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On the numerical range of sectorial forms

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ABSTRACT. We provide a sharp and optimal generic bound for the angle of the sectorial form associated to a non-symmetric second-order elliptic differential operator with various boundary conditions. Consequently this gives an, in general, sharper \mathcal{H}^∞ -angle for the \mathcal{H}^∞ -calculus on L_p for all $p \in (1, \infty)$ if the coefficients are real valued.

1. INTRODUCTION

In the L_2 -theory of second-order divergence form operators it is classical that the numerical range of the sesquilinear form $\mathfrak{a}: W^{1,2}(\Omega) \times W^{1,2}(\Omega) \rightarrow \mathbb{C}$ given by

$$\mathfrak{a}(u, v) = \int_{\Omega} \mu \nabla u \cdot \overline{\nabla v}$$

is contained in the sector with (half-)angle $\arctan \frac{M}{m}$, if the coefficient function μ admits the uniform bound M and ellipticity constant m . Moreover, it is well-known that the angle of the numerical range sector has implications for resolvent estimates and for the holomorphic calculus, both for the L_2 -realisation of the elliptic operator and the L_p -realisation, see below. Hence the question arises whether the above angle is optimal. In this paper we show that one can improve the angle. The expression we find is completely explicit in M and m , uniform in all matrices with uniform bound M and ellipticity constant m . Moreover, it is optimal, see Example 2.5. All of this allows in Corollary 2.4 to give a sharper estimate for the angle of the sector containing the numerical range.

Further, we provide resolvent decay for the operator A_p which is associated with the form \mathfrak{a} on $L_p(\Omega)$, where $p \in (1, \infty)$. Uniform resolvent estimates for the elliptic operators are important for the treatment of nonautonomous parabolic equations, see for example [19], [13], [18], [1], [22]. For an alternative approach, not using the evolution system, see [17]. This use of uniform resolvent estimates is standard nowadays, see [2, Chapter II], [15, Section 6.1].

In Section 3 we prove that the operator A_p admits a bounded \mathcal{H}^∞ -calculus with (half-)angle smaller than $\pi/2$. Using the better numerical range on $L_2(\Omega)$, we obtain a better \mathcal{H}^∞ -angle on $L_p(\Omega)$ by applying the Crouzeix–Delyon theorem and a theorem of Kalton–Kunzmann–Weis. This enables sharper estimates for the purely imaginary powers of the operators. Applying the Dore–Venni theorem one obtains, as a byproduct, even maximal parabolic regularity on $L_p(\Omega)$ for all $p \in (1, \infty)$.

In Section 4 we consider as in [10] or [20] an elliptic operator subject to mixed boundary conditions and domain inhomogeneities supported on a lower dimensional hypersurface. Such results are of use when treating parabolic problems with dynamical boundary conditions, compare also [21].

2. NUMERICAL RANGE AND SPECTRAL THEORETIC CONSEQUENCES

Let H be a Hilbert space with $H \neq \{0\}$ and let T be an operator in H with domain $\text{dom}(T)$. The **numerical range** $\Lambda(T)$ of T is defined by

$$\Lambda(T) = \{(Tu, u)_H : u \in \text{dom}(T) \text{ and } \|u\|_H = 1\}.$$

A classical theorem of Hausdorff says that the numerical range is a convex set. For all $\theta \in [0, \pi)$ define

$$\Sigma(\theta) = \{r e^{i\varphi} : r \in [0, \infty) \text{ and } \varphi \in [-\theta, \theta]\}.$$

Then $\Sigma(\theta)$ is closed and $0 \in \Sigma(\theta)$.

Proposition 2.1. *Let T be a bounded operator in a Hilbert space H . Let $E = \frac{1}{2i}(T - T^*)$ be the imaginary part of T . Suppose that T is coercive and let $m > 0$ be such that*

$$\operatorname{Re}(Tu, u)_H \geq m \|u\|_H^2.$$

for all $u \in H$. Then

$$(1) \quad \|E\| \leq \sqrt{\|T\|^2 - m^2}$$

and $\Lambda(T) \subset \Sigma(\arctan \sqrt{(\frac{\|T\|}{m})^2 - 1})$.

Proof. Define $S = \frac{1}{2}(T + T^*)$, the real part of T . Then S is self-adjoint and $(Su, u)_H = \operatorname{Re}(Tu, u)_H \geq m \|u\|_H^2$ for all $u \in H$. Hence $\|Su\| \geq m \|u\|_H$ for all $u \in H$.

Note that the operator E is self-adjoint. First suppose that the operator E has an eigenvalue such that the modulus is equal to $\|E\|$, that is, there exist $u \in H$ and $\lambda \in \mathbb{R}$ such that $Eu = \lambda u$, $\|u\|_H = 1$ and $|\lambda| = \|E\|$. Then

$$\begin{aligned} \|T\|^2 &\geq \|Tu\|^2 = ((S + iE)u, (S + iE)u)_H \\ &= \|Su\|^2 + \|Eu\|^2 - i(Su, Eu)_H + i(Eu, Su)_H \\ &= \|Su\|^2 + \|E\|^2 - i\lambda(Su, u)_H + i\lambda(u, Su)_H \\ &= \|Su\|^2 + \|E\|^2 \geq m^2 + \|E\|^2, \end{aligned}$$

which implies (1).

Now we consider the general case. Let $\varepsilon > 0$. It follows from the spectral theorem that there exists a self-adjoint bounded operator P such that $\|P\| \leq \varepsilon$ and the operator $E + P$ has an eigenvalue such that the modulus equals $\|E + P\|$. Apply the above to the operator $S + i(E + P)$ and note that this operator has the same coercivity constant m . One obtains the estimate

$$\|E\| - \varepsilon \leq \|E + P\| \leq \sqrt{\|T + iP\|^2 - m^2} \leq \sqrt{(\|T\| + \varepsilon)^2 - m^2}.$$

Finally take the limit $\varepsilon \downarrow 0$.

The inclusion is easy since

$$|\operatorname{Im}(Tu, u)_H| = |(Eu, u)_H| \leq \|E\| \|u\|^2 \leq \frac{\sqrt{\|T\|^2 - m^2}}{m} \operatorname{Re}(Tu, u)$$

for all $u \in H$. □

The estimate (1) is sharp. Equality occurs for example if $T = I + iE$, where E is a bounded self-adjoint operator.

We apply Proposition 2.1 to sectorial forms associated to second-order differential operators.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ be open and $\mu: \Omega \rightarrow \mathbb{C}^{d \times d}$ be a bounded measurable function. Suppose that there exists an $m > 0$ such that $\operatorname{Re} \mu(x) \xi \cdot \bar{\xi} \geq m |\xi|^2$ for all $\xi \in \mathbb{C}^d$.*

Define the sesquilinear form $\mathfrak{a}: W^{1,2}(\Omega) \times W^{1,2}(\Omega) \rightarrow \mathbb{C}$ by

$$\mathfrak{a}[u, v] = \int_{\Omega} \mu \nabla u \cdot \bar{\nabla} v.$$

Then

$$(2) \quad \mathfrak{a}[u] \in \Sigma(\kappa)$$

for all $u \in \text{dom}(\mathfrak{a})$, where $\kappa = \arctan \sqrt{\left(\frac{M}{m}\right)^2 - 1}$ and $M = \text{ess sup}_{x \in \Omega} \|\mu(x)\|$. Here $\|\cdot\|$ is the usual operator norm on $\mathcal{L}(\mathbb{C}^d)$.

Proof. Let $u \in W^{1,2}(\Omega)$. If $x \in \Omega$, then one can apply Proposition 2.1 to the operator $\mu(x)$ on \mathbb{C}^d to deduce that $(\mu \nabla u \cdot \overline{\nabla u})(x) \in \Sigma(\kappa)$. Now integrate over $x \in \Omega$. \square

The angle of the sector for a sectorial form gives resolvent bounds for the associated operator, see [14] Theorem V.3.2. The following situation for j -elliptic forms gives a particularly nice description. For j -elliptic forms and the associated m -sectorial operators we refer to [3] Section 2.

Theorem 2.3. *Let V, H be Hilbert spaces, $\mathfrak{a}: V \times V \rightarrow \mathbb{C}$ a sesquilinear form, $j: V \rightarrow H$ a continuous linear operator and $\kappa \in [0, \frac{\pi}{2})$. Suppose that $\mathfrak{a}[u] \in \Sigma(\kappa)$ for all $u \in \text{dom}(\mathfrak{a})$ and that j has dense range. Further suppose that \mathfrak{a} is j -elliptic. Let A be the operator associated with (\mathfrak{a}, j) . Then $\sigma(A) \subset \overline{\Lambda(A)} \subset \Sigma(\kappa)$ and*

$$\|(A + \lambda I)^{-1}\|_{H \rightarrow H} \leq \frac{1}{d(-\lambda, \Sigma(\kappa))}$$

for all $\lambda \in \mathbb{C}$ with $-\lambda \notin \Sigma(\kappa)$.

Proof. Since \mathfrak{a} is j -elliptic, the operator A is m -sectorial. Hence $\sigma(A) \subset \overline{\Lambda(A)}$ by [14] Theorem V.3.2. Let $f \in \text{dom}(A)$ with $\|f\|_H = 1$. Then there exists a $u \in V$ such that $j(u) = f$ and $\mathfrak{a}[u, v] = (Af, j(v))_H$ for all $v \in V$. Therefore $(Af, f)_H = \mathfrak{a}[u, u] \in \Sigma(\kappa)$. So $\Lambda(A) \subset \Sigma(\kappa)$. It follows from [14] Theorem V.3.2 that $\|(A + \lambda I)^{-1}\|_{H \rightarrow H} \leq \frac{1}{d(-\lambda, \overline{\Lambda(A)})}$ for all $\lambda \in \mathbb{C}$ with $-\lambda \notin \overline{\Lambda(A)}$. This implies the inequality in the theorem. \square

We return to second-order differential operators.

Corollary 2.4. *Adopt the assumptions and notation as in Theorem 2.2. Let $V \subset W^{1,2}(\Omega)$ be a closed subspace such that $C_c^\infty(\Omega) \subset V$ and let $\mathfrak{a}_V = \mathfrak{a}|_{V \times V}$. Let A_V be the m -sectorial operator associated with the form \mathfrak{a}_V . Then $\sigma(A_V) \subset \overline{\Lambda(A_V)} \subset \Sigma(\kappa)$. Moreover, let $\theta \in (\kappa, \frac{\pi}{2})$. Then*

$$\|(A_V + \lambda I)^{-1}\|_{2 \rightarrow 2} \leq \frac{M}{m \sin \theta - \sqrt{M^2 - m^2} \cos \theta} \frac{1}{|\lambda|}$$

for all $\lambda \in \Sigma(\pi - \theta)$.

Proof. Evidently, $\mathfrak{a}_V[u] \in \Sigma(\kappa)$ for all $u \in V$ by Theorem 2.2. Now apply Theorem 2.3 with $j: V \rightarrow L_2(\Omega)$ the identity map. Then $\sigma(A_V) \subset \overline{\Lambda(A_V)} \subset \Sigma(\kappa)$ and

$$\|(A_V + \lambda I)^{-1}\|_{2 \rightarrow 2} \leq \frac{1}{d(-\lambda, \Sigma(\kappa))}$$

for all $\lambda \in \mathbb{C}$ with $-\lambda \notin \Sigma(\kappa)$. Then the assertion follows by elementary trigonometry. \square

Example 2.5. We present an example of an elliptic differential operator with real coefficients such that the angle κ in (2) is optimal.

Let $\Omega \subset \mathbb{R}^2$ be a non-empty open bounded set. Choose $\mu(x) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ for all $x \in \Omega$ and $V = W^{1,2}(\Omega)$. Define $u \in W^{1,2}(\Omega)$ by $u(x, y) = -x + y + i(x + y)$. A straightforward calculation gives $\mathfrak{a}[u] = (4 - 4i)|\Omega|$. Also $m = 1$ and $M = \sqrt{2}$. So $\kappa = \frac{\pi}{4}$ and it cannot be improved.

3. BOUNDED \mathcal{H}^∞ -CALCULUS AND MAXIMAL PARABOLIC REGULARITY

Let $\Omega \subset \mathbb{R}^d$ be open connected and $\mu: \Omega \rightarrow \mathbb{R}^{d \times d}$ be a bounded measurable function. Suppose that there exists an $m > 0$ such that $\operatorname{Re} \mu(x) \xi \cdot \bar{\xi} \geq m |\xi|^2$ for all $\xi \in \mathbb{C}^d$. Let $M = \operatorname{ess\,sup}_{x \in \Omega} \|\mu(x)\|$ and set

$$\kappa = \arctan \sqrt{\left(\frac{M}{m}\right)^2 - 1}.$$

In this section we consider realisations of the elliptic operator with (real) coefficients μ with mixed boundary conditions on the space $L_p(\Omega)$.

Define the sesquilinear form $\mathfrak{a}: W^{1,2}(\Omega) \times W^{1,2}(\Omega) \rightarrow \mathbb{C}$ by

$$\mathfrak{a}[u, v] = \int_{\Omega} \mu \nabla u \cdot \bar{\nabla} v.$$

Let D be a closed subset of $\partial\Omega$. We denote by $W_D^{1,2}(\Omega)$ the closure in $W^{1,2}(\Omega)$ of the set

$$\{u|_{\Omega} : u \in C_c^\infty(\mathbb{R}^d) \text{ and } D \cap \operatorname{supp} u = \emptyset\}.$$

Moreover, define $\mathcal{W}_D^{1,2}(\Omega)$ to be the closure of

$$\{u : u \in W^{1,2}(\Omega) \text{ and } D \cap \operatorname{supp} u = \emptyset\}$$

in $W^{1,2}(\Omega)$. Define $\mathfrak{a}_W = \mathfrak{a}|_{W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)}$ and $\mathfrak{a}_{\mathcal{W}} = \mathfrak{a}|_{\mathcal{W}_D^{1,2}(\Omega) \times \mathcal{W}_D^{1,2}(\Omega)}$. Then \mathfrak{a}_W and $\mathfrak{a}_{\mathcal{W}}$ are closed sectorial forms. Let A_W and $A_{\mathcal{W}}$ be the operators associated with the forms \mathfrak{a}_W and $\mathfrak{a}_{\mathcal{W}}$, respectively. Roughly speaking, A_W and $A_{\mathcal{W}}$ are two versions of elliptic operators with Dirichlet boundary conditions on D and Neumann boundary conditions on $\partial\Omega \setminus D$.

Theorem 3.1. *For all $p \in [1, \infty]$ the semigroups generated by $-A_W$ and $-A_{\mathcal{W}}$ extend consistently to contraction semigroups on $L_p(\Omega)$, which are C_0 -semigroups if $p \in [1, \infty)$ and they are bounded holomorphic if $p \in (1, \infty)$.*

Proof. It suffices to show that the semigroups generated by $-A_W$ and $-A_{\mathcal{W}}$ are submarkovian. Then the other statements follow by duality, interpolation, and [16] Proposition 3.12.

The semigroup generated by $-A_W$ is submarkovian by [16] Corollary 4.10. It follows from [16] Corollary 4.10 and Theorem 2.13, applied to the operator with Neumann boundary conditions, that $(1 \wedge |u|) \operatorname{sgn} u \in W^{1,2}(\Omega)$ and

$$\operatorname{Re} \mathfrak{a}[(1 \wedge |u|) \operatorname{sgn} u, (|u| - 1)^+ \operatorname{sgn} u] \geq 0$$

for all $u \in W^{1,2}(\Omega)$. Hence one deduces that $(1 \wedge |u|) \operatorname{sgn} u \in \mathcal{W}_D^{1,2}(\Omega)$ for all $u \in W^{1,2}(\Omega)$ with $D \cap \operatorname{supp} u = \emptyset$. Since $\{u \in W^{1,2}(\Omega) : D \cap \operatorname{supp} u = \emptyset\}$ is dense in $\mathcal{W}_D^{1,2}(\Omega)$, it follows from [16] Theorem 2.13 $3' \Rightarrow 1$ that the semigroup generated by $-A_{\mathcal{W}}$ is submarkovian. \square

For all $p \in [1, \infty)$ we denote by $-A_{W,p}$ and $-A_{\mathcal{W},p}$ the generator of the C_0 -semigroup on $L_p(\Omega)$ which is consistent with the semigroup generated by $-A_W$ and $-A_{\mathcal{W}}$.

We suppose that the reader is familiar with the concept of bounded \mathcal{H}^∞ -calculus and refer for details to [9] and [5]. We wish to prove upper bounds for the \mathcal{H}^∞ -angle of the operators $A_{W,p}$ and $A_{\mathcal{W},p}$, first for $p = 2$ and then for all $p \in (1, \infty)$.

Theorem 3.2. *Adopt the assumptions and notation as in the beginning of this section.*

- (a) Suppose that $\mathbb{1}_\Omega \notin W_D^{1,2}(\Omega)$. Then for all $\varepsilon > 0$ the operator A_W admits an \mathcal{H}^∞ -calculus on the sector $\Sigma(\kappa + \varepsilon)^\circ$. Stronger,

$$\|f(A_W)\|_{2 \rightarrow 2} \leq \left(2 + \frac{2}{\sqrt{3}}\right) \operatorname{ess\,sup}_{z \in \Sigma(\kappa + \varepsilon)^\circ} |f(z)|$$

for all $f \in \mathcal{H}^\infty(\Sigma(\kappa + \varepsilon)^\circ)$.

- (b) Suppose that $\mathbb{1}_\Omega \in W_D^{1,2}(\Omega)$. Then for all $\delta, \varepsilon > 0$ the operator $A_W + \delta I$ admits an \mathcal{H}^∞ -calculus on the sector $\Sigma(\kappa + \varepsilon)^\circ$. Stronger,

$$\|f(A_W + \delta I)\|_{2 \rightarrow 2} \leq \left(2 + \frac{2}{\sqrt{3}}\right) \operatorname{ess\,sup}_{z \in \Sigma(\kappa + \varepsilon)^\circ} |f(z)|$$

for all $f \in \mathcal{H}^\infty(\Sigma(\kappa + \varepsilon)^\circ)$.

Similar statements are valid for the operator $A_{\mathcal{W}}$ instead of A_W .

Proof. ‘(a)’. Since Ω is connected, the operator A_W is injective. Moreover $\Lambda(A_W) \subset \Sigma(\kappa)$ by Corollary 2.4. Then the claim follows from the Crouzeix–Delyon theorem [4] Theorem 1.

‘(b)’. Let $\delta > 0$. Then the operator $A_W + \delta I$ is injective and $\Lambda(A_W + \delta I) \subset \Sigma(\kappa)$. Then one can argue as in Statement (a).

The proof for $A_{\mathcal{W}}$ is word-by-word the same. □

Next we consider the operators on $L_p(\Omega)$.

Theorem 3.3. *Adopt the assumptions and notation as in the beginning of this section. Let $p \in (1, \infty)$ and set $\kappa_p = (1 - |1 - \frac{2}{p}|)\kappa + |1 - \frac{2}{p}| \frac{\pi}{2}$. Then one has the following.*

- (a) Suppose that $\mathbb{1}_\Omega \notin W_D^{1,2}(\Omega)$. Then for all $\varepsilon > 0$ the operator $A_{W,p}$ admits an \mathcal{H}^∞ -calculus on the sector $\Sigma(\kappa_p + \varepsilon)^\circ$.
- (b) Suppose that $\mathbb{1}_\Omega \in W_D^{1,2}(\Omega)$. Then for all $\delta, \varepsilon > 0$ the operator $A_{W,p} + \delta I$ admits an \mathcal{H}^∞ -calculus on the sector $\Sigma(\kappa_p + \varepsilon)^\circ$.

Similar statements are valid for the operator $A_{\mathcal{W},p}$ instead of $A_{W,p}$.

Proof. ‘(a)’. The semigroup $(e^{-tA_W})_{t>0}$ is positive by [16] Corollary 4.3. Hence by consistency and density the semigroup $(e^{-tA_{W,p}})_{t>0}$ is positive for all $p \in [1, \infty)$. Since $-A_{W,p}$ is injective and the generator of a positive contraction semigroup, it follows from Duong [7] Theorem 2 that for all $\theta \in (\frac{\pi}{2}, \pi)$ the operator $A_{W,p}$ admits an \mathcal{H}^∞ -calculus on the sector $\Sigma(\theta)^\circ$.

Finally we use interpolation. Suppose $p \in (2, \infty)$ and $\varepsilon > 0$. For all $n \in \mathbb{N}$ with $n > p$ we interpolate between 2 and n . Let $\theta_n \in (0, 1)$ be such that $\frac{1}{p} = \frac{\theta_n}{2} + \frac{1-\theta_n}{n}$. Then $\lim_{n \rightarrow \infty} \theta_n = \frac{2}{p}$. Now the operator $A_{W,2}$ admits an \mathcal{H}^∞ -calculus on the sector $\Sigma(\kappa + \varepsilon)^\circ$ and the operator $A_{W,n}$ admits an \mathcal{H}^∞ -calculus on the sector $\Sigma(\frac{\pi}{2} + \varepsilon)^\circ$. One deduces from [12] Proposition 4.9 that the operator $A_{W,p}$ admits an \mathcal{H}^∞ -calculus on the sector $\Sigma(\theta_n \kappa + (1 - \theta_n) \frac{\pi}{2} + \varepsilon)^\circ$. Taking n large enough, the statement follows.

If $p \in (1, 2)$, then the proof is similar, or one can use duality.

‘(b)’. The proof is similar.

For the operators $A_{\mathcal{W},p}$ the argument is almost the same. The only thing that is not immediately clear is the positivity of the semigroup. But that can be deduced as at the end of the proof of Theorem 3.1. □

We emphasise that $\kappa_p < \frac{\pi}{2}$ for all $p \in (1, \infty)$. As an application of Theorem 3.3 we obtain bounded imaginary powers with constant smaller than $\frac{\pi}{2}$.

Corollary 3.4. *Adopt the assumptions and notation as in the beginning of this section. Let $p \in (1, \infty)$ and let $\kappa_p \in [\kappa, \frac{\pi}{2})$ be as in Theorem 3.3. Then for all $\varepsilon > 0$ there exists a $c > 0$ such that $\|(A_{W,p} + I)^{is}\|_{p \rightarrow p} \leq c e^{(\kappa_p + \varepsilon)|s|}$ and $\|(A_{\mathcal{W},p} + I)^{is}\|_{p \rightarrow p} \leq c e^{(\kappa_p + \varepsilon)|s|}$ for all $s \in \mathbb{R}$.*

Proof. Apply Theorem 3.3 to the function $z \mapsto z^{is}$. □

Corollary 3.5. *Adopt the assumptions and notation as in the beginning of this section. For all $p \in (1, \infty)$ the operators $A_{W,p}$ and $A_{\mathcal{W},p}$ satisfy maximal parabolic regularity in $L_p(\Omega)$.*

Proof. This follows from the Dore–Venni theorem [6] and Corollary 3.4. □

We emphasise that Ω is merely an open connected (non-empty) set. It does not need to have the doubling property, nor to be bounded.

4. AN OUTLOOK TO MORE GENERAL MEASURE SPACES

In this section we consider a bounded domain and an elliptic operator with complex coefficients subject to mixed boundary conditions and domain inhomogeneities supported on a lower dimensional hypersurface, enforcing a jump in the conormal derivative.

Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected. Let $D \subset \partial\Omega$ be closed. Further let Γ_0 be a Borel subset of Ω which is a $(d - 1)$ -set in the sense of Jonsson–Wallin (see [11] Subsection VII.1.1), that is, there are $c_1, c_2 > 0$ such that

$$c_1 r^{d-1} \leq \mathcal{H}_{d-1}(B(x, r) \cap \Gamma_0) \leq c_2 r^{d-1}$$

for all $x \in \Gamma_0$ and $r \in (0, 1]$, where \mathcal{H}_{d-1} is the $(d - 1)$ -dimensional Hausdorff measure. We emphasise that Γ_0 does not have to be closed. Finally, we suppose that every element of $\overline{\partial\Omega} \setminus \overline{D}$ admits a bi-Lipschitz chart.

Set $\Gamma = \Gamma_0 \cup (\partial\Omega \setminus D)$. Let ρ be the restriction of \mathcal{H}_{d-1} to the set Γ . Define $\mathbb{L}_2 = L_2(\Omega \cup \Gamma, dx + d\rho)$. There is a natural isomorphism from \mathbb{L}_2 onto $L_2(\Omega, dx) \oplus L_2(\Gamma, d\rho)$. We identify \mathbb{L}_2 with $L_2(\Omega, dx) \oplus L_2(\Gamma, d\rho)$ in the natural way. In this section we consider an m -sectorial operator in \mathbb{L}_2 .

For all $u \in L_{1,\text{loc}}(\Omega)$ define the function $\text{Tr } u$ by

$$\text{dom}(\text{Tr } u) = \left\{ x \in \Gamma : \lim_{r \downarrow 0} \frac{1}{|\Omega \cap B(x, r)|} \int_{\Omega \cap B(x, r)} u(y) dy \text{ exists} \right\}$$

and

$$(\text{Tr } u)(x) = \lim_{r \downarrow 0} \frac{1}{|\Omega \cap B(x, r)|} \int_{\Omega \cap B(x, r)} u(y) dy$$

for all $x \in \text{dom}(\text{Tr})$. (See [11] Section VIII.1.1.) It follows from [20] Proposition 2.8 that $\text{Tr } u \in L_2(\Gamma, d\rho)$ for all $u \in W_D^{1,2}(\Omega)$. Define $j: W_D^{1,2}(\Omega) \rightarrow \mathbb{L}_2$ by

$$j(u) = (u, \text{Tr } u) \in L_2(\Omega, dx) \oplus L_2(\Gamma, d\rho).$$

By [20] Lemma 2.10(i) the map j is continuous and has dense range.

Theorem 4.1. *Adopt the above notation and assumptions. Let $\mu: \Omega \rightarrow \mathbb{C}^{d \times d}$ be a bounded measurable function. Suppose that there exists an $m > 0$ such that $\operatorname{Re} \mu(x) \xi \cdot \bar{\xi} \geq m |\xi|^2$ for all $\xi \in \mathbb{C}^d$. Define the sesquilinear form $\mathfrak{a}: W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega) \rightarrow \mathbb{C}$ by*

$$\mathfrak{a}[u, v] = \int_{\Omega} \mu \nabla u \cdot \overline{\nabla v}.$$

Define $\kappa = \arctan \sqrt{\left(\frac{M}{m}\right)^2 - 1}$ and $M = \operatorname{ess\,sup}_{x \in \Omega} \|\mu(x)\|$. Let A be the m -sectorial operator in \mathbb{L}_2 associated with (\mathfrak{a}, j) . Then $\sigma(A) \subset \overline{\Lambda(A)} \subset \Sigma(\kappa)$. Moreover, let $\theta \in (\kappa, \frac{\pi}{2})$. Then

$$\|(A + \lambda I)^{-1}\|_{\mathbb{L}_2 \rightarrow \mathbb{L}_2} \leq \frac{M}{m \sin \theta - \sqrt{M^2 - m^2} \cos \theta} \frac{1}{|\lambda|}$$

for all $\lambda \in \Sigma(\pi - \theta)$.

Proof. This follows from Theorems 2.2 and 2.3. □

Let us mention that this provides an adequate functional analytic instrument to give equations with dynamical boundary conditions like

$$\begin{aligned} \partial_t u - \nabla \cdot \mu \nabla u &= f_{\Omega} && \text{on } I \times (\Omega \setminus \Gamma_0), \\ u &= 0 && \text{on } I \times D, \\ \partial_t u + \nu \cdot \mu \nabla u &= f_1 && \text{on } I \times (\partial\Omega \setminus D), \\ \partial_t u + [\nu_{\Gamma_0} \cdot \mu \nabla u] &= f_0 && \text{on } I \times \Gamma_0, \\ u(0) &= u_0 && \text{on } \Omega \cup (\partial\Omega \setminus D), \end{aligned}$$

a precise meaning, inclusively its quasilinear variants, see [20]. For a strict derivation of dynamical boundary conditions in various physical contexts see [8].

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REFERENCES

- [1] P. Acquistapace and B. Terreni. Linear parabolic equations in banach spaces with variable domains but constant interpolation spaces. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 13:75–107, 1986.
- [2] H. Amann. *Linear and quasilinear parabolic problems*. Number 89 in Monographs in Mathematics. Birkhäuser, Boston, 1995.
- [3] W. Arendt and A. F. M. ter Elst. Sectorial forms and degenerate differential operators. *J. Operator Theory*, 67:33–72, 2012.
- [4] M. Crouzeix and B. Delyon. Some estimates for analytic functions of strip or sectorial operators. *Arch. Math.*, 81:559–566, 2003.
- [5] R. Denk, M. Hieber, and J. Prüss. ∇ -boundedness, fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788), 2003.
- [6] G. Dore and A. Venni. On the closedness of the sum of two closed operators. *Math. Z.*, 196:189–201, 1987.
- [7] X. T. Duong. h_{∞} functional calculus of elliptic partial differential operators on l^p spaces. In *Miniconference on operators in analysis*, volume 24 of *Proceedings of the Centre for Mathematics and its Applications*, pages 91–102, Canberra, 1990. Australian National University.
- [8] G. R. Goldstein. Derivation and physical interpretation of general boundary conditions. *Adv. Differential Equations*, 11:457–480, 2006.

- [9] M. Haase. *The functional calculus for sectorial operators*. Number 169 in Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 2006.
- [10] D. Hömberg, K. Krumbiegel, and J. Rehberg. Optimal control of a parabolic equation with dynamic boundary condition. *Appl. Math. Optim.*, 67:3–31, 2013.
- [11] A. Jonsson and H. Wallin. Function spaces on subsets of \mathbb{R}^n . *Math. Rep.*, 2(1), 1984.
- [12] N. Kalton, P. Kunstmann, and L. Weis. Perturbation and interpolation theorems for the h^∞ -calculus with applications to differential operators. *Math. Ann.*, pages 747–801, 2006.
- [13] T. Kato. Abstract evolution equations of parabolic type in banach and hilbert spaces. *Nagoya Math. J.*, 19:93–125, 1961.
- [14] T. Kato. *Perturbation theory for linear operators*. Number 132 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin etc., second edition, 1980.
- [15] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Number 16 in Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Verlag, Basel, 1995.
- [16] E.-M. Ouhabaz. *Analysis of heat equations on domains*. Number 31 in London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, 2005.
- [17] G. D. Prato and E. Sinestrari. Hölder regularity for nonautonomous abstract parabolic equations. *Israel J. Math.*, 42:1–19, 1982.
- [18] P. E. Sobolevskii. Equations of parabolic type in a banach space (russian). *Trudy Moskov. Mat. Obšč.*, 10:297–350, 1961.
- [19] H. Tanabe. On the equations of evolution in a banach space. *Osaka Math. J.*, 12:363–376, 1960.
- [20] A. F. M. ter Elst, M. Meyries, and J. Rehberg. Parabolic equations with dynamical boundary conditions and source terms on interfaces. *Ann. Mat. Pura Appl.*, 193:1295–1318, 2014.
- [21] H. Vogt and J. Voigt. Wentzell boundary conditions in the context of dirichlet forms. *Adv. Differential Equations*, 8:821–842, 2003.
- [22] K. Yosida. *Functional Analysis*. Number 123 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York etc., sixth edition, 1980.