für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Enhanced policy iteration for American options via scenario selection

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submitted: 17th November 2005

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No. 1071
Berlin 2005

2000 *Mathematics Subject Classification.* 60G40, 62L15, 91B28.

Key words and phrases. American options, Monte Carlo simulation, optimal stopping, policy improvement.

Supported by the DFG Research Center MATHEON (FZT 86) in Berlin.

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Abstract

In Kolodko & Schoenmakers [9] and Bender & Schoenmakers [2] a policy iteration was introduced which allows to achieve tight lower approximations of the price for early exercise options via a nested Monte-Carlo simulation in a Markovian setting. In this paper we enhance the algorithm by a scenario selection method. It is demonstrated by numerical examples that the scenario selection can significantly reduce the number of actually performed inner simulations, and thus can heavily speed up the method (up to factor 10 in some examples). Moreover, it is shown that the modified algorithm retains the desirable properties of the original one such as the monotone improvement property, termination after a finite number of iteration steps, and numerical stability.

1 Introduction

In recent years the pricing of American options on a high-dimensional system of underlyings via Monte Carlo has become an ever growing field of interest. While, in principle, the backward dynamic program provides a recursive representation of the (time-discretized) price process of an American option, it requires the evaluation of high order nestings of conditional expectations. Therefore Monte Carlo estimators for regression functions, which do not run into explosive cost when nested several times, have been proposed by several authors, see Longstaff & Schwartz [10], Broadie & Glasserman [5], and Bouchard et *al.* [4]. None of the methods can be generically applied, as they depend on a sophisticated choice of basis functions, knowledge of the transition densities, or the numerical evaluation of iterated Skorohod integrals, respectively.

An alternative to solving the backward dynamic program recursively are policy iterations for dynamic programming. The main advantage of policy iterations is, that they yield lower approximations of the price process for any given order of nested conditional expectations, which are typically of increasing quality the higher the order. (The latter property is referred to as monotone improvement property.) In a Markovian setting, this methodology allows to apply the plain Monte Carlo estimator to evaluate the conditional expectations, at least for nestings of order one. As one can only obtain the approximations corresponding to low order nestings this way, the quality of a single improvement step is of prime importance. A new policy improvement algorithm was developed in Kolodko & Schoenmakers [9] and Bender

& Schoenmakers [2] which outperforms for instance the more classical Howard improvement (e.g. [12]). Although this new algorithm is applicable rather generically, the nested Monte Carlo simulation makes it still quite costly.

In the present paper we enhance the policy improvement algorithm by a scenario selection method, while retaining the monotone improvement property of the original procedure. In this way the number of actually performed inner simulation can be reduced, which in some of our numerical examples speeds up the procedure by factor 10. The basic idea is as follows: Suppose the holder of an American option has some pre-information, for example he knows good closed form approximations of the price for the corresponding European options. Such approximations are typically available in the literature for practically relevant options. Given a trajectory of the underlying system, the investor rules out some time points, at which an optimal strategy cannot (or at least is very unlikely to) exercise, by the pre-information. Then the policy improvement is run only at the remaining time points. (Here, the set of remaining time points depends on the state of the underlying system. Hence we do not simply reduce to an other American option with a smaller set of exercise dates.)

After a short recap of American options and optimal stopping in discrete time (Section 2), we introduce the enhanced algorithm in Section 3.1 and verify the monotone improvement property. We also prove that the algorithm terminates after a finite number of iteration steps. The latter result is of theoretical interest rather, since in practice only one or two iterations can be calculated by plain Monte Carlo. We also estimate the additional error when time points are ruled out which are only unlikely but not impossible to be in the range of an optimal policy. In Section 3.2 we provide a pseudo-code for the implementation of the algorithm and prove its numerical stability in Section 3.3. Examples for American basket-call options on dividend paying stocks and American basket-put options are presented in Section 4. In particular the enhanced version of the algorithm is compared to the original version showing that the scenario selection may drastically increase the efficiency. Some proofs are postponed to the Appendix.

2 Optimal stopping in discrete time

It is well known that by the no arbitrage principle the pricing of American options is equivalent to the optimal stopping problem of the discounted derivative under a pricing measure. We now recall some facts about the optimal stopping problem in discrete time.

Suppose $(Z(i)\colon\ i = 0, 1, \dots, k)$ is a nonnegative stochastic process in discrete time on a probability space $(\Omega, \mathcal{F}, P)$ adapted to some filtration $(\mathcal{F}_i :\ 0 \le i \le k)$ which satisfies

$$\sum_{i=1}^{k} E|Z(i)| < \infty.$$

We may think of the process Z as a cashflow, which an investor may exercise once.

The investors' problem is to maximize his expected gain by choosing the optimal time for exercising. This problem is known as optimal stopping in discrete time.

To formalize the stopping problem we define $\mathcal{S}_i$ as the set of $\mathcal{F}_i$ stopping times taking values in $\{i, \ldots, k\}$. The stopping problem can now be stated as follows: Find stopping times $\tau^*(i) \in \mathcal{S}_i$ such that for $0 \leq i \leq k$

$$E^{\mathcal{F}_i}\left[Z(\tau^*(i))\right] = \text{esssup}_{\tau \in \mathcal{S}_i} E^{\mathcal{F}_i}\left[Z(\tau)\right]. \tag{1}$$

The process on the right hand side is called the *Snell envelope* of Z and we denote it by $Y^*(i)$. We collect some facts, which can be found in Neveu [11] for example.

1. The Snell envelope Y^* of Z is the smallest supermartingale that dominates Z. It can be constructed recursively by backward dynamic programming:

$$\begin{aligned} Y^*(k) &= Z(k) \\ Y^*(i) &= \max\{Z(i),\ E^{\mathcal{F}_i}[Y^*(i+1)]\}. \end{aligned}$$

2. A family of optimal stopping times is given by

$$\tilde{\tau}^*(i) = \inf\{i \leq j \leq k :\ Z(j) \geq Y^*(j)\}.$$

   If several optimal stopping families exist, then the above family is the family of first optimal stopping times. In that case

$$\hat{\tau}^*(i) = \inf\{i \leq j \leq k :\ Z(j) > Y^*(j)\}$$

   is the family of last optimal stopping times.

For the remainder of the paper we assume that

$$P(Z(k) > 0) > 0. \tag{2}$$

Clearly, this is no loss of generality: Let $\tilde{k} = \max\{0 \leq i \leq k;\ P(Z(i) > 0) > 0\}$. Then exercising at $i > \tilde{k}$ cannot be optimal and hence the stopping problem is equivalent to the one with exercise set $\{0, \ldots, \tilde{k}\}$.

3 Enhancing the policy iteration method

3.1 Definition and monotone improvement property

Suppose the buyer of the option chooses ad hoc a family of stopping times $(\tau(i) : 0 \leq i \leq k)$ taking values in the set $\{0, \ldots, k\}$. We interpret $\tau(i)$ as the time, at which the buyer will exercise his option, provided he has not exercised prior to time i. This interpretation requires the following consistency condition:

Definition 3.1. A family of integer-valued stopping times $(\tau(i): \ 0 \le i \le k)$ is said to be *consistent*, if

$$\begin{aligned} i \le \tau(i) \le k, \ \tau(k) \equiv k, \\ \tau(i) > i \Rightarrow \tau(i) = \tau(i+1), \qquad 0 \le i < k. \end{aligned} \tag{3}$$

Indeed, suppose $\tau(i) > i$, i.e. according to our interpretation the investor has not exercised the first right prior to time $i+1$. Then he has not exercised the first right prior to time i, either. This means he will exercise the first right at times $\tau(i)$ and $\tau(i+1)$, which requires $\tau(i) = \tau(i+1)$. A typical example of a consistent stopping family can be obtained by comparison with the still-alive European options, i.e.

$$\tau(i) := \inf \left\{ j : \ i \le j \le k, \ Z(j) \ge \max_{j+1 \le p \le k} E^{\mathcal{F}_j} \left[Z(p) \right] \right\}. \tag{4}$$

In addition to the algorithm introduced in Kolodko & Schoenmakers [9] and further developed in Bender & Schoenmakers [2] we suppose that the investor has in some sense a-priori knowledge about an optimal exercise strategy. We consider a random set A, $A \subset \{0, \ldots, k\}$, for which $\mathbf{1}_A(i)$ is $\mathcal{F}_i$-adapted, and $k \in A$ almost surely. Henceforth we will call such a set *an adapted random set*. Given some consistent stopping family τ we then consider a new stopping family by

$$\tilde{\tau}(i) := \inf \left\{ j : \ i \le j \le k, \ (Z(j) \ge \max_{j+1 \le p \le k} E^{\mathcal{F}_j} \left[Z(\tau(p)) \right]) \wedge (j \in A) \right\}. \tag{5}$$

Note that the stopping family $\tilde{\tau}$ is consistent. In particular $\tilde{\tau}(k) = k$, since $\max \emptyset = -\infty$ and $k \in A$. Moreover, by the definition of $\tilde{\tau}$, we have for all $0 \le i \le k$,

$$\tilde{\tau}(i) \in A. \tag{6}$$

In (5) the investor exploits his 'a-priori knowledge' by not exercising outside the set $A(\omega)$. If there exists some optimal stopping family τ^* such that

$$\tau^*(i) \in A, \quad i = 0, \ldots, k, \ P - a.s. \tag{7}$$

we call A an *a-priori set*. For instance, given any $\mathcal{F}_i$-adapted lower bound $L(i)$ of the Snell envelope $Y^*(i)$,

$$A(\omega) = \{ i : \ 0 \le i \le k, \ Z(i, \omega) \ge L(i, \omega) \} \tag{8}$$

is an a-priori set. For an a-priori set A, (5) means that, due to the new family $\tilde{\tau}$, the investor will not exercise at a date j which is either suboptimal or for optimality not necessary to exercise since $\tau^*(j) \in A$.

We call $\tilde{\tau}$ a *one-step improvement* of τ for the following reason: Denote by $Y(i;\tau)$ the value process corresponding to the stopping family τ, namely

$$Y(i;\tau) = E^{\mathcal{F}_i} \left[Z(\tau(i)) \right]. \tag{9}$$

Then due to the next theorem which is in fact a generalization of Theorem 3.6[1] in Bender & Schoenmakers [2], the one-step improvement yields a higher value than the given family, provided $\tau(i) \in A$.

Theorem 3.2. *Suppose A is an adapted random set, τ is a consistent stopping family such that $\tau(i) \in A$ a.s. for all $0 \le i \le k$. Consider*

$$\hat{\tau}(i) := \inf\left\{ j : \ i \le j \le k, \ (Z(j) > \max_{j+1 \le p \le k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right]) \wedge (j \in A) \right\}, \qquad (10)$$

and let $\bar{\tau}$ be a consistent stopping family such that

$$\tilde{\tau}(i) \le \bar{\tau}(i) \le \hat{\tau}(i), \quad 0 \le i \le k. \qquad (11)$$

Then,

$$Y(i;\bar{\tau}) \ge Y(i;\tilde{\tau}) \ge \max_{p \ge i} E^{\mathcal{F}_i}\left[Z(\tau(p))\right] \ge Y(i;\tau), \quad 0 \le i \le k. \qquad (12)$$

Moreover, $Y(i;\bar{\tau}) \ge Z(i)$ on $\{i \in A\}$.

Proof. Define $Z_A(i) := \mathbf{1}_A(i) Z(i)$. Since $\tau(i) \in A$, we have by (2)

$$\max_{p \ge i} E^{\mathcal{F}_i}\left[Z_A(\tau(p))\right] = \max_{p \ge i} E^{\mathcal{F}_i}\left[Z(\tau(p))\right] > 0, \quad 0 \le i \le k.$$

Consequently,

$$\tilde{\tau}(i) = \inf\left\{ j : \ i \le j \le k, \ Z_A(j) \ge \max_{j+1 \le p \le k} E^{\mathcal{F}_j}\left[Z_A(\tau(p))\right] \right\}, \qquad (13)$$

$$\hat{\tau}(i) = \inf\left\{ j : \ i \le j \le k, \ Z_A(j) > \max_{j+1 \le p \le k} E^{\mathcal{F}_j}\left[Z_A(\tau(p))\right] \right\}. \qquad (14)$$

We now apply Theorem 3.6, Proposition 3.11, and Remark 3.10 from [2] to the cashflow Z_A and obtain

$$E^{\mathcal{F}_i}\left[Z_A(\bar{\tau}(i))\right] \ge E^{\mathcal{F}_i}\left[Z_A(\tilde{\tau}(i))\right] \ge \max\left\{ Z_A(i), \ \max_{p \ge i} E^{\mathcal{F}_i}\left[Z_A(\tau(p))\right] \right\}.$$

So,

$$Y(i;\bar{\tau}) = E^{\mathcal{F}_i}\left[Z(\bar{\tau}(i))\right] \ge E^{\mathcal{F}_i}\left[Z_A(\bar{\tau}(i))\right] \ge E^{\mathcal{F}_i}\left[Z_A(\tilde{\tau}(i))\right] = Y(i;\tilde{\tau})$$

by (6), and

$$Y(i;\tilde{\tau}) \ge \max\left\{ Z_A(i), \ \max_{p \ge i} E^{\mathcal{F}_i}\left[Z(\tau(p))\right] \right\}$$

since $\tau(i) \in A$. □

Remark 3.1. It is interesting to note, that $\bar{\tau}(i)$ need not take values in A. For the case $A \equiv \{1, \ldots, k\}$ the results coincide with [2].

[1] Numbering according to the preprint version of the cited article

The following example shows that assumption $\tau(i) \in A$ cannot be dispensed with in Theorem 3.2.

Example 3.3. Suppose ξ is a binary trial with $P(\xi = 1) = P(\xi = -1)$. Define the process Z by $Z(0) = 9/4$, $Z(1) = Z(3) = 2$ and $Z(2) = 2 + \xi$. The filtration $\mathcal{F}_i$ is assumed to be generated by Z. Then $\sigma = 5/2 - \xi/2$ is a stopping time which yields an expected payoff $E[Z(\sigma)] = 5/2$. Consequently, immediate exercise at $t = 0$ cannot be optimal. With this knowledge we define an a-priori set $A(\omega) \equiv \{1, 2, 3\}$. We want to improve upon the trivial starting family $\tau(i) = i$, which obviously violates the condition $\tau(0) \in A$. We define

$$\tilde{\tau}(0) = \inf\{j \geq 0; (Z(j) \geq \max_{p=1,2,3} E[Z(p)]) \wedge (j \in A)\}.$$

A simple calculation gives $\tilde{\tau}(0) = 1$ and hence $E[Z(\tilde{\tau}(0))] = 2 < 9/4 = E[Z(\tau(0))]$. Hence, $\tilde{\tau}(0)$ does not improve upon $\tau(0)$.

It is natural to iterate the policy improvement (5). Suppose A is an adapted random set, and τ_0 is some consistent stopping family satisfying $\tau_0(i) \in A$ for all $0 \leq i \leq k$. Define, recursively,

$$\begin{aligned} \tau_m &= \tilde{\tau}_{m-1}, \\ Y_m(i) &= Y(i; \tau_m), \quad m = 1, 2, \ldots \end{aligned}$$

By Theorem 3.2, Y_m is an increasing sequence. Taking (13) and Proposition 4.4 in Kolodko & Schoenmakers [9] into account, we observe that the algorithm terminates after at most k steps.

Proposition 3.4. *Suppose $m \geq k - i$. Then,*

$$\tau_m(i) = \tilde{\tau}_A^*(i), \quad Y_m(i) = Y_A^*(i),$$

where $\tilde{\tau}_A^*$ and Y_A^* denote the first optimal stopping family and the Snell envelope for the stopping problem with cashflow $Z_A(i) = \mathbf{1}_A(i)Z(i)$. In particular, it follows that

$$Y_m(i) = Y^*(i)$$

for $m \geq k - i$, if A is an a-priori set.

If A is an a-priori set, the proposition states that the policy improvement algorithm terminates at the Snell envelope as fast as backward dynamic programming does. Most importantly, in every iteration step we obtain increased lower approximations of the Snell envelope, simultaneously at all exercise dates. If A is only an adapted random set, but not an a-priori set, the algorithm terminates at the Snell envelope Y_A^* of the cashflow $Z_A(i) = \mathbf{1}_A(i)Z(i)$ and not at Y^*. As we will demonstrate by numerical examples in Section 4, it can be numerically more efficient to choose an adapted random set which contains the image of an optimal stopping family with only high probability. The following theorem estimates the difference between the two Snell envelopes in such a situation. The proof is postponed to the Appendix.

Theorem 3.5. *Let A be an adaptive random set containing k a.s., and suppose that for some $q > 1$, $E[|Z(i)|^q] < \infty$ for all $0 \leq i \leq k$. Then for every consistent stopping family τ^* which is optimal for the cashflow Z the following estimate holds:*

$$E[Y^*(i) - Y_A^*(i)] \leq K_{q,i} P\left(\{\tau^*(i) \notin A\}\right)^{1-1/q},$$

where

$$K_{q,i} = (k-i)^{1/q} \max_{i \leq j \leq k-1} \left(E[|Z(j) 1_{\{0,\ldots,k\}\setminus A}(j)|^q]\right)^{1/q}.$$

In the case $A(\omega) \equiv \{1, \ldots, k\}$ the policy iteration presented in this subsection coincides with the one suggested in [9]. As will be explained in Section 3.2 and exemplified in Section 4, an appropriate choice of A can significantly reduce the computational cost for a Monte Carlo implementation of this policy improvement. In this respect the following corollary, which follows directly from Proposition A.1, is interesting.

Corollary 3.6. *Suppose τ is a consistent stopping family and $A_1 \subset A_2$ are adapted random sets. Define*

$$\begin{aligned} \tilde{\tau}(i) &:= \inf\left\{ j : i \leq j \leq k,\ (Z(j) \geq \max_{j+1 \leq p \leq k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right]) \wedge (j \in A_1) \right\}, \\ \tilde{\sigma}(i) &:= \inf\left\{ j : i \leq j \leq k,\ (Z(j) \geq \max_{j+1 \leq p \leq k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right]) \wedge (j \in A_2) \right\}. \end{aligned}$$

Then obviously $\tilde{\tau}$ and $\tilde{\sigma}$ are consistent and $\tilde{\tau}(i) \geq \tilde{\sigma}(i)$, $0 \leq i \leq k$. So by Proposition A.1 and Jensen's inequality we have

$$\begin{aligned} (Y(i;\tilde{\tau}) - Y(i;\tilde{\sigma}))_- &\leq \sum_{j=i}^{k-1} E^{\mathcal{F}_i}\left[\mathbf{1}_{\{\tilde{\tau}(i)>j\}} \mathbf{1}_{\{\tilde{\sigma}(i)=j\}} \left(Y(j;\tilde{\tau}) - Z(j)\right)_-\right], \qquad (15) \\ (Y(i;\tilde{\tau}) - Y(i;\tilde{\sigma}))_+ &\leq \sum_{j=i}^{k-1} E^{\mathcal{F}_i}\left[\mathbf{1}_{\{\tilde{\tau}(i)>j\}} \mathbf{1}_{\{\tilde{\sigma}(i)=j\}} \left(Y(j;\tilde{\tau}) - Z(j)\right)_+\right]. \end{aligned}$$

As a special case we may compare one step of the plain version of the algorithm, $\tilde{\sigma}$ with $A_2(\omega) = \{0, \ldots, k\}$, with a modified version (5) due to a non-trivial adaptive random set A_1 containing k a.s. Obviously, constructing $\tilde{\tau}$ is generally cheaper than constructing $\tilde{\sigma}$, and the quality loss with respect to $\tilde{\sigma}$, due to $\tilde{\tau}$ may be estimated by (15). In fact (15) means that $\tilde{\tau}$ may be worse than $\tilde{\sigma}$ only if $Y(i;\tilde{\tau})$ can be below the cashflow at a time where $\tilde{\sigma}$ says 'exercise' but $\tilde{\tau}$ refuses to do so.

3.2 On the implementation

We now give some comments on the practical Monte Carlo implementation of the policy iteration. Henceforth, we suppose that the cashflow Z is of the form $Z(i) =$

$f(i, X(i))$ where $f(i, x)$ is a deterministic function and $(X(i), \mathcal{F}_i)$ is a – possibly high-dimensional – Markovian chain. Note that one improvement step of an initial lower bound $Y(0; \tau_0)$ requires a nested Monte Carlo simulation provided there are no closed form expressions for the conditional expectations in (5). The introduction of an adapted random set can significantly reduce the number of actually performed inner simulations and consequently increases the efficiency of the method. We suggest to implement an improvement step as follows.

Step 1: Choose an adapted random set A such that $(j \in A)$ can be checked in closed form given a trajectory of X up to time j. This means, there are Borel sets $B_j \subset \mathbb{R}^{j+1}$, $B_k = \mathbb{R}^{k+1}$,which are explicitly known to the investor, such that $(j \in A)$ if and only if $(X_0, \ldots, X_j) \in B_j$. For instance good closed form approximations of the price processes of still alive Europeans are often available for practically relevant products. In such situation let $L(i)$ be a closed form approximation of their maximum, $\max_{i+1 \le p \le k} E^{\mathcal{F}_i}[Z(p)]$. Typically $L(i) = g(i, X_i)$ for some explicitly known deterministic function $g(i, x)$. Define

$$\begin{aligned} A(\omega) &= \{i : \; Z(i, \omega) \ge L(i, \omega)\} \\ &= \{i : \; f(i, X(i, \omega)) \ge g(i, X(i, \omega))\}. \end{aligned} \tag{16}$$

Obviously, A is in closed form in the above sense. One can just define $B_j = \mathbb{R}^j \times (f(j, \cdot) - g(j, \cdot))^{-1}([0, \infty))$. Clearly, A is an a-priori set, if L is a lower approximation of the maximum of still alive Europeans, but only an adapted random set in general.

Step 2: Choose an initial stopping family τ_0 such that $\tau_0(i) \in A$ for all $0 \le i \le k$. A natural choice is

$$\tau_0(i) = \inf\{i \le j \le k; \; j \in A\}.$$

One can alternatively define

$$\tau_0(i) = \inf\{i \le j \le k; \; (f(j, X(j)) \ge H(j)) \wedge (j \in A)\}$$

where H is a close-to-optimal deterministic vector determined by pre-simulation as suggested in Andersen [1].

Step 3: Construct a lower bound $Y(0; \tau_1)$ due to the improved policy $\tau_1 := \tilde{\tau}_0$ using the following pseudo-code.

Simulate M trajectories $X^{(m)}$, $m = 1, \ldots, M$, starting at $X(0)$;

Along each trajectory $X^{(m)}$ we compute $\eta^{(m)} \approx \tau_1^{(m)}(0)$ as follows:

$i := 0$;

A: Search the first exercise date $\eta \ge i$ such that η belongs to A, which formally means $(X^{(m)}(0), \ldots, X^{(m)}(\eta)) \in B_\eta$, since A is in closed form.
If $\eta = k$ (i.e. $\tau_1^{(m)}(0) = k$) then set $\eta^{(m)} := k$, else:
Consider η as a candidate for $\tau_1^{(m)}(0)$.

To decide whether $\eta \approx \tau_1^{(m)}(0)$ or not we do the following:

Simulate M_1 trajectories $(X^{(m,p)}(q),\ q = \eta, \ldots, k)$, $p = 1, \ldots, M_1$, under the conditional measure $P^{X^{(m)}(\eta)}$ (hence $X^{(m,p)}(\eta) = X^{(m)}(\eta)$);

Along each trajectory (m,p) search all exercise dates $\geq \eta$ where the policy τ_0 says 'exercise'. From these dates we can detect easily (an approximation of) the family $(\tau_0^{(m,p)}(q),\ q \geq \eta)$ along the path (m,p);

Then, for $q = \eta, \ldots, k$ compute

$$\begin{aligned} Dummy[q] &:= \frac{1}{M_1} \sum_{p=1}^{M_1} f(\tau_0^{(m,p)}(q), X^{(m,p)}(\tau_0^{(m,p)}(q))) \\ &\approx E^{X^{(m)}(\eta)} Z(\tau_0(q)); \end{aligned}$$

Next determine

$$Max\_Dummy := \max_{\eta \leq q \leq k} Dummy[q] \approx \max_{\eta \leq q \leq k} E^{X^{(m)}(\eta)} Z(\tau_0(q));$$

Check whether $f(\eta, X^{(m)}(\eta)) \geq Max\_Dummy$:

If yes, set $\eta^{(m)} := \eta \approx \tau_1^{(m)}(0)$;

If no, do $i := \eta + 1$ and go to **(A)**;

We so end up with $\eta^{(m)} \approx \tau_1^{(m)}(0)$;

Finally compute $\displaystyle\frac{1}{M} \sum_{m=1}^{M} f(\eta^{(m)}, X^{(m)}(\eta^{(m)})) \approx EZ(\tau_1(0)) = Y(0; \tau_1)$.

Step 4: Given a consistent stopping family τ_0 as in Step 2, $Y(0; \tau_0)$ is a lower bound of $Y^*(0)$. We recommend to construct an upper bound from this lower bound by the duality method developed by Rogers [13] and Haugh & Kogan [6]. Define,

$$Y_{up}(0; \tau) = E\left[\max_{0 \leq j \leq k} (Z(j) - M(j))\right], \tag{17}$$

where $M(0) = 0$ and, for $1 \leq i \leq k$,

$$M(i) = \sum_{p=1}^{i} \left(Y(p; \tau) - E^{\mathcal{F}_{p-1}}\left[Y(p; \tau)\right]\right).$$

Approximation of this upper bound due to $Y(\cdot; \tau_0)$ by Monte Carlo also requires nested simulation. It is, thus, roughly as expensive as the improvement of τ_0 described in Step 3. For a detailed treatment of efficient computation of dual upper bounds see for example Kolodko & Schoenmakers [8]. A multiplicative analogon of the duality method is due to Jamshidian [7].

3.3 Stability

As described in the previous section, for practical implementation one typically has to approximate the conditional expectations in the exercise criterion. We now extend a stability result from Bender & Schoenmakers [2] to the case of a nontrivial adapted random set. Let A be an adapted random set and τ be a consistent stopping family which satisfies $\tau(i) \in A$ for all $0 \leq i \leq k$. Further suppose $\epsilon^{(N)}(i)$ is a sequence of $\mathcal{F}_i$-adapted processes $\epsilon^{(N)}(i)$ such that

$$\lim_{N \to \infty} \epsilon^{(N)}(i) = 0, \quad P - a.s.$$

A perturbed version of the one step improvement is then defined by

$$\begin{aligned} \tilde{\tau}^{(N)}(i) \quad &:= \quad \inf\Big\{ j : \ i \leq j \leq k, \ (Z(j) \geq \max_{j+1 \leq p \leq k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right] + \epsilon^{(N)}(j)) \\ &\qquad \wedge (j \in A) \Big\}. \end{aligned} \tag{18}$$

The sequence $\epsilon^{(N)}$ accounts for the errors when approximating the conditional expectation. We may and will assume that $\epsilon^{(N)}(k) = 0$, since no conditional expectation is to be evaluated at $j = k$. In accordance with the previous section we suppose that the criterion $j \in A$ can be checked in closed form. We first recall that even with a trivial a-priori set we can neither expect

$$\tilde{\tau}^{(N)}(i) \to \tilde{\tau}(i) \quad \text{in probability}$$

nor

$$Y(0; \tilde{\tau}^{(N)}) \to Y(0; \tilde{\tau})$$

in general. For corresponding counterexamples we refer to [2]. However, we can generalize a stability result from [2] where the error is measured in terms of the shortfall instead of the absolute value. As emphasized in [2], preventing shortfall (viz. change to the worse) is the relevant criterion to look at since our goal is improvement.

Theorem 3.7. *For all* $0 \leq i \leq k$,

$$\lim_{N \to \infty} \left(Y(i; \tilde{\tau}^{(N)}) - Y(i; \tilde{\tau}) \right)_- = 0,$$

where the limit is P*-almost surely and in* $L^1(P)$.

Proof. Since $\epsilon^{(N)}(k) = 0$ we may write as in (13),

$$\tilde{\tau}^{(N)}(i) = \inf\Big\{ j : \ i \leq j \leq k, \ Z_A(j) \geq \max_{j+1 \leq p \leq k} E^{\mathcal{F}_j}\left[Z_A(\tau(p))\right] + \epsilon^{(N)}(j) \Big\},$$

where $Z_A(i) = \mathbf{1}_A(i) Z(i)$. The assertion now follows from Theorem 4.4 in [2] applied to the cashflow Z_A. □

Remark 3.2. Theorem 3.7 provides stability of one improvement step. More generally, one can prove that the shortfall of the expected gain corresponding to m perturbed steps of the algorithm below the expected gain corresponding to m theoretical steps converges to zero. In the case of the trivial a-priori set $A = \{0, \ldots, k\}$ this statement is made precise and proved in [2], Section 4.2. This result carries over to the case of a general adapted random set. Here it is crucial that the criterion $(j \in A)$ involves no approximation so that it is guaranteed that e.g. $\tilde{\tau}^{(N)}(i) \in A$ for all $0 \le i \le k$.

4 Numerical examples

We now illustrate our algorithm with two examples: Bermudan basket-call and basket-put options on 5 assets. We assume, that each asset is governed under the risk-neutral measure by the following SDE:

$$dS_i(t) = (r - \delta)S_i(t)dt + \sigma S_i(t)dW_i(t), \quad 1 \le i \le 5,$$

where $(W_1(t), \ldots, W_5(t))$ is a standard 5-dimensional Brownian motion. Suppose that an option can be exercised at $k+1$ dates $T_0, \ldots, T_k$, where $0 = T_0, \ldots, T_k = 3$ are uniformly distributed at $[0, 3]$. The price of the Bermudan option is given by (1) with

$$Z(i) = e^{-rT_i}(\frac{S_1(T_i) + \ldots + S_5(T_i)}{5} - K)^+ \quad \text{for the call option and}$$
$$Z(i) = e^{-rT_i}(K - \frac{S_1(T_i) + \ldots + S_5(T_i)}{5})^+ \quad \text{for the put option.}$$

For our simulation, we take the following parameter values,

$$r = 0.05, \quad \sigma = 0.2, \quad S_1(0) = \ldots S_5(0) = S_0, \quad K = 100,$$
$$\delta = 0.1 \text{ for call option}, \quad \delta = 0 \text{ for put option.}$$

We consider a call and a put option 'out-the-money', 'at-the-money', and 'in-the-money' at $t = 0$. For an adapted random set A, given by (16) due to a particular process, we consider the initial stopping family $\tau(i) = \inf\{j \ge i : j \in A\}$ and construct the lower bound $Y(0; \tau)$, the improved lower bound $Y(i; \tilde{\tau})$ with $\tilde{\tau}$ given by (5), and the dual upper bound $Y_{up}(0; \tau)$. For comparison, we also compute the standard one-step improvement of the lower bound $Y(0; \check{\tau})$, where

$$\check{\tau}(i) := \inf\left\{j : \ i \le j \le k, \ Z(j) \ge \max_{j+1 \le p \le k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right]\right\}.$$

4.1 Basket-call

For this example we take an a-priori set A given by (16) due to

$$L(j) := \max_{j+1 \le p \le k} e^{-rT_p} E^{\mathcal{F}_j}((S_1(T_p) \cdots S_5(T_p))^{1/5} - K), \quad 0 \le j \le k, \qquad (19)$$

which is a lower approximation of the Snell envelope since

$$L(j) \leq \max_{j+1 \leq p \leq k} E^{\mathcal{F}_j} Z(p) \leq Y^*(j), \quad 0 \leq j \leq k.$$

The process $G(t) := (S_1(t) \cdots S_5(t))^{1/5}$ has a log-normal distribution and can be represented as

$$G(t) = e^{(r - \frac{1}{2}\sigma^2)t + \frac{\sigma}{5}(W_1(t) + \cdots + W_5(t))}.$$

Then $\tilde{W}(t) := \frac{1}{\sqrt{5}}(W_1(t) + \cdots + W_5(t))$ is a standard Brownian motion and we thus have

$$G(t) = e^{(r - \frac{1}{2}\tilde{\sigma}^2)t + \tilde{\sigma}\tilde{W}(t)} \cdot e^{-0.4\sigma^2 t} \quad \text{with} \quad \tilde{\sigma} = \frac{\sigma}{\sqrt{5}}.$$

So the right hand side of (19) can be given in closed form by the well-known Black-Scholes formula (BS),

$$\begin{aligned} e^{-rT_p} E^{\mathcal{F}_j} (G(T_p) - K)^+ &= e^{-rT_p} E^{\mathcal{F}_j} (e^{(r - \frac{1}{2}\tilde{\sigma}^2)T_p + \tilde{\sigma}\tilde{W}(T_p)} - K e^{0.4\sigma^2 T_p}) e^{-0.4\sigma^2 T_p} \\ &= e^{-0.4\sigma^2 T_p - rT_j} BS(G(T_j), r, \delta, \tilde{\sigma}, K e^{0.4\sigma^2 T_p}, T_p - T_j). \end{aligned}$$

First we simulate $Y(0;\tau)$ by 10^7 Monte Carlo trajectories and next simulate $Y(0;\tilde{\tau})$ and $Y(0;\check{\tau})$. To reduce the number of nested (inner) Monte Carlo paths, we use the representations

$$\begin{aligned} Y(0;\tilde{\tau}) &= Y(0;\tau) + E[Z(\tilde{\tau}(0)) - Z(\tau(0))] \quad \text{and} \\ Y(0;\check{\tau}) &= Y(0;\tau) + E[Z(\check{\tau}(0)) - Z(\tau(0))] \end{aligned} \tag{20}$$

Here we simulate the second term using $2 \cdot 10^5$ outer Monte Carlo trajectories and 1000 inner trajectories. Further, $Y_{up}(0;\tau) - Y(0;\tau)$ is simulated by 20 000 outer and 1000 inner trajectories. The results are given in Table 1, where we can see that although the initial stopping family gives a rather crude lower bound (the gap between $Y(0;\tau)$ and its dual upper bound $Y_{up}(0;\tau)$ is 4%-17% relative to the value), the improvements $Y(0;\tilde{\tau})$ and $Y(0;\check{\tau})$ are pretty close to the Bermudan price (the error is less then 1.7% relative to the value).

It is important to note the following. For simulating $Y(0;\check{\tau})$ we need to estimate the conditional expectations by nested Monte Carlo simulation at each exercise date until the decision to exercise is made. However, since a closed form expression of the process L is available, we can avoid the nested Monte Carlo simulation at many exercise dates by rejecting the dates, which are not in A, see Section 3.2. In Table 1, columns 7 and 8, we display the average number of points (per trajectory), where the nested Monte Carlo simulation has been carried out for constructing $Y(0;\tilde{\tau})$ and $Y(0;\check{\tau})$ respectively. We see, that pre-selecting exercise dates by checking $Z(i) < L(i)$ for each i reduces the number of nested Monte Carlo simulations up to 7 times. However, the values of $Y(0;\tilde{\tau})$ and $Y(0;\check{\tau})$ are the same within one standard deviation.

Table 1.

k	S_0	$Y(0;\tau)$ (SD)	$Y(0;\tilde{\tau})$ (SD)	$Y(0;\check{\tau})$ (SD)	$Y_{up}(0;\tau)$ (SD)	$\tilde{N}$	$\check{N}$
	90	0.369(0.001)	0.383(0.001)	0.387(0.002)	0.384(0.001)	1.0	2.0
3	95	0.906(0.001)	0.938(0.002)	0.935(0.002)	0.934(0.001)	1.1	1.9
	100	1.978(0.001)	2.025(0.003)	2.023(0.003)	2.027(0.002)	1.1	1.8
	103	3.000(0.000)	3.057(0.003)	3.053(0.003)	3.057(0.002)	1.1	1.7
	90	0.376(0.000)	0.415(0.002)	0.415(0.002)	0.416(0.002)	1.1	4.9
6	95	0.934(0.001)	1.028(0.003)	1.025(0.003)	1.030(0.002)	1.1	4.7
	100	2.140(0.001)	2.290(0.004)	2.294(0.004)	2.298(0.003)	1.2	4.1
	103	3.000(0.000)	3.160(0.004)	3.158(0.004)	3.173(0.003)	1.3	3.5
	90	0.368(0.000)	0.425(0.002)	0.430(0.002)	0.431(0.002)	1.1	7.8
9	95	0.917(0.001)	1.053(0.003)	1.050(0.003)	1.071(0.003)	1.2	7.3
	100	2.136(0.001)	2.366(0.005)	2.361(0.005)	2.395(0.004)	1.4	6.1
	103	3.000(0.000)	3.247(0.005)	3.245(0.005)	3.282(0.004)	1.4	5.0

4.2 Basket-put

In our next example we determine a process $L(j)$ by a moment-matching procedure. Let us define $f(T_j) := (S_1(T_j) + \ldots + S_5(T_j))/5$ for $0 \le j \le k$ and take j, p with $j \le p \le k$. First, we approximate $f(T_p)$ by

$$f_j(T_p) := f(T_j)\exp\left((r_j - \frac{1}{2}\sigma_j^2)(T_p - T_j) + \sigma_j(W(T_p) - W(T_j))\right),$$

where the parameters r_j and σ_j are taken in such a way that the first two moments of $f(T_p)$ and $f_j(T_p)$ are equal conditional $\mathcal{F}_j$:

$$r_j = r,$$

$$\sigma_j = \frac{1}{T_p - T_j}\ln\left(\frac{\sum_{m,n=1}^{5} S_m(T_j)S_n(T_j)\exp(1_{m=n}\sigma^2(T_p - T_j))}{(\sum_{m=1}^{5} S_m(T_j))^2}\right),$$

see, e.g., Brigo et *al.* [3]. Then, we approximate $E^{\mathcal{F}_j}Z(p)$ by $E^{\mathcal{F}_j}[e^{-rT_p}(K - f_j(T_p))^+]$ using the Black-Scholes formula,

$$E^{\mathcal{F}_j}[e^{-rT_p}(K - f_j(T_p))^+] = e^{-rT_j}BS(f(T_j), r, \sigma_j, K, T_p - T_j),$$

and define

$$L(j) := e^{-rT_j}\max_{j+1\le p\le k} BS(f(T_j), r, \sigma_j, K, T_p - T_j), \quad 0 \le j \le k.$$

The process L thus provides a close approximation for the maximum of still alive Europeans. Even the initial stopping family τ leads to a reasonable lower approximation $Y(0;\tau)$ of the Bermudan price (less then 4% relative). Although we can not claim that L is a lower bound of the Snell envelope Y^* (and thus A is not necessarily an a-priori set), the improved lower bound $Y(0;\tilde{\tau})$ coincide with $Y(0;\check{\tau})$ and with the dual upper bound $Y_{up}(0;\tau)$ within one standard deviation. See Table 2, where we used 10^7 Monte Carlo trajectories for $Y(0;\tau)$ and 2000 trajectories (with 1000 nested trajectories) for $Y_{up}(0;\tau) - Y(0;\tau)$. To simulate $Y(0;\tilde{\tau})$ and $Y(0;\check{\tau})$ we apply representation (20), where the second term is estimated with 10^5 outer and 1000 inner trajectories.

As in the previous example, using an a-priori information for improved stopping family reduces the number of nested simulations up to 7 times, see Table 2, columns 7-8. Here we denote $\tilde{N}$ and $\check{N}$ the average number of the nested Monte Carlo simulation (per trajectory), which have been performed for constructing $Y(0;\tilde{\tau})$ and $Y(0;\check{\tau})$ respectively.

Table 2.

k	S_0	$Y(0;\tau)$ (SD)	$Y(0;\tilde{\tau})$ (SD)	$Y(0;\check{\tau})$ (SD)	$Y_{up}(0;\tau)$ (SD)	$\tilde{N}$	$\check{N}$
	97	3.000(0.000)	3.001(0.002)	3.005(0.003)	3.006(0.001)	1.0	1.7
3	100	2.156(0.001)	2.162(0.003)	2.160(0.003)	2.160(0.001)	1.0	1.8
	105	1.095(0.001)	1.099(0.002)	1.100(0.002)	1.100(0.001)	1.0	1.9
	110	0.537(0.001)	0.537(0.001)	0.536(0.001)	0.538(0.001)	1.0	2.0
	97	3.000(0.000)	3.051(0.005)	3.052(0.005)	3.050(0.004)	1.2	3.5
6	100	2.361(0.001)	2.400(0.005)	2.395(0.005)	2.406(0.004)	1.1	4.0
	105	1.170(0.002)	1.190(0.003)	1.191(0.003)	1.190(0.002)	1.1	4.6
	110	0.571(0.001)	0.581(0.002)	0.579(0.002)	0.578(0.001)	1.0	4.9
	97	3.000(0.000)	3.119(0.006)	3.104(0.006)	3.120(0.006)	1.3	5.0
9	100	2.386(0.001)	2.481(0.006)	2.475(0.005)	2.482(0.006)	1.3	6.2
	105	1.180(0.001)	1.226(0.004)	1.223(0.004)	1.228(0.004)	1.1	7.3
	110	0.580(0.001)	0.603(0.003)	0.600(0.003)	0.602(0.003)	1.1	7.7

A Proof of Theorem 3.5

We first prove the next Proposition.

Proposition A.1. *Let τ and σ be two consistent stopping families, such that $\sigma(i) \leq \tau(i)$, $0 \leq i \leq k$. Then,*

$$Y(i;\tau) - Y(i;\sigma) = \sum_{j=i}^{k-1} E^{\mathcal{F}_i}[1_{\tau(i)>j} 1_{\sigma(i)=j} (Y(j;\tau) - Z(j))].$$

Proof. We have,

$$
\begin{aligned}
Y(i;\tau) - Y(i;\sigma) &= E^{\mathcal{F}_i} 1_{\tau(i)>\sigma(i)} (Z(\tau(i)) - Z(\sigma(i))) \\
&= \sum_{j=i}^{k-1} E^{\mathcal{F}_i} 1_{\tau(i)>j} 1_{\sigma(i)=j} (Z(\tau(i)) - Z(j)) \\
&= \sum_{j=i}^{k-1} E^{\mathcal{F}_i} 1_{\tau(i)>j} 1_{\sigma(i)=j} (Z(\tau(j)) - Z(j)) \\
&= \sum_{j=i}^{k-1} E^{\mathcal{F}_i} 1_{\tau(i)>j} 1_{\sigma(i)=j} (E^{\mathcal{F}_j} Z(\tau(j)) - Z(j)) \\
&= \sum_{j=i}^{k-1} E^{\mathcal{F}_i} 1_{\tau(i)>j} 1_{\sigma(i)=j} (Y(j;\tau) - Z(j)).
\end{aligned}
$$

Here we use that, due to the consistency, if $i \le j \le \tau(i)$, then $\tau(j) = \tau(i)$. ☐

As a second preliminary result for the proof of Theorem 3.5 we have the following lemma.

Lemma A.2. *Suppose τ^* is some optimal stopping family for the cashflow Z. Then $A^*(\omega) = \{\tau^*(i,\omega),\ 0 \le i \le k\}$ is an a-priori set. Moreover,*

$$
\tau^*(i) = \inf\left\{ j \ge i;\ (Z(j) \ge \max_{j+1 \le p \le k} E^{\mathcal{F}_j} Z(\tau^*(p))) \wedge (j \in A^*) \right\}
$$

provided τ^* is consistent.

Proof. Since $\{i \in A^*\} = \bigcup_{0 \le j \le i} \{\tau^*(j) = i\} \in \mathcal{F}_i$ and $\tau^*(i) \in A^*$, A^* is an a-priori set. Suppose now that additionally τ^* is consistent. Then, by consistency and optimality of $\tilde{\tau}$, and by the supermartingal property of the Snell envelope, we have

$$
\begin{aligned}
& \inf\left\{ j \ge i;\ (Z(j) \ge \max_{j+1 \le p \le k} E^{\mathcal{F}_j} Z(\tau^*(p))) \wedge (j \in A^*) \right\} \\
= & \inf\left\{ j \ge i;\ (Z(j) \ge \max_{j \le p \le k} E^{\mathcal{F}_j} Z(\tau^*(p))) \wedge (j \in A^*) \right\} \\
= & \inf\left\{ j \ge i;\ (Z(j) \ge \max_{j \le p \le k} E^{\mathcal{F}_j} Y^*(p)) \wedge (j \in A^*) \right\} \\
= & \inf\left\{ j \ge i;\ (Z(j) \ge Y^*(j)) \wedge (j \in A^*) \right\}.
\end{aligned}
$$

Moreover, by consistency, $j \in A^*$ if and only if $\tau^*(j) = j$. Hence,

$$
\begin{aligned}
& \inf\left\{ j \ge i;\ (Z(j) \ge Y^*(j)) \wedge (j \in A^*) \right\} \\
= & \inf\left\{ j \ge i;\ (Z(\tau^*(j)) \ge Y^*(\tau^*(j))) \wedge (\tau^*(j) = j) \right\} \\
= & \inf\left\{ j \ge i;\ \tau^*(j) = j \right\} = \tau^*(i).
\end{aligned}
$$

Note, for the second identity we applied the well-known fact, that evaluated at any optimal stopping time the Snell envelope Y^* equals the cashflow Z. Thus, $(Z(\tau^*(j)) \geq Y^*(\tau^*(j)))$ is always satisfied for all $0 \leq j \leq k$. □

After these preparations we prove Theorem 3.5.

Proof of Theorem 3.5. Let τ be a consistent and optimal stopping family for the cashflow Z. We define $\tilde{\tau}$ and $\tilde{\sigma}$ as in Corollary 3.6 for $A_2(\omega) = \{\tau(i,\omega),\ 0 \leq i \leq k\}$ and $A_1(\omega) = A_2(\omega) \cap A(\omega)$. Then $\tilde{\tau} \geq \tilde{\sigma}$ and $\tilde{\sigma} = \tau$ is optimal due to Lemma A.2. Hence by Proposition A.1 we have,

$$E[Y^*(i) - Y(i;\tilde{\tau})] = \sum_{j=i}^{k-1} E[1_{\tilde{\tau}(i)>j} 1_{\tau(i)=j} (Z(j) - Y(j;\tilde{\tau}))]$$

As $\tilde{\tau}$ takes values in A and is possibly suboptimal for the cashflow Z_A, we obtain

$$Y(i;\tilde{\tau}) = E^{\mathcal{F}_i}[Z_A(\tilde{\tau}(i))] \leq Y_A^*(i).$$

Consequently, by Hölder's inequality,

$$\begin{aligned}
& E[Y^*(i) - Y_A^*(i)] \leq E[Y^*(i) - Y(i;\tilde{\tau})] \\
\leq\ & \sum_{j=i}^{k-1} E\left[1_{\{j \notin A\}} 1_{\{\tau(i)=j\}} Z(j)\right] \\
\leq\ & \max_{i \leq j \leq k-1} \left(E[|Z(j) 1_{\{0,\ldots,k\}\setminus A}(j)|^q]\right)^{1/q} \sum_{j=i}^{k-1} P\left(\{\tau(i) = j\} \cap \{j \notin A\}\right)^{1-1/q} \\
\leq\ & \max_{i \leq j \leq k-1} \left(E[|Z(j) 1_{\{0,\ldots,k\}\setminus A}(j)|^q]\right)^{1/q} (k-i)^{1/q} \\
& \times \left(\sum_{j=i}^{k-1} P\left(\{\tau(i) = j\} \cap \{j \notin A\}\right)\right)^{1-1/q}.
\end{aligned}$$

The obvious equation

$$\sum_{j=i}^{k-1} P\left(\{\tau(i) = j\} \cap \{j \notin A\}\right) = P\left(\{\tau(i) \notin A\}\right)$$

concludes. □

References

[1] Andersen, L. (1999) A simple approach to the pricing of Bermudan swaptions in a multifactor LIBOR market model. *Journal of Computational Finance*, **3**, 5-32.

[2] Bender, C., Schoenmakers, J. (2004) An iterative procedure for the multiple stopping problem. Preprint.

[3] Brigo, D., Mercurio, F., Rapisarda, F., Scotti, R. (2004) Approximated moment-matching dynamics for basket-options pricing. *Quantitative Finance*, **4**, 1-16.

[4] Bouchard, B., Ekeland, I., Touzi, N. (2004) On the Malliavin approach to Monte Carlo approximation of conditional expectations. *Finance and Stochastics*, **8**, 45-71.

[5] Broadie, M., Glasserman, P. (2004) A stochastic mesh method for pricing high-dimensional American options. *Journal of Computational Finance*, **7**(4), 35-72.

[6] Haugh, M. B., Kogan, L. (2004) Pricing american options: a duality approach. *Operations Research*, **52**, 258-270.

[7] Jamshidian, F. (2004) Numeraire-invariant option pricing & american, bermudan, and trigger stream rollover. Working paper

[8] Kolodko, A., Schoenmakers, J. (2004) Upper Bound for Bermudan Style Derivatives. WIAS Preprint 877 (2003), *Monte Carlo Methods and Applications,* **10**(3-4), 331-343.

[9] Kolodko, A., Schoenmakers, J. (2004) Iterative construction of the optimal Bermudan stopping time. WIAS Preprint 926 (2004), *Finance and Stochastics*, forthcoming.

[10] Longstaff, F. A., Schwartz, R. S. (2001) Valuing American options by simulation: a simple least-squares approach. *Review of Financial Studies*, **14**, 113-147.

[11] Neveu, J. (1975) *Discrete parameter martingales.* North-Holland: Amsterdam.

[12] Puterman, M. (1994) *Markov decision processes.* New York: Wiley.

[13] Rogers, L. C. G. (2002) Monte Carlo valuation of American options. *Math. Finance*, **12**, 271-286.