 1, \ldots, m, \Big(\sum_{j=1}^m |\xi_j \eta_j|^2 + \sum_{j=1}^{N-m} |w_{2j-1} + i w_{2j}|\Big)^{\frac{1}{2}} < \tilde{\rho} \wedge \\ \Big(\sum_{j=1}^m \Big|\xi_j \frac{1}{\eta_j}\Big|^2 + \sum_{j=1}^{N-m} |w_{2j-1} - i w_{2j}|\Big)^{\frac{1}{2}} < \tilde{\rho} \Big\}.\end{aligned}$$

Now let us define the function $\tilde{f}_1 : T_{2N,\tilde{\rho}} \to \mathbb{C}$, like in equation (3.14). Then $\tilde{f}_1$ takes the form

$$
\begin{aligned}
&\tilde{f}_1(z_1, z_2, \ldots, z_{2m}, w_1, w_2, \ldots, w_{2N-2m}) \\
&\quad = \tilde{f}_1\left(\frac{\xi_1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \frac{\xi_1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \ldots, \frac{\xi_m}{2i}\Big(\eta_m - \frac{1}{\eta_m}\Big), w_1, w_2, \ldots, w_{2N-2m}\right) \\
&\quad = g\Big(\xi_1\eta_1, \ldots, \xi_m\eta_m, w_1 + iw_2, w_3 + iw_4, \ldots, w_{2N-2m-1} + iw_{2N-2m}\Big) \\
&\qquad\qquad g_-\Big(\xi_1\frac{1}{\eta_1}, \ldots, \xi_m\frac{1}{\eta_m}, w_1 - iw_2, w_3 - iw_4, \ldots, w_{2N-2m-1} - iw_{2N-2m}\Big),
\end{aligned}
$$

if $z = (z_1, z_2, \ldots, z_{2m}, w_1, w_2, \ldots, w_{2N-2m}) \in \tilde{S}_{2N,\tilde{\rho}}$.
As $\tilde{f}_1 \equiv F|_{\overline{B}_{2N}}$ Lemma 2.12 and Remark 2.6 ensure that $\tilde{f}_1$ can be continued analytically to $L_{2N,\rho}$. Hence we may proceed quite similar to the case $\hat{z}_j \neq 0$ for $j = 1, 2, \ldots, N$.
We define for $\varepsilon_1 > 0$ the set

$$
\begin{aligned}
U_{N,\varepsilon_1} := \Big\{&(z, w) = (z_1, z_2, \ldots, z_m, w_2, w_4, \ldots, w_{2N-2m}) \in \mathbb{C}^N : z_j \neq 0, j = 1, \ldots, m, \\
&\max_{1\le j\le m}\Big|z_j - \frac{|\hat{z}_j|}{\hat{\rho}}\Big| < \varepsilon_1 \ \wedge \max_{1\le j\le N-m}|w_{2j}| < \varepsilon_1\Big\}.
\end{aligned}
$$

If now $\eta_j = \hat{z}_j/\xi_j$ for $j = 1, 2, \ldots, m$ and $w_{2l-1} = -iw_{2l}$ for $l = 1, \ldots, N-m$, where $(\xi_1, \xi_2, \ldots, \xi_m, w_2, w_4, \ldots, w_{2N-2m}) \in U_{N,\varepsilon_1}$, we will see that for $\varepsilon_1 > 0$ sufficiently small

$$
\hat{z} = (\hat{z}_1, \ldots, \hat{z}_m, 0, \ldots, 0) = (\xi_1\eta_1, \ldots, \xi_m\eta_m, w_1 + iw_2, \ldots, w_{2N-2m-1} + iw_{2N-2m})
$$

is a non–removable singularity of g such that the following conditions are fulfilled:

(a) $$\left(\xi_1\frac{1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \xi_1\frac{1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \ldots, \xi_m\frac{1}{2i}\Big(\eta_m - \frac{1}{\eta_m}\Big), w_1, w_2, \ldots, w_{2N-2m}\right) \in L_{2N,\rho}$$

(b) $$\left(\xi_1\frac{1}{\eta_1}, \xi_2\frac{1}{\eta_2}, \ldots, \xi_m\frac{1}{\eta_m}, w_1 - iw_2, w_3 - iw_4, \ldots, w_{2N-2m-1} - iw_{2N-2m}\right) \in \mathcal{B}_{N,\tilde{\rho}}$$

To (a): By Lemma 3.4 we obtain for $(\xi_1, \xi_2, \ldots, \xi_m, w_2, \ldots, w_{2N-2m}) \in U_{N,\varepsilon_1}$

$$
\begin{aligned}
&\sum_{j=1}^{m}|\xi_j|^2\left(\Big|\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\Big|^2 + \Big|\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\Big|^2\right) + \sum_{j=1}^{2N-2m}|w_j|^2 + \\
&\qquad\left|\sum_{j=1}^{m}\xi_j^2\left(\Big(\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\Big)^2 + \Big(\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\Big)^2\right) + \sum_{j=1}^{2N-2m}w_j^2 - 1\right| \le \\
&\sum_{j=1}^{m}|\xi_j|^2\left(\Big|\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\Big|^2 + \Big|\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\Big|^2\right) + 2\sum_{j=1}^{2N-2m}|w_j|^2 + \\
&\qquad\qquad\left|\sum_{j=1}^{m}\xi_j^2\left(\Big(\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\Big)^2 + \Big(\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\Big)^2\right) - 1\right| \le \\
&\qquad\qquad \frac{1}{2}\hat{\rho}^2 + \frac{1}{2}\sum_{j=1}^{m}\frac{|\xi_j|^4}{\hat{\rho}_j^2} + \Big|\sum_{j=1}^{m}\xi_j^2 - 1\Big| + 4(N-m)\varepsilon_1^2 < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big),
\end{aligned}
$$

if $\varepsilon_1 > 0$ is sufficiently small.
To (b): For $(\xi_1, \xi_2, \dots, \xi_m, w_2, \dots, w_{2N-2m}) \in U_{N,\varepsilon_1}$ and $\varepsilon_1 > 0$ sufficiently small we estimate as in equation (3.17)

$$\sum_{j=1}^{m} \Big|\xi_j \frac{1}{\eta_j}\Big|^2 + \sum_{j=1}^{N-m} |w_{2j-1} - iw_{2j}|^2 < 1 + 4(N-m)\varepsilon_1^2 < \tilde{\rho}^2.$$

Now, item (a) and (b) combined with the proof of the claim in case (i), yield

$$g_{\text{-}}\Big(\xi_1^2 \frac{1}{\hat{z}_1}, \xi_2^2 \frac{1}{\hat{z}_2}, \dots, \xi_m^2 \frac{1}{\hat{z}_m}, -i2w_2, -i2w_4, \dots, -i2w_{2N-2m}\Big) = 0$$

for $(\xi_1, \xi_2, \dots, \xi_m, w_2, \dots, w_{2N-2m}) \in U_{N,\varepsilon_1}$ if $\varepsilon_1 > 0$ sufficiently small. However this would imply

$$g(z) = 0 \qquad \text{for} \quad z \in \mathcal{B}_{N,\tilde{\rho}},$$

which is clearly impossible.
Proof of (ii):
Observe, Lemma 2.12 and Theorem 2.2 ensure

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N}, F)} \le \frac{1}{\rho}$$

if and only if F has an analytic continuation to

$$\mathcal{L}_{2N,\rho} = \Big\{ z = (z_1, z_2, \dots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1\le j\le N} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1|\big) < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big)\Big\}.$$

Next, we define the set

$$\mathcal{T}_{2N,\tilde{\rho}} = \Big\{ z = (z_1, z_2, \dots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1\le j\le N} |z_{2j-1} + iz_{2j}| < \tilde{\rho} \;\wedge\; \max_{1\le j\le N} |z_{2j-1} - iz_{2j}| < \tilde{\rho}\Big\},$$

where $\tilde{\rho} \in (1, \infty)$ is so chosen that $g \in \mathcal{H}(\mathcal{D}_{N,\tilde{\rho}}) \backslash \mathcal{H}(\overline{\mathcal{D}}_{N,\tilde{\rho}})$. If g is holomorphic in $\mathbb{C}^N$ we set $\tilde{\rho} = \infty$ and consider $\mathcal{T}_{2N,\infty} = \mathbb{C}^{2N}$.
Then the function $f_1 : \mathcal{T}_{2N,\tilde{\rho}} \to \mathbb{C}$ defined by

$$f_1(z_1, z_2, \dots, z_{2N}) := g(z_1 + iz_2, z_3 + iz_4, \dots, z_{2N-1} + iz_{2N}) \\ g_{\text{-}}(z_1 - iz_2, z_3 - iz_4, \dots, z_{2N-1} - iz_{2N})$$

is holomorphic in $\mathcal{T}_{2N,\tilde{\rho}}$ and

$$f_1(x_1, y_1, x_2, y_2, \dots, x_N, y_N) = F(x_1, x_2, \dots, x_N, y_1, y_2, \dots, y_N)$$

for $(x_1, y_1, x_2, y_2, \dots, x_N, y_N) \in \overline{D}_{2N}$.
"$\Leftarrow$": By hypothesis we have $g \in \mathcal{H}(\mathcal{D}_{N,\rho})$. Consequently, $\rho \le \tilde{\rho}$ and f_1 is holomorphic in $\mathcal{T}_{2N,\rho}$. Since $\mathcal{L}_{2N,\rho} \subset \mathcal{T}_{2N,\rho}$ by Lemma 1.8, we see that f_1 is a holomorphic extension of F to $\mathcal{L}_{2N,\rho}$ which implies

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N}, F)} \le \frac{1}{\rho}.$$

"$\Rightarrow$": This direction is proved by contradiction. We suppose

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N}, F)} \le \frac{1}{\rho}$$

but g has no holomorphic extension to $\mathcal{D}_{N,\rho}$. Then f_1 has a uniquely determined holomorphic extension to $L_{2N,\rho}$ and $\tilde{\rho}$ is a number of the interval $(1,\rho)$. Hence there exists a non–removable singularity $\hat{z}$ of g such that $\hat{\rho} := |\hat{z}| \in [\tilde{\rho},\rho) \cap [\tilde{\rho},\tilde{\rho}+\varepsilon_0)$, where $\varepsilon_0 > 0$ is an arbitrary number. Without loss of generality we may assume

$$\hat{z}_j \neq 0 \qquad \text{for} \quad j = 1,2,\ldots,m, \quad m \leq N,$$

and

$$\hat{z}_j = 0 \qquad \text{for} \quad j = m+1, m+2,\ldots,N.$$

Now, we write the non–removable singularity $\hat{z}$ of g in the form

$$\hat{z} = \left(\xi_1 \frac{\hat{z}_1}{\xi_1}, \xi_2 \frac{\hat{z}_2}{\xi_2}, \ldots, \xi_m \frac{\hat{z}_m}{\xi_m}, w_1 + iw_2, \ldots, w_{2N-2m-1} + iw_{2N-2m}\right),$$

where $\xi_j \in \mathbb{C}\backslash\{0\}$, $j = 1,2,\ldots,m$, and $w_{2j-1} = -iw_{2j}$, $w_j \in \mathbb{C}$, $j = 1,\ldots,N-m$.
Next we define for $\tilde{\varepsilon} > 0$ the set

$$U_{N,\tilde{\varepsilon}} = \big\{z = (z_1,z_2,\ldots,z_N) \in \mathbb{C}^N : \max_{1\leq j\leq m} |z_j - |\hat{z}_j|/\hat{\rho}| < \tilde{\varepsilon}, \max_{m+1\leq j\leq N} |z_j| < \tilde{\varepsilon}\big\}.$$

Further, let for the rest of the proof

$$(\xi_1,\ldots,\xi_m, w_2, w_4,\ldots, w_{2N-2m}) \in U_{N,\tilde{\varepsilon}}, \quad \eta_j = \hat{z}_j/\xi_j, \quad \xi_j \neq 0, \quad j = 1,2,\ldots,m,$$

and

$$w_{2j-1} = -iw_{2j}, \qquad j = 1,\ldots,N-m.$$

Then we deduce from Lemma 3.2 that for $\tilde{\varepsilon} > 0$ sufficiently small

$$\begin{aligned}\Big(\xi_1 \frac{1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \xi_1 \frac{1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \xi_2 \frac{1}{2}\Big(\eta_2 + \frac{1}{\eta_2}\Big), \xi_2 \frac{1}{2i}\Big(\eta_2 - \frac{1}{\eta_2}\Big), \ldots,\\ \xi_m \frac{1}{2}\Big(\eta_m + \frac{1}{\eta_m}\Big), \xi_m \frac{1}{2i}\Big(\eta_m - \frac{1}{\eta_m}\Big), w_1, w_2, \ldots, w_{2N-2m}\Big) \in \mathcal{L}_{2N,\rho}.\end{aligned}$$

Moreover, we have for $\tilde{\varepsilon} > 0$ sufficiently small

$$\max\left\{\max_{1\leq j\leq m}\Big|\xi_j \frac{1}{\eta_j}\Big|, \max_{1\leq l\leq N-m} |w_{2l-1} - iw_{2l}|\right\} \leq \max\left\{\max_{1\leq j\leq m}\Big|\xi_j^2 \frac{1}{\hat{z}_j}\Big|, \max_{1\leq l\leq N-m} 2|w_{2l}|\Big|\right\} < \tilde{\rho}.$$

Observe, f_1 takes the form

$$\begin{aligned} f_1(&z_1, z_2, \ldots, z_{2m}, w_1, w_2, \ldots, w_{2N-2m}) \\ &= f_1\left(\frac{\xi_1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \frac{\xi_1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \ldots, \frac{\xi_m}{2i}\Big(\eta_m - \frac{1}{\eta_m}\Big), w_1, w_2, \ldots, w_{2N-m}\right) \\ &= g\Big(\xi_1\eta_1, \ldots, \xi_m\eta_m, w_1 + iw_2, w_3 + iw_4, \ldots, w_{2N-2m-1} + iw_{2N-2m}\Big) \\ &\qquad\qquad g_-\Big(\xi_1 \frac{1}{\eta_1}, \ldots, \xi_m \frac{1}{\eta_m}, w_1 - iw_2, w_3 - iw_4, \ldots, w_{2N-2m-1} - iw_{2N-2m}\Big)\end{aligned}$$

if $z_{2j-1} = \frac{\xi_j}{2}\big(\eta_j + \frac{1}{\eta_j}\big)$, $z_{2j} = \frac{\xi_j}{2i}\big(\eta_j - \frac{1}{\eta_j}\big)$, $\xi_j, \eta_j \neq 0$, $j = 1,2,\ldots,m$. Thus, by similar arguments as in the proof of Theorem 1.6, we conclude

$$g(z) = 0 \qquad \text{for} \quad z \in \mathcal{D}_{N,\tilde{\rho}},$$

which is in contrast to our assumption that g has no analytic continuation to $\overline{\mathcal{D}}_{N,\tilde{\rho}}$. ■

The methods we used to prove Theorems 1.6 can also be applied to prove Theorem 1.7.

Proof of Theorem 1.7:

Since the proof of this theorem can be established by slight modifications of the proof of Theorem 1.6 we only give a rough sketch.
(i): The "if"–part follows immediately from the proof of the "$\Leftarrow$"–direction of Theorem 1.6 if we replace g_- by h_-, where h_- is specified as in Definition 3.5. Thus let us concentrate on the "if and only if"–part. We suppose

$$\limsup_{n\to\infty} \sqrt[n]{E_n^c(\overline{B}_{2N}, F)} \leq \frac{1}{\rho}$$

but g or h has no holomorphic extension to $\mathcal{B}_{N,\rho}$.

We first consider the case that $g \in \mathcal{H}(\mathcal{B}_{N,\tilde{\rho}})\backslash\mathcal{H}(\overline{\mathcal{B}}_{N,\tilde{\rho}})$ and $h \in \mathcal{H}(\mathcal{B}_{N,\tilde{\rho}})$, where $\tilde{\rho} \in (1, \rho)$. Then, we may proceed as in the proof of the "$\Rightarrow$"–direction of Theorem 1.6, if we choose h_- instead of g_-. Hence we obtain that $h \equiv 0$ on $\mathcal{B}_N$ which is a contradiction to the hypothesis.

Now, let $h \in \mathcal{H}(\mathcal{B}_{N,\hat{\rho}})\backslash\mathcal{H}(\overline{\mathcal{B}}_{N,\hat{\rho}})$ and $g \in \mathcal{H}(\mathcal{B}_{N,\hat{\rho}})$, where $\hat{\rho} \in (1, \rho)$. Then we define

$$G(x_1, y_1, x_2, ..., y_N) := h(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)\overline{g(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)}.$$

As

$$\limsup_{n\to\infty} \sqrt[n]{E_n^c(\overline{B}_{2N}, G)} = \limsup_{n\to\infty} \sqrt[n]{E_n^c(\overline{B}_{2N}, F)}$$

we may argue as above (take G instead of F and h for g). Thus we derive that $g \equiv 0$ on $\mathcal{B}_N$ in contrast to our assumption and we are done.
(ii) This statement can be verified by similar arguments as in item (i). ■

Observe, the proofs of the "if"–directions of Theorem 1.6 and Theorem 1.7 are based on Lemma 3.2 and Lemma 3.3. A different approach shows the proof of Theorem 1.9.

Lemma 3.2 and 3.3 play the key role for Theorem 1.8.

Proof of Theorem 1.8:

Without loss of generality we may assume that $r = 1$. Otherwise we can take the scaled function $\hat{F}(x, y) = F(rx, ry) = |g(r(x + iy))|^2$ for $(x, y) \in \mathbb{R}^{2N}$.
(i): Ad (a)$\Leftrightarrow$(b): This equivalence follows immediately from the Theorems 1.7 and Lemma 2.12. Ad (b)$\Leftrightarrow$(c)$\Leftrightarrow$(d): Firstly, we assume that F has no holomorphic extension to $\mathbb{C}^{2N}$. Therefore we can choose $\rho \in (1, \infty)$ such that $F \in \mathcal{H}(T_{2N,\rho})\backslash\mathcal{H}(\overline{T}_{2N,\rho})$, see Lemma 3.3 for the definition of $T_{2N,\rho}$. Lemma 2.12 and the proof of the "if"–part of Theorem 1.6 combined, shows

$$F \in \mathcal{H}(T_{2N,\rho})\backslash\mathcal{H}(\overline{T}_{2N,\rho}) \qquad \text{if and only if} \qquad F \in \mathcal{H}(L_{2N,\rho})\backslash\mathcal{H}(\overline{L}_{2N,\rho}).$$

Hence, since $T_{2N,\rho} \supset L_{2N,\rho}$ by Lemma 3.3, there exists a singular point $\hat{z}$ of F satisfying

$$\hat{z} \in \partial T_{2N,\rho} \qquad \text{if and only if} \qquad \hat{z} \in \partial L_{2N,\rho}.$$

Consequently, we obtain that F has a holomorphic extension to $T_{2N,\rho}$ if F has no singular points on $M_{2N,\rho}$ (and vice versa), where

$$\begin{aligned} M_{2N,\rho} := \Bigg\{ & z = (z_1, \dots, z_{2N}) \in \mathbb{C}^{2N} : \\ & \Bigg(\|z\|^2 + \Bigg| \sum_{j=1}^{2N} z_j^2 - 1 \Bigg| = \frac{1}{2}\Big(R^2 + \frac{1}{R^2}\Big) \wedge \sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2 = R^2 \Bigg) \ \vee \\ & \Bigg(\|z\|^2 + \Bigg| \sum_{j=1}^{2N} z_j^2 - 1 \Bigg| = \frac{1}{2}\Big(R^2 + \frac{1}{R^2}\Big) \wedge \sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2 = R^2 \Bigg), \quad R \in (1, \rho) \Bigg\}. \end{aligned}$$

Now, from the proof of Lemma 3.3 we conclude $z = (z_1, \ldots, z_{2N}) \in M_{2N,\rho}$ if and only if $z = (z_1, \ldots, z_{2N})$ has the form

$$z_{2j-1} = \frac{R_j}{R}\frac{1}{2}\Big(Re^{it_j} + \frac{1}{Re^{it_j}}\Big), \qquad z_{2j} = \pm\frac{R_j}{R}\frac{1}{2i}\Big(Re^{it_j} - \frac{1}{Re^{it_j}}\Big), \quad j = 1, \ldots, N,$$

where $\sum_{j=1}^{N} R_j^2 = R^2$, $R_j \in [0, R]$, $t_j \in [0, 2\pi]$, $j = 1, \ldots, N$, and $R \in (1, \rho)$.
If F has a holomorphic extension to $\mathbb{C}^{2N}$ then the statement is quite obvious, if we regard it as the limiting case "$\rho = \infty$". This finishes item (i).
(ii): We skip the proof of this result as it can be obtained quite similar to (i). ■

3.3 Proof of Theorem 1.9

Theorem 1.9 demonstrates that the "if"–direction of Theorem 1.7 can be extended to a much larger class of domains than closed balls in $\mathbb{R}^{2N}$, whereas the "if and only if"–direction fails to be true in general.

Proof of Theorem 1.9:

Due to Theorem 1.4 we can choose two sequences $\{p_{1,n}\}_{n\in\mathbb{N}}$ and $\{p_{2,n}\}_{n\in\mathbb{N}}$ of polynomials $p_{1,n}, p_{2,n}$ of degree $\le n$ such that for an arbitrary $R \in (1, \rho)$ and all $n \in \mathbb{N}$ the estimate

$$\max\big\{\|g - p_{1,n}\|_L, \|h - p_{2,n}\|_L\big\} \le \frac{M}{R^n} \tag{3.18}$$

holds, where $M > 0$ is some constant independent of n. Consequently, we have

$$\begin{aligned}
\|g\overline{h} - p_{1,n}\overline{p_{2,n}}\|_L &\le \|g\overline{h} - p_{1,n}\overline{h}\|_L + \|p_{1,n}\overline{h} - p_{1,n}\overline{p_{2,n}}\|_L \\
&\le \|h\|_L\|g - p_{1,n}\|_L + \|p_{1,n}\|_L\|\overline{h} - \overline{p_{2,n}}\|_L \\
&\le \frac{M_1}{R^n}, \qquad M_1 := 3M\max\{\|g\|_L, \|h\|_L\}.
\end{aligned}$$

Next we put

$$q_{1,0}(z) := p_{1,0}(z), \quad q_{1,k}(z) := p_{1,k}(z) - p_{1,k-1}(z), \qquad k \in \mathbb{N}, \quad z \in \mathbb{C}^N,$$

and

$$q_{2,0}(z) := p_{2,0}(z), \quad q_{2,k}(z) := p_{2,k}(z) - p_{2,k-1}(z), \qquad k \in \mathbb{N}, \quad z \in \mathbb{C}^N,$$

Further, let us define the polynomials

$$Q_n(x,y) := \sum_{\substack{k,l=0\\ k+l\le n}}^{n} q_{1,k}(z)\overline{q_{2,l}(z)}, \quad z = x + iy, \quad x, y \in \mathbb{R}^N, \quad n \in \mathbb{N},$$

as well as

$$P_{2n}(x,y) := \sum_{k,l=0}^{n} q_{1,k}(z)\,\overline{q_{2,l}(z)} = p_{1,n}(z)\overline{p_{2,n}(z)}, \quad z = x + iy, \quad x, y \in \mathbb{R}^N, \quad n \in \mathbb{N}.$$

Then we get

$$P_{2n}(x,y) - Q_n(x,y) = \sum_{\substack{k,l=0\\ k+l>n}}^{n} q_{1,k}(z)\overline{q_{2,l}(z)} = \sum_{k=1}^{n} q_{1,k}(z)\big(\overline{p_{2,n}(z) - p_{2,n-k}(z)}\big).$$

In view of equation (3.18) we obtain

$$|p_{2,l}(z) - p_{2,k}(z)| \le |h(z) - p_{2,l}(z)| + |h(z) - p_{2,k}(z)| \le \frac{2M}{R^k} \qquad \text{for} \quad k < l, \quad z \in K.$$

From the definition of $q_{1,k}$ and the last estimate we conclude

$$|q_{1,k}(z)\big(\overline{p_{2,n}(z) - p_{2,n-k}(z)}\big)| \le \frac{2M}{R^{k-1}}\frac{2M}{R^{n-k}} = \frac{4M^2}{R^{n-1}} \qquad \text{for} \quad z \in K.$$

This gives

$$|P_{2n}(x,y) - Q_n(x,y)| \le \frac{4nM^2}{R^{n-1}} \qquad \text{for} \quad (x,y) \in L$$

and in consequence,

$$\begin{aligned} |F(x,y) - Q_n(x,y)| &\le |F(x,y) - P_{2n}(x,y)| + |P_{2n}(x,y) - Q_n(x,y)| \\ &\le \frac{M_1}{R^n} + \frac{4nM^2}{R^{n-1}} \qquad \text{for} \quad (x,y) \in L. \end{aligned}$$

Thus the result

$$\limsup_{n\to\infty} \sqrt[n]{E_n^c(L,F)} \le \frac{1}{\rho}$$

follows as $R < \rho$ was arbitrary. ■

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