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Abstract

In this paper we introduce a novel approach for an important problem of break detection. Specifically, we are interested in detection of an abrupt change in the covariance structure of a high-dimensional random process – a problem, which has applications in many areas e.g., neuroimaging and finance. The developed approach is essentially a testing procedure involving a choice of a critical level. To that end a non-standard bootstrap scheme is proposed and theoretically justified under mild assumptions. Theoretical study features a result providing guaranties for break detection. All the theoretical results are established in a high-dimensional setting (dimensionality $p \gg n$). Multiscale nature of the approach allows for a trade-off between sensitivity of break detection and localization. The approach can be naturally employed in an on-line setting. Simulation study demonstrates that the approach matches the nominal level of false alarm probability and exhibits high power, outperforming a recent approach.

1 Introduction

The analysis of high dimensional time series is crucial for many fields including neuroimaging and financial engineering. There, one often has to deal with processes involving abrupt structural changes which necessitate a corresponding adaptation of a model and/or a strategy. Structural break analysis comprises determining if an abrupt change is present in the given sample and if so, estimating the change-point, namely the moment in time when it takes place. In literature both problems may be referred to as *change-point* or *break detection*. In this study we will be using terms *break detection* and *change-point localization* respectively in order to distinguish between them. The majority of approaches to the problem consider only a univariate process [13] [1]. However, in recent years the interest for multi-dimensional approaches has increased. Most of them cover the case of fixed dimensionality [28] [26] [2] [36] [37]. Some approaches [10, 25, 11] feature *high-dimensional* theoretical guaranties but only the case of dimensionality polynomially growing in sample size is covered. The case of exponential growth has not been considered so far.

In order to detect a break, a test statistic is usually computed for each point t (e.g. [28]). The break is detected if the maximum of these values exceeds a certain *threshold*. A proper choice of the latter may be a tricky issue. Consider a pair of plots (Figure 1) of the statistic $A(t)$ defined in Section 2. It is rather difficult to see how many breaks are there, if any. The classic approach to the problem is based on the asymptotic behaviour of the statistic [13] [1] [2] [25] [6] [37]. As an alternative, permutation [25] [28] or parametric bootstrap may be used [25]. Clearly, it seems attractive to choose the threshold in a solely data-driven way as it is suggested in the recent paper [10], but a careful bootstrap validation is still an open question.

In the current study we are interested in a particular kind of a break – an abrupt transformation in the covariance matrix – which is motivated by applications to neuroimaging. The covariance structure of data in functional Magnetic Resonance Imaging has recently drawn a lot of interest, as it encodes

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so-called functional connectivity networks [35] which refer to the explicit influence among neural systems [19]. A rather popular approach to inferencing these networks is based on estimating inverse covariance or precision matrices [18]. The technique generally makes use of the observation that functional connectivity networks are of small-world type [35], which makes sparsity assumptions feasible. Analysing the dynamics of these networks is particularly important for the research on neural diseases and also in the context of brain development with emphasis on characterizing the re-configuration of the brain during learning [4].

A similar problem is found in finance: the dynamics of the covariance structure of a high-dimensional process modelling exchange rates and market indexes is crucial for a proper asset allocation in a portfolio [12, 5, 14, 30].

One approach to the change-point localization is developed in [26], the corresponding significance testing problem is considered in [2]. However, neither of these papers address the high-dimensional case.

A widely used break detection approach (named CUSUM) [11, 2, 25] suggests to compute a statistic at a point t as a distance of estimators of some parameter of the underlying distributions obtained using all the data before and after that point. This technique requires the whole sample to be known in advance, which prevents it from being used in *online* setting. In order to overcome this drawback we propose the following augmentation: choose a window size $n \in \mathbb{N}$ and compute parameter estimators using only n points before and n points after the *central point* t (see Section 2 for formal definition). Window size n is an important parameter and its choice is case-specific (see Section 4 for theoretical treatment of this issue). Using a small window results in high variability and low sensitivity, while a large window implies higher uncertainty in change-point localization yielding the issue of a proper choice of window size. The *multiscale* nature of the proposed method enables us to incorporate the advantages of narrower and wider windows by considering multiple window sizes at once in order for wider windows to provide higher sensitivity while narrower ones improve change-point localization.

The contribution of our study is the development of a novel break detection approach which is

- high-dimensional, allowing for up to exponential growth of the dimensionality with the window size
- suitable for on-line setting
- multiscale, attaining trade-off between break detection sensitivity and change-point localization accuracy
- using a fully data-driven threshold selection algorithm rigorously justified under mild assumptions
- featuring formal sensitivity guaranties in high-dimensional setting

We consider the following setup. Let $X_1, \dots X_N \in \mathbb{R}^p$ denote an independent sample of zero-mean vectors (the on-line setting is discussed in Section 3) and we want to test a hypothesis

$$\mathbb{H}_0 := \{\forall i : \operatorname{Var}[X_i] = \operatorname{Var}[X_{i+1}]\}$$

versus an alternative suggesting the existence of a break:

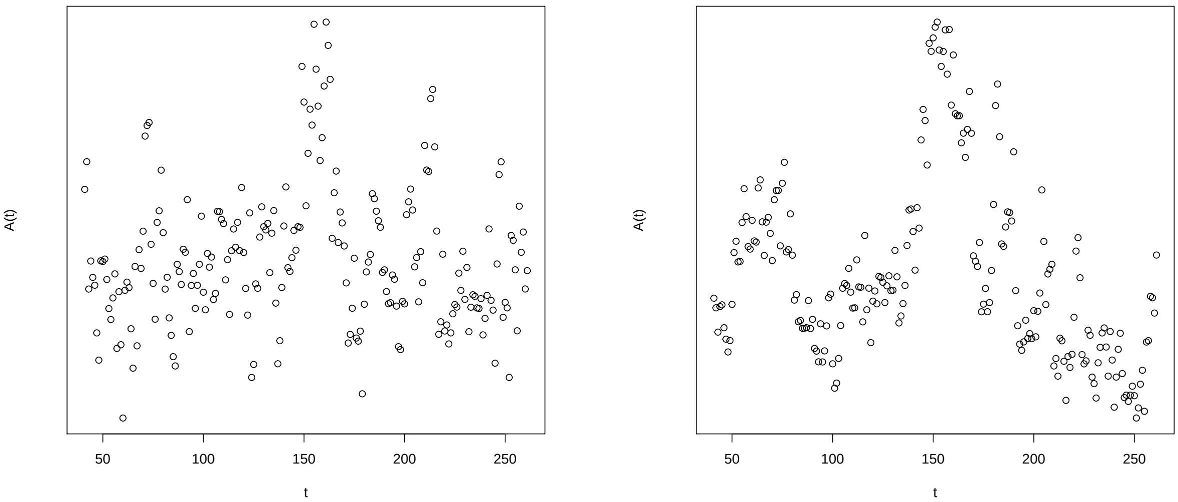


Figure 1: Plots of test statstics $A(t)$ computed on synthetically generated data without (left) and with a single change-point at $t = 150$ (right). Clearly, the choice of a threshold is not obvious.

$$\mathbb{H}_1 \coloneqq \{\exists \tau : \mathrm{Var}\,[X_1] = \mathrm{Var}\,[X_2] = ... = \mathrm{Var}\,[X_\tau] \neq \mathrm{Var}\,[X_{\tau+1}] = ... = \mathrm{Var}\,[X_N]\}$$

and localize the change-point τ as precisely as possible or (in online setting) to detect a break as soon as possible.

The approach proposed in the paper focuses on applications in neuroimagin

In the current study it is also assumed that some subset of indices $\mathcal{I}_s \subseteq 1..N$ of size s (possibly, $s = N$) is chosen. The threshold is chosen relying on the sub-sample $\{X_i\}_{i \in \mathcal{I}_s}$ while the test-statistic is computed based on the whole sample.

To this end we define a family of test statistics in Section 2.1 which is followed by Section 2.2 describing a data-driven (bootstrap) calibration scheme and Section 2.3 describing change-point localization procedure. The theoretical part of the paper justifies the proposed procedure in a high-dimensional setting. The result justifying the validity of the proposed calibration scheme is stated in Section 3. Section 4 is devoted to the sensitivity result yielding a bound for the window size n necessary to reliably detect a break of a given extent and hence bounding the uncertainty of the change-point localization (or the delay of detection in online setting). The theoretical study is supported by a comparative simulation study (described in Section 5) demonstrating conservativeness of the proposed test and higher sensitivity compared to the other algorithms. Appendix A contains a finite-sample version of sensitivity result along with the proofs. Appendix B provides a a finite-sample version of bootstrap sensitivity result which is followed by the proofs. Finally, Appendix H lists results which were essential for our theoretical study.

2 Proposed approach

This section describes the proposed approach along with a data-driven calibration scheme. Informally the proposed statistic can be described as follows. Provided that the break may happen only at moment t, one could estimate some parameter of the distribution using n data-points to the left of t, estimate it again using n data-points to the right and use the norm of their difference as a test-statistic $A_n(t)$. Yet, in practice one does not usually possess such knowledge, therefore we propose to maximize these statistics over all possible locations t yielding A_n. Finally, in order to attain a trade-off between break detection sensitivity and change-point localization accuracy we build a multiscale approach: consider a family of test statistics $\{A_n\}_{n\in\mathfrak{N}}$ for multiple window sizes $n \in \mathfrak{N} \subset \mathbb{N}$ at once.

2.1 Definition of the test statistic

Now we present a formal definition of the test statistic. In order to detect a break we consider a set of window sizes $\mathfrak{N} \subset \mathbb{N}$. Denote the size of the widest window as n_+ and of the narrowest as n_-. Given a sample of length N, for each window size $n \in \mathfrak{N}$ define a set of central points $\mathbb{T}_n := \{n+1, ..., N-n+1\}$. Next, for all $n \in \mathfrak{N}$ define a set of indices which belong to the window on the left side of the central point $t \in \mathbb{T}_n$ as $\mathcal{I}_n^l(t) := \{t-n, ..., t-1\}$ and correspondingly for the window on the right side define $\mathcal{I}_n^r(t) := \{t, ..., t+n-1\}$. Denote the sum of numbers of central points for all window sizes $n \in \mathfrak{N}$ as

$$T := \sum_{n\in\mathfrak{N}} |\mathbb{T}_n| \, .$$

For each window size $n \in \mathfrak{N}$, each central point $t \in \mathbb{T}_n$ and each side $\mathfrak{S} \in \{l, r\}$ we define a de-sparsified estimator of precision matrix [24] [23] as

$$\hat{T}_n^{\mathfrak{S}}(t) := \hat{\Theta}_n^{\mathfrak{S}}(t) + \hat{\Theta}_n^{\mathfrak{S}}(t)^T - \hat{\Theta}_n^{\mathfrak{S}}(t)^T \hat{\Sigma}_n^{\mathfrak{S}}(t) \hat{\Theta}_n^{\mathfrak{S}}(t)$$

where

$$\hat{\Sigma}_n^{\mathfrak{S}}(t) = \frac{1}{n} \sum_{i\in\mathcal{I}_n^{\mathfrak{S}}(t)} X_i X_i^T$$

and $\hat{\Theta}_n^{\mathfrak{S}}(t)$ is a consistent estimator of precision matrix which can be obtained by graphical lasso [32] or node-wise procedure [23] (see Definition 3.1 and Appendix H.5 for details).

Now define a matrix of size $p \times p$ with elements

$$Z_{i,uv} := \Theta_u^* X_i \Theta_v^* X_i - \Theta_{uv}^* \tag{2.1}$$

where $\Theta^* := \mathbb{E}\left[X_i X_i^T\right]^{-1}$ for $i \le \tau$, Θ_u^* stands for the u-th row of Θ^*. Denote their variances as $\sigma_{uv}^2 := \operatorname{Var}\left[Z_{1,uv}\right]$ and introduce the diagonal matrix $S = diag(\sigma_{1,1}, \sigma_{1,2}...\sigma_{p,p-1}\sigma_{p,p})$. Denote a consistent estimator (see Definition 3.1 for details) of the precision matrix Θ^* obtained based on the sub-sample $\{X_i\}_{i\in\mathcal{I}_s}$ as $\hat{\Theta}$. In practice, the variances σ_{uv}^2 are unknown, but under normality assumption one can plug in $\hat{\sigma}_{uv}^2 := \hat{\Theta}_{uu}\hat{\Theta}_{vv} + \hat{\Theta}_{uv}^2$ which have been proven to be consistent (uniformly for all u and v) estimators of σ_{uv}^2 [24] [3]. If the node-wise procedure is employed, the uniform consistency of an empirical estimate of σ_{uv}^2 has been shown under some mild assumptions (not including normality) [23].

For each window size $n \in \mathfrak{N}$ and a central point $t \in \mathbb{T}_n$ we define a statistic

$$A_n(t) := \left\| \sqrt{\frac{n}{2}} S^{-1} \overline{(\hat{T}_n^l(t) - \hat{T}_n^r(t))} \right\|_\infty \tag{2.2}$$

where we write $\overline{M}$ for a vector composed of stacked columns of matrix M. Finally we define our family of test statistics for all $n \in \mathfrak{N}$ as

$$A_n = \max_{t \in \mathbb{T}_n} A_n(t).$$

Our approach heavily relies on the following expansion under $\mathbb{H}_0$

$$\sqrt{n}(\hat{T}_n^{\mathfrak{S}}(t) - \Theta^*) = \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{I}_n^{\mathfrak{S}}(t)} Z_i + r_n^{\mathfrak{S}}(t)\sqrt{n}, \tag{2.3}$$

where the residual term

$$r_n^{\mathfrak{S}}(t) := \hat{T}_n^{\mathfrak{S}}(t) - \left(\Theta^* - \Theta^* \left(\hat{\Sigma}_n^{\mathfrak{S}}(t) - \Sigma^*\right) \Theta^*\right)$$

can be controlled under mild assumptions [24] [23] [3].

This expansion might have been used in order to investigate the asymptotic properties of A_n and obtain the threshold, however we propose a data-driven scheme.

Remark 2.1. *A different test statistic $A_n(t)$ can be defined as the maximum distance between elements of empirical covariance matrices $\hat{\Sigma}(t)_n^l$ and $\hat{\Sigma}(t)_n^r$. However, application to neuroimaging motivates the search for a structural change in a functional connectivity network which is encoded by the structure of the corresponding precision matrix. Clearly, a change in the precision matrix also means a change in the covariance matrix, though we believe that the definition* (2.2) *increases the sensitivity to this kind of alternative.*

2.2 Bootstrap calibration

Our approach rejects $\mathbb{H}_0$ in favor of $\mathbb{H}_1$ if at least one of statistics A_n exceeds the corresponding threshold $x_n^\flat(\alpha)$ or formally if $\exists n \in \mathfrak{N} : A_n > x_n^\flat(\alpha)$.

In order to properly choose the thresholds, we define bootstrap statistics $A_n^\flat$ in the following non-standard way. Note, that we cannot use an ordinary scheme with replacement or weighted bootstrap since in a high-dimensional case ($|\mathcal{I}_s| \leq p$) the covariance matrix of bootstrap distribution would be singular which would made inverse covariance matrix estimation procedures meaningless.

First, draw with replacement a sequence $\{\varkappa_i\}_{i=1}^N$ of indices from $\mathcal{I}_s$ and denote

$$X_i^\flat = X_{\varkappa_i} - \mathbb{E}_{\mathcal{I}_s}[X_j]$$

where $\mathbb{E}_{\mathcal{I}_s}[\cdot]$ stands for averaging over values of index belonging to $\mathcal{I}_s$ e.g., $\mathbb{E}_{\mathcal{I}_s}[X_j] = \frac{1}{|\mathcal{I}_s|}\sum_{j \in \mathcal{I}_s} X_j$. Denote the measure $X_i^\flat$ are distributed with respect to as $\mathbb{P}^\flat$. In accordance with (2.1) define

$$Z_{i,uv}^\flat := \hat{\Theta}_u X_i^\flat \hat{\Theta}_v X_i^\flat - \hat{\Theta}_{uv}$$

and for technical purposes define

$$\hat{Z}_{i,uv} := \hat{\Theta}_u X_i \hat{\Theta}_v X_i - \hat{\Theta}_{uv}.$$

Now for all central point t define a bootstrap counterpart of $A_n(t)$

$$A_n^\flat(t) := \left\| \frac{1}{\sqrt{2n}} S^{-1} \overline{\left(\sum_{i \in \mathcal{I}_n^l(t)} Z_i^\flat - \sum_{i \in \mathcal{I}_n^r(t)} Z_i^\flat \right)} \right\|_\infty \tag{2.4}$$

which is intuitively reasonable due to expansion (2.3). And finally we define the bootstrap counterpart of A_n as

$$A_n^\flat = \max_{t \in \mathbb{T}_n} A_n^\flat(t).$$

Now for each given $\mathrm{x} \in [0, 1]$ we can define quantile functions $z_n^\flat(\mathrm{x})$ such that

$$z_n^\flat(\mathrm{x}) := \inf \left\{ z : \mathbb{P}^\flat \left\{ A_n^\flat > z \right\} \le \mathrm{x} \right\}.$$

Next for a given significance level α we apply multiplicity correction choosing α^* as

$$\alpha^* := \sup \left\{ \mathrm{x} : \mathbb{P}^\flat \left\{ \exists n \in \mathfrak{N} : A_n^\flat > z_n^\flat(\mathrm{x}) \right\} \le \alpha \right\}$$

and finally choose thresholds as $x_n^\flat(\alpha) := z_n^\flat(\alpha^*)$.

Remark 2.2. *One can choose $\mathcal{I}_s = 1, 2, ..., N$ and use the whole given sample for calibration as well as for detection. In fact, it would improve the bounds in Theorem 3.1 and Theorem 4.1, since it effectively means $s = N$. However, in practise such a decision might lead to reduction of sensitivity due to overestimation of the thresholds.*

2.3 Change-point localization

In order to localize a change-point we have to assume that $\mathcal{I}_s \subseteq 1..\tau$. Consider the narrowest window detecting a change-point as $\hat{n}$:

$$\hat{n} := \min \left\{ n \in \mathfrak{N} : A_n > x_n^\flat(\alpha) \right\} \tag{2.5}$$

and the central point where this window detects a break for the first time as

$$\hat{\tau} := \min \left\{ t \in \mathbb{T}_{\hat{n}} : A_{\hat{n}}(t) > x_{\hat{n}}^\flat(\alpha) \right\}.$$

By construction of the family of test statistics we conclude (up to the confidence level α) that the change-point τ is localized in the interval

$$[\hat{\tau} - \hat{n}; \hat{\tau} + \hat{n} - 1].$$

Clearly, if a non-multiscale version of the approach is employed, i.e. $|\mathfrak{N}| = \{n\}$, $n = \hat{n}$ and the precision of localization (delay of the detection in online setting) equals n.

3 Bootstrap validity

This section states and discusses the theoretical result demonstrating the validity of the proposed bootstrap scheme i.e.

$$\mathbb{P}\left\{\forall n \in \mathfrak{N} : A_n \le x_n^\flat(\alpha)\right\} \approx 1 - \alpha.$$

Our theoretical results require the tails of the underlying distributions to be light. Specifically, we impose Sub-Gaussianity vector condition.

Assumption 3.1 (Sub-Gaussianity vector condition)**.**

$$\exists L : \forall i \in 1..N \sup_{\substack{a \in \mathbb{R}^p \\ ||a||_2 \le 1}} \mathbb{E}\left[\exp\left(\left(\frac{a^T X_i}{L}\right)^2\right)\right] \le 2.$$

Naturally, in order to establish a theoretical result we have to assume that a method featuring theoretical guarantees was used for estimating the precision matrices. Such methods include graphical lasso [32], adaptive graphical lasso [38] and thresholded de-sparsified estimator based on node-wise procedure [23]. These approaches overcome the high dimensionality of the problem by imposing a sparsity assumption, specifically bounding the maximum number of non-zero elements in a row: $d := \max_i \left|\{j | \Theta^*_{ij} \neq 0\}\right|$. These approaches are guaranteed to yield a root-n consistent estimate revealing the sparsity pattern of the precision matrix [32, 3, 23] or formally

Definition 3.1. *Consider an i.i.d. sample $x_1, x_2, ...x_n \in \mathbb{R}^p$. Denote their precision matrix as $\Theta^* = \mathbb{E}\left[X_1\right]^{-1}$. Let p and d grow with n. A positive-definite matrix $\hat{\Theta}^n$ is a consistent estimator of the high-dimensional precision matrix if*

$$\left|\left|\Theta^* - \hat{\Theta}^n\right|\right|_\infty = O_p\left(\sqrt{\frac{\log p}{n}}\right)$$

and

$$\forall i, j \in 1..p \text{ and } \Theta^*_{ij} = 0 \Rightarrow \hat{\Theta}^n_{ij} = 0.$$

Graphical lasso and its adaptive versions impose an assumption, common for ℓ_1-penalized approaches.

Assumption 3.2 (Irrepresentability condition)**.** *Denote an active set*

$$\mathcal{S} := \left\{(i, j) \in 1..p \times 1..p : \Theta^*_{ij} \neq 0\right\}$$

and define a $p^2 \times p^2$ matrix $\Gamma^* := \Theta^* \otimes \Theta^*$ where $\otimes$ denotes Kronecker product. Irrepresentability condition holds if there exists $\psi \in (0, 1]$ such that

$$\max_{e \notin \mathcal{S}} \left|\left|\Gamma^*_{e\mathcal{S}}(\Gamma^*_{\mathcal{S}\mathcal{S}})^{-1}\right|\right|_1 \le 1 - \psi.$$

The interpretation of irrepresentability condition under normality assumption is given in [24] [32]. Particularly, Assumption 3.2 requires low correlation between the elements of empirical covariance matrix from the active set $\mathcal{S}$ and from its complement. The higher the constant ψ is, the stricter upper bound is assumed.

These observations give rise to the two following assumptions.

Assumption 3.3.A. *Suppose, either graphical lasso or its adaptive version was used with regularization parameter* $\lambda_n \asymp \sqrt{\log p/n}$ *and also impose Assumption 3.2.*

Assumption 3.3.B. *Suppose, thresholded de-sparsified estimator based on node-wise procedure was used with regularization parameter* $\lambda_n \asymp \sqrt{\log p/n}$*.*

Now we are ready to establish a result which guarantees that the suggested bootstrap procedure yields proper thresholds.

Theorem 3.1. *Assume* $\mathbb{H}_0$ *holds and furthermore, let* $X_1, X_2, ...X_N \in \mathbb{R}^p$ *be i.i.d. Let Assumption 3.1 and either Assumption 3.3.A or Assumption 3.3.B hold. Also assume, the spectrum of* Θ^* *is bounded. Allow the parameters* $d, s, p, |\mathfrak{N}|, n_-, n_+$ *grow with* N*. Further let* $N > 2n_+$*,* $n_+ \geq n_-$ *and also impose the sparsity assumption*

$$d = o\left(\frac{\sqrt[4]{\min\{s, n_-^2\}}}{|\mathfrak{N}|^3 \log^{10}(pN)}\right).$$

Then

$$\left|\mathbb{P}\left\{\forall n \in \mathfrak{N} : A_n \leq x_n^\flat(\alpha)\right\} - (1-\alpha)\right| = o_P(1).$$

The finite-sample version of this result, namely, Theorem B.1, is given in Appendix B along with the proofs.

Bootstrap validity result discussion Theorem 3.1 guarantees under mild assumptions (Assumption 3.2 seems to be the most restrictive one, yet it may be dropped if the node-wise procedure is employed) that the first-type error rate meets the nominal level α if the narrowest window size n_- and the set $\mathcal{I}_s$ are large enough. Clearly, the dependence on dimensionality p is logarithmic which establishes applicability of the approach in a high-dimensional setting. It is worth noticing that, unusually, the sparsity bound gets stricter with N but the dependence is only logarithmic. Indeed, we gain nothing from longer samples, since we use only $2n$ data points each time.

On-line setting As one can easily see, the theoretical result is stated in off-line setting, when the whole sample of size N is acquired in advance. In on-line setting we suggest to control the probability α to raise a false alarm for at least one central point t among N data points (which differs from the classical techniques controlling the mean distance between false alarms [33]). Having α and N chosen one should acquire s data-points (the set $\mathcal{I}_s$), use the proposed bootstrap scheme with bootstrap samples of length N in order to obtain the thresholds. Next the approach can be naturally applied in on-line setting and Theorem 3.1 guarantees the capability of the proposed bootstrap scheme to control the aforementioned probability to raise a false alarm.

Proofs The proof of the bootstrap validity result, presented in Appendix B, mostly relies on the high-dimensional central limit theorems obtained in [9], [8]. These papers also present bootstrap justification results, yet do not include a comprehensive bootstrap validity result. The theoretical treatment is complicated by the randomness of $x_n^\flat(\alpha)$. We overcome it by applying the so-called "sandwiching" proof technique (see Lemma C.1), initially used in [34].

4 Sensitivity result

Consider the following setting. Let there be index τ, such that $\{X_i\}_{i\le\tau}$ are i.i.d. and $\{X_i\}_{i>\tau}$ are i.i.d. as well. Denote precision matrices $\Theta_1^{-1} := \mathbb{E}\left[X_1 X_1^T\right]$ and $\Theta_2^{-1} := \mathbb{E}\left[X_{\tau+1} X_{\tau+1}^T\right]$. Define the break extent Δ as

$$\Delta := ||\Theta_1 - \Theta_2||_\infty .$$

The question is, how large the window size n should be in order to reliably reject $\mathbb{H}_0$ and how firmly can we localize the change-point.

Theorem 4.1. *Let Assumption 3.1 and either Assumption 3.3.A or Assumption 3.3.B hold. Also assume, the spectrums of Θ_1 and Θ_2 are bounded. Allow the parameters $d, s, p, |\mathfrak{N}|, n_-, n_+$ grow with N and let Δ decay with N. Further let $N > 2n_+, n_+ \ge n_-$,*

$$d = o\left(\frac{\sqrt{\max\{s, n_-\}}}{d\log^7(pN\,|\mathfrak{N}|\,n_+)}\right) \tag{4.1}$$

and

$$\frac{\log^2(pN\,|\mathfrak{N}|)}{n_+\Delta} = o(1). \tag{4.2}$$

Then $\mathbb{H}_0$ will be rejected with probability approaching 1.

This result is a direct corollary of the finite-sample sensitivity result established and discussed in Appendix A.

The assumption $\mathcal{I}_s \subseteq 1..\tau$ is only technical. The result may be proven without relying on it by methodologically the same argument.

Sensitivity result discussion Assumptions (4.1) and (4.2) are essentially a sparsity bound and a bound for the largest window size n_+. Clearly, they do not yield a particular value n_+ necessary to detect a break, since it depends on the underlying distributions, however, the result includes dimensionality p only under the sign of logarithm, which guarantees high sensitivity of the test in high-dimensional setting.

Online setting Theorem 4.1 is established in offline setting as well. In online setting it guarantees that the proposed approach can reliably detect a break of an extent not less than Δ with a delay at most n_+ bounded by (4.2).

Change-point localization guaranties Theorem 4.1 implies by construction of statistic A_n that the change-point can be localized with precision up to n_+. Hence condition (4.2) provides the bound for change-point localization accuracy.

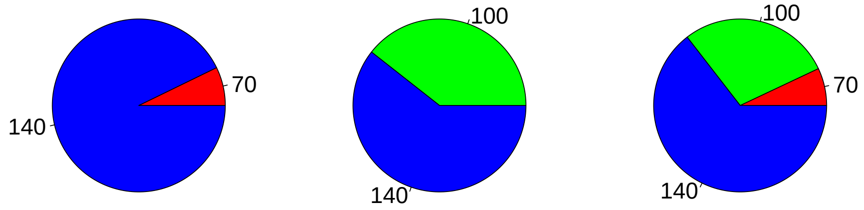
5 Simulation study

5.1 Design

In our simulation we test

$$\mathbb{H}_0 = \left\{\{X_i\}_{i=1}^N \sim \mathcal{N}(0, I)\right\}$$

Figure 2: Pie charts representing distribution of narrowest detecting window $\hat{n}$ and the precision of localization in cases of $|\mathfrak{N}| = \{70, 140\}$, $|\mathfrak{N}| = \{100, 140\}$ and $|\mathfrak{N}| = \{70, 100, 140\}$ respectively



versus an alternative

$$\mathbb{H}_1 = \left\{ \exists \tau : \{X_i\}_{i=1}^{\tau} \sim \mathcal{N}(0, I) \text{ and } \{X_i\}_{i=\tau+1}^{N} \sim \mathcal{N}(0, \Sigma_1) \right\}$$

The alternative covariance matrix Σ_1 was generated in the following way. First we draw $k \sim Poiss(3)$. The matrix Σ_1 is composed as a block-diagonal matrix of k matrices of size 2×2 with ones on their diagonals and their off-diagonal element drawn uniformly from $[-0.6; -0.3] \cup [0.3; 0.6]$ and an identity matrix of size $(p - 2k) \times (p - 2k)$. The dimensionality of the problem is chosen as $p = 50$, the length of the sample $N = 1000$ and we choose the set $\mathcal{I}_s = [1, 2, ..100]$, e.g. $\tau > 100$. The absence of positive effect of large sample size N is discussed in Sections 3 and 4. Moreover, in all the simulations under alternative the sample was generated with the change point in the middle: $\tau = N/2$ but the algorithm was oblivious about this as well as about either of the covariance matrices. The significance level $\alpha = 0.05$ was chosen. In all the experiments graphical lasso with penalization parameter $\lambda_n = \sqrt{\frac{\log p}{n}}$ was used in order to obtain $\hat{\Theta}_n^{\mathfrak{S}}(t)$. In the same way, graphical lasso with penalization parameter λ_s was used in order to obtain $\hat{\Theta}$.

We have also come up with an approach to the same problem not involving bootstrap. The paper [27] defines a high-dimensional two-sample test for equality of matrices. Moreover, the authors prove asymptotic normality of their statistic which makes computing p-value possible. We suggest to run this test for every $t \in \mathbb{T}_n$ and every $n \in \mathfrak{N}$, adjust the obtained p-values using Holm method [20] and eventually compare them against α.

The paper [28] suggests an approach based on comparing characteristic functions of random variables. The critical values were chosen with permutation test as proposed by the authors. In our experiments the method was allowed to consider all the sample at once. The R-package ecp [22] was used.

The first type error rate and power for our approach are reported in Table 1. As one can see, our approach allows to properly control first type error rate. As expected, its power is higher for larger windows and it is decreased by adding narrower windows into consideration which is the price to be paid for better localization of a change point.

In our study the approach proposed in [28] and the one based on the two sample test [27] turned out to be conservative, but neither of them exhibited power above 0.1.

Also, in order to justify application of multiscale approach (i. e. $|\mathfrak{N}| > 1$) for the sake of better change-point localization we report the distribution of the narrowest detecting window $\hat{n}$ (defined by (2.5)) over $\mathfrak{N}$ in Figure 2. The Table 1 represents average precision of change-point localization for various choices of set of window sizes $\mathfrak{N}$. One can see, that multiscale approach significantly improves the precision of localization.

Table 1: First type error rate, power and precision of change-point localization of the proposed approach for various sets of window sizes $\mathfrak{N}$

$\mathfrak{N}$	I type error rate	Power	Localization precision
$\{70\}$	0.02	0.09	70
$\{100\}$	0.00	0.37	100
$\{140\}$	0.01	0.81	140
$\{70, 140\}$	0.01	0.76	135
$\{100, 140\}$	0.01	0.75	124
$\{70, 100, 140\}$	0.01	0.74	123

A Proof of sensitivity result

Proof of Theorem 4.1. Proof consists in applying of the finite-sample Theorem A.1. Its applicability is guaranteed by the consistency results given in papers [32, 3, 23] and by the results from [24, 23, 3] bounding the term $R_{\hat{T}}$. High probability of set $\mathcal{T}_{\leftrightarrow}$ is ensured by Lemma G.1. □

Theorem A.1. *Let $\mathcal{I}_s \subseteq 1..\tau$. Let $\hat{\Theta}$ denote a symmetric estimator of Θ_1 s.t. for some $r \in \mathbb{R}$ it holds that*

$$\left|\left|\Theta_1 - \hat{\Theta}\right|\right|_\infty < r$$

and $(\Theta_1)_{ij} = 0 \Rightarrow \hat{\Theta}_{ij} = 0$. Suppose Assumption 3.1 holds and there exists $R_{\hat{T}}$ such that $\left|\left|r^{\mathfrak{S}}_{n_+}(t)\right|\right|_\infty \leq R_{\hat{T}}$ for all $\mathfrak{S} \in \{l, r\}$ and $t \in \mathbb{T}_{n_+}$ on some set

$$\mathcal{T}_{\leftrightarrow} := \left\{\forall t \leq \tau - n_+ : \left|\left|\hat{\Sigma}^{\mathfrak{S}}_n(t) - \Sigma^*_1\right|\right|_\infty \leq \delta_{n_+}\right\}$$
$$\bigcap \left\{\forall t \geq \tau + n_+ : \left|\left|\hat{\Sigma}^{\mathfrak{S}}_n(t) - \Sigma^*_2\right|\right|_\infty \leq \delta_{n_+}\right\}.$$

Moreover, let the residual R_{A^b} defined in Lemma F.2 be bounded:

$$R_{A^b} \leq \frac{\alpha}{6\,|\mathfrak{N}|}.$$

Also let

$$\sqrt{\frac{n_+}{2}}\, ||S||_\infty \left(\Delta - 2R_{\hat{T}}\right) \geq \mathrm{q}, \tag{A.1}$$

where

$$\mathrm{q} := \sqrt{2\left(1 + \Delta_Y(r)\right)\log\left(\frac{2N\,|\mathfrak{N}|\,p^2}{\alpha - 3\,|\mathfrak{N}|\,R_{A^b}}\right)} \tag{A.2}$$

and Δ_Y is defined in Lemma G.2. Then on set $\mathcal{T}_{\leftrightarrow}$ with probability at least

$$1 - p_s^{\Sigma_Y}(x, q),$$

where $p_s^{\Sigma_Y}(x, q)$ is defined in Lemma G.2, $\mathbb{H}_0$ will be rejected.

Discussion of finite-sample sensitivity result The assumption (A.1) is rather complicated. Here we note that if either graphical lasso [32], adaptive graphical lasso [38] or thresholded de-sparsified estimator based on node-wise procedure [23] with penalization parameter chosen as $\lambda_s \asymp o(\sqrt{\log p/n})$ was used, given $d, s, p, N, n_-, n_+ \to \infty$, $N > 2n_+$, $n_+ \geq n_-$, $s \geq n_-$ and $d = o(\sqrt{n_+})$ it boils down to

$$n_+ \geq D_6 \frac{1}{\Delta} \left(\left|\left|S^{-1}\right|\right|_\infty \log(N \left|\mathfrak{N}\right| p^2) \right)^2$$

for some positive constant D_6 independent of $N, \mathfrak{N}, p, d, S$ while the parameters q, γ and x may be chosen as in (B.5), (B.4), (B.3) (high probability of $\mathcal{T}_\leftrightarrow$ is ensured by Lemma G.1). At the same time the remainder $R_{A^\flat}$ can be bounded by (B.2).

As expected, the bound for sufficient window size decreases with growth of the break extent Δ and the size of the set $\mathcal{I}_s$, but increases with dimensionality p. It is worth noticing, that the latter dependence is only logarithmic. And again, in the same way as with Theorem 3.1, the bound increases with the sample size N (only logarithmically) since we use only $2n$ data points.

Proof of Theorem A.1. Consider a pair of centered normal vectors

$$\eta := \begin{pmatrix} \eta^1 & \eta^2 & \dots & \eta^{|\mathfrak{N}|} \end{pmatrix} \sim \mathcal{N}(0, \Sigma_Y^*),$$

$$\zeta := \begin{pmatrix} \zeta^1 & \zeta^2 & \dots & \zeta^{|\mathfrak{N}|} \end{pmatrix} \sim \mathcal{N}(0, \hat{\Sigma}_Y),$$

$$\Sigma_Y^* := \frac{1}{2n_+} \sum_{j=1}^{2n_+} \operatorname{Var}\left[Y_{\cdot j}^n \right],$$

$$\hat{\Sigma}_Y := \frac{1}{2n_+} \sum_{j=1}^{2n_+} \operatorname{Var}\left[Y_{\cdot j}^{n\flat} \right],$$

where vectors $Y_{\cdot j}^n$ and $Y_{\cdot j}^{n\flat}$ are defined in proofs of Lemma E.2 and Lemma F.1 respectively. Lemma A.2 applies here and yields for all positive q

$$\mathbb{P}\left\{ ||\zeta^{n_+}||_\infty \geq \mathrm{q} \right\} \leq 2 \left|\mathbb{T}_{n_+}\right| p^2 \exp\left(-\frac{\mathrm{q}^2}{2 \left|\left|\hat{\Sigma}_Y\right|\right|_\infty} \right),$$

where $\hat{\Sigma}_Y = \operatorname{Var}[\zeta]$ and $\left|\mathbb{T}_{n_+}\right|$ is the number of central points for window of size n_+. Applying Lemma G.2 on a set of probability at least $1 - p_s^{\Sigma_Y}(x, q)$ yields $\left|\left|\Sigma_Y^* - \hat{\Sigma}_Y\right|\right|_\infty \leq \Delta_Y$, and hence, due to the fact that $||\Sigma_Y^*||_\infty = 1$ by construction,

$$\mathbb{P}\left\{ ||\zeta^{n_+}||_\infty \geq \mathrm{q} \right\} \leq 2 \left|\mathbb{T}_{n_+}\right| p^2 \exp\left(-\frac{\mathrm{q}^2}{2(1 + \Delta_Y)} \right).$$

Due to Lemma F.2 and continuity of Gaussian c.d.f.

$$\mathbb{P}^\flat \left\{ A_{n_+}^\flat \geq x_{n_+}^\flat(\alpha) \right\} \geq \alpha / \left|\mathfrak{N}\right| - 2R_{A^\flat}$$

and due to Lemma F.2 along with the fact that $\left|\mathbb{T}_{n_+}\right| < N$, choosing q as proposed by equation (A.2) we ensure that $x_{n_+}^\flat(\alpha) \leq \mathrm{q}$.

Now by assumption of the theorem and by construction of the test statistics A_n

$$A_{n_+} \geq \sqrt{\frac{n_+}{2}} \left|\left|S\right|\right|_\infty (\Delta - 2R_{\hat{T}}).$$

Finally, we notice that due to assumption (A.1) $A_{n_+} > \mathrm{q}$ and therefore, $\mathbb{H}_0$ will be rejected. □

Lemma A.1. *Consider a centered random Gaussian vector $\xi \in \mathbb{R}^p$ with arbitrary covariance matrix Σ. For any positive q it holds that*

$$\mathbb{P}\left\{\max_i \xi_i \geq \mathrm{q}\right\} \leq p \exp\left(-\frac{\mathrm{q}^2}{2\left|\left|\Sigma\right|\right|_\infty}\right).$$

Proof. By convexity we obtain the following chain of inequalities for any t

$$e^{t\mathbb{E}[t \max_i \xi_i]} \leq \mathbb{E}\left[e^{t \max_i \xi_i}\right] \leq \mathbb{E}\left[e^{t \sum_i \xi_i}\right] \leq p e^{t^2 ||\Sigma||_\infty /2}.$$

Chernoff bound yields for any t

$$\mathbb{P}\left\{\max_i \xi_i \geq \mathrm{q}\right\} \leq \frac{p e^{t^2 ||\Sigma||_\infty /2}}{e^{t\mathrm{q}}}.$$

Finally, optimization over t yields the claim. □

As a trivial corollary, one obtains

Lemma A.2. *Consider a centered random Gaussian vector $\xi \in \mathbb{R}^p$ with arbitrary covariance matrix Σ. For any positive q it holds that*

$$\mathbb{P}\left\{\left|\left|\xi\right|\right|_\infty \geq \mathrm{q}\right\} \leq 2p \exp\left(-\frac{\mathrm{q}^2}{2\left|\left|\Sigma\right|\right|_\infty}\right).$$

B Proof of bootstrap validity result

Proof of Theorem 3.1. Proof consists in applying of the finite-sample Theorem B.1. Its applicability is guaranteed by the consistency results given in papers [32, 3, 23] and by the results from [24, 23, 3] bounding the term $R_{\hat{T}}$. High probability of set $\mathcal{T}_T$ is ensured by Lemma G.1. □

Theorem B.1. *Assume $\mathbb{H}_0$ holds and furthermore, let $X_1, X_2, ...X_N$ be i.i.d. Let $\hat{\Theta}$ denote a symmetric estimator of Θ^* s.t. for some positive r*

$$\left|\left|\Theta^* - \hat{\Theta}\right|\right|_\infty < r$$

and $\Theta^*_{ij} = 0 \Rightarrow \hat{\Theta}_{ij} = 0$. Suppose Assumption 3.1 holds and there exists $R_{\hat{T}}$ such that $\sqrt{n}\left|\left|r_n^{\mathfrak{S}}(t)\right|\right|_\infty \leq R_{\hat{T}}$ for all $\mathfrak{S} \in \{l, r\}$, $n \in \mathfrak{N}$ and $t \in \mathbb{T}_n$ on set

$$\mathcal{T}_T := \left\{\forall \mathfrak{S} \in \{l, r\}, n \in \mathfrak{N}, t \in \mathbb{T}_n : \left|\left|\hat{\Sigma}_n^{\mathfrak{S}}(t) - \mathbb{E}\left[X_1 X_1^T\right]\right|\right|_\infty \leq \delta_n\right\}.$$

Moreover, let

$$R := (3 + 2\left|\mathfrak{N}\right|)\left(2R_A(R_{\hat{T}}) + 2R_{A^b} + R_\Sigma^{\pm}(r)\right) \leq \frac{\alpha}{2},$$

where the remainders R_A, $R_{A^\flat}$, $R_\Sigma^\pm$ are defined in Lemma E.1, Lemma F.2 and Lemma C.1 respectively and the mis-tie Δ_Y involved in the definition of $R_\Sigma^\pm$ comes from Lemma G.2. Then on set $\mathcal{T}_T$ it holds that

$$\left|\mathbb{P}\left\{\forall n \in \mathfrak{N} : A_n \le x_n^\flat(\alpha)\right\} - (1-\alpha)\right| \le R + 2(1-q).$$

where

$$q = 1 - p_s^{\Sigma_Y}(x,q) - p^\Sigma(\gamma) - p_s^M(x) \tag{B.1}$$

and the terms $p_s^{\Sigma_Y}(x,q)$, $p^\Sigma(\gamma)$ and $p_s^M(x)$ are defined in Lemma G.2, Lemma G.1 and Lemma F.2 respectively.

Discussion of finite-sample bootstrap validity result The terms Δ_Y, R_A, $R_{A^\flat}$ and $R_\Sigma^\pm$ involved in the statement of Theorem B.1 are rather complicated. The exact expressions for them are provided by Lemma G.2, Lemma E.1, Lemma F.2 and Lemma C.1 respectively, 3rd and 4th moments M_3^3 and M_4^4 involved therein are bounded by Lemma E.4 and Lemma G.3 while asymptotic bounds for $R_{\hat{T}}$ are provided in [23] (for node-wise procedure) and [24] (for graphical lasso). For the case of graphical lasso an explicit form of $R_{\hat{T}}$ is given in [3].

Here we just note that if $\hat{\Theta}$ is a root-n consistent estimator, recovering sparsity pattern (graphical lasso [32], adaptive graphical lasso [38] or thresholded de-sparsified estimator based on node-wise procedure [23]), then for $d, s, p, N, n_-, n_+ \to \infty$, $N > 2n_+$, $n_+ \ge n_-$, $s \ge n_-$ and $\frac{d^2}{n_-} = o(1)$ given the spectrum of Θ^* is bounded

$$R_{A^\flat} \le D_1 \left(\frac{L^4 d \log^7(2p^2Tn_+)}{n_-}\right)^{1/6} \log^2(ps). \tag{B.2}$$

If either graphical lasso, adaptive graphical lasso or node-wise procedure [29] is used with $\lambda_n \asymp \sqrt{\frac{\log p}{n}}$ in order to obtain $\hat{\Theta}_n^{\mathfrak{S}}(t)$, then on set $\mathcal{T}_T$ it holds that

$$R_A \le D_2 \left(\frac{L^4 d \log^7(2p^2Tn_+)}{n_-}\right)^{1/6} + D_3 \sqrt{\frac{\log 2p^2T}{n_-}} d \log p.$$

The high probability of $\mathcal{T}_T$ may be ensured by means of Lemma G.1 e.g., choosing $\gamma = \log(500T)$ for $\mathbb{P}\{\mathcal{T}_T\} \ge 0.99$. Further

$$\Delta_Y \le D_4 \frac{L^4 d^2}{\sqrt{s}},$$

$$R_\Sigma^\pm \le D_5 \left(\frac{L^4 d^2}{\sqrt{s}}\right)^{1/3} \log^{2/3}(2p^2T).$$

Here D_1, ..., D_5 are positive constants independent of N, $\mathfrak{N}$, d, p and s. We also note that the proper choice of x, γ and q in (B.1) is

$$x = 6, \tag{B.3}$$

$$\gamma = \log(500T), \tag{B.4}$$

$$q = 7 + 4\log(p) \tag{B.5}$$

which ensures the probability defined by (B.1) to be above 0.99. For exact expression of $p_s^{\Sigma_Y}(x, q)$, $p^{\Sigma}(\gamma)$ and $p_s^M(x)$ see Lemma G.2, Lemma G.1 and Lemma F.2.

Proof of Theorem B.1. The proof consists in application of Lemma F.1, Lemma E.2 and Lemma D.1 justifying applicability of Lemma C.1. □

C Sandwiching lemma

Lemma C.1. *Consider a normal multivariate vector η with a deterministic covariance matrix and a normal multivariate vector ζ with a possibly random covariance matrix such that*

$$\sup_{\{x_n\}_{n\in\mathfrak{N}}\subset\mathbb{R}} \left|\mathbb{P}\left\{\forall n \in \mathfrak{N} : A_n \le x_n\right\} - \mathbb{P}\left\{\forall n \in \mathfrak{N} : ||\eta_n||_\infty \le x_n\right\}\right| \le R_A, \tag{C.1}$$

$$\sup_{\{x_n\}_{n\in\mathfrak{N}}\subset\mathbb{R}} \left|\mathbb{P}^\flat\left\{\forall n \in \mathfrak{N} : A_n^\flat \le x_n\right\} - \mathbb{P}^\flat\left\{\forall n \in \mathfrak{N} : ||\zeta_n||_\infty \le x_n\right\}\right| \le R_{A^\flat}, \tag{C.2}$$

$$\sup_{\{x_n\}_{n\in\mathfrak{N}}\subset\mathbb{R}} \left|\mathbb{P}\left\{\forall n \in \mathfrak{N} : A_n \le x_n\right\} - \mathbb{P}^\flat\left\{\forall n \in \mathfrak{N} : A_n^\flat \le x_n\right\}\right| \le R. \tag{C.3}$$

where η_n and ζ_n are sub-vectors of η and ζ respectively. Then

$$\left|\mathbb{P}\left\{\forall n \in \mathfrak{N} : A_n \le x_n^\flat(\alpha)\right\} - (1-\alpha)\right| \le (3 + 2\,|\mathfrak{N}|)\,(R + R_A + R_{A^\flat})\,.$$

Proof. Let us introduce some notation. Denote multivariate cumulative distribution function of A_n, $A_n^\flat$, $||\eta_n||_\infty, ||\zeta_n||_\infty$ as $P, P^\flat, \mathcal{N}, \mathcal{N}^\flat : \mathbb{R}^{|\mathfrak{N}|} \to [0,1]$ respectively. Define the following sets for all $\delta \in [0, \alpha]$

$$\mathcal{Z}_+(\delta) := \{z : \mathcal{N}(z) \ge 1 - \alpha - \delta\}\,,$$

$$\mathcal{Z}_-(\delta) := \{z : \mathcal{N}(z) \le 1 - \alpha + \delta\}$$

and their boundaries

$$\partial\mathcal{Z}_+(\delta) := \{z : \mathcal{N}(z) = 1 - \alpha - \delta\}\,, \tag{C.4}$$

$$\partial\mathcal{Z}_-(\delta) := \{z : \mathcal{N}(z) = 1 - \alpha + \delta\}\,.$$

Consider $\delta = R + R_A + R_{A^\flat}$ and denote sets $\mathcal{Z}_+ = \mathcal{Z}_+(\delta)$, $\mathcal{Z}_- = \mathcal{Z}_-(\delta)$, $\partial\mathcal{Z}_- = \partial\mathcal{Z}_-(\delta)$, $\partial\mathcal{Z}_+ = \partial\mathcal{Z}_+(\delta)$ Define a set of thresholds satisfying the confidence level

$$\mathcal{Z}^\flat := \left\{z : P^\flat(z) \ge 1 - \alpha \;\&\; \forall z_1 < z : P^\flat(z_1) < 1 - \alpha\right\}$$

here and below comparison of vectors should be understood element-wise. Notice that due to continuity of multivariate normal distribution and assumption (C.2) $\forall z^\flat \in \mathcal{Z}^\flat$

$$\left|P^\flat(z^\flat) - (1-\alpha)\right| \le R_{A^\flat}. \tag{C.5}$$

Now for all $z_- \in \partial \mathcal{Z}_-$ and for all $z^\flat \in \mathcal{Z}^\flat$ it holds that

$$\begin{aligned}
P^\flat(z_-) &\leq P(z_-) + R \\
&\leq \mathcal{N}(z_-) + R + R_A \\
&\leq 1 - \alpha - R_{A^\flat} \\
&\leq P^\flat(z^\flat)
\end{aligned}$$

where we have consequently used (C.3), (C.1), (C.4) and (C.5). In the same way one obtains for all $z_+ \in \partial \mathcal{Z}_+$ and for all $z^\flat \in \mathcal{Z}^\flat$

$$P^\flat(z_+) \geq P^\flat(z^\flat)$$

which implies that $\mathcal{Z}^\flat \subset \mathcal{Z}_- \cap \mathcal{Z}_+$.

Now denote quantile functions of $||\eta_n||_\infty$ as $z^N : [0,1] \to \mathbb{R}^{|\mathfrak{N}|}$:

$$\forall n \in \mathfrak{N} : \mathbb{P}\left\{||\eta_n||_\infty \geq z_n^N(\mathrm{x})\right\} = \mathrm{x}.$$

In exactly the same way define quantile functions $z^{N^\flat} : [0,1] \to \mathbb{R}^{|\mathfrak{N}|}$ of $||\zeta_n||_\infty$. Clearly for all $\mathrm{x} \in [0,1]$,

$$z^N(\mathrm{x}+\delta) \leq z^\flat(\mathrm{x}) \leq z^N(\mathrm{x}-\delta)$$

and hence

$$\begin{gathered}
z^\flat(\alpha^*) \leq z^N(\alpha^* - \delta) \leq z^\flat(\alpha^* - 2\delta), \\
1 - \alpha \leq P^\flat(z^N(\alpha^* - \delta)) \leq P^\flat(z^\flat(\alpha^* - 2\delta)).
\end{gathered}$$

Using Taylor expansion with Lagrange remainder term we obtain for some $0 \leq \kappa \leq 2\delta$

$$\begin{aligned}
\mathcal{N}^\flat\left(z^\flat(\alpha^* - 2\delta)\right) &\leq \mathcal{N}^\flat\left(z^{N^\flat}(\alpha^* - 2\delta)\right) + \delta \\
&= \mathcal{N}^\flat\left(z^{N^\flat}(\alpha^*)\right) + \sum_{n \in \mathfrak{N}} \partial_{z_n^\flat} \mathcal{N}^\flat(z^{N^\flat}(\alpha^*)) \partial_\alpha z_n^{N^\flat}(\alpha^*)\kappa + \delta \\
&\leq 1 - \alpha + \sum_{n \in \mathfrak{N}} \partial_{z_n^\flat} \mathcal{N}^\flat(z^{N^\flat}(\alpha^*)) \partial_\alpha z_n^{N^\flat}(\alpha^*)\kappa + 3\delta.
\end{aligned}$$

Next successively using Lemma C.2 and the fact that the quantile function is an inverse function of c.d.f. we obtain

$$\mathcal{N}^\flat\left(z^\flat(\alpha^* - 2\delta)\right) \leq 1 - \alpha + 3\delta + 2\delta\,|\mathfrak{N}|$$

and therefore

$$\begin{gathered}
1 - \alpha \leq P^\flat\left(z^\flat(\alpha^* - 2\delta)\right) \leq 1 - \alpha + \delta\,(3 + 2\,|\mathfrak{N}|)\,, \\
1 - \alpha \leq P^\flat\left(z^N(\alpha^* - \delta)\right) \leq 1 - \alpha + \delta\,(3 + 2\,|\mathfrak{N}|)\,.
\end{gathered}$$

In the same way one obtains

$$1 - \alpha - \delta\,(3 + 2\,|\mathfrak{N}|) \leq P^\flat\left(z^N(\alpha^* + \delta)\right) \leq 1 - \alpha.$$

Next, by the argument used in the beginning of the proof we obtain

$$z^N(\alpha^* + \delta), z^N(\alpha^* - \delta) \in \mathcal{Z}_-(\delta\,(3 + 2\,|\mathfrak{N}|)) \cap \mathcal{Z}_+\,(\delta\,(3 + 2\,|\mathfrak{N}|))\,.$$

As the final ingredient, we need to choose deterministic α^+ and α^- such that

$$N(z^N(\alpha^- + \delta)) = 1 - \alpha - \delta\,(3 + 2\,|\mathfrak{N}|)\,,$$

$$N(z^N(\alpha^+ - \delta)) = 1 - \alpha + \delta\,(3 + 2\,|\mathfrak{N}|)$$

(which is possible due to continuity), so $\alpha^- \le \alpha^* \le \alpha^+$ and hence by monotonicity

$$z^N(\alpha^- + \delta) \le z^N(\alpha^* + \delta) \le z^\flat(\alpha^*) \le z^N(\alpha^* - \delta) \le z^N(\alpha^+ - \delta)$$

and finally

$$\begin{aligned}
1 - \alpha - \delta\,(3 + 2\,|\mathfrak{N}|) &\le P(z^N(\alpha^- + \delta)) \\
&\le P(z^\flat(\alpha^*)) \\
&\le P(z^N(\alpha^+ - \delta)) \\
&\le 1 - \alpha + \delta\,(3 + 2\,|\mathfrak{N}|)\,.
\end{aligned}$$

□

Lemma C.2. *Consider a random variable ξ and an event A defined on the same probability space. Let c.d.f. $\mathbb{P}\{\xi \le x\}$ and $\mathbb{P}\{\xi \le x \& A\}$ be differentiable. Then*

$$\frac{\partial_x \mathbb{P}\{\xi \le x \& A\}}{\partial_x \mathbb{P}\{\xi \le x\}} \le 1$$

Proof. Indeed, denoting the complement of set A as $\overline{A}$ we obtain,

$$\begin{aligned}
\frac{\partial_x \mathbb{P}\{\xi \le x \& A\}}{\partial_x \mathbb{P}\{\xi \le x\}} &= \frac{\partial_x \mathbb{P}\{\xi \le x \& A\}}{\partial_x \left(\mathbb{P}\{\xi \le x \& A\} + \mathbb{P}\left\{\xi \le x \& \overline{A}\right\}\right)} \\
&= \frac{\partial_x \mathbb{P}\{\xi \le x \& A\}}{\partial_x \mathbb{P}\{\xi \le x \& A\} + \partial_x \mathbb{P}\left\{\xi \le x \& \overline{A}\right\}} \\
&= \frac{1}{1 + \frac{\partial_x \mathbb{P}\left\{\xi \le x \& \overline{A}\right\}}{\partial_x \mathbb{P}\{\xi \le x \& A\}}}
\end{aligned}$$

Using the fact that derivative of c.d.f. is non-negative we finalize the proof. □

D Similarity of joint distributions of $\{A_n\}_{n \in \mathfrak{N}}$ and $\{A_n^\flat\}_{n \in \mathfrak{N}}$

Lemma D.1. *Under assumptions of Theorem 3.1 it holds that on set $\mathcal{T}$ with probability at least*

$$1 - p_s^{\Sigma_Y}(x, q) - p^{\Sigma}(\gamma) - p_s^M(x)$$

that

$$\sup_{\{x_n\}_{n \in \mathfrak{N}} \subset \mathbb{R}} \left| \mathbb{P}\{\forall n \in \mathfrak{N} : A_n \le x_n\} - \mathbb{P}^\flat\left\{\forall n \in \mathfrak{N} : A_n^\flat \le x_n\right\} \right| \le R_A + R_{A^\flat} + R_\Sigma^\pm.$$

Proof. The proof consists in applying Lemma F.1, Lemma E.2, Lemma G.2 and Lemma H.3. □

E Gaussian approximation result for A_n

Lemma E.1. *Suppose there exists $R_{\hat{T}}$ such that $\sqrt{n}\left|\left|r^{\mathfrak{S}}(t)\right|\right|_\infty \le R_{\hat{T}}$ for all $\mathfrak{S}$ and t on some set $\mathcal{T}$. Then on set $\mathcal{T}$ it holds that*

$$\sup_x \left|\mathbb{P}\left\{\forall n \in \mathfrak{N} : A_n \le x_n\right\} - \mathbb{P}\left\{\forall n \in \mathfrak{N} : ||\eta^n||_\infty \le x_n\right\}\right| \le R_A$$
$$:= C_A\left(\left(F \log^7(p^2 T n_+)\right)^{1/6} + 4R_{\hat{T}}\sqrt{\log(2p^2T)}\right).$$

where F is defined by (E.2) *and η^n by* (E.1).

Proof. Substituting (2.3) to (2.2) yields

$$A_n(t) = \left|\left|\underbrace{\frac{1}{\sqrt{2n}}S^{-1}\left(\sum_{i\in\mathcal{I}_n^l(t)}\overline{Z}_i - \sum_{i\in\mathcal{I}_n^r(t)}\overline{Z}_i\right)}_{S_Z^n(t)} + \frac{1}{\sqrt{2}}(\overline{r_n^l} - \overline{r_n^r})\right|\right|_\infty.$$

Now denote stacked $S_Z^n(t)$ for all $n \in \mathfrak{N}$ and as S_Z^n and for all n as S_Z. Lemma E.2 bounds the c.d.f. of $||S_Z||_\infty$ as

$$\sup_x \left|\mathbb{P}\left\{\forall n \in \mathfrak{N} : ||S_Z^n||_\infty \le x_n\right\} - \mathbb{P}\left\{\forall n \in \mathfrak{N} : ||\eta^n||_\infty \le x_n\right\}\right| \le C_A\left(F \log^7(p^2 T n_+)\right)^{1/6}.$$

But clearly on set $\mathcal{T}$

$$\left|A_n - ||S_Z^n||_\infty\right| \le \sqrt{2}R_{\hat{T}}$$

And hence for all $\{x_n\}_{n\in\mathfrak{N}} \subset \mathbb{R}$

$$\begin{aligned}\left|\mathbb{P}\left\{\forall n \in \mathfrak{N} : A_n < x_n | \mathcal{T}\right\} - \mathbb{P}\left\{\forall n \in \mathfrak{N} : ||\eta^n||_\infty \le x_n\right\}\right| &\le C_A\left(F \log^7(p^2 T n_+)\right)^{1/6} \\ &+ \mathbb{P}\left\{\forall n \in \mathfrak{N} : ||\eta^n||_\infty \le x_n + \sqrt{2}R_{\hat{T}}\right\} \\ &- \mathbb{P}\left\{\forall n \in \mathfrak{N} : ||\eta^n||_\infty \le x_n - \sqrt{2}R_{\hat{T}}\right\}.\end{aligned}$$

Now notice that $\forall i : (\Sigma_Y^*)_{ii} = 1$ and bound the latter two terms by means of Lemma H.2:

$$\begin{aligned}\sup_{\{x_n\}_{n\in\mathfrak{N}}\subset\mathbb{R}^{|\mathfrak{N}|}} \left|\mathbb{P}\left\{\forall n \in \mathfrak{N} : A_n < x_n | \mathcal{T}\right\} - \mathbb{P}\left\{\forall n \in \mathfrak{N} : ||\eta^n||_\infty \le x_n\right\}\right| &\le C_A\left(F \log^7(p^2 T n_+)\right)^{1/6} \\ &+ 4R_{\hat{T}}(\sqrt{\log(2p^2T)})\end{aligned}$$

□

Lemma E.2. *Let Assumption 3.1 hold. Then*

$$\sup_x |\mathbb{P}\{\forall n \in \mathfrak{N} : ||S_Z^n||_\infty \le x_n\} - \mathbb{P}\{\forall n \in \mathfrak{N} : ||\eta^n||_\infty \le x_n\}| \le C_A \left(F \log^7(2p^2 T n_+)\right)^{1/6}$$

Where

$$\begin{pmatrix} \eta^1 & \eta^2 & \dots & \eta^{|\mathfrak{N}|} \end{pmatrix} \sim \mathcal{N}(0, \Sigma_Y^*), \tag{E.1}$$

$$\Sigma_Y^* = \frac{1}{N} \sum_{i=1}^{N} \mathrm{Var}\left[Y_{\cdot i}\right],$$

$$F = \frac{1}{2n_-}\left(\beta \log 2 \vee \frac{\sqrt{2}}{\sqrt{2}-1}\gamma\right)^2 \vee \frac{1}{2n_+}\left(\frac{n_+}{n_-}\right)^{1/3} M_3^2 \vee \sqrt{\frac{1}{2n_+ n_-}} M_4^2 \tag{E.2}$$

with γ defined by (E.5), β by (E.6) and Y by (E.3) and an independent constant C_A .

Proof. Consider a matrix Y_n with $2n_+$ columns

$$Y_n^T := \sqrt{\frac{n_+}{n}} \times$$
$$\begin{pmatrix}
Z_1^S & O & \dots & O & -Z_{2n_++1}^S & \dots \\
Z_2^S & Z_2^S & \dots & \dots & \dots & \dots \\
\dots & Z_3^S & \dots & \dots & \dots & \dots \\
Z_n^S & \dots & \dots & \dots & \dots & \dots \\
-Z_{n+1}^S & Z_{n+1}^S & \dots & \dots & \dots & \dots \\
-Z_{n+2}^S & -Z_{n+2}^S & \dots & \dots & \dots & \dots \\
\dots & -Z_{n+3}^S & \dots & O & \dots & \dots \\
-Z_{2n}^S & \dots & \dots & Z_{2n_+-2n+1}^S & O & \dots \\
O & -Z_{2n+1}^S & \dots & Z_{2n_+-2n+2}^S & Z_{2n_+-2n+2}^S & \dots \\
O & O & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & -Z_{2n_+-1}^S & -Z_{2n_+-1}^S & \dots \\
O & O & \dots & -Z_{2n_+}^S & -Z_{2n_+}^S & \dots
\end{pmatrix}$$

where $Z_i^S := (S^{-1}\overline{Z}_i)^T$. Clearly, columns of the matrix are independent and

$$S_Z^n = \frac{1}{\sqrt{2n_+}} \sum_{l=0}^{2n_+} (Y_n)_{\cdot l}$$

Next define a block matrix composed of Y_n matrices:

$$Y := \left(\begin{array}{c} Y_1 \\ \hline Y_2 \\ \hline \dots \\ \hline Y_{|\mathfrak{N}|} \end{array}\right) \tag{E.3}$$

Clearly vectors $Y_{\cdot l}$ are independent and

$$S_Z = \frac{1}{\sqrt{2n_+}} \sum_{l=0}^{2n_+} Y_{\cdot l}$$

In order to complete the proof we make use of Lemma H.1. Denote

$$B_{n_+} = \sqrt{\frac{n_+}{n_-}} \left(\beta \log 2 \vee \frac{\sqrt{2}}{\sqrt{2}-1} \gamma \right) \vee \left(\frac{n_+}{n_-} \right)^{1/6} M_3 \vee \left(\frac{n_+}{n_-} \right)^{1/4} M_4 \tag{E.4}$$

By means of Lemma G.3 one shows that the assumptions of Lemma E.3 hold for components of Z_i^S with

$$\gamma := 12 L^2 \sqrt{d} \Lambda \left(\Theta^* \right) ||\Theta^*||_\infty \left|\left| S^{-1} \right|\right|_\infty \tag{E.5}$$

$$\beta := \left(\frac{9}{2} L^2 \sqrt{d} \Lambda \left(\Theta^* \right) + 1 \right) ||\Theta^*||_\infty \left|\left| S^{-1} \right|\right|_\infty \tag{E.6}$$

where $\Lambda \left(\Theta^* \right)$ denotes the maximal eigen value of Θ^*. Therefore condition (H.1) holds with B_n defined by equation (E.4).

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[(Y_{ij}^n)^2 \right] \geq \min_j \operatorname{Var} \left[Z_{1j}^S \right] = 1$$

Hence, Assumption H.1 is fulfilled with $b = 1$. Next notice that for some k-th component of Z_i^S and central point t (both defined by j):

$$\begin{aligned}
\frac{1}{2n_+} \sum_{i=1}^{2n_+} \mathbb{E} \left[\left| Y_{ij}^n \right|^3 \right] &= \frac{1}{2n_+} \sum_{i \in \mathcal{I}_n^l(t) \cup \mathcal{I}_n^r(t)} \mathbb{E} \left[\left(\sqrt{\frac{n_+}{n}} \left| Z_{ik}^S \right| \right)^3 \right] \\
&= \frac{1}{2n_+} \sum_{i \in \mathcal{I}_n^l(t) \cup \mathcal{I}_n^r(t)} \left(\frac{n_+}{n} \right)^{3/2} \mathbb{E} \left[\left| Z_{ik}^S \right|^3 \right] \\
&= \frac{2n}{2n_+} \left(\frac{n_+}{n} \right)^{3/2} \mathbb{E} \left[\left| Z_{ik}^S \right|^3 \right] \\
&= \sqrt{\frac{n_+}{n}} \mathbb{E} \left[\left| Z_{ik}^S \right|^3 \right] \\
&\leq \sqrt{\frac{n_+}{n_-}} M_3^3
\end{aligned}$$

and in the same way:

$$\frac{1}{2n_+} \sum_{i=1}^{N} \mathbb{E} \left[\left| Y_{ij}^n \right|^4 \right] \leq \frac{n_+}{n_-} M_4^4$$

Therefore Assumption H.2 holds with B_{n_+} so Lemma H.1 applies here and provides us with the claimed bound. Moreover, C_A depends only on b which equals one which implies that the constant C_A depends on nothing.

□

Lemma E.3. *Consider a random variable ξ. Suppose the following bound holds $\forall x \geq 0$:*

$$\mathbb{P}\left\{|\xi| \geq \gamma x + \beta\right\} \leq e^{-x}$$

Then

$$\mathbb{E}\left[\exp\left(\frac{|\xi|}{B}\right)\right] \leq 2$$

for

$$B = \beta \log 2 \vee \frac{\sqrt{2}}{\sqrt{2}-1}\gamma$$

Proof. Integration by parts yields

$$\mathbb{E}\left[\exp\left(\frac{|\xi|}{B}\right)\right] \leq \exp\left(\frac{\beta}{B}\right) + \frac{\gamma}{B}\int_0^{+\infty} \exp\left(\frac{\gamma x + \beta}{B}\right) e^{-x} dx$$

$$\int_0^{+\infty} \exp\left(\frac{\gamma x + \beta}{B}\right) e^{-x} dx = \frac{B}{B-\gamma}\exp\left(\frac{\beta}{B}\right)$$

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{|\xi|}{B}\right)\right] &\leq \frac{B}{B-\gamma}\exp\left(\frac{\beta}{B}\right)\\ &\leq 2\end{aligned}$$

□

By the same technique the following lemma can be proven

Lemma E.4. *Under assumptions of Lemma E.3*

$$\mathbb{E}\left[|\xi|^3\right] \leq \beta^3 + 3\gamma\beta^2 + 6\beta\gamma^2 + 2\gamma^3,$$

$$\mathbb{E}\left[\xi^4\right] \leq \beta^4 + 4\gamma\beta^3 + 12\beta^2\gamma^2 6\beta\gamma^3 + 24\gamma^4.$$

F Gaussian approximation result for $A_n^\flat$

Lemma F.1.

$$\sup_{\{x_n\}_{n\in\mathfrak{N}}\subset\mathbb{R}} \left|\mathbb{P}^\flat\left\{\forall n \in \mathfrak{N} : A^\flat \leq x_n\right\} - \mathbb{P}^\flat\left\{\forall n \in \mathfrak{N} : ||\zeta^n||_\infty \leq x_n\right\}\right| \leq \hat{C}_{A^\flat}\left(F^\flat \log^7(2p^2Tn_+)\right)^{1/6}.$$

Where

$$\begin{pmatrix} \zeta^1 & \zeta^2 & \dots & \zeta^{|\mathfrak{N}|} \end{pmatrix} \sim \mathcal{N}(0, \hat{\Sigma}_Y),$$

$$\hat{\Sigma}_Y = \frac{1}{N} \sum_{i=1}^{N} \mathrm{Var}\left[Y^\flat_{\cdot i}\right],$$

$$F^\flat = \left(\frac{1}{2n_- \log^2 2} \vee \frac{1}{2n_+} \left(\frac{n_+}{n_-} \right)^{1/3} \vee \sqrt{\frac{1}{2n_+ n_-}} \right) \left|\left|S^{-1}\right|\right|_\infty^2 (M^\flat)^2$$

$$M^\flat = \max_{i \in \mathcal{I}_s} \left|\left|\hat{Z}_i\right|\right|_\infty$$

$Y^\flat_n$ *are defined by* (F.1)*, and* $\hat{C}_{A^\flat}$ *depends only on* $\min_{1 \le k \le p} (\hat{\Sigma}_Y)_{kk}$

Proof. Denote the term under the sign of $||\cdot||_\infty$ in (2.4) as $S_Z^{n\flat}$

$$S_Z^{n\flat} := \frac{1}{\sqrt{2n}} \left(\sum_{i \in \mathcal{I}_n^l(t)} Z_i^{S\flat} - \sum_{i \in \mathcal{I}_n^r(t)} Z_i^{S\flat} \right)^T$$

where $Z_i^{S\flat} := (S^{-1} \overline{Z_i^\flat})^T$ and let $S_Z^\flat$ be a vector composed of stacked vectors $S_Z^{n\flat}$ for all $n \in \mathfrak{N}$.

Consider a matrix

$$(Y_n^\flat)^T := \sqrt{\frac{n_+}{n}} \times \begin{pmatrix} Z_1^{S\flat} & O & \dots & O & -Z_{2n_++1}^{S\flat} & \dots \\ Z_2^{S\flat} & Z_2^{S\flat} & \dots & \dots & \dots & \dots \\ \dots & Z_3^{S\flat} & \dots & \dots & \dots & \dots \\ Z_n^{S\flat} & \dots & \dots & \dots & \dots & \dots \\ -Z_{n+1}^{S\flat} & Z_{n+1}^{S\flat} & \dots & \dots & \dots & \dots \\ -Z_{n+2}^{S\flat} & -Z_{n+2}^{S\flat} & \dots & \dots & \dots & \dots \\ \dots & -Z_{n+3}^{S\flat} & \dots & O & \dots & \dots \\ -Z_{2n}^{S\flat} & \dots & \dots & Z_{2n_+-2n+1}^{S\flat} & O & \dots \\ O & -Z_{2n+1}^{S\flat} & \dots & Z_{2n_+-2n+2}^{S\flat} & Z_{2n_+-2n+2}^{S\flat} & \dots \\ O & O & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & -Z_{2n_+-1}^{S\flat} & -Z_{2n_+-1}^{S\flat} & \dots \\ O & O & \dots & -Z_{2n_+}^{S\flat} & -Z_{2n_+}^{S\flat} & \dots \end{pmatrix} \tag{F.1}$$

which is a bootstrap counterpart of Y_n from the proof of Lemma E.2 and construct a block matrix $Y^\flat$:

$$Y^\flat = \begin{pmatrix} Y_1^\flat \\ \hline Y_2^\flat \\ \hline \dots \\ \hline Y_{|\mathfrak{N}|}^\flat \end{pmatrix}$$

Clearly vectors $Y^\flat_{\cdot l}$ are independent and

$$S_Z^\flat = \frac{1}{\sqrt{2n_+}} \sum_{l=0}^{N} Y_{\cdot l}^\flat$$

Now notice

$$\frac{1}{2n_+} \sum_{i=1}^{N} \mathbb{E}\left[|Y_{ij}|^3\right] \leq \sqrt{\frac{n_+}{n_-}} \max_{i \in \mathcal{I}_s} \left|\left|\hat{Z}_i\right|\right|_\infty^3 \left|\left|S^{-1}\right|\right|_\infty^3$$

$$\frac{1}{2n_+} \sum_{i=1}^{N} \mathbb{E}\left[|Y_{ij}|^4\right] \leq \frac{n_+}{n_-} \max_{i \in \mathcal{I}_s} \left|\left|\hat{Z}_i\right|\right|_\infty^4 \left|\left|S^{-1}\right|\right|_\infty^4$$

And finally apply Lemma H.1.

□

Lemma F.2. *Let $\hat{\Theta}$ denote an estimator of Θ^* s.t. for some positive r*

$$\left|\left|\Theta^* - \hat{\Theta}\right|\right|_\infty < r$$

and $\Theta^*_{ij} = 0 \Rightarrow \hat{\Theta}_{ij} = 0$, furthermore, let $\Delta_Y(r) < 1/2$, also suppose Assumption 3.1 holds. Then at least with probability $1 - p_s^M(x) - p_s^{\Sigma_Y}(x, q)$

$$\sup_{\{x_n\}_{n \in \mathfrak{N}} \subset \mathbb{R}} \left|\mathbb{P}^\flat\left\{\forall n \in \mathfrak{N} : A^\flat \leq x_n\right\} - \mathbb{P}^\flat\left\{\forall n \in \mathfrak{N} : ||\zeta^n||_\infty \leq x_n\right\}\right| \leq R_{A^\flat} := C_{A^\flat}\left(\hat{F}\log^7(2p^2 T n_+)\right)^{1/6}$$

where

$$\hat{F} = \left(\frac{1}{2n_- \log^2 2} \vee \frac{1}{2n_+}\left(\frac{n_+}{n_-}\right)^{1/3} \vee \sqrt{\frac{1}{2n_+ n_-}}\right) \left|\left|S^{-1}\right|\right|_\infty^2 (C^\flat)^2$$

$$C^\flat := \mathcal{Z}_s(x) + (3(dx)^2 + 1)r$$

and constant $C_{A^\flat}$ depends only on Δ_Y.

Proof. The proof consists in subsequently applying Lemma F.1 and Lemma F.3 ensuring $C^\flat \geq M^\flat = \max_{i \in \mathcal{I}_s} \left|\left|\hat{Z}_i\right|\right|_\infty$ with probability at least $1 - p_s^M(x)$ and applying Lemma G.2 providing that $\left|\left|\Sigma_Y^* - \hat{\Sigma}_Y\right|\right|_\infty \leq \Delta_Y \leq 1 = \min_{1 \leq k \leq p}(\Sigma_Y^*)_{kk}$ with probability at least $1 - p_s^{\Sigma_Y}(x, q)$ which implies the existence of a deterministic constant $C_{A^\flat} > \hat{C}_{A^\flat}$. □

Lemma F.3. *Let $\hat{\Theta}$ denote an estimator of Θ^* s.t. for some positive r*

$$\left|\left|\Theta^* - \hat{\Theta}\right|\right|_\infty < r$$

and $\Theta^*_{ij} = 0 \Rightarrow \hat{\Theta}_{ij} = 0$. Also let Assumption 3.1 hold. Then with probability at least $1 - p_s^M(x)$

$$M^\flat \leq \mathcal{Z}_s(x) + \Delta_Z(x) \tag{F.2}$$

where $p_s^M(x) := p_{\mathcal{Z}_s}(x) + p_s^X(x)$.

Proof. Direct application of Lemma G.4 yields

$$\mathbb{P}\left\{\forall i \in \mathcal{I}_s : ||Z_i||_\infty \leq \mathcal{Z}_s(x)\right\} \geq 1 - p_{\mathcal{Z}_s}(x)$$

which in combination with the fact (provided by Lemma G.6) that $\left|\left|\hat{Z}_i - Z_i\right|\right|_\infty \leq \Delta_Z(x)$ implies (F.2).

□

G $\hat{\Sigma}_Y \approx \Sigma_Y^*$

First of all, if $\Sigma_Z^* := \operatorname{Var}\left[\overline{Z_i}\right] \approx \operatorname{Var}_\flat\left[\overline{Z_i^\flat}\right]$, then $\Sigma_Y^* \approx \hat{\Sigma}_Y$ as well (Lemma G.2). The idea is to notice that

$$\operatorname{Cov}\left[\overline{Z_i^\flat}\right] = \hat{\Sigma}_{\hat{Z}} := \mathbb{E}_{\mathcal{I}_s}\left[\left(\overline{\hat{Z}_i} - \mathbb{E}_{\mathcal{I}_s}\left[\overline{\hat{Z}_i}\right]\right)\left(\overline{\hat{Z}_i} - \mathbb{E}_{\mathcal{I}_s}\left[\overline{\hat{Z}_i}\right]\right)^T\right]$$

due to the choice of the bootstrap scheme. Next we show that

$$\Sigma_Z^* \approx \hat{\Sigma}_Z := \mathbb{E}_{\mathcal{I}_s}\left[\left(\overline{Z_i} - \mathbb{E}_{\mathcal{I}_s}\left[\overline{Z_i}\right]\right)\left(\overline{Z_i} - \mathbb{E}_{\mathcal{I}_s}\left[\overline{Z_i}\right]\right)^T\right]$$

(Lemma G.5) and finalize the proof by proving that $\hat{\Sigma}_Z \approx \hat{\Sigma}_{\hat{Z}}$ (Lemma G.7).

The results of this section rely on a lemma which is a trivial corollary of Lemma 6 by [24] providing the concentration result for the empirical covariance matrix

Lemma G.1. *Let Assumption 3.1 hold for some $L > 0$. Then for any positive γ*

$$\delta_n(\chi) := 2L^2\left(\frac{2\log p + \chi}{n} + \sqrt{\frac{4\log p + 2\chi}{n}}\right)$$

$$\mathbb{P}\left\{\left|\left|\hat{\Sigma} - \Sigma^*\right|\right|_\infty \geq \delta_n(\gamma)\right\} \leq p^{\Sigma}(\gamma) := 2e^{-\chi}.$$

Lemma G.2. *Assume, Assumption 3.1 holds. Moreover, let*

$$\left|\left|\mathbb{E}_{\mathcal{I}_s}\left[X_i X_i^T\right] - \Sigma^*\right|\right|_\infty \leq \delta_s$$

and let $\hat{\Theta}$ denote a symmetric estimator of Θ^* s.t.

$$\left|\left|\Theta^* - \hat{\Theta}\right|\right|_\infty < r$$

and $\Theta_{ij}^* = 0 \Rightarrow \hat{\Theta}_{ij} = 0$. Then for positive x and q

$$\mathbb{P}\left\{\left|\left|\hat{\Sigma}_Y - \Sigma_Y^*\right|\right|_\infty \geq \Delta_Y\right\} \leq p_s^{\Sigma_Y}(x, q)$$

where

$$p_s^{\Sigma_Y}(x, q) := p_s^{\Sigma_{Z1}}(x, q) + p_s^{\Sigma_{Z2}}(x)$$

$$\Delta_Y := ||S^{-1}||_\infty^2 \left(\Delta_{\Sigma_Z}^{(1)} + \Delta_{\Sigma_Z}^{(2)}\right)$$

and $\Delta_{\Sigma_Z}^{(1)}$ and $\Delta_{\Sigma_Z}^{(2)}$ along with the probabilities $p_s^{\Sigma_{Z1}}(x,q)$ and $p_s^{\Sigma_{Z2}}(x)$ are defined in Lemma G.5 and Lemma G.7 respectively.

Proof. Notice that

$$\left|\left|\hat{\Sigma}_Y - \Sigma_Y^*\right|\right|_\infty = \left|\left|S^{-1}\hat{\Sigma}_{\hat{Z}}S^{-1} - S^{-1}\Sigma_Z^* S^{-1}\right|\right|_\infty \leq ||S^{-1}||_\infty^2 \left|\left|\hat{\Sigma}_{\hat{Z}} - \Sigma_Z^*\right|\right|_\infty$$

because the matrices $\hat{\Sigma}_Y$ and Σ_Y^* are composed of blocks $S^{-1}\hat{\Sigma}_Z S^{-1}$ and $S^{-1}\Sigma_Z^* S^{-1}$ respectively, each block multiplied by some positive value not greater than 1 (which can be verified by simple algebra).

By Lemma G.7 and Lemma G.5

$$\left|\left|\hat{\Sigma}_{\hat{Z}} - \Sigma_Z^*\right|\right|_\infty \leq \Delta_{\Sigma_Z}^{(1)} + \Delta_{\Sigma_Z}^{(2)}$$

and hence

$$\left|\left|\hat{\Sigma}_Y - \Sigma_Y^*\right|\right|_\infty \leq ||S^{-1}||_\infty^2 (\Delta_{\Sigma_Z}^{(1)} + \Delta_{\Sigma_Z}^{(2)})$$

with probability at least

$$1 - p_s^{\Sigma_{Z1}}(x,q) - p_s^{\Sigma_{Z2}}(x)$$

□

Lemma G.3. *Under Assumption 3.1 it holds for arbitrary $1 \leq u, v \leq p$ and positive x that*

$$\mathbb{P}\left\{|Z_{1,uv}| \leq \left(3L^2\sqrt{d}\Lambda\left(\Theta^*\right)\left(\frac{3}{2} + 4x\right) + 1\right) ||\Theta^*||_\infty\right\} \geq 1 - e^{-x}$$

Proof. Re-write the definition (2.1) of an element $Z_{i,uv}$ for arbitrary $1 \leq u, v \leq p$

$$\begin{aligned} Z_{i,uv} &= \Theta_u^* X_i \Theta_v^* X_i - \Theta_{uv}^* \\ &= X_i^T \left[\Theta_u^* (\Theta_v^*)^T\right] X_i - \Theta_{uv}^*. \end{aligned}$$

The first term is clearly a value of a quadratic form defined by the matrix $B = \Theta_u^*(\Theta_v^*)^T$. Note that $rank B = 1$ which implies that it is either positive semi-definite or negative semi-definite. Next we apply Lemma H.4 and obtain for all positive x

$$\mathbb{P}\left\{\left|X_i^T B X_i\right| \geq 3L^2\left(|trB| + 2\sqrt{tr(B^2)x} + 2\left|\Lambda\left(B\right)\right|x\right)\right\} \leq e^{-x}. \tag{G.1}$$

Again, due to the fact that B is a rank-1 matrix

$$trB = \Lambda\left(B\right) = \sqrt{trB^2} \tag{G.2}$$

and by construction of matrix B

$$
\begin{aligned}
|trB| &= \left|\Theta_u^* (\Theta_v^*)^T\right| \\
&\leq ||\Theta_u^*||_1 \, ||\Theta^*||_\infty \\
&\leq \sqrt{d} ||\Theta_u^*||_2 \, ||\Theta^*||_\infty \\
&\leq \sqrt{d} \Lambda \left(\Theta^*\right) ||\Theta^*||_\infty .
\end{aligned} \tag{G.3}
$$

Substitution of (G.2) and (G.3) to (G.1) yields

$$
\mathbb{P}\left\{\left|X_i^T B X_i\right| \geq 3L^2 \sqrt{d} \Lambda \left(\Theta^*\right) ||\Theta^*||_\infty \left(1 + 2\sqrt{x} + 2x\right)\right\} \leq e^{-x}.
$$

And since $\sqrt{x} \leq x + \frac{1}{4}$

$$
\mathbb{P}\left\{\left|X_i^T B X_i\right| \geq 3L^2 \sqrt{d} \Lambda \left(\Theta^*\right) ||\Theta^*||_\infty \left(\frac{3}{2} + 4x\right)\right\} \leq e^{-x}.
$$

Finally, we obtain a bound for $Z_{i,uv}$ as

$$
\mathbb{P}\left\{|Z_{i,uv}| \geq \left(3L^2 \sqrt{d} \Lambda \left(\Theta^*\right) \left(\frac{3}{2} + 4x\right) + 1\right) ||\Theta^*||_\infty \right\} \leq e^{-x}.
$$

$\square$

Correction for all i, u and v establishes the following result

Lemma G.4. *Consider an i.i.d. sample X_i of length n. Under Assumption 3.1 for positive x it holds that*

$$
\mathbb{P}\left\{\forall i \in \{1..n\} : ||Z_i||_\infty \leq \mathcal{Z}_n(x)\right\} \geq 1 - p_{\mathcal{Z}_n}(x)
$$

where

$$
\mathcal{Z}_n(x) := \left(3L^2 \sqrt{d} \Lambda \left(\Theta^*\right) \left(\frac{3}{2} + 4 \log p^2 n + 4x\right) + 1\right) ||\Theta^*||_\infty ,
$$

$$
p_{\mathcal{Z}_n}(x) := e^{-x}.
$$

Lemma G.5. *Under Assumption 3.1 for positive x and q*

$$
\mathbb{P}\left\{\left|\left|\hat{\Sigma}_Z - \Sigma_Z^*\right|\right|_\infty \geq \Delta_{\Sigma_Z}^{(1)}\right\} \leq p_s^{\Sigma_{Z1}}(x, q)
$$

where

$$
\Delta_{\Sigma_Z}^{(1)} := \frac{s}{s-1} \frac{\left(4\mathcal{Z}_s^2(x) + \frac{s-1}{s} ||\Sigma_Z^*||_\infty\right) q}{3s} \left(1 + \sqrt{1 + \frac{9s\sigma_W^2}{q\left(4\mathcal{Z}_s^2(x) + \frac{s-1}{s} ||\Sigma_Z^*||_\infty\right)^2}}\right)
$$

$$
p_s^{\Sigma_{Z1}}(x, q) := p^4 e^{-q} + p_{\mathcal{Z}_s}(x)
$$

Proof. Denote

$$W^{(i)} := (\overline{Z}_i - \mathbb{E}_{\mathcal{I}_s}\left[\overline{Z}_i\right])(\overline{Z}_i - \mathbb{E}_{\mathcal{I}_s}\left[\overline{Z}_i\right])^T - \frac{s-1}{s}\Sigma_Z^*$$

and note that

$$\frac{s-1}{s}\left(\hat{\Sigma}_Z - \Sigma_Z^*\right) = \frac{1}{s}\sum_{i \in \mathcal{I}_s} W^{(i)}.$$

By Lemma G.4 we have $||Z_i||_\infty \le \mathcal{Z}_s(x)$ with probability at least $1 - p_{\mathcal{Z}_s}(x)$ which implies $\left|\left|W^{(i)}\right|\right|_\infty \le 4\mathcal{Z}_s^2(x) + \frac{s-1}{s}\left|\left|\Sigma_Z^*\right|\right|_\infty$. Since $W_{kl}^{(i)}$ are i.i.d., bounded and centered, Bernstein inequality applies here:

$$\mathbb{P}\left\{\mathbb{E}_{\mathcal{I}_s}\left[W_{kl}^{(i)}\right] \ge \frac{\left(4\mathcal{Z}_s^2(x) + \frac{s-1}{s}\left|\left|\Sigma_Z^*\right|\right|_\infty\right)q}{3s}\left(1 + \sqrt{1 + \frac{9s\sigma_W^2}{q\left(4\mathcal{Z}_s^2(x) + \frac{s-1}{s}\left|\left|\Sigma_Z^*\right|\right|_\infty\right)^2}}\right)\right\} \le e^{-q}$$

where σ_W^2 is the smallest variance of components of $W^{(i)}$. Therefore

$$\mathbb{P}\left\{\left|\left|\mathbb{E}_{\mathcal{I}_s}\left[W^{(i)}\right]\right|\right|_\infty \ge \frac{\left(4\mathcal{Z}_s^2(x) + \frac{s-1}{s}\left|\left|\Sigma_Z^*\right|\right|_\infty\right)q}{3s}\left(1 + \sqrt{1 + \frac{9s\sigma_W^2}{q\left(4\mathcal{Z}_s^2(x) + \frac{s-1}{s}\left|\left|\Sigma_Z^*\right|\right|_\infty\right)^2}}\right)\right\} \le p^4 e^{-q}.$$

□

The following lemma bounds the mis-tie between Z_i and $\hat{Z}_i$.

Lemma G.6. *Let Assumption 3.1 holds and let $\hat{\Theta}$ be a symmetric estimator of Θ^* s.t.*

$$\left|\left|\Theta^* - \hat{\Theta}\right|\right|_\infty < r$$

and $\Theta_{ij}^* = 0 \Rightarrow \hat{\Theta}_{ij} = 0$. Then for positive x

$$\mathbb{P}\left\{\forall i \in \mathcal{I}_s : \left|\left|Z_i - \hat{Z}_i\right|\right|_\infty \le \Delta_Z(x)\right\} \ge 1 - p_s^X(x)$$

where

$$\Delta_Z(x) := 2rd^{3/2}x^2\left|\left|\Theta^*\right|\right|_\infty + (rdx)^2$$

$$p_s^X(x) := se^{-x^2/L^2}$$

Proof. Due to sub-Gaussianity,

$$\forall \alpha \in \mathbb{R}^p : \mathbb{P}\left\{\left|\alpha^T X_i\right| \le x\right\} \ge 1 - se^{-x^2/L^2} \tag{G.4}$$

Now consider the mis-tie of arbitrary elements $Z_{i,uv}$ and $\hat{Z}_{i,uv}$:

$$\left|Z_{i,uv} - \hat{Z}_{i,uv}\right| = \left|\Theta^*_u X_i \Theta^*_v X_i + \Theta^*_{uv} - \hat{\Theta}_u X_i \hat{\Theta}_v X_i - \hat{\Theta}_{uv}\right|$$
$$\leq \left|(\Theta^*_u - \hat{\Theta}_u) X_i \Theta^*_v X_i\right| + \left|(\Theta^*_u - \hat{\Theta}_u) X_i \hat{\Theta}_v X_i\right| + r$$

Now note that due to (G.4) and assumptions imposed on Θ^*

$$|\Theta^*_v X_i| \leq \sqrt{d}\, ||\Theta^*||_\infty x$$

$$\left|(\Theta^*_v - \hat{\Theta}_v) X_i\right| \leq rdx$$

$$\left|\hat{\Theta}_v X_i\right| \leq |\Theta^*_v X_i| + \left|(\Theta^*_v - \hat{\Theta}_v) X_i\right| \leq \sqrt{d}\, ||\Theta^*||_\infty x + rdx$$

And hence

$$\left|Z_{i,uv} - \hat{Z}_{i,uv}\right| \leq 2rd^{3/2}x^2\, ||\Theta^*||_\infty + (rdx)^2$$

□

Lemma G.7. *Assume Assumption 3.1 holds. Let $\hat{\Theta}$ be a symmetric estimator of Θ^* s.t.*

$$\left|\left|\Theta^* - \hat{\Theta}\right|\right|_\infty < r$$

and $\Theta^*_{ij} = 0 \Rightarrow \hat{\Theta}_{ij} = 0$. Then for positive x

$$\mathbb{P}\left\{\left|\left|\hat{\Sigma}_Z - \hat{\Sigma}_{\hat{Z}}\right|\right|_\infty \geq \Delta^{(2)}_{\Sigma_Z}\right\} \leq p^{\Sigma_{Z2}}_s(x)$$

where

$$p^{\Sigma_{Z2}}_s(x) := p^X_s(x) + p_{\mathcal{Z}_s}(x)$$

$$\Delta^{(2)}_{\Sigma_Z} = \Delta_Z(x)(2\mathcal{Z}_s(\mathrm{x}) + \Delta_Z(x))$$

Proof. By Lemma G.4 with probability at least $1 - p_{\mathcal{Z}_s}(\mathrm{x})$ we have $||Z_i||_\infty \leq \mathcal{Z}_s(\mathrm{x})$ and in combination with Lemma G.6 we obtain $\left|\left|\overline{\hat{Z}_i}\right|\right|_\infty \leq \mathcal{Z}_s(\mathrm{x}) + \Delta_Z(x)$ with probability at least $1 - p_{\mathcal{Z}_s}(x) - p^X_s(x)$. Now denote

$$\xi_i := \overline{Z_i} - \mathbb{E}_{\mathcal{I}_s}\left[\overline{Z_i}\right] \text{ and } \hat{\xi}_i := \overline{\hat{Z}_i} - \mathbb{E}_{\mathcal{I}_s}\left[\overline{\hat{Z}_i}\right]$$

And deliver the bound

$$\left|\left|\hat{\Sigma}_Z - \hat{\Sigma}_{\hat{Z}}\right|\right|_\infty \leq \mathbb{E}_{\mathcal{I}_s}\left[\xi_i(\xi_i - \hat{\xi}_i)^T + (\xi_i - \hat{\xi}_i)\hat{\xi}_i^T\right]$$
$$\leq \left(\left|\left|\hat{\xi}_i\right|\right|_\infty + ||\xi_i||_\infty\right)\left|\left|\xi_i - \hat{\xi}_i\right|\right|_\infty$$
$$\leq \Delta_Z(x)(2\mathcal{Z}_s(\mathrm{x}) + \Delta_Z(x))$$

□

H Known results

H.1 Gaussian approximation result

In this section we briefly describe the result obtained in [9].

Throughout this section consider an independent sample $x_1, ..., x_n \in \mathbb{R}^p$ of centered random variables. Define their Gaussian counterparts $y_i \sim \mathcal{N}(0, \mathrm{Var}[x_i])$ and denote their scaled sums as

$$S_n^X := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i$$

$$S_n^Y := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} y_i$$

Definition H.1. *We call a set A of the form $A = \{w \in \mathbb{R}^p : a_i \leq w_i \leq b_i \ \forall i \in \{1..p\}\}$ a hyperrectangle. The family of all hyperrectangles is denoted as A^{re}.*

Assumption H.1. *$\exists b > 0$ such that*

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[x_{ij}^2\right] \geq b \text{ for all } j \in 1..p$$

Assumption H.2. *$\exists G_n \geq 1$ such that*

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[|x_{ij}|^{2+k}\right] \leq G_n^{2+k} \text{ for all } j \in 1..p \text{ and } k \in \{1, 2\}$$

$$\mathbb{E}\left[\exp\left(\frac{|x_{ij}|}{G_n}\right)\right] \leq 2 \text{ for all } j \in 1..p \text{ and } i \in 1..n \tag{H.1}$$

Lemma H.1 (Proposition 2.1 by [9])**.** *Let Assumption H.1 hold for some b and Assumption H.2 hold for some G_n. Then*

$$\sup_{A \in A^{re}} \left|\mathbb{P}\left\{S_n^X \in A\right\} - \mathbb{P}\left\{S_n^Y \in A\right\}\right| \leq C \left(\frac{G_n^2 \log^7(pn)}{n}\right)^{1/6}$$

and the constant C depends only on b.

H.2 Anti-concentration result

Lemma H.2 (Nazarov's inequality [31])**.** *Consider a normal p-dimensional vector $X \sim \mathcal{N}(0, \Sigma)$ and let $\forall i : \Sigma_{ii} = 1$. Then for any $y \in \mathbb{R}^p$ and any positive a*

$$\mathbb{P}\{X \leq y + a\} - \mathbb{P}\{X \leq y\} \leq Ca\sqrt{\log p},$$

where C is an independent constant.

H.3 Gaussian comparison result

By the technique given in the proof of Theorem 4.1 by [9] one obtains the following generalization of the result given in [7]

Lemma H.3. *Consider a pair of covariance matrices Σ_1 and Σ_2 of size $p \times p$ such that*

$$||\Sigma_1 - \Sigma_2||_\infty \leq \Delta$$

and $\forall k : C_1 \geq \Sigma_{1,kk} \geq c_1 > 0$. Then for random vectors $\eta \sim \mathcal{N}(0, \Sigma_1)$ and $\zeta \sim \mathcal{N}(0, \Sigma_2)$ it holds that

$$\sup_{A \in A^{re}} |\mathbb{P}\{\eta \in A\} - \mathbb{P}\{\zeta \in A\}| \leq C\Delta^{1/3} \log^{2/3} p,$$

where C is a positive constant which depends only on C_1 and c_1.

H.4 Tail inequality for quadratic forms

The following result is a direct corollary of Theorem 1 in [21]

Lemma H.4. *Consider a positive semi-definite or negative semi-definite matrix B and suppose Assumption 3.1 holds. Then for all $t > 0$*

$$\mathbb{P}\left\{\left|X_1^T B X_1\right| \geq 3L^2 \left(|trB| + 2\sqrt{tr(B^2)t} + 2\left|\Lambda\left(B\right)\right| t\right)\right\} \leq e^{-t}$$

H.5 High-dimensional precision matrix estimation

In order to address the problem of high-dimensional precision matrix estimation one has to assume its sparsity. Below we describe two approaches exploiting this assumption. In both of them we assume that an i.i.d. sample $X_1, ...X_n \in \mathbb{R}^p$ is supplied.

H.5.1 Graphical lasso

In [17] the graphical lasso approach was suggested. An estimate may be obtained as the solution of the following optimization problem over a positive-definite cone S_{++}^p of $p \times p$ dimensional matrices.

$$\hat{\Theta}^{GL} := \arg \min_{\Theta \in S_{++}^p} \left[tr(\Theta\hat{\Sigma}) - \log det\Theta + \lambda \left|\left|\Theta\right|\right|_1\right] \tag{H.2}$$

where $\hat{\Sigma}$ stands for the empirical covariance matrix

$$\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} X_i X_i^T.$$

The theoretical treatment of the approach keeps track on the following Schatten norms: $\kappa_{\Sigma^*} = |||\Sigma^*|||_\infty$ and $\kappa_{\Gamma^*} = |||(\Gamma^*_{SS})^{-1}|||_\infty$. The following result establishes consistency of the estimator in the sense of Definition 3.1.

Lemma H.5 (Theorem 1, [32]). *Consider a distribution satisfying Assumption 3.2 with some $\phi \in (0,1]$, let $\hat{\Theta}$ be a solution of the optimization problem (H.2) with tuning parameter $\lambda_n = \frac{8}{\psi}\delta_n$. Furthermore, impose the following sparsity assumption:*

$$d \leq \frac{1}{6(\delta_n + \lambda_n)\max\{\kappa_{\Gamma^*}\kappa_{\Sigma^*}, \kappa^2_{\Gamma^*}\kappa^3_{\Sigma^*}\}}.$$

Then on the set $\mathcal{T} = \left\{ \left|\left| \hat{\Sigma} - \Sigma^* \right|\right|_\infty < \delta_n \right\}$ the following holds:

$$\left|\left| \hat{\Theta}^{GL} - \Theta^* \right|\right|_\infty \leq r_\lambda := 2\kappa_{\Gamma^*}(\delta_n + \lambda_n)$$

and

$$\Theta^*_{ij} = 0 \Rightarrow \hat{\Theta}_{ij} = 0.$$

A rather similar result is provided in paper [3] for adaptive versions of graphical lasso suggested and studied in [39] [15] [16] [38] .

H.5.2 Node-wise lasso

This section describes the node-wise lasso approach which was suggested in [29].

For each $1 \leq j \leq n$ define a vector

$$\hat{\Gamma}_j := (\hat{\gamma}_{j\,1}, ..., \hat{\gamma}_{j\,j-1}, 1, \hat{\gamma}_{j\,j+1}, ..., \hat{\gamma}_{j\,p})$$

where $\hat{\gamma}_j$ is defined as a solution of the following lasso regression:

$$\hat{\gamma}_j := \arg\max_{\gamma \in \mathbb{R}^{p-1}} \frac{1}{n} \sum_{1 \leq i \leq n} \left(X_{ij} - X^T_{i,-j}\gamma\right)^2 + 2\lambda \left|\left|\gamma\right|\right|_1$$

and

$$\hat{\tau}^2_j := \frac{1}{n} \sum_{1 \leq i \leq n} \left(X_{ij} - X^T_{i,-j}\hat{\gamma}_j\right)^2 + \lambda \left|\left|\gamma\right|\right|_1.$$

Finally the j-th column of the estimator is defined as

$$\hat{\Theta}^{MB}_j := \hat{\Gamma}_j / \hat{\tau}^2_j.$$

Note, that this estimator might not be symmetric, so one cannot use it as an estimator $\hat{\Theta}$ based on the sub-sample $\{X_i\}_{i \in \mathcal{I}_s}$. The paper [23] suggests to construct a de-sparsified estimator $\hat{T}(\hat{\Theta}^{MB})$ where

$$\hat{T}(\hat{\Theta}) := \hat{\Theta} + \hat{\Theta}^T - \hat{\Theta}^T \hat{\Sigma} \hat{\Theta}$$

and threshold elements of $\hat{T}$ obtaining a positive-definite estimate.

Under Assumption 3.1, the sparsity assumption $\frac{d \log p}{n} = o(1)$ and the assumption of the bounded spectrum (Assumption H.3) the paper [23] establishes the root-n consistency of such an estimator (see Definition 3.1).

Assumption H.3.

$$\exists E : \frac{1}{E} \leq \lambda\left(\Theta^*\right) \leq \Lambda\left(\Theta^*\right) \leq E.$$

H.5.3 Bounds for r

While graphical lasso and node-wise estimate are point estimates, de-sparsified estimators have been suggested in order to obtain confidence intervals [24] [23].

The analysis of these estimators relies on the bounds for the residual term r:

$$r := \hat{T} - \left(\Theta^* - \Theta^*(\Sigma^* - \hat{\Sigma})\Theta^*\right)$$

The next two lemmas bound the remainder r for the case of graphical lasso and node-wise estimator.

Lemma H.6 (by [24])**.** *Impose Assumption 3.1, Assumption 3.2 and Assumption H.3. Then under the sparsity assumption*

$$\frac{d \log p}{\sqrt{n}} = o(1) \tag{H.3}$$

it holds that

$$||r||_\infty = O_p\left(\frac{d \log p}{n}\right).$$

A finite sample-size bound for r along with its adaptations for the case of adaptive graphical lasso may be found in [3]

Lemma H.7 (by [23])**.** *Let $\hat{\Theta}$ be yielded by the node-wise procedure with $\lambda_n \asymp \sqrt{\frac{\log p}{n}}$. Then under Assumption 3.1, Assumption H.3 and sparsity assumption* (H.3)

$$||r||_\infty = O_p\left(\frac{d \log p}{n}\right).$$

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