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A temperature-dependent phase-field model for phase separation and damage

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Abstract

In this paper we study a model for phase separation and damage in thermoviscoelastic materials. The main novelty of the paper consists in the fact that, in contrast with previous works in the literature (cf., e.g., [21, 22]), we encompass in the model thermal processes, nonlinearly coupled with the damage, concentration and displacement evolutions. More in particular, we prove the existence of "entropic weak solutions", resorting to a solvability concept first introduced in [10] in the framework of Fourier-Navier-Stokes systems and then recently employed in [9, 38] for the study of PDE systems for phase transition and damage. Our global-in-time existence result is obtained by passing to the limit in a carefully devised time-discretization scheme.

1 Introduction and modeling

In this paper we propose and analyze a model for phase separation and damage in a thermoviscoelastic body, occupying a spatial domain $\Omega \subset \mathbb{R}^d$, where $d \in \{2, 3\}$. We shall consider here a suitable weak formulation of the following PDE system

$$c_t = \operatorname{div}(m(c,z)\nabla\mu), \tag{1.1a}$$

$$\mu = -\Delta_p(c) + \phi'(c) + \frac{1}{2}\big(b(c,z)\mathbb{C}(\varepsilon(\mathbf{u}) - \varepsilon^*(c)) : (\varepsilon(\mathbf{u}) - \varepsilon^*(c))\big)_{,c} - \vartheta + c_t, \tag{1.1b}$$

$$z_t + \partial I_{(-\infty,0]}(z_t) - \Delta_p(z) + \partial I_{[0,\infty)}(z) + \sigma'(z) \ni -\frac{1}{2} b_{,z}(c,z)\mathbb{C}(\varepsilon(\mathbf{u}) - \varepsilon^*(c)) : (\varepsilon(\mathbf{u}) - \varepsilon^*(c)) + \vartheta, \tag{1.1c}$$

$$\vartheta_t + c_t\vartheta + z_t\vartheta + \rho\vartheta \operatorname{div}(\mathbf{u}_t) - \operatorname{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g + |c_t|^2 + |z_t|^2 + a(c,z)\varepsilon(\mathbf{u}_t) : \mathbb{V}\varepsilon(\mathbf{u}_t) + m(c,z)|\nabla\mu|^2, \tag{1.1d}$$

$$\mathbf{u}_{tt} - \operatorname{div}\big(a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(c,z)\mathbb{C}(\varepsilon(\mathbf{u}) - \varepsilon^*(c)) - \rho\vartheta\mathbb{1}\big) = \mathbf{f} \tag{1.1e}$$

posed in $\Omega \times (0,T)$. The system couples

- the viscous Cahn-Hilliard equation (1.1a)–(1.1b) ruling the evolution of the concentration c;
- the damage flow rule (1.1c) for the local proportion of the damage z;
- the internal energy balance (1.1d) for the absolute temperaure ϑ ;
- the momentum balance (1.1e) describing the dynamics for the displacement $\mathbf{u}$.

The symbol $(\cdot)_t$ denotes the partial derivative with respect to time. In the Cahn-Hilliard equation (1.1a) m denotes the mobility of the system and μ the chemical potential, whose expression is given in (1.1b). There, $\Delta_p(\cdot) := \operatorname{div}(|\nabla\cdot|^{p-2}\nabla\cdot)$ denotes the p-Laplacian, ϕ is a mixing potential, b is an elastic coefficient function depending possibly on both c and z, $\mathbb{C}$ represents the elasticity tensor, ε^* a residual strain tensor, and $(\cdot)_{,c}$ the partial derivate with respect to the variable c (with an analogous notation for the other variables). In the damage flow rule (1.1c) $\partial I_{(-\infty,0]} : \mathbb{R} \rightrightarrows \mathbb{R}$ denotes the subdifferential of the indicator function of the set $(-\infty, 0]$, given by

$$\partial I_{(-\infty,0]}(v) = \begin{cases} \{0\} & \text{for } v < 0, \\ [0,+\infty) & \text{for } v = 0 \end{cases}$$

while $\partial I_{[0,\infty)} : \mathbb{R} \rightrightarrows \mathbb{R}$ is the subdifferential of the indicator function of the set $[0,\infty)$, i.e.

$$\partial I_{[0,\infty)}(z) = \begin{cases} (-\infty, 0] & \text{for } z = 0, \\ \{0\} & \text{for } z > 0\,. \end{cases}$$

The presence of these two maximal monotone graphs, enforcing in particular the irreversibility of the damage phenomenon, entails the constraint $z(t) \in [0,1]$ for $t \in (0,T)$ as soon as $z(0) \in [0,1]$. This is physically meaningful because z denotes the damage parameter which is set to be equal to 0 in case the material is completely damaged and it is equal to 1 in the completely safe case, while $z \in (0,1)$ indicates partial damage. The function σ in (1.1c) represents a smooth function, possibly non-convex, of the damage variable z. In the temperature equation (1.1d), ρ denotes a positive thermal expansion coefficient, K the heat conductivity of the system, g a given heat source and a a viscosity coefficient possibly depending on c and z, while $\mathbb{V}$ is the viscosity tensor. Finally, in the momentum balance (1.1e) $\mathbf{f}$ denotes a given volume force.

We will supplement system (1.1) with the initial-boundary conditions

$$c(0) = c^0, \quad z(0) = z^0, \quad \vartheta(0) = \vartheta^0, \quad \mathbf{u}(0) = \mathbf{u}^0, \quad \mathbf{u}_t(0) = \mathbf{v}^0 \quad \text{a.e. in } \Omega, \tag{1.2a}$$

$$\nabla c \cdot \mathbf{n} = 0, \quad m(c,z)\nabla\mu \cdot \mathbf{n} = 0, \quad \nabla z \cdot \mathbf{n} = 0, \quad \mathsf{K}(\vartheta)\nabla\vartheta \cdot \mathbf{n} = h, \quad \mathbf{u} = \mathbf{d} \quad \text{a.e. on } \partial\Omega \times (0,T), \tag{1.2b}$$

where $\mathbf{n}$ indicates the outer unit normal to $\partial\Omega$, while h and $\mathbf{d}$ denote, respectively, a given boundary heat source and displacement.

The PDE system (1.1) may be written in the more compact form

$$c_t = \operatorname{div}(m(c,z)\nabla\mu), \tag{1.3a}$$

$$\mu = -\Delta_p(c) + \phi'(c) + W_{,c}(c, \varepsilon(\mathbf{u}), z) - \vartheta + c_t, \tag{1.3b}$$

$$z_t + \partial I_{(-\infty,0]}(z_t) - \Delta_p(z) + \partial I_{[0,\infty)}(z) + \sigma'(z) \ni -W_{,z}(c, \varepsilon(\mathbf{u}), z) + \vartheta, \tag{1.3c}$$

$$\vartheta_t + c_t\vartheta + z_t\vartheta + \rho\vartheta \operatorname{div}(\mathbf{u}_t) - \operatorname{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g + |c_t|^2 + |z_t|^2 + a(c,z)\varepsilon(\mathbf{u}_t) : \mathbb{V}\varepsilon(\mathbf{u}_t) + m(c,z)|\nabla\mu|^2, \tag{1.3d}$$

$$\mathbf{u}_{tt} - \operatorname{div}\big(a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t) + W_{,\varepsilon}(c, \varepsilon(\mathbf{u}), z) - \rho\vartheta\mathbb{1}\big) = \mathbf{f}, \tag{1.3e}$$

with the following choice of the elastic energy density

$$W(c, \varepsilon, z) = \frac{1}{2} b(c,z)\mathbb{C}(\varepsilon - \varepsilon^*(c)) : (\varepsilon - \varepsilon^*(c)). \tag{1.4}$$

The expression of W is typically quadratic as a function of the strain tensor $\varepsilon(\mathbf{u})$, whereas the coefficient b can depend on c and z. This accounts for possible inhomogeneity of elasticity on the one hand, and is characteristic for damage on the other hand. Indeed, the natural choice would be that b vanishes for $z = 0$, i.e. when the material is completely damaged.

Derivation of the model Let us briefly discuss the thermodynamically consistent derivation of the PDE-system (1.1).

The *state variables* that determine the local thermodynamic state of the material and the *dissipative variables* whose evolution describes the way along which the system tends to dissipate energy are as follows:
State variables

$$\vartheta,\ c,\ \nabla c,\ \varepsilon(\mathbf{u}),\ z,\ \nabla z$$

Dissipation variables

$$\nabla\vartheta,\ c_t,\ \varepsilon(\mathbf{u}_t),\ z_t$$

By classical principles of thermodynamics, the evolution of the system is based on the free energy $\mathscr{F}$ and the pseudopotential of dissipation $\mathscr{P}$, for which we assume the following general form:

$$\mathscr{F}(c, z, \vartheta, \varepsilon(\mathbf{u})) = \int_\Omega F(c, \nabla c, z, \nabla z, \vartheta, \varepsilon(\mathbf{u}))\,\mathrm{d}x \quad \text{and} \quad \mathscr{P}(\nabla\vartheta, c_t, \varepsilon(\mathbf{u}_t), z_t) = \int_\Omega P(\nabla\vartheta, c_t, \varepsilon(\mathbf{u}_t), z_t)\,\mathrm{d}x. \tag{1.5}$$

Our evolutionary system has been obtained by the principle of virtual power and by balance equations of micro-forces, a generalization of the approaches by Fremond [15] and Gurtin [20]. In addition, we also include temperature-dependent effects by means of the balance equation of energy.
The system relies altogether on the balance equations of mass, forces, micro-forces and energy:
Evolution system

Mass balance
$$c_t + \operatorname{div} \boldsymbol{J} = 0, \tag{1.6a}$$
Force balance
$$\mathbf{u}_{tt} - \operatorname{div} \boldsymbol{\sigma} = \mathbf{f}, \tag{1.6b}$$
Micro-force balance
$$B - \operatorname{div} \mathbf{H} = 0, \tag{1.6c}$$
$$\Pi - \operatorname{div} \boldsymbol{\xi} = 0, \tag{1.6d}$$
Energy balance
$$U_t + \operatorname{div} \boldsymbol{q} \; = g + \boldsymbol{\sigma} : \varepsilon(\mathbf{u}_t) + \operatorname{div}(\mathbf{H}) z_t + \mathbf{H} \cdot \nabla z_t + \operatorname{div}(\boldsymbol{\xi}) c_t + \boldsymbol{\xi} \cdot \nabla c_t - \operatorname{div}(\boldsymbol{J})\, \mu - \boldsymbol{J} \cdot \nabla \mu \tag{1.6e}$$

where the internal energy density is given by $U = F - \vartheta \partial_\vartheta F$.
Note that the system is not closed for the variables. Therefore, constitutive laws have to be imposed for the mass flux $\mathbf{J}$, the stress tensor $\boldsymbol{\sigma}$, the internal microforce B for z, the microstress $\mathbf{H}$ for z, the internal microforce Π for c, the microstress $\boldsymbol{\xi}$ for c and the heat flux $\mathbf{q}$.
Constitutive relations
Following Frémond's perspective, we assume that the stress tensor $\boldsymbol{\sigma}$, the microforce B and the microstress $\mathbf{H}$, may be additively decomposed into their non-dissipative and dissipative components, i.e.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{nd} + \boldsymbol{\sigma}^{d} \qquad \text{with} \qquad \boldsymbol{\sigma}^{nd} = \partial_{\varepsilon(\mathbf{u})} F, \qquad \boldsymbol{\sigma}^{d} = \partial_{\varepsilon(\mathbf{u}_t)} P, \tag{1.7}$$
$$B = B^{nd} + B^{d} \qquad \text{with} \qquad B^{nd} \in \partial_z F, \qquad B^{d} \in \partial_{z_t} P, \tag{1.8}$$
$$\mathbf{H} = \mathbf{H}^{nd} + \mathbf{H}^{d} \qquad \text{with} \qquad \mathbf{H}^{nd} = \partial_{\nabla z} F, \qquad \mathbf{H}^{d} = \partial_{\nabla z_t} P = 0. \tag{1.9}$$

In a similar way, by choosing Gurtin's approach, cf. [20] equations (3.19)-(3.23), we get the constitutive relations:

$$\mathbf{J} = -m(c,z)\nabla\mu, \qquad \Pi = \partial_c F + \partial_{c_t} P - \mu, \qquad \boldsymbol{\xi} = \partial_{\nabla c} F. \tag{1.10}$$

The heat source is given by the standard constitutive relation:

$$\boldsymbol{q} = -\frac{\partial P}{\partial \nabla \vartheta}.$$

In the framework of the formulation of the damage and phase separation theory [15, 20], we choose for our system the following free energy and dissipation potential:

$$\begin{aligned} \mathscr{F}(c, z, \vartheta, \varepsilon(\mathbf{u})) := & \int_\Omega \frac{1}{p}|\nabla c|^p + \frac{1}{p}|\nabla z|^p + W(c, \varepsilon(\mathbf{u}), z) + \phi(c) + \sigma(z) + I_{[0,+\infty)}(z)\,\mathrm{d}x \\ & + \int_\Omega -\vartheta \log \vartheta - \vartheta\big(c + z + \rho \operatorname{div}(\mathbf{u})\big)\,\mathrm{d}x, \end{aligned} \tag{1.11}$$
$$\mathscr{P}(\nabla\vartheta, c_t, \varepsilon(\mathbf{u}_t), z_t) := \int_\Omega \frac{1}{2}\mathsf{K}(\vartheta)|\nabla\vartheta|^2 + \frac{1}{2}z_t^2 + \frac{1}{2}c_t^2 + \frac{1}{2}a(c,z)\varepsilon(\mathbf{u}_t) : \mathbb{V}\varepsilon(\mathbf{u}_t) + I_{(-\infty,0]}(z_t)\,\mathrm{d}x\,. \tag{1.12}$$

The first two gradient terms in (1.11) represent the nonlocal interactions in phase separation and damage processes. The analytical study of gradient theories goes back to [28, 35], where phase separation processes were investigated. A typical choice for W has been introduced in (1.4). The functions ϕ and σ represent the mixing potentials. The term $\vartheta(c + z + \rho \operatorname{div} \mathbf{u})$ models the phase and thermal expansion processes in the system. It may also be regarded as linear approximation near to the thermodynamical equilibrium. In the following lines we will get further insight into the choices of these functionals. Exploiting (1.6)-(1.12) results in system (1.1), for which the Clausius-Duhem inequality is satisfied.

As discussed, our approach is based on a gradient theory of phase separation and damage processes due to [15, 20, 6]. For a non-gradient approach to damage models we refer to [17, 18, 1]. There, the damage variable z takes only two distinct values, i.e. $\{0, 1\}$, in contrast to phase-field models where intermediate values $z \in [0, 1]$ are also allowed. In addition, the mechanical properties of damage phenomena are described in [17, 18, 1] differently. They choose a z-mixture of a linearly elastic strong and weak material with two different elasticity tensors. We also refer to [14], where a non-gradient damage model was studied by means of Young measures.

Mathematical difficulties. The main mathematical difficulties attached with the proof of existence of solutions to such a PDE system are related to the presence of the quadratic dissipative terms on the right-hand side in the internal energy balance (1.3d), as well as the doubly nonlinear and possibly nonsmooth carachter of the damage relation (1.3c). This is the reason why we shall resort here to a weak solution notion for (1.3) coupled with (1.2). In this solution concept, partially drawn from [38], the Cahn-Hilliard system (1.3a–1.3b) and the balance of forces (1.3e) (read a.e. in $\Omega \times (0, T)$) are coupled with an *"entropic" formulation* of the heat equation (1.3d) and a weak formulation of the damage flow rule (1.3c) taken from [21, 22]. Let us briefly illustrate them.

The "entropic" formulation of the heat equation. It consists of a *weak entropy inequality*

$$
\begin{aligned}
&\int_s^t \int_\Omega (\log(\vartheta) + c + z)\varphi_t \,\mathrm{d}x\,\mathrm{d}r - \rho \int_s^t \int_\Omega \mathrm{div}(\mathbf{u}_t)\varphi \,\mathrm{d}x\,\mathrm{d}r - \int_s^t \int_\Omega \mathsf{K}(\vartheta)\nabla \log(\vartheta) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}r \\
&\leq \int_\Omega (\log(\vartheta(t)) + c(t) + z(t))\varphi(t)\,\mathrm{d}x - \int_\Omega (\log(\vartheta(s)) + c(s) + z(s))\varphi(s)\,\mathrm{d}x \\
&\qquad\qquad\qquad\qquad - \int_s^t \int_\Omega \mathsf{K}(\vartheta)|\nabla \log(\vartheta)|^2 \varphi \,\mathrm{d}x\,\mathrm{d}r \\
&\quad - \int_s^t \int_\Omega \left(g + |c_t|^2 + |z_t|^2 + a(c,z)\varepsilon(\mathbf{u}_t) : \mathbb{V}\varepsilon(\mathbf{u}_t) + m(c,z)|\nabla\mu|^2\right)\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r - \int_s^t \int_{\partial\Omega} h\frac{\varphi}{\vartheta}\,\mathrm{d}S\,\mathrm{d}r
\end{aligned}
\tag{1.13}
$$

required to be valid for almost all $0 \leq s \leq t \leq T$ and for $s = 0$, and for all sufficiently regular and positive test functions φ, coupled with a *total energy inequality*:

$$
\begin{aligned}
\mathscr{E}(c(t), z(t), \vartheta(t), \mathbf{u}(t), \mathbf{u}_t(t)) \leq{}& \mathscr{E}(c(s), z(s), \vartheta(s), \mathbf{u}(s), \mathbf{u}_t(s)) \\
&+ \int_s^t \int_\Omega g \,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_{\partial\Omega} h \,\mathrm{d}S\,\mathrm{d}r \\
&+ \int_s^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_t \,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_{\partial\Omega} (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{d}_t \,\mathrm{d}S\,\mathrm{d}r,
\end{aligned}
\tag{1.14}
$$

valid for almost all $0 \leq s \leq t \leq T$, and for $s = 0$, where the total energy $\mathscr{E}$ is the sum of the internal energy and the kinetic energy, i.e.

$$
\mathscr{E}(c, z, \vartheta, \mathbf{u}, \mathbf{u}_t) := \mathscr{U}(c, z, \vartheta, \varepsilon(\mathbf{u})) + \int_\Omega \frac{1}{2}|\mathbf{u}_t|^2 \,\mathrm{d}x, \tag{1.15}
$$

being the internal energy $\mathscr{U}$ specified by (cf. also (1.11)):

$$
\begin{aligned}
\mathscr{U}(c, z, \vartheta, \varepsilon(\mathbf{u})) &:= \mathscr{F}(c, z, \vartheta, \varepsilon(\mathbf{u})) - \vartheta \cdot \partial_\vartheta \mathscr{F}(c, z, \vartheta, \varepsilon(\mathbf{u})) \\
&= \int_\Omega \frac{1}{p}|\nabla c|^p + \frac{1}{p}|\nabla z|^p + W(c, \varepsilon(\mathbf{u}), z) + \phi(c) + \sigma(z) + I_{[0,+\infty)}(z) + \vartheta \,\mathrm{d}x.
\end{aligned}
\tag{1.16}
$$

From an analytical viewpoint, observe that the entropy inequality (1.13) has the advantage that all the quadratic terms on the right-hand side of (1.3d) are multiplied by a negative test function, which, together with the fact that we are only requiring an *inequality* and not an equation, will allow us to apply upper semicontinuity arguments for the limit passage in the time-discrete approximation of system (1.3) set up in Section 4.

The *"entropic" formulation*, first introduced in [10] in the framework of heat conduction in fluids, and then applied to a phase separation model derived according to FRÉMOND's approach [15] in [9], has been successively used also in models for different kinds of special materials. Besides the aforementioned work on damage [38], we may mention the papers [11], [12], and [13] on liquid crystals, and more recently the analysis of a model for the evolution of non-isothermal binary incompressible immiscible fluids (cf. [7]).

Let us also mention that other approaches to treat PDE systems with an L^1-right-hand side are available in the literature: among others, we refer to [45], resorting to the notion of *renormalized solution*, and [40] where the coupling of rate-independent and thermal processes is considered. The heat equation therein, with an L^1-right-hand side, is tackled by means of Boccardo-Gallouët type techniques.

The weak formulation of the damage flow rule. Following the lines of [21, 22], we replace the damage inclusion (1.3c) by the *damage energy-dissipation inequality*

$$\begin{aligned}\int_s^t\int_\Omega |z_t|^2\,\mathrm{d}x\,\mathrm{d}r + \int_\Omega\left(\frac{1}{p}|\nabla z(t)|^p+\sigma(z(t))\right)\mathrm{d}x\\ \leq \int_\Omega\left(\frac{1}{p}|\nabla z(s)|^p+\sigma(z(s))\right)\mathrm{d}x + \int_s^t\int_\Omega z_t\left(-W_{,z}(c,\varepsilon(\mathbf{u}),z)+\vartheta\right)\mathrm{d}x\,\mathrm{d}r,\end{aligned}\tag{1.17}$$

imposed for all $t\in(0,T]$, for $s=0$, and for almost all $0<s\le t$ and the *one-sided variational inequality for the damage process*

$$\int_\Omega\left(z_t\zeta+|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta+\xi\zeta+\sigma'(z(t))\zeta+W_{,z}(c,\varepsilon(\mathbf{u}),z)\zeta-\vartheta\zeta\right)\mathrm{d}x\ge 0\quad\text{a.e. in }(0,T),\tag{1.18}$$

required to be valid for all sufficiently regular test functions ζ, where $\xi\in\partial I_{[0,+\infty)}(z)$ a.e. in Q, and $z(x,t)\in[0,1]$, $z_t(x,t)\in(-\infty,0]$ a.e. in Q.

Entropic weak solutions. In what follows, we shall refer to the formulation consisting of (1.3a–1.3b), (1.3e), (1.13), (1.14), (1.17), (1.18), supplemented with the initial and boundary conditions (1.2), as the *entropic weak formulation* of (the initial-boundary value problem for) system (1.3). Let us point out that, in case of regular solutions, it can be seen that the *"entropic" formulation* is equivalent to the internal energy balance (1.3d) (cf. Remark 2.5 as well as [38, Rmk. 2.6] for more details). Likewise, the *weak formulation of the damage flow rule* would give rise to the damage inclusion (1.3c) for sufficiently regular solutions. In this sense, we can observe that our formulation is consistent with the PDE system (1.1).

Our results and related literature. In this paper we prove existence of global-in-time entropic weak solutions under the following assumptions on the data:

- the mixing potential ϕ is the sum of a convex possibly non-smooth part and a regular λ-concave part (cf. Hyp. (I)). Hence, both the sum of a logarithmic potential (e.g. $(1+c)\log(1+c)+(1-c)\log(1-c)$) or an indicator function (e.g. $I_{[-1,1]}(c)$) and a smooth concave perturbation (e.g. $-c^2$) are allowed as choices of ϕ, cf. Remark 2.1 ahead);
- the mobility m is a smooth function bounded from below by a positive constant;
- the function σ is regular;
- the heat conductivity K is a continuous function growing like a power of ϑ. This choice is motivated by mathematics, indeed it is needed in order to get suitable estimates on the temperature ϑ, but it is also justified by the physical behavior of certain materials (cf. [25, 44]);

- the function a is bounded away from zero and bounded from above as well as its partial derivatives with respect to both c and z. These assumptions are mainly made in order to prevent the full degeneracy of the momentum balance (1.3e) and in order to obtain from it the sufficient regularity on $\mathbf{u}$ needed to handle the nonlinear coupling with the temperature and damage relations. Instead, the coefficient b in the elastic energy density (1.4) can possibly vanish, and both b and the eigenstrain ε^* are required to be sufficiently regular functions;
- the thermal expansion coefficient ρ is assumed to be a positive constant. For more general behavior of ρ possibly depending on the damage parameter z the reader can refer to [24], while the fact that ρ is chosen to be independent of ϑ is justified by the fact that we assume to have a constant specific heat c_v (equal to 1 in (1.3d) for simplicity): indeed they are related (by thermodynamical laws) by the relation $\partial_\vartheta c_v = \vartheta \partial_\vartheta \rho$;
- the initial data are taken in the energy space, except for the initial displacement and velocity which, jointly with the boundary Dirichlet datum for $\mathbf{u}$, must enjoy the regularity needed in order to perform elliptic regularity estimates on the momentum balance (1.3e).

Furthermore, we consider a *gradient theory* for damage. From the physical viewpoint, the term $\frac{1}{p}|\nabla z|^p$ contributing to (1.11) models nonlocality of the damage process, since the gradient of z accounts for the influence of damage at a material point, undamaged in its neighborhood. The mathematical advantages attached to the presence of this term, and of the analogous contribution $\frac{1}{p}|\nabla c|^p$, are rather obvious. Let us mention that, in fact, throughout the paper we shall assume that the exponent p in (1.3b) and (1.3c) fulfills $p > d$. This assumption is mainly mathematically motivated by the fact that it ensures that c and z are estimated in $W^{1,p}(\Omega) \subset \mathrm{C}^0(\overline{\Omega})$, and has been adopted for the analysis of other damage models (cf., e.g., [5, 31, 32, 27]).

Regarding the previous results on this type of problems in the literature, let us point out that, by now, several contributions on systems coupling rate-dependent damage and thermal processes (cf., e.g. [4, 37, 38, 24]) as well as rate-dependent damage and phase separation (cf., e.g., [21, 22]) are available in the literature. Up to our knowledge, this is one of the first contributions on the analysis of a model encompassing all of the three processes (temperature evolution, damage, phase separation) in a thermoviscoelastic material. Recently, a thermodynamically consistent, quite general model describing diffusion of a solute or a fluid in a solid undergoing possible phase transformations and rate-independent damage, beside possible visco-inelastic processes, has been studied in [43]. Let us highlight the main difference to our own model: the evolution of the damage process is therein considered *rate-independent*, which clearly affects the weak solution concept adopted in [43]. In particular, we may point out that dealing with a *rate-dependent* flow rule for the damage variable is one of the challenges of our own analysis, due to the presence of the quadratic nonlinearity in $\varepsilon(\mathbf{u})$ on the right-hand side of (1.3c).

Let us conclude by mentioning some open problems which are currently under study, such as uniqueness of solutions, at least for the isothermal case, and the global-in-time existence analysis for the complete damage (degenerating) case, in which the coefficient a in the momentum balance (1.3e) is allowed to vanish in some parts of the domain (cf. [37] for the case without phase separation and [23] for the isothermal case).

Plan of the paper. In Section 2, after listing all the assumptions on the data of the problem, we rigorously state the *entropic weak* formulation of the problem and give the main result of the paper, i.e. Theorem 2.6 ensuring the global-in-time existence of entropic weak solutions.

In Section 3 we (formally) derive all the a priori estimates on system (1.3) which will be at the core of our existence analysis.

As previously mentioned, Thm. 2.6 is proved by passing to the limit in a carefully devised time-discretization scheme, also coupled with regularization procedures, which could also be of interest in view of possible numerical simulations on the model. To its analysis, the whole Section 4 is devoted. While postponing more detailed comments on its features, let us mention here that our time-discrete scheme will be *thermodynamically consistent*, in that it will ensure the validity of the discrete versions of the entropy and energy inequalities (1.13) and (1.14). This will play a crucial role in the limit passage, developed in Section 5, where the proof of Theorem 2.6 will be carried out.

2 Weak formulation and statement of the main result

In this section, first of all we recall some notation and preliminary results that will be used throughout the paper. Next, we list all of the conditions on the nonlinearities featuring in system (1.3), as well as on the data f, g, h and on the initial data. We are thus in the position to give our notion of weak solution to the initial-boundary value problem for system (1.3) and state our main existence result, Theorem 2.6.

2.1 Preliminaries

In what follows, we will suppose that

$$\Omega \subset \mathbb{R}^d, \quad d \in \{2, 3\}, \quad \text{is a bounded domain with } \mathrm{C}^2\text{-boundary } \partial\Omega. \tag{2.1}$$

This smoothness requirement will allow us to apply regularity results for elliptic systems, at the basis of a regularity estimate that we shall perform on the momentum equation and that will have a key role in the proof of our existence result for system (1.3).

Notation for function spaces, norms, operators Given a Banach space X, we will use the symbol $\langle \cdot, \cdot \rangle_X$ for the duality pairing between X' and X. Moreover, we shall denote by $\mathrm{BV}([0,T]; X)$ (by $\mathrm{C}^0_{\mathrm{weak}}([0,T]; X)$, respectively), the space of functions from $[0,T]$ with values in X that are defined at every $t \in [0,T]$ and have bounded variation on $[0,T]$ (and are *weakly* continuous on $[0,T]$, resp.).

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $d \in \{2,3\}$. We set $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$. We identify both $L^2(\Omega)$ and $L^2(\Omega; \mathbb{R}^d)$ with their dual spaces, and denote by $(\cdot, \cdot)$ the scalar product in $\mathbb{R}^d$, by $(\cdot, \cdot)_{L^2(\Omega)}$ both the scalar product in $L^2(\Omega)$ and in $L^2(\Omega; \mathbb{R}^d)$, and by $H^1_0(\Omega; \mathbb{R}^d)$, $H^2_{\mathrm{Dir}}(\Omega; \mathbb{R}^d)$ and $H^2_N(\Omega)$ the spaces

$$\begin{aligned}
&H^1_0(\Omega; \mathbb{R}^d) := \left\{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \,:\, \mathbf{v} = 0 \ \text{ on } \partial\Omega \right\}, \text{ endowed with the norm } \|\mathbf{v}\|^2_{H^1_0(\Omega;\mathbb{R}^d)} := \int_\Omega \varepsilon(\mathbf{v}) : \varepsilon(\mathbf{v}) \,\mathrm{d}x, \\
&H^2_{\mathrm{Dir}}(\Omega; \mathbb{R}^d) := H^1_0(\Omega; \mathbb{R}^d) \cap H^2(\Omega; \mathbb{R}^d) = \left\{ \mathbf{v} \in H^2(\Omega; \mathbb{R}^d) \,:\, \mathbf{v} = 0 \ \text{ on } \partial\Omega \right\}, \\
&H^2_N(\Omega) := \left\{ v \in H^2(\Omega) \,:\, \partial_n v = 0 \text{ on } \partial\Omega \right\}.
\end{aligned}$$

Note that by Korn's inequality $\|\cdot\|_{H^1_0(\Omega;\mathbb{R}^d)}$ is a norm equivalent to the standard one on $H^1(\Omega; \mathbb{R}^d)$. We denote by $\mathcal{D}(\overline{Q})$ the space of the C^∞-functions with compact support on Q. For $q \geq 1$ we will adopt the notation

$$W^{1,q}_+(\Omega) := \left\{ \zeta \in W^{1,q}(\Omega) \,:\, \zeta(x) \geq 0 \quad \text{for a.a. } x \in \Omega \right\}, \qquad \text{and analogously for } W^{1,q}_-(\Omega). \tag{2.2}$$

Finally, throughout the paper we shall denote by the symbols c, c', C, C' various positive constants depending only on known quantities. Furthermore, the symbols I_i, $i = 0, 1, \ldots$, will be used as place-holders for several integral terms popping in the various estimates: we warn the reader that we will not be self-consistent with the numbering, so that, for instance, the symbol I_1 will occur several times with different meanings.

Preliminaries of mathematical elasticity We postpone to Sec. 2.2 the precise statement of all assumptions on the *elastic* contribution $W_{,\varepsilon}(c, \varepsilon(\mathbf{u}), z)$ to the elliptic operator in (1.3e). Concerning the stiffness tensor $\mathbb{C}$ (we will take the viscosity tensor to be a multiple of $\mathbb{C}$, cf. (2.20) ahead), we suppose that

$$\mathbb{C} = (c_{ijkh}) \in \mathrm{C}^1(\Omega; \mathbb{R}^{d \times d \times d \times d}) \tag{2.3}$$

with coefficients satisfying the classical symmetry and ellipticity conditions (with the usual summation convention)

$$c_{ijkh} = c_{jikh} = c_{khij}, \qquad \exists\, \nu_0 > 0 \,:\quad c_{ijkh}\xi_{ij}\xi_{kh} \geq \nu_0 \xi_{ij}\xi_{ij} \quad \forall\, \xi_{ij} : \xi_{ij} = \xi_{ji}. \tag{2.4}$$

Observe that with (2.4), we also encompass in our analysis the case of an anisotropic and inhomogeneous material. Thanks to (2.4) and to the C^2-regularity of Ω we have the following elliptic regularity result (cf. e.g. [36, Lemma 3.2, p. 260]) or [29, Chap. 6, p. 318]):

$$\exists\, c_1,\, c_2>0 \quad \forall\, \mathbf{u}\in H^2_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)\ : \qquad c_1\|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)} \le \|\operatorname{div}(\mathbb{C}\varepsilon(\mathbf{u}))\|_{L^2(\Omega;\mathbb{R}^d)} \le c_2\|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)}\,. \tag{2.5}$$

Under the assumption that $\mathbf{u}$ has prescribed boundary values $\mathbf{d}\in H^2(\Omega;\mathbb{R}^d)$, i.e. $\mathbf{u}=\mathbf{d}$ a.e. on $\partial\Omega$, we obtain by applying (2.5) to $\mathbf{u}-\mathbf{d}$

$$\begin{aligned} &\exists\, \widetilde{c}_1,\, \widetilde{c}_2>0 \quad \forall\, \mathbf{u}\in H^2(\Omega;\mathbb{R}^d) \text{ with } \mathbf{u}=\mathbf{d} \text{ a.e. on } \partial\Omega\ : \\ &\qquad \widetilde{c}_1\|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)} \le \|\operatorname{div}(\mathbb{C}\varepsilon(\mathbf{u}))\|_{L^2(\Omega;\mathbb{R}^d)} + \|\mathbf{d}\|_{H^2(\Omega;\mathbb{R}^d)} \le \widetilde{c}_2\big(\|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)} + \|\mathbf{d}\|_{H^2(\Omega;\mathbb{R}^d)}\big)\,. \end{aligned} \tag{2.6}$$

Useful inequalities For later reference, we recall here the Gagliardo-Nirenberg inequality in a particular case: for all $r,\, q\in[1,+\infty]$, and for all $v\in L^q(\Omega)$ such that $\nabla v\in L^r(\Omega)$, there holds

$$\|v\|_{L^s(\Omega)} \le C_{\mathrm{GN}}\|v\|^{\theta}_{W^{1,r}(\Omega)}\|v\|^{1-\theta}_{L^q(\Omega)} \qquad \text{with } \frac{1}{s}=\theta\left(\frac{1}{r}-\frac{1}{d}\right)+(1-\theta)\frac{1}{q},\quad 0\le\theta\le 1, \tag{2.7}$$

the positive constant C_{GN} depending only on d, r, q, θ.

We will also make use of the following interpolation inequality from [30, Thm. 16.4, p. 102]

$$\forall\varrho>0 \quad \exists\, C_\varrho>0 \quad \forall u\in X: \qquad \|u\|_Y \le \varrho\|u\|_X + C_\varrho\|u\|_Z, \tag{2.8}$$

where $X\subseteq Y\subseteq Z$ are Banach spaces with compact embedding $X\Subset Y$.

Combining this with the compact embedding

$$H^2_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)\Subset W^{1,d^\star-\eta}(\Omega;\mathbb{R}^d), \quad \text{with } d^\star=\begin{cases}\infty & \text{if } d=2,\\ 6 & \text{if } d=3,\end{cases} \quad \text{for all } \eta>0, \tag{2.9}$$

(where for $d=2$ we mean that $H^2_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)\Subset W^{1,q}(\Omega;\mathbb{R}^d)$ for all $1\le q<\infty$), we have

$$\forall\,\varrho>0\ \ \exists\, C_\varrho>0\ \ \forall\,\eta>0\ \ \forall\,\mathbf{u}\in H^2_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)\ :\ \ \|\varepsilon(\mathbf{u})\|_{L^{d^\star-\eta}(\Omega;\mathbb{R}^{d\times d})} \le \varrho\|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)} + C_\varrho\|\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^d)}. \tag{2.10}$$

We also obtain by interpolation

$$\forall\,\varrho>0\ \ \exists\, C_\varrho>0\ \ \forall\,\eta>0\ \ \forall\,\mathbf{u}\in H^1(\Omega;\mathbb{R}^d)\ :\ \ \|u\|_{L^{d^\star-\eta}(\Omega;\mathbb{R}^d)} \le \varrho\|\mathbf{u}\|_{H^1(\Omega;\mathbb{R}^d)} + C_\varrho\|\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^d)}. \tag{2.11}$$

We will also resort to the following *nonlinear* Poincaré-type inequality (proved in, e.g., [19, Lemma 2.2]), with $\mathfrak{m}(w)$ the mean value of w:

$$\forall\, q>0 \quad \exists\, C_q>0 \quad \forall\, w\in H^1(\Omega)\ : \qquad \||w|^q w\|_{H^1(\Omega)} \le C_q(\|\nabla(|w|^q w)\|_{L^2(\Omega)} + |\mathfrak{m}(w)|^{q+1})\,. \tag{2.12}$$

2.2 Assumptions

We now collect all the conditions on the functions ϕ, m, σ, K, a, W, $\mathbb{V}$ in system (1.3).

Hypothesis (I). Concerning the potential ϕ for the concentration variable c, we require that

$$\begin{gathered} \phi=\widehat{\beta}+\gamma \quad \text{with } \widehat{\beta}:\mathbb{R}\to[0,+\infty] \text{ proper, convex, and l.s.c., with } \widehat{\beta}(0)=0, \text{ and} \\ \gamma\in\mathrm{C}^1(\mathbb{R}), \qquad \gamma\ \lambda_\gamma\text{-concave for some } \lambda_\gamma\ge 0, \text{ and} \\ \text{such that } \exists\, C_\phi\in\mathbb{R}\ \ \forall\, c\in\operatorname{dom}(\phi):\ \ \phi(c)\ge C_\phi\,. \end{gathered} \tag{2.13}$$

In what follows, we will denote the convex-analysis subdifferential $\partial\widehat{\beta}:\mathbb{R}\rightrightarrows\mathbb{R}$ by β, and by $\operatorname{dom}(\beta)$ the set $\{c\in\mathbb{R}\ :\ \beta(c)\neq\emptyset\}$. From $0\in\operatorname{Argmin}_{r\in\mathbb{R}}\widehat{\beta}(r)$, it follows that $0\in\beta(0)$.

Remark 2.1 (Consequences of Hypothesis (I)). For later use we observe that, since the map $c \mapsto \gamma(c) - \lambda_\gamma \frac{c^2}{2}$ is concave, we have the following convex-concave decomposition for ϕ:

$$\phi(c) = \underbrace{\widehat{\beta}(c) + \lambda_\gamma \frac{c^2}{2}}_{\text{convex}} + \underbrace{\gamma(c) - \lambda_\gamma \frac{c^2}{2}}_{\text{concave}} . \tag{2.14}$$

Example 2.2. Admissible choices for $\widehat{\beta}$ are both the physically meaningful potentials $\widehat{\beta}(c) = (1+c)\log(1+c) + (1-c)\log(1-c)$ and $\widehat{\beta}(c) = I_{[-1,1]}(c)$, while γ can be a general smooth concave perturbation, e.g. $\gamma(c) = -\lambda_\gamma c^2$.

Hypothesis (II). As for the nonlinear functions m and σ, we suppose that

$$m \in \mathrm{C}^1(\mathbb{R}\times\mathbb{R}) \ \text{ and } \ \exists\, m_0 > 0\ \forall\,(c,z) \in \mathbb{R}\times\mathbb{R} : \ m(c,z) \geq m_0, \tag{2.15}$$

$$\sigma \in \mathrm{C}^2(\mathbb{R}). \tag{2.16}$$

Hypothesis (III) The heat conductivity function

$$\begin{aligned} &\mathsf{K} : [0,+\infty) \to (0,+\infty) \ \text{ is continuous and} \\ &\exists\, c_0,\, c_1 > 0 \quad \exists \kappa > 1 \quad \forall \vartheta \in [0,+\infty) : \quad c_0(1+\vartheta^\kappa) \leq \mathsf{K}(\vartheta) \leq c_1(1+\vartheta^\kappa)\,. \end{aligned} \tag{2.17}$$

We will denote by $\widehat{\mathsf{K}}$ the primitive $\widehat{\mathsf{K}}(x) := \int_0^x \mathsf{K}(r)\,\mathrm{d}r$ of K.

Hypothesis (IV). We require

$$\begin{aligned} a \in \mathrm{C}^1(\mathbb{R}\times\mathbb{R}) \quad \text{and} \quad &\exists\, a_0, a_1 > 0 \quad \forall c, z \in \mathbb{R} : \qquad a_0 \leq a(c,z) \leq a_1, \\ &\exists\, a_2 > 0 \quad \forall c, z \in \mathbb{R} : \qquad |a_{,c}(c,z)| + |a_{,z}(c,z)| \leq a_2. \end{aligned} \tag{2.18}$$

Hypothesis (V). We suppose that

$$W(x,c,\varepsilon,z) = \frac{1}{2} b(c,z)\mathbb{C}(x)(\varepsilon - \varepsilon^*(c)) : (\varepsilon - \varepsilon^*(c)), \tag{2.19}$$

where we recall that $b(c,z)$ models the influence of the concentration and damage on the stiffness tensor $\mathbb{C}$ and ε^* models the eigenstrain. We assume

$$\varepsilon^* \in \mathrm{C}^2(\mathbb{R}), \qquad b \in \mathrm{C}^2(\mathbb{R}\times\mathbb{R}) \quad \text{and} \quad \exists\, b_0 > 0 \quad \forall c, z \in \mathbb{R} : \quad 0 \leq b(c,z) \leq b_0, \quad \mathbb{V} = \omega\mathbb{C}, \quad \omega > 0. \tag{2.20}$$

The tensor function $\mathbb{C}$ should satisfy conditions (2.3) and (2.4). Let us mention in advance that the last condition on $\mathbb{V}$ will play a crucial role in the proof of $H^2(\Omega;\mathbb{R}^d)$-regularity for the discrete displacements, cf. Lemma 4.16 ahead.

For notational convenience, from now on we shall neglect the x-dependence of W. For later reference, we observe that

$$\begin{aligned}
&W_{,c}(c,\varepsilon,z) = \frac{1}{2} b_{,c}(c,z)\mathbb{C}(\varepsilon - \varepsilon^*(c)) : (\varepsilon - \varepsilon^*(c)) - b(c,z)(\varepsilon^*)'(c)\mathbb{C} : (\varepsilon - \varepsilon^*(c)), \\
&W_{,cc}(c,\varepsilon,z) = \frac{1}{2} b_{,cc}(c,z)\mathbb{C}(\varepsilon - \varepsilon^*(c)) : (\varepsilon - \varepsilon^*(c)) - b_{,c}(c,z)(\varepsilon^*)'(c)\mathbb{C} : (\varepsilon - \varepsilon^*(c)) \\
&\qquad\qquad - b(c,z)(\varepsilon^*)''(c)\mathbb{C} : (\varepsilon - \varepsilon^*(c)) + b(c,z)(\varepsilon^*)'(c)\mathbb{C} : (\varepsilon^*)'(c), \\
&W_{,z}(c,\varepsilon,z) = \frac{1}{2} b_{,z}(c,z)\mathbb{C}(\varepsilon - \varepsilon^*(c)) : (\varepsilon - \varepsilon^*(c)), \\
&W_{,zz}(c,\varepsilon,z) = \frac{1}{2} b_{,zz}(c,z)\mathbb{C}(\varepsilon - \varepsilon^*(c)) : (\varepsilon - \varepsilon^*(c)), \\
&W_{,\varepsilon}(c,\varepsilon,z) = b(c,z)\mathbb{C}(\varepsilon - \varepsilon^*(c)), \\
&W_{,\varepsilon c}(c,\varepsilon,z) = b_{,c}(c,z)\mathbb{C}(\varepsilon - \varepsilon^*(c)) - b(c,z)(\varepsilon^*)'(c)\mathbb{C}, \\
&W_{,\varepsilon z}(c,\varepsilon,z) = b_{,z}(c,z)\mathbb{C}(\varepsilon - \varepsilon^*(c))\,.
\end{aligned} \tag{2.21}$$

Finally, we will suppose throughout the work that $p > d$ and that the data $\mathbf{d}$, $\mathbf{f}$, g, and h comply with

$$\mathbf{d} \in H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^1(\Omega;\mathbb{R}^d)), \tag{2.22a}$$
$$\mathbf{f} \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{2.22b}$$
$$g \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)'), \quad g \geq 0 \quad \text{a.e. in } Q\,, \tag{2.22c}$$
$$h \in L^1(0,T;L^2(\partial\Omega)), \quad h \geq 0 \quad \text{a.e. in } \Sigma\,, \tag{2.22d}$$

and that the initial data fulfill

$$c^0 \in W^{1,p}(\Omega), \quad \widehat{\beta}(c^0) \in L^1(\Omega), \quad \mathfrak{m}_0 := \mathfrak{m}(c^0) \text{ belongs to the interior of } \mathrm{dom}(\beta), \tag{2.23a}$$
$$z^0 \in W^{1,p}(\Omega), \quad 0 \leq z^0 \leq 1 \text{ in } \Omega, \tag{2.23b}$$
$$\vartheta^0 \in L^1(\Omega), \quad \log\vartheta^0 \in L^1(\Omega), \quad \exists\, \vartheta_* > 0 \,:\, \vartheta^0 \geq \vartheta_* > 0 \text{ a.e. in } \Omega, \tag{2.23c}$$
$$\mathbf{u}^0 \in H^2(\Omega;\mathbb{R}^d) \text{ with } \mathbf{u}^0 = \mathbf{d}(0) \ \text{ a.e. on } \partial\Omega, \tag{2.23d}$$
$$\mathbf{v}^0 \in H^1(\Omega;\mathbb{R}^d). \tag{2.23e}$$

Remark 2.3. Let us point out explicitly that, if we choose ϕ as the logarithmic potential from Example 2.2, or with ϕ given by the sum $I_{[0,1]} + \gamma$, we enforce the (physically meaningful) property that $c \in (0,1)$ ($c \in [0,1]$, respectively) in Ω. From (2.23a) we read that this constraint has to be enforced on the initial datum c^0 as well, in the same was as we require $z^0 \in [0,1]$ with (2.23b).

The latter condition, combined with the information that $z(\cdot, x)$ is nonincreasing for almost all $x \in \Omega$ thanks to the term $\partial I_{(-\infty,0]}(z_t)$ in (1.3c), will yield that the solution component z is in $[0,1]$ a.e. in Q. This property, albeit not needed for the analysis of (1.3), is in accordance with the physical meaning of the damage parameter.

Clearly, in the case the concentration variable c is forced to vary between two fixed values, and z is forced to be in $[0,1]$, values of the functions m, σ, a and b outside these ranges do not affect the PDE system.

2.3 Entropic solutions and main result

Prior to the precise statement of our weak solution notion for the initial-boundary value problem for system (1.3), we shortly introduce and motivate its main ingredients, namely a suitable weak formulation of the flow rule (1.3c) for the damage variable and the "entropic" formulation of the heat equation (1.3d). To them, the standard weak formulation of the Cahn-Hilliard equation, and the pointwise (a.e. in Q) momentum equation will be coupled.

Entropy and total energy inequalities for the heat equation Along the footsteps of [10, 9], cf. also [38] in the case of a PDE system in thermoviscoelasticity, we will weakly formulate (1.3d) by means of an "entropy inequality", and of a "total energy (in)equality". The former is obtained by testing (1.3d) by φ/ϑ, with φ a *positive* smooth test function. Integrating over space and time leads to

$$\begin{aligned}
&\int_0^T \int_\Omega \big(\partial_t \log(\vartheta) + c_t + z_t + \rho\mathrm{div}(\mathbf{u}_t)\big)\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \mathsf{K}(\vartheta)\nabla\log(\vartheta)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad - \int_0^T \int_\Omega \mathsf{K}(\vartheta)\frac{\varphi}{\vartheta}\nabla\log(\vartheta)\cdot\nabla\vartheta\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \big(g + |c_t|^2 + |z_t|^2 + a(c,z)\varepsilon(\mathbf{u}_t) : \mathbb{V}\varepsilon(\mathbf{u}_t) + m(c,z)|\nabla\mu|^2\big)\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} h\frac{\varphi}{\vartheta}\,\mathrm{d}S\,\mathrm{d}t
\end{aligned} \tag{2.24}$$

for all $\varphi \in \mathcal{D}(\overline{Q})$. Then, the entropy inequality (2.45) below follows.

The total energy inequality (cf. the forthcoming (2.46)) associated with system (1.3) corresponds to its standard *energy* estimate. Formally, it is indeed obtained by testing (1.3a) by μ, (1.3b) by c_t, (1.3c) by z_t, (1.3d) by 1, and (1.3e) by $\mathbf{u}_t$, and it features the total energy (1.15) of the system.

Weak flow rule for the damage parameter We will adopt the solution notion from [21, 22], which can be motivated by observing that, due to the convexity of $I_{(-\infty,0]}$, the flow rule (1.3c) reformulates as $z_t \leq 0$ a.e. in Q and

$$\Big(z_t - \Delta_p(z) + \xi + \sigma'(z) + W_{,z}(c, \varepsilon(\mathbf{u}), z) - \vartheta\Big)\zeta \geq 0 \qquad \text{a.e. in } Q, \text{ for all } \zeta \leq 0, \tag{2.25a}$$

$$\Big(z_t - \Delta_p(z) + \xi + \sigma'(z) + W_{,z}(c, \varepsilon(\mathbf{u}), z) - \vartheta\Big)z_t \leq 0 \qquad \text{a.e. in } Q, \tag{2.25b}$$

with $\xi \in \partial I_{[0,+\infty)}(z)$ in $\Omega \times (0,T)$. Our weak formulation of (1.3c) in fact consists of the condition $z_t \leq 0$, of the integrated version of (2.25a), with negative test functions from $W^{1,p}(\Omega)$, and of the *damage energy-dissipation* inequality obtained by integrating (2.25b).

We are now in the position to give the following notion of weak solution:

Definition 2.4 (Entropic weak formulation)**.** Given data $(\mathbf{d}, \mathbf{f}, g, h)$ fulfilling (2.22) and initial values $(c^0, z^0, \vartheta^0, \mathbf{u}^0, \mathbf{v}^0)$ fulfilling (2.23), we call a quintuple $(c, \mu, z, \vartheta, \mathbf{u})$ an *entropic weak solution* to the PDE system (1.3), supplemented with the initial and boundary conditions (1.2), if

$$c \in L^\infty(0,T; W^{1,p}(\Omega)) \cap H^1(0,T; L^2(\Omega)),\ \Delta_p(c) \in L^2(0,T; L^2(\Omega)), \tag{2.26}$$

$$\mu \in L^2(0,T; H^2_N(\Omega)), \tag{2.27}$$

$$z \in L^\infty(0,T; W^{1,p}(\Omega)) \cap H^1(0,T; L^2(\Omega)), \tag{2.28}$$

$$\vartheta \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^1(\Omega)),\ \vartheta^{\frac{\kappa+\alpha}{2}} \in L^2(0,T; H^1(\Omega)) \text{ for all } \alpha \in (0,1), \tag{2.29}$$

$$\mathbf{u} \in H^1(0,T; H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T; H^1(\Omega;\mathbb{R}^d)) \cap H^2(0,T; L^2(\Omega;\mathbb{R}^d)), \tag{2.30}$$

and subgradients (specified in (2.37) and (2.42) below)

$$\eta \in L^2(0,T; L^2(\Omega)), \tag{2.31}$$

$$\xi \in L^2(0,T; L^2(\Omega)), \tag{2.32}$$

where $(c, z, \vartheta, \mathbf{u})$ comply the initial conditions (note that the initial condition for ϑ is implicitly formulated in (2.46) below)

$$c(0) = c^0, \qquad z(0) = z^0, \qquad \mathbf{u}(0) = \mathbf{u}^0, \qquad \mathbf{u}_t(0) = \mathbf{v}^0 \qquad \text{a.e. in } \Omega, \tag{2.33}$$

the Dirichlet condition

$$\mathbf{u} = \mathbf{d} \qquad \text{a.e. on } \partial\Omega \times (0,T) \tag{2.34}$$

and the following relations:

(i) Cahn-Hilliard system:

$$c_t = \operatorname{div}(m(c,z)\nabla\mu) \qquad \text{a.e. in } Q, \tag{2.35}$$

$$\mu = -\Delta_p(c) + \eta + \gamma'(c) + W_{,c}(c, \varepsilon(\mathbf{u}), z) - \vartheta + c_t \qquad \text{a.e. in } Q, \tag{2.36}$$

$$\eta \in \partial\hat{\beta}(c) \qquad \text{a.e. in } Q; \tag{2.37}$$

(ii) balance of forces:

$$
\begin{aligned}
&\mathbf{u}_{tt} - \operatorname{div}\boldsymbol{\sigma} = \mathbf{f} && \text{a.e. in } Q, && (2.38)\\
&\boldsymbol{\sigma} = a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t) + W_{,\varepsilon}(c,\varepsilon(\mathbf{u}),z) - \rho\vartheta\mathbb{1} && \text{a.e. in } Q; && (2.39)
\end{aligned}
$$

(iii) weak formulation of the damage flow rule:
damage energy-dissipation inequality for all $t \in (0,T]$, for $s = 0$, and for almost all $0 < s \le t$

$$
\begin{aligned}
&\int_s^t \int_\Omega |z_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \left(\frac{1}{p}|\nabla z(t)|^p + \sigma(z(t))\right) \mathrm{d}x \\
&\qquad \le \int_\Omega \left(\frac{1}{p}|\nabla z(s)|^p + \sigma(z(s))\right) \mathrm{d}x + \int_s^t \int_\Omega z_t\left(-W_{,z}(c,\varepsilon(\mathbf{u}),z) + \vartheta\right) \mathrm{d}x\,\mathrm{d}r
\end{aligned}
\tag{2.40}
$$

and the *one-sided variational inequality for the damage process*

$$
\begin{aligned}
&\int_\Omega \left(z_t\zeta + |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + \xi\zeta + \sigma'(z(t))\zeta + W_{,z}(c,\varepsilon(\mathbf{u}),z)\zeta - \vartheta\zeta\right)\mathrm{d}x \ge 0 \\
&\qquad\qquad\qquad \text{for all } \zeta \in W^{1,p}_-(\Omega), \quad \text{a.e. in } (0,T),
\end{aligned}
\tag{2.41}
$$

where

$$
\xi \in \partial I_{[0,+\infty)}(z) \qquad \text{a.e. in } Q,
\tag{2.42}
$$

as well as the constraints

$$
z \in [0,1], \qquad z_t \in (-\infty,0] \qquad \text{a.e. in } Q;
\tag{2.43}
$$

(iv) strict positivity and entropy inequality:

$$
\exists\, \underline{\vartheta} > 0 \text{ for a.a. } (x,t) \in Q: \quad \vartheta(x,t) \ge \underline{\vartheta} > 0
\tag{2.44}
$$

and for almost all $0 \le s \le t \le T$, and for $s = 0$ the entropy inequality holds:

$$
\begin{aligned}
&\int_s^t \int_\Omega (\log(\vartheta) + c + z)\varphi_t \,\mathrm{d}x\,\mathrm{d}r - \rho\int_s^t\int_\Omega \operatorname{div}(\mathbf{u}_t)\varphi\,\mathrm{d}x\,\mathrm{d}r - \int_s^t\int_\Omega \mathsf{K}(\vartheta)\nabla\log(\vartheta)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}r \\
&\le \int_\Omega (\log(\vartheta(t)) + c(t) + z(t))\varphi(t)\,\mathrm{d}x - \int_\Omega (\log(\vartheta(s)) + c(s) + z(s))\varphi(s)\,\mathrm{d}x \\
&\qquad\qquad - \int_s^t\int_\Omega \mathsf{K}(\vartheta)|\nabla\log(\vartheta)|^2\varphi\,\mathrm{d}x\,\mathrm{d}r \\
&\quad - \int_s^t\int_\Omega \left(g + |c_t|^2 + |z_t|^2 + a(c,z)\varepsilon(\mathbf{u}_t) : \mathbb{V}\varepsilon(\mathbf{u}_t) + m(c,z)|\nabla\mu|^2\right)\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r - \int_s^t\int_{\partial\Omega} h\frac{\varphi}{\vartheta}\,\mathrm{d}S\,\mathrm{d}r
\end{aligned}
\tag{2.45}
$$

for all $\varphi \in \mathrm{C}^0([0,T]; W^{1,d+\epsilon}(\Omega)) \cap H^1(0,T; L^{(d^\star)'}(\Omega))$ for some $\epsilon > 0$, with $\varphi \ge 0$;

(v) total energy inequality for almost all $0 \le s \le t \le T$, and for $s = 0$:

$$
\begin{aligned}
\mathscr{E}(c(t), z(t), \vartheta(t), \mathbf{u}(t), \mathbf{u}_t(t)) &\le \mathscr{E}(c(s), z(s), \vartheta(s), \mathbf{u}(s), \mathbf{u}_t(s)) \\
&\quad + \int_s^t\int_\Omega g\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}r \\
&\quad + \int_s^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\partial\Omega} (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{d}_t\,\mathrm{d}S\,\mathrm{d}r,
\end{aligned}
\tag{2.46}
$$

where for $s = 0$ we read $\vartheta(0) = \vartheta^0$, and $\mathscr{E}$ is given by (1.15).

Remark 2.5. A few comments on Definition 2.4 are in order:

– First of all, observe that inequalities (2.41) and (2.40) yield the *damage variational inequality* (with ξ fulfilling (2.42))

$$\begin{aligned}\int_s^t \int_\Omega |\nabla z|^{p-2}\nabla z \cdot \nabla \zeta \,\mathrm{d}x\,\mathrm{d}r - \int_\Omega \frac{1}{p}|\nabla z(t)|^p \,\mathrm{d}x + \int_\Omega \frac{1}{p}|\nabla z(s)|^p \,\mathrm{d}x \\ + \int_s^t \int_\Omega \Big(z_t(\zeta - z_t) + \sigma'(z)(\zeta - z_t) + \xi(\zeta - z_t)\Big)\,\mathrm{d}x\,\mathrm{d}r \\ \geq \int_s^t \int_\Omega \Big(- W_{,z}(c, \varepsilon(\mathbf{u}), z)(\zeta - z_t) + \vartheta(\zeta - z_t)\Big)\,\mathrm{d}x\,\mathrm{d}r\end{aligned} \tag{2.47}$$

for all $t \in (0,T]$, for $s = 0$, and for almost all $0 < s \leq t$ and for all test functions $\zeta \in L^p(0,T; W^{1,p}_-(\Omega)) \cap L^\infty(0,T; L^\infty(\Omega))$.

– Concerning the *entropic* formulation (=entropy+total energy inequalities) of the heat equation, we point out that it is consistent with the classical one. Namely, if the functions ϑ, c, z are sufficiently smooth, then inequalities (2.45) and (2.46), combined with (1.3a)–(1.3c) and (1.3e) yield the pointwise formulation of (1.3d), cf. [38, Rmk. 2.6] for all details.

– Observe that the *damage energy-dissipation* inequality (2.40) is required to hold for all $t \in (0,T]$ and for almost all $0 \leq s < t$, and $s = 0$. Indeed we will not be able to improve it to an equality, or to an inequality holding on *every* subinterval $[s,t] \subset [0,T]$. This is due to the fact that we will obtain (2.40) by passing to the limit in its time-discrete version (cf. Lemma 4.5), exploiting lower semicontinuity arguments to take the limit of the left-hand side, and pointwise, almost everywhere in $(0,T)$, convergences to take the limit of the right-hand side. Analogous considerations apply to the *entropy* and *total energy* inequalities (2.45) and (2.46).

– We remark that the *damage energy-dissipation* and the *total energy* inequalities are obtained independently one of another: while this will be clear from the proof of Theorem 2.6 below, we refer to [38, Rmk. 2.8] and [37, Sec. 2.4] for further comments.

– The quasi-linear p-Laplacian operator $\Delta_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)'$ with homogeneous Neumann conditions occurring in (2.36) is defined in the distributional sense as

$$\langle -\Delta_p(v), w\rangle_{W^{1,p}(\Omega)} = \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla w \,\mathrm{d}x.$$

However, since $\Delta_p(c) \in L^2(0,T; L^2(\Omega))$ due to (2.26), the Cahn-Hilliard system can be interpreted in a pointwise formulation. In view of the regularity result [41, Thm. 2, Rmk. 3.5], we infer the enhanced regularity

$$c \in L^2(0,T; W^{1+\sigma,p}(\Omega)) \qquad \text{for all } 1 \leq \sigma < \frac{1}{p}.$$

– All the terms in the total energy inequality (2.46) have a physical interpretation: The second and the third term on the right-hand side of (2.46) describe energy changes due to external heat sources. The integrand $\mathbf{f} \cdot \mathbf{u}_t$ in the fourth term on the right-hand side of (2.46) specifies the power expended by the external volume force $\mathbf{f}$, whereas the integrand $\big(\boldsymbol{\sigma} n\big) \cdot \mathbf{d}_t$ of the fifth term indicates the power expended by the time-dependent Dirichlet data $\mathbf{d}$ on the boundary $\partial\Omega$ (remember that $\boldsymbol{\sigma}$ is the stress tensor given in (2.39)).

We can now state our existence result for the entropic formulation of system (1.3). Observe that, while the basic time-regularity for ϑ (in fact for $\log(\vartheta)$) is poor in the general case, under an additional restriction on the exponent κ from Hypothesis (III) we will be able to obtain BV-time regularity for ϑ.

Theorem 2.6. *Assume **Hypotheses (I)–(V)**, and let the data* $(\mathbf{d}, \mathbf{f}, g, h)$ *comply with* (2.22)*. Then, for any quintuple* $(c^0, z^0, \vartheta^0, \mathbf{u}^0, \mathbf{v}^0)$ *fulfilling* (2.23) *there exists an entropic weak solution* $(c, \mu, z, \vartheta, \mathbf{u})$ *to the PDE system* (1.3)*, supplemented with the initial and boundary conditions* (1.2)*, such that*

$$\log(\vartheta) \in L^\infty(0,T; W^{1,d+\epsilon}(\Omega)') \qquad \text{for all } \epsilon > 0. \tag{2.48}$$

Furthermore, if in addition the exponent κ *in* (2.17) *satisfies*

$$\kappa \in (1, 5/3) \quad \text{if } d = 3 \text{ and } \kappa \in (1,2) \quad \text{if } d = 2\ , \tag{2.49}$$

then we have

$$\vartheta \in \mathrm{BV}([0,T]; W^{2,d+\epsilon}(\Omega)') \qquad \text{for every } \epsilon > 0, \tag{2.50}$$

and the total energy inequality (2.46) *holds <u>for all</u>* $t \in [0,T]$*, for* $s = 0$*, and for almost all* $s \in (0,t)$*.*

We will prove Theorem 2.6 throughout Sections 4 & 5 by passing to the limit in a carefully devised time discretization scheme and several regularizations. Namely, in Section 4 we are going to set up our time discretization scheme for system (1.3) and perform on it all the a priori estimates allowing us to prove, in Sec. 5, that (along a suitable subsequence) the approximate solutions converge to an entropic weak solution to (1.3). However, to enhance the readability of the paper in Section 3 we will (formally) perform all estimates on the time-continuous level, i.e. on system (1.3) itself.

3 Formal a priori estimates

Let us briefly outline all the estimates that will be formally developed on the time-continuous system (1.3):

- in the **<u>First estimate</u>**, from the (formally written) *total energy identity* (cf. (3.6) below) we will derive a series of bounds on the *non-dissipative* variables c, z, ϑ, $\mathbf{u}$, as well as on $\|\mathbf{u}_t\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))}$.
- Then, with the **<u>Second estimate</u>**, we shall adapt some calculations first developed in [9] (see also [38]) to derive a bound for $\|\vartheta\|_{L^2(0,T;H^1(\Omega))}$ via a clever test of the heat equation (1.3d).
- Exploiting the previously obtained estimates, in the **<u>Third estimate</u>** we will obtain bounds for the *dissipative* variables c_t, z_t, $\varepsilon(\mathbf{u}_t)$, as well as for $\nabla\mu$.
- The **<u>Fourth estimate</u>** is an elliptic regularity estimate on the momentum equation, along the footsteps of [3] where it was developed in the case of a *scalar* displacement variable. With this, in particular we gain a (uniform in time) bound on $\|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)}$ which translates into an (uniform in time) $L^2(\Omega)$-bound for the term $W_{,c}(c, \varepsilon(\mathbf{u}), z)$ in (1.3b).
- Using this, in the **<u>Fifth estimate</u>** we obtain a bound on the $L^2(0,T;H^1(\Omega))$-norm of μ from a bound on its mean value $\fint_\Omega \mu \,\mathrm{d}x$, combined with the previously obtained bound for $\nabla\mu$ via the Poincaré inequality. To develop the related calculations, we will momentarily suppose that

$$\begin{aligned} &\widehat{\beta} \in \mathrm{C}^1(\mathbb{R}) \text{ and satisfies the following property:} \\ &\forall\, \mathfrak{m} \in \mathbb{R}\ \exists\, C_{\mathfrak{m}},\, C'_{\mathfrak{m}} > 0 \quad |\beta(c+\mathfrak{m})| \le C_{\mathfrak{m}} \beta(c+\mathfrak{m})c + C'_{\mathfrak{m}}\,. \end{aligned} \tag{3.1}$$

- We are then in the position to obtain a $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$-estimate for each single term in (1.3b) in the **<u>Sixth estimate</u>**.
- With the **<u>Seventh</u>** and **<u>Eighth</u>** estimates we gain some information on the (BV-)time regularity of $\log(\vartheta)$ and ϑ, respectively (in the latter case, under the further condition (2.49) on the growth exponent κ of K).
- Finally, in the **<u>Ninth estimate</u>** we resort to higher elliptic regularity results to gain a uniform bound on $\|\mu\|_{L^2(0,T;H^2(\Omega))}$.

In the proof of the forthcoming Proposition 4.18 we will discuss how to make all of the following calculations rigorous in the context of the time-discretization scheme from Definition 4.1 (let us mention in advance that, for the *Fifth estimate* we will need the analogue of (3.1) on the level of the Yosida regularization of β), with the exception of the computations related to the ensuing **Seventh a priori estimate**. Indeed, while in the present time-continuous context this formal estimate will provide a BV-in-time bound for $\log(\vartheta)$, on the time-discrete level it will be possible to render it only in a *weaker* form, albeit still useful for the compactness arguments developed in Section 5.

In the following calculations, at several spots we will follow the footsteps of [38], hence we will give the main ideas, skipping some details and and referring to the latter paper. In comparison to [38], the additional coupling with the Cahn-Hilliard system (1.3a)–(1.3b) requires new a priori estimates (see the **Fifth**, **Sixth** and **Ninth estimates** below). Beyond this the remaining system (1.3c)–(1.3e) also depends on the phase field variable c and the estimation techniques used in [38] need to be adapted to this situation. And, finally, the time-dependent Dirichlet boundary conditions for $\mathbf{u}$ requires substantial modifications especially in the **First**, but also in the **Third** and **Fourth estimates** below.

Strict positivity of ϑ Along the lines of [9], we rearrange terms in (1.3d) and (formally, disregarding the -positive- boundary datum h) we obtain

$$\begin{aligned}\vartheta_t - \operatorname{div}(\mathsf{K}(\vartheta)\nabla\vartheta) &= g + |c_t|^2 + |z_t|^2 + a(c,z)\varepsilon(\mathbf{u}_t) : \mathbb{V}\varepsilon(\mathbf{u}_t) + m(c,z)|\nabla\mu|^2 - c_t\vartheta - z_t\vartheta - \rho\vartheta\operatorname{div}(\mathbf{u}_t)\\ &\geq g + \frac{1}{2}|c_t|^2 + \frac{1}{2}|z_t|^2 + c|\varepsilon(\mathbf{u}_t)|^2 + m(c,z)|\nabla\mu|^2 - C\vartheta^2 \geq -C\vartheta^2 \quad \text{a.e. in } Q.\end{aligned} \tag{3.2}$$

Here, for the first inequality we have used that $\mathbb{V}$ is positive definite by (2.20) and (2.4), that a is strictly positive thanks to (2.18), and that

$$|\operatorname{div}(\mathbf{u}_t)| \leq c(d)|\varepsilon(\mathbf{u}_t)| \quad \text{a.e. in } Q \tag{3.3}$$

with $c(d)$ a positive constant only depending on the space dimension d. The second inequality in (3.2) also relies on the fact that $g \geq 0$ a.e. in Q. Therefore we conclude that v solving the Cauchy problem

$$v_t = -\frac{1}{2}v^2, \quad v(0) = \vartheta_* > 0$$

is a subsolution of (1.3d), and a comparison argument yields that there exists $\underline{\vartheta} > 0$ such that

$$\vartheta(\cdot,t) \geq v(t) > \underline{\vartheta} > 0 \quad \text{for all } t \in [0,T]\,. \tag{3.4}$$

First estimate: We test (1.3a) by μ, (1.3b) by c_t, (1.3c) by z_t, (1.3d) by 1, (1.3e) by $\mathbf{u}_t$, add the resulting relations and integrate over the time interval $(0,t)$, $t \in (0,T]$. Here the second term in the force balance equation is treated by integration by parts in space as follows (notice that $\mathbf{u}_t = \mathbf{d}_t$ a.e. on $\partial\Omega \times (0,T)$):

$$\begin{aligned}&\int_0^t \int_\Omega -\operatorname{div}\left(a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t) + W_{,\varepsilon}(c,\varepsilon(\mathbf{u}),z) - \rho\vartheta\mathbb{1}\right)\cdot\mathbf{u}_t \,\mathrm{d}x\,\mathrm{d}s\\ &\quad = \int_0^t \int_\Omega a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t) : \varepsilon(\mathbf{u}_t) + W_{,\varepsilon}(c,\varepsilon(\mathbf{u}),z) : \varepsilon(\mathbf{u}_t) - \rho\vartheta\operatorname{div}(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_{\partial\Omega} (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{d}_t\,\mathrm{d}S\,\mathrm{d}s.\end{aligned} \tag{3.5}$$

Furthermore, we use that, by the chain rule,

$$\text{(i)} \quad \begin{aligned}&\int_0^t \int_\Omega W_{,c}(c,\varepsilon(\mathbf{u}),z)c_t + W_{,z}(c,\varepsilon(\mathbf{u}),z)z_t + W_{,\varepsilon}(c,\varepsilon(\mathbf{u}),z) \colon \varepsilon(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_\Omega W(c(t),z(t),\varepsilon(\mathbf{u}(t)))\,\mathrm{d}x - \int_\Omega W(c(0),z(0),\varepsilon(\mathbf{u}(0)))\,\mathrm{d}x,\end{aligned}$$

$$\text{(ii)} \quad \int_0^t \int_\Omega (\eta + \gamma'(c))\,c_t\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \phi(c(t))\,\mathrm{d}x - \int_\Omega \phi(c(0))\,\mathrm{d}x,$$

$$\text{(iii)} \quad \int_0^t \int_\Omega \left(\partial I_{[0,+\infty)}(z) + \sigma'(z)\right) z_t\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega I_{[0,+\infty)}(z(t)) + \sigma(z(t))\,\mathrm{d}x - \int_\Omega I_{[0,+\infty)}(z(0)) + \sigma(z(0))\,\mathrm{d}x,$$

as well as the identity $\int_0^t\int_\Omega \partial I_{(-\infty,0]}(z_t)z_t \,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_\Omega I_{(-\infty,0]}(z_t)\,\mathrm{d}x\,\mathrm{d}s = 0$ due to the positive 1-homogeneity of $\partial I_{(-\infty,0]}$. Also taking into account the cancellation of a series of terms, we arrive at the *total energy identity*

$$\begin{aligned}\mathscr{E}(c(t), z(t), \vartheta(t), \mathbf{u}(t), \mathbf{u}_t(t)) = \mathscr{E}(c_0, z_0, \vartheta_0, \mathbf{u}_0, \mathbf{v}_0) + \int_0^t\int_\Omega g\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}s \\ + \int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}(\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{d}_t\,\mathrm{d}S\,\mathrm{d}s\,,\end{aligned} \tag{3.6}$$

which incorporates the initial conditions (2.33).

We estimate the second, third and fourth terms on the right-hand side of (3.6) via (2.22) and obtain

$$\begin{aligned}&\left|\int_0^t\int_\Omega g\,\mathrm{d}x\,\mathrm{d}s\right| \overset{(2.22c)}{\le} C, \qquad \left|\int_0^t\int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}s\right| \overset{(2.22d)}{\le} C,\\ &\left|\int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}s\right| \overset{(2.22b)}{\le} C + \|\mathbf{u}_t\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}.\end{aligned}$$

We now carefully handle the last term on the right-hand side of (3.6). Since no viscous term of the type $\varepsilon(\mathbf{u}_t)$ occurs on its left-hand side, to absorb the last term on the right-hand side and close the estimate we will extensively make use of integration by parts in space, as well as of the force balance equation (1.3e) of integration by parts in time, and of Young's inequality ($\delta > 0$ will be chosen later):

$$\begin{aligned}\int_0^t\int_{\partial\Omega}(\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{d}_t\,\mathrm{d}S\,\mathrm{d}s &= \int_0^t\int_\Omega \mathrm{div}(\boldsymbol{\sigma})\cdot\mathbf{d}_t\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \boldsymbol{\sigma}:\varepsilon(\mathbf{d}_t)\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t\int_\Omega(-\mathbf{f}+\mathbf{u}_{tt})\cdot\mathbf{d}_t\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \boldsymbol{\sigma}:\varepsilon(\mathbf{d}_t)\,\mathrm{d}x\,\mathrm{d}s\\ &\le \|\mathbf{f}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\|\mathbf{d}_t\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} + \int_0^t \|\mathbf{u}_t\|_{L^2(\Omega;\mathbb{R}^d)}\|\mathbf{d}_{tt}\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\\ &\quad + \delta\|\mathbf{u}_t(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + C_\delta\|\mathbf{d}_t(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\mathbf{v}^0\|_{L^2(\Omega;\mathbb{R}^d)}\|\mathbf{d}_t(0)\|_{L^2(\Omega;\mathbb{R}^d)}\\ &\quad + \underbrace{\int_0^t\int_\Omega a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t):\varepsilon(\mathbf{d}_t)\,\mathrm{d}x\,\mathrm{d}s}_{\doteq I_1} + \underbrace{\int_0^t\int_\Omega b(c,z)\mathbb{C}(\varepsilon(\mathbf{u})-\varepsilon^*(c)):\varepsilon(\mathbf{d}_t)\,\mathrm{d}x\,\mathrm{d}s}_{\doteq I_2}\\ &\quad + \rho\|\,\mathrm{div}(\mathbf{d}_t)\|_{L^\infty(Q)}\int_0^t\int_\Omega|\vartheta|\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Moreover, by using integration by parts in space again, the properties of the coefficient functions a and b stated in Hypothesis (IV) and (V), and by using (2.22a) on $\mathbf{d}$, $\mathbf{u}_t = \mathbf{d}_t$ a.e. on $\partial\Omega\times(0,T)$ and the trace theorem we obtain

$$
\begin{aligned}
I_1 =& -\int_0^t\int_\Omega \mathbf{u}_t\cdot \operatorname{div}\big(a(c,z)\mathbb{V}\varepsilon(\mathbf{d}_t)\big)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\partial\Omega}\mathbf{u}_t\cdot\big(a(c,z)\mathbb{V}\varepsilon(\mathbf{d}_t)\mathbf{n}\big)\,\mathrm{d}S\,\mathrm{d}s\\
=& -\int_0^t\int_\Omega \mathbf{u}_t\cdot\Big(\big(a_{,c}(c,z)\nabla c+a_{,z}(c,z)\nabla z\big)\cdot\mathbb{V}\varepsilon(\mathbf{d}_t)\Big)\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_\Omega \mathbf{u}_t\cdot\left(a(c,z)\mathbb{V}\operatorname{div}(\varepsilon(\mathbf{d}_t))\right)\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\int_{\partial\Omega}\mathbf{d}_t\cdot\big(a(c,z)\mathbb{V}\varepsilon(\mathbf{d}_t)\mathbf{n}\big)\,\mathrm{d}S\,\mathrm{d}s\\
\leq& C\|\varepsilon(\mathbf{d}_t)\|_{L^\infty(Q;\mathbb{R}^{d\times d})}\Big(\int_0^t\|\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s+\|a_{,c}(c,z)\|^2_{L^\infty(0,T;L^\infty(\Omega))}\int_0^t\|\nabla c\|^2_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\\
&\qquad+\|a_{,z}(c,z)\|^2_{L^\infty(0,T;L^\infty(\Omega))}\int_0^t\|\nabla z\|^2_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\Big)\\
&+C\int_0^t\|\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s+C\|a(c,z)\|^2_{L^\infty(0,T;L^\infty(\Omega))}\|\varepsilon(\mathbf{d}_t)\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^{d\times d}))}\\
&+C\|\mathbf{d}_t\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))}\|\varepsilon(\mathbf{d}_t)\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^{d\times d}))}\|a(c,z)\|_{L^\infty(0,T;L^\infty(\partial\Omega))}\\
\leq& C\int_0^t\Big(\|\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^d)}+\|\nabla c\|^2_{L^2(\Omega;\mathbb{R}^d)}+\|\nabla z\|^2_{L^2(\Omega;\mathbb{R}^d)}\Big)\,\mathrm{d}s+C,\\
I_2 \leq& C\int_0^t\int_\Omega b(c,z)^2\mathbb{C}(\varepsilon(\mathbf{u})-\varepsilon^*(c)):(\varepsilon(\mathbf{u})-\varepsilon^*(c))+\mathbb{C}\varepsilon(\mathbf{d}_t):\varepsilon(\mathbf{d}_t)\,\mathrm{d}x\,\mathrm{d}s\\
\leq& C\|b(c,z)\|_{L^\infty(0,T;L^\infty(\Omega))}\int_0^t\int_\Omega W(c,\varepsilon(\mathbf{u}),z)\,\mathrm{d}x\,\mathrm{d}s+C\|\varepsilon(\mathbf{d}_t)\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}.
\end{aligned}
$$

All in all, again taking into account (2.22), we gain the estimate

$$
\begin{aligned}
&\mathscr{E}(c(t),z(t),\vartheta(t),\mathbf{u}(t),\mathbf{u}_t(t))\\
&\quad\leq C_\delta+\delta\|\mathbf{u}_t(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}+\int_0^t C\big(\|\mathbf{d}_{tt}\|^2_{L^2(\Omega;\mathbb{R}^d)}+1\big)\times\\
&\qquad\times\int_0^t\Big(\int_\Omega W(c,\varepsilon(\mathbf{u}),z)\,\mathrm{d}x+\|\nabla c\|^2_{L^2(\Omega;\mathbb{R}^d)}+\|\nabla z\|^2_{L^2(\Omega;\mathbb{R}^d)}+\int_\Omega|\vartheta|\,\mathrm{d}x+\|\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^d)}\Big)\,\mathrm{d}s\\
&\quad\leq C_\delta+\delta\|\mathbf{u}_t(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}+\int_0^t C\big(\|\mathbf{d}_{tt}\|^2_{L^2(\Omega;\mathbb{R}^d)}+1\big)\mathscr{E}(c(s),z(s),\vartheta(s),\mathbf{u}(s),\mathbf{u}_t(s))\,\mathrm{d}s.
\end{aligned}
$$

Choosing $\delta=1/4$, using Gronwall Lemma together with (2.22a) and taking the positivity of ϑ into account, we conclude

$$
\|\vartheta\|_{L^\infty(0,T;L^1(\Omega))}+\|\mathbf{u}\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))}+\|c\|_{L^\infty(0,T;W^{1,p}(\Omega))}+\|\nabla z\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^d))}\leq C. \tag{3.7}
$$

Note that we have also used the Poincaré inequality to obtain the boundedness for c in $L^\infty(0,T;W^{1,p}(\Omega))$ because it holds $\int_\Omega c(t)\,\mathrm{d}x\equiv \mathit{const}$ for all $t\in[0,T]$ (this follows from (1.3a) and the no-flux condition for μ in (1.2b)).

Second estimate: Let $F(\vartheta)=\vartheta^\alpha/\alpha$, with $\alpha\in(0,1)$. We test (1.3d) by $F'(\vartheta):=\vartheta^{\alpha-1}$, and integrate on $(0,t)$ with $t\in(0,T]$, thus obtaining

$$
\begin{aligned}
&\int_\Omega F(\vartheta_0)\,\mathrm{d}x+\int_0^t\int_\Omega gF'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\partial\Omega}hF'(\vartheta)\,\mathrm{d}S\,\mathrm{d}s+\int_0^t\int_\Omega(|c_t|^2+|z_t|^2)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\int_\Omega a(c,z)\varepsilon(\mathbf{u}_t):\mathbb{V}\varepsilon(\mathbf{u}_t)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_\Omega m(c,z)|\nabla\mu|^2F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s\\
&\;=\int_\Omega F(\vartheta(t))\,\mathrm{d}x+\int_0^t\int_\Omega(c_t+z_t)\vartheta F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s+\rho\int_0^t\int_\Omega\vartheta\operatorname{div}(\mathbf{u}_t)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s\int_0^t\int_\Omega\mathsf{K}(\vartheta)\nabla\vartheta\cdot\nabla(F'(\vartheta))\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

By the positivity of g and h we can neglect the second and third terms on the left-hand side, whereas, taking into account the ellipticity condition (2.4) and the positivity (2.15) and (2.18) of m and a, we infer

$$\begin{aligned} &\frac{4(1-\alpha)}{\alpha^2}\int_0^t\int_\Omega \mathsf{K}(\vartheta)|\nabla(\vartheta^{\alpha/2})|^2\,\mathrm{d}x\,\mathrm{d}s + \bar{c}\int_0^t\int_\Omega (|\varepsilon(\mathbf{u}_t)|^2+|\nabla\mu|^2)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t\int_\Omega (|c_t|^2+|z_t|^2)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s \le \int_\Omega |F(\vartheta_0)|\,\mathrm{d}x + I_1 + I_2 + I_3, \end{aligned} \tag{3.8}$$

with $\bar{c}>0$ depending on ν_0, m_0, and a_0, where $I_3 \doteq |\rho|\int_0^t\int_\Omega |\vartheta\,\mathrm{div}(\mathbf{u}_t)F'(\vartheta)|\,\mathrm{d}x\,\mathrm{d}s$. We estimate

$$I_1 = \int_\Omega |F(\vartheta(t))|\,\mathrm{d}x \le \frac{1}{\alpha}\int_\Omega \max\{\vartheta(t),1\}^\alpha\,\mathrm{d}x \le \frac{1}{\alpha}\int_\Omega \max\{\vartheta(t),1\}\,\mathrm{d}x \le C$$

since $\alpha < 1$ and taking into account the previously obtained inequality (3.7). Analogously we can estimate $\int_\Omega |F(\vartheta_0)|\,\mathrm{d}x$ thanks to (2.23c); moreover,

$$I_2 = \int_0^t\int_\Omega |(c_t+z_t)\vartheta F'(\vartheta)|\,\mathrm{d}x\,\mathrm{d}s \le \frac{1}{4}\int_0^t\int_\Omega \left(|c_t|^2+|z_t|^2\right)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s + 2\int_0^t\int_\Omega F'(\vartheta)\vartheta^2\,\mathrm{d}x\,\mathrm{d}s.$$

Using inequality (3.3) and Young's inequality, we have that

$$I_3 = |\rho|\int_0^t\int_\Omega |\vartheta\,\mathrm{div}(\mathbf{u}_t)F'(\vartheta)|\,\mathrm{d}x\,\mathrm{d}s \le \frac{\bar{c}}{4}\int_0^t\int_\Omega |\varepsilon(\mathbf{u}_t)|^2F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_\Omega F'(\vartheta)\vartheta^2\,\mathrm{d}x\,\mathrm{d}s\,.$$

All in all, we conclude

$$\begin{aligned} &\frac{4(1-\alpha)}{\alpha^2}\int_0^t\int_\Omega \mathsf{K}(\vartheta)|\nabla(\vartheta^{\alpha/2})|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{3\bar{c}}{4}\int_0^t\int_\Omega (|\varepsilon(\mathbf{u}_t)|^2+|\nabla\mu|^2)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s \\ &+\frac{3}{4}\int_0^t\int_\Omega (|c_t|^2+|z_t|^2)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s \le C + C\int_0^t\int_\Omega \vartheta^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

Observe that

$$\int_0^t\int_\Omega \mathsf{K}(\vartheta)|\nabla(\vartheta^{\alpha/2})|^2\,\mathrm{d}x\,\mathrm{d}s \ge c_1\int_0^t\int_\Omega \vartheta^\kappa|\nabla(\vartheta^{\alpha/2})|^2\,\mathrm{d}x\,\mathrm{d}s = \tilde{c}_1\int_0^t\int_\Omega |\nabla(\vartheta^{(\kappa+\alpha)/2})|^2\,\mathrm{d}x\,\mathrm{d}s\,.$$

Hence, from (3.9) we infer the estimate

$$\tilde{c}_1\int_0^t\int_\Omega |\nabla(\vartheta^{(\kappa+\alpha)/2})|^2\,\mathrm{d}x\,\mathrm{d}s \le C_0 + C_0\int_0^t\int_\Omega \vartheta^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}s. \tag{3.10}$$

We now repeat the very same calculations as in the *Second* and *Third* estimates in [38, Sec. 3], to which we refer for all details. Namely, we introduce the auxiliary quantity $w := \max\{\vartheta^{(\kappa+\alpha)/2},1\}$ and observe that

$$\int_0^t\int_\Omega |\nabla(\vartheta^{(\kappa+\alpha)/2})|^2\,\mathrm{d}x\,\mathrm{d}s \ge \int_0^t\int_{\{\vartheta(s)\ge 1\}} |\nabla(\vartheta^{(\kappa+\alpha)/2})|^2\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_\Omega |\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s, \tag{3.11}$$

$$\vartheta^{\alpha+1} = \left(\vartheta^{(\alpha+1)/q}\right)^q \le w^q \qquad \text{a.e. in } Q, \tag{3.12}$$

for all $q\ge 1$ such that

$$\frac{\kappa+\alpha}{2} \ge \frac{\alpha+1}{q} \Leftrightarrow\ q \ge 2 - 2\frac{\kappa-1}{\kappa+\alpha}. \tag{3.13}$$

Therefore from (3.10) we infer that

$$\tilde{c}_1\int_0^t\int_\Omega |\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s \le C_0 + C_0\int_0^t \|w\|^q_{L^q(\Omega)}\,\mathrm{d}s. \tag{3.14}$$

We now apply the Gagliardo-Nirenberg inequality for $d = 3$, yielding

$$\|w\|_{L^q(\Omega)} \le c_1 \|\nabla w\|^{\theta}_{L^2(\Omega;\mathbb{R}^d)} \|w\|^{1-\theta}_{L^r(\Omega)} + c_2 \|w\|_{L^r(\Omega)} \tag{3.15}$$

with $1 \le r \le q$ and θ satisfying $1/q = \theta/6 + (1-\theta)/r$. Hence $\theta = 6(q-r)/q(6-r)$. Observe that $\theta \in (0,1)$ if $q < 6$. Applying the Young inequality with exponents $2/(\theta q)$ and $2/(2-\theta q)$ we infer

$$C_0 \int_0^t \|w\|^q_{L^q(\Omega)} \,\mathrm{d}s \le \frac{\tilde{c}_1}{2} \int_0^t \int_\Omega |\nabla w|^2 \,\mathrm{d}x \,\mathrm{d}s + C \int_0^t \|w\|^{2q(1-\theta)/(2-q\theta)}_{L^r(\Omega)} \,\mathrm{d}s + C' \int_0^t \|w\|^q_{L^r(\Omega)} \,\mathrm{d}s. \tag{3.16}$$

We then plug (3.16) into (3.14), and obtain

$$\frac{\tilde{c}_1}{2} \int_0^t \int_\Omega |\nabla w|^2 \,\mathrm{d}x \,\mathrm{d}s \le C_0 + C \int_0^t \|w\|^{2q(1-\theta)/(2-q\theta)}_{L^r(\Omega)} \,\mathrm{d}s + C' \int_0^t \|w\|^q_{L^r(\Omega)} \,\mathrm{d}s. \tag{3.17}$$

Hence, we choose $1 \le r \le 2/(\kappa+\alpha)$ so that for almost all $t \in (0,T)$

$$\|w(t)\|_{L^r(\Omega)} = \left(\int_\Omega \max\{\vartheta(t)^{r(\kappa+\alpha)/2}, 1\} \,\mathrm{d}x \right)^{1/r} \le C \left(\|\vartheta(t)\|_{L^1(\Omega)} + |\Omega| \right) \le C' \tag{3.18}$$

where we have used the bound for $\|\vartheta\|_{L^\infty(0,T;L^1(\Omega))}$ from estimate (3.7). Observe that the inequalities

$$\begin{cases} \theta q \le 2 \;\Leftrightarrow\; 6\frac{q-r}{6-r} \le 2 \;\Leftrightarrow\; q \le 2 + \frac{2}{3} r, \\ r \le \frac{2}{\kappa+\alpha} \end{cases}$$

lead to $q \le 2 + \frac{4}{3(\kappa+\alpha)}$ which is still compatible with (3.13), since $\frac{\kappa-1}{\kappa+\alpha} < 1$. Inserting (3.18) into (3.17) we ultimately deduce $\int_0^t \int_\Omega |\nabla w|^2 \,\mathrm{d}x \,\mathrm{d}s \le C$. Taking also (3.16) and (3.18) into account we then conclude $\int_0^t \|w\|^q_{L^q(\Omega)} \,\mathrm{d}s \le C$. By using this as well as estimates (3.10) and (3.12) we see that

$$c \int_0^t \int_\Omega \vartheta^{\kappa+\alpha-2} |\nabla \vartheta|^2 \,\mathrm{d}x \,\mathrm{d}s = \int_0^t \int_\Omega |\nabla(\vartheta^{(\kappa+\alpha)/2})|^2 \,\mathrm{d}x \,\mathrm{d}s \le C. \tag{3.19}$$

From (3.19) and the strict positivity of ϑ (see (3.4)) it follows that $\int_0^t \int_\Omega |\nabla \vartheta|^2 \,\mathrm{d}x \,\mathrm{d}s \le C$, provided that $\kappa + \alpha - 2 \ge 0$. Observe that, since $\kappa > 1$ we can choose $\alpha \in (0,1)$ such that this inequality holds. Hence, in view of estimate (3.7) and applying Poincaré inequality, we gather

$$\|\vartheta\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{3.20}$$

With the very same calculations as in [38, Sec. 3] we also obtain

$$\|\vartheta\|_{L^q(Q)} \le C \quad \text{with } q = 8/3 \quad \text{if } d = 3, \quad q = 3 \quad \text{if } d = 2 \tag{3.21}$$

interpolating between estimate (3.20) and estimate (3.7) for $\|\vartheta\|_{L^\infty(0,T;L^1(\Omega))}$ and using the Gagliardo-Nirenberg inequality (2.7). Furthermore, we observe

$$\int_\Omega |\nabla \vartheta^{(\kappa-\alpha)/2}|^2 \,\mathrm{d}x = c \int_\Omega \vartheta^{\kappa-\alpha-2} |\nabla \vartheta|^2 \,\mathrm{d}x \le \frac{c}{\underline{\vartheta}^{2\alpha}} \int_\Omega \vartheta^{\kappa+\alpha-2} |\nabla \vartheta|^2 \,\mathrm{d}x \le C, \tag{3.22}$$

thanks to the positivity property (3.4) and estimate (3.19). Resorting to a nonlinear version of the Poincaré inequality (cf. e.g. (2.12)), we then infer

$$\|\vartheta^{(\kappa-\alpha)/2}\|_{L^2(0,T;H^1(\Omega))},\ \|\vartheta^{(\kappa+\alpha)/2}\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{3.23}$$

Third estimate: We test (1.3d) by 1, integrate in time, and subtract the resulting relation from the total energy balance (3.6). We thus obtain

$$
\begin{aligned}
&\int_0^t \int_\Omega |c_t|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \tfrac{1}{p}|\nabla c(t)|^p + \phi(c(t))\,\mathrm{d}x + \int_0^t \int_\Omega m(c,z)|\nabla\mu|^2 \,\mathrm{d}x\,\mathrm{d}s + \\
&+ \int_0^t \int_\Omega |z_t|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \tfrac{1}{p}|\nabla z(t)|^p + I_{[0,+\infty)}(z(t)) + \sigma(z(t))\,\mathrm{d}x \\
&+ \frac{1}{2}\int_\Omega |\mathbf{u}_t(t)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t) \colon \varepsilon(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega W(c(t),\varepsilon(\mathbf{u}(t)),z(t))\,\mathrm{d}x \\
&= \int_\Omega \tfrac{1}{p}|\nabla c_0|^p + \phi(c_0)\,\mathrm{d}x + \int_\Omega \tfrac{1}{p}|\nabla z_0|^p + I_{[0,+\infty)}(z_0) + \sigma(z_0)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2 \,\mathrm{d}x \\
&+ \int_\Omega W(c_0,\varepsilon(\mathbf{u}_0),z_0)\,\mathrm{d}x + \int_0^t \int_\Omega \vartheta\left(\rho \operatorname{div}\mathbf{u}_t + c_t + z_t\right)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t \,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\partial\Omega} (\boldsymbol{\sigma}\mathbf{n})\cdot \mathbf{d}_t \,\mathrm{d}S\,\mathrm{d}s.
\end{aligned}
\tag{3.24}
$$

Observe that the first, second, third, and fourth integral terms on the right-hand side are bounded thanks to conditions (2.23) on $(c_0, z_0, \mathbf{u}_0, \mathbf{v}_0)$. As in the First estimate we deduce boundedness of the last and the last but one integral terms on the right-hand side. Since ϕ, $I_{[0,+\infty)} + \sigma$, and W are bounded from below, exploiting (2.4), (2.18), and (2.20) to deduce that $\int_0^t \int_\Omega a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t) \colon \varepsilon(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}x \geq c\int_0^t \int_\Omega |\varepsilon(\mathbf{u}_t)|^2 \,\mathrm{d}x\,\mathrm{d}s$, and using (2.15) to deduce that $\int_0^t \int_\Omega m(c,z)|\nabla\mu|^2 \,\mathrm{d}x\,\mathrm{d}s \geq m_0 \int_0^t \int_\Omega |\nabla\mu|^2 \,\mathrm{d}x\,\mathrm{d}s$, we find that

$$
\int_0^t \int_\Omega (|c_t|^2 + |z_t|^2 + |\nabla\mu|^2 + |\varepsilon(\mathbf{u}_t)|^2)\,\mathrm{d}x\,\mathrm{d}s \leq C + \int_0^t \int_\Omega \vartheta\left(\rho \operatorname{div}\mathbf{u}_t + c_t + z_t\right)\,\mathrm{d}x\,\mathrm{d}s. \tag{3.25}
$$

Then, we can estimate the integral term on the right-hand side by

$$
\varrho \int_0^t \int_\Omega \left(|\varepsilon(\mathbf{u}_t)|^2 + |c_t|^2 + |z_t|^2\right)\,\mathrm{d}x\,\mathrm{d}s + C_\varrho \int_0^t \int_\Omega |\vartheta|^2 \,\mathrm{d}x\,\mathrm{d}s,
$$

for a sufficiently small constant $\varrho > 0$, in such a way as to absorb the first integral term into the left-hand side of (3.25). Exploiting (3.20) on ϑ, we thus conclude, also with the aid of Korn's inequality and condition (2.22a) on the boundary value $\mathbf{d}$,

$$
\|c_t\|_{L^2(Q)} + \|\nabla\mu\|_{L^2(Q;\mathbb{R}^d)} + \|z_t\|_{L^2(Q)} + \|\mathbf{u}_t\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))} \leq C\,. \tag{3.26}
$$

Furthermore, taking into account the previously proved bound (3.7), we also gather

$$
\|z\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|\mathbf{u}\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))} \leq C. \tag{3.27}
$$

Fourth estimate: We test (1.3e) by $-\operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))$ and integrate in time. This leads to

$$
\begin{aligned}
&-\int_0^t \mathbf{u}_{tt} \cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \operatorname{div}(a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t)) \cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s \\
&= -\int_0^t \int_\Omega \operatorname{div}(W_{,\varepsilon}(c,\varepsilon(\mathbf{u}),z)) \cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s + \rho\int_0^t \int_\Omega \nabla\vartheta\cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s \\
&-\int_0^t \int_\Omega \mathbf{f} \cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}
\tag{3.28}
$$

The following calculations are based on [38, Sec. 5], to which we refer for details. However in the present case we have to take care of the non-homogeneous Dirichlet boundary condition for $\mathbf{u}$. The first term on the left-hand side of (3.28) gives

$$
\begin{aligned}
&-\int_0^t \int_\Omega \mathbf{u}_{tt} \cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t)) \,\mathrm{d}x \,\mathrm{d}s \\
&\quad = -\int_0^t \int_{\partial\Omega} \mathbf{u}_{tt} \cdot \big(\mathbb{V}\varepsilon(\mathbf{u}_t)\mathbf{n}\big) \,\mathrm{d}S \,\mathrm{d}s + \int_\Omega \frac{1}{2}\varepsilon(\mathbf{u}_t(t)) : \mathbb{V}\varepsilon(\mathbf{u}_t(t)) \,\mathrm{d}x - \int_\Omega \frac{1}{2}\varepsilon(\mathbf{u}_t(0)) : \mathbb{V}\varepsilon(\mathbf{u}_t(0)) \,\mathrm{d}x.
\end{aligned}
\tag{3.29}
$$

On the boundary cylinder $\partial\Omega \times (0,T)$ we find $\mathbf{u}_{tt} = \mathbf{d}_{tt}$ a.e. (note that not necessarily $\varepsilon(\mathbf{u}_t) = \varepsilon(\mathbf{d}_t)$ a.e. on $\partial\Omega \times (0,T)$) which yields by using the trace theorem and Young's inequality ($\delta > 0$ will be chosen later)

$$
\begin{aligned}
\Big|\int_0^t \int_{\partial\Omega} \mathbf{u}_{tt} \cdot \big(\mathbb{V}\varepsilon(\mathbf{u}_t)\mathbf{n}\big) \,\mathrm{d}S \,\mathrm{d}s\Big| &= \Big|\int_0^t \int_{\partial\Omega} \mathbf{d}_{tt} \cdot \big(\mathbb{V}\varepsilon(\mathbf{u}_t)\mathbf{n}\big) \,\mathrm{d}S \,\mathrm{d}s\Big| \\
&\le \delta \|\mathbf{u}_t\|^2_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))} + C_\delta \|\mathbf{d}_{tt}\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))}.
\end{aligned}
$$

The last term on the right-hand side can be estimated by using (2.22a).

For the second term on the left-hand side of (3.28) we find

$$
\begin{aligned}
&\int_0^t \int_\Omega \operatorname{div}(a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t)) \cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t)) \,\mathrm{d}x \,\mathrm{d}s \\
&= \int_0^t \int_\Omega a(c,z)\operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t)) \cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t)) \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_\Omega (\nabla a(c,z) \cdot \mathbb{V}\varepsilon(\mathbf{u}_t)) \cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t)) \,\mathrm{d}x \,\mathrm{d}s \\
&\ge c \int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}s - \|\mathbf{d}_t\|^2_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))} + I_1,
\end{aligned}
\tag{3.30}
$$

where the second inequality follows from (2.6). The second term on the right-hand side is bounded due to (2.22a). We move I_1 to the right-hand side of (3.28) and estimate

$$
\begin{aligned}
|I_1| &= \Big|\int_0^t \int_\Omega (\nabla a(c,z)\mathbb{V}\varepsilon(\mathbf{u}_t)) \cdot \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\Big| \,\mathrm{d}x \,\mathrm{d}s \\
&\le C \int_0^t \|\nabla a(c,z)\|_{L^{d+\zeta}(\Omega;\mathbb{R}^d)} \|\varepsilon(\mathbf{u}_t)\|_{L^{d^\star-\eta}(\Omega;\mathbb{R}^{d\times d})} \|\operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\|_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}s \\
&\le \delta \int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}s + C_\delta \int_0^t \|\nabla a(c,z)\|^2_{L^{d+\zeta}(\Omega;\mathbb{R}^d)} \|\varepsilon(\mathbf{u}_t)\|^2_{L^{d^\star-\zeta}(\Omega;\mathbb{R}^{d\times d})} \,\mathrm{d}s \\
&\le \delta \int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}s + C_\delta \varrho^2 \int_0^t \big(\|c\|^2_{W^{1,p}(\Omega)} + \|z\|^2_{W^{1,p}(\Omega)}\big) \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}s \\
&\quad + C_\delta C_\varrho \int_0^t \big(\|c\|^2_{W^{1,p}(\Omega)} + \|z\|^2_{W^{1,p}(\Omega)}\big) \|\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}s.
\end{aligned}
\tag{3.31}
$$

In the first line, we have chosen $\zeta > 0$ fulfilling $p \ge d+\zeta$, and $\eta > 0$ such that $\frac{1}{d+\zeta} + \frac{1}{d^\star-\eta} + \frac{1}{2} \le 1$, with $d^\star$ from (2.9), in order to apply the Hölder inequality. Moreover, we have exploited (2.18), giving $\|\nabla a(c,z)\|_{L^{d+\zeta}(\Omega;\mathbb{R}^d)} \le C(\|c\|_{W^{1,p}(\Omega)} + \|z\|_{W^{1,p}(\Omega)})$, as well as (2.10) to estimate $\|\varepsilon(\mathbf{u}_t)\|_{L^{d^\star-\zeta}(\Omega;\mathbb{R}^{d\times d})}$. Finally, δ and ϱ are positive constants that we will choose later, accordingly determining C_δ, $C_\varrho > 0$ via the Young inequality. For the right-hand side of (3.28) we proceed as follows

$$
\begin{aligned}
&-\int_0^t\int_\Omega \operatorname{div}(W_{,\varepsilon}(c,\varepsilon(\mathbf{u}),z)) \,\cdot\, \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s\\
&= -\int_0^t\int_\Omega W_{,\varepsilon c}(c,\varepsilon(\mathbf{u}),z)\nabla c \,\cdot\, \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega W_{,\varepsilon z}(c,\varepsilon(\mathbf{u}),z)\nabla z \,\cdot\, \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s\\
&\quad - \int_0^t\int_\Omega \big(W_{,\varepsilon\varepsilon}(c,\varepsilon(\mathbf{u}),z) :\cdot \nabla(\varepsilon(\mathbf{u}))\big) \,\cdot\, \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s\\
&\le C_4\int_0^t\int_\Omega \big(|\nabla c|+|\nabla z|\big)\big(|\varepsilon(\mathbf{u})|+1\big)|\operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))|\,\mathrm{d}x\,\mathrm{d}s + C_4\int_0^t\int_\Omega |\nabla(\varepsilon(\mathbf{u}))||\operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))|\,\mathrm{d}x\,\mathrm{d}s \qquad (3.32)\\
&\le C_4\int_0^t \big(\|\nabla c\|_{L^{d+\zeta}(\Omega;\mathbb{R}^d)} + \|\nabla z\|_{L^{d+\zeta}(\Omega;\mathbb{R}^d)}\big)\big(\|\varepsilon(\mathbf{u})\|_{L^{d^\star-\eta}(\Omega;\mathbb{R}^{d\times d})}+1\big)\|\operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\\
&\quad + C_4'\int_0^t \|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)}\|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\\
&\le \sigma\int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s + C_\sigma\int_0^t \Big(\Big(\|c\|^2_{W^{1,p}(\Omega)}+\|z\|^2_{W^{1,p}(\Omega)}+1\Big)\Big(\|\mathbf{u}\|^2_{H^2(\Omega;\mathbb{R}^d)}+1\Big)\|\mathbf{u}\|^2_{H^2(\Omega;\mathbb{R}^d)}\Big)\,\mathrm{d}s\,.
\end{aligned}
$$

Here, the positive constants ζ and η again fulfill $p \ge d+\zeta$ and $\frac{1}{d+\zeta}+\frac{1}{d^\star-\eta}+\frac{1}{2}\le 1$, and we have exploited inequality (2.10) with a constant σ that we will choose later, and some $C_\sigma>0$. Moreover, we have used the structural assumption (2.19) on W (cf. also (2.21)), and estimates (3.7) and (3.27), yielding $\|c\|_{L^\infty(Q)}+\|z\|_{L^\infty(Q)}\le C$, whence

$$
\|b(c,z)\|_{L^\infty(Q)} + \|b_{,c}(c,z)\|_{L^\infty(Q)} + \|b_{,z}(c,z)\|_{L^\infty(Q)} + \|\varepsilon^*(c)\|_{L^\infty(Q)} + \|(\varepsilon^*)'(c)\|_{L^\infty(Q)} \le C.
$$

Finally, we estimate

$$
\left|\rho\int_0^t\int_\Omega \nabla\vartheta\cdot\operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}s\right| \le \eta\int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s + C_\eta\int_0^t \|\nabla\vartheta\|^2_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s \qquad (3.33)
$$

for some positive constant η to be fixed later and for some $C_\eta>0$. Combining estimates (3.29)–(3.33) with (3.28) taking into account the previously proved estimates (3.7), (3.20), and exploiting (2.22b) on $\mathbf{f}$ to estimate the last term on the right-hand side of (3.28), we obtain

$$
\begin{aligned}
&\frac{\nu_0}{2}\int_\Omega |\varepsilon(\mathbf{u}_t(t))|^2\,\mathrm{d}x + c\int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\\
&\qquad\le C\int_\Omega |\varepsilon(\mathbf{v}^0)|^2\,\mathrm{d}x + C\|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} + C\|\mathbf{d}\|^2_{H^1(0,T;H^2(\Omega;\mathbb{R}^d))\cap H^2(0,T;H^1(\Omega;\mathbb{R}^d))}\\
&\qquad\quad + \frac{c}{2}\int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s + C\left(1+\|\mathbf{u}^0\|^2_{H^2(\Omega;\mathbb{R}^d)} + \int_0^t\int_0^s \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}r\,\mathrm{d}s\right),
\end{aligned}
$$

with β_0 from (2.4) (cf. also (2.20)), where we have used the fact that $\int_0^t \|\mathbf{u}\|^2_{H^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s \le \|\mathbf{u}_0\|^2_{H^2(\Omega;\mathbb{R}^d)} + \int_0^t\int_0^s \|\mathbf{u}_t\|^2_{H^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}r\,\mathrm{d}s$ and chosen δ, ϱ, σ, and η sufficiently small. Therefore, using the standard Gronwall lemma and conditions (2.23d)–(2.23e) on the initial data $\mathbf{u}^0$ and $\mathbf{v}^0$, we conclude

$$
\|\mathbf{u}_t\|_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))\cap L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))} \le C \quad \text{whence} \quad \|\mathbf{u}\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^d))} \le C. \qquad (3.34)
$$

By comparison in (1.3e) we also get

$$
\|\mathbf{u}_{tt}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \le C. \qquad (3.35)
$$

In the end, taking into account the form (2.19) of W, we infer from (3.27) and (3.34), taking into account the continuous embedding $H^2(\Omega;\mathbb{R}^d)\subset W^{1,d^\star}(\Omega;\mathbb{R}^d)$, that

$$
\|W_{,c}(c,\varepsilon(\mathbf{u}),z)\|_{L^\infty(0,T;L^2(\Omega))} + \|W_{,z}(c,\varepsilon(\mathbf{u}),z)\|_{L^\infty(0,T;L^2(\Omega))} \le C. \qquad (3.36)
$$

Fifth estimate: Recall that, for the time being we suppose $\widehat{\beta} \in \mathrm{C}^1(\mathbb{R})$, and we will use the notation $\phi' = \beta+\gamma'$. It follows from (1.3a) and the no-flux boundary conditions on c that $\fint_\Omega c_t \,\mathrm{d}x = 0$ a.e. in $(0,T)$, hence there exists $\mathfrak{m}_0 \in \mathbb{R}$ with

$$\fint_\Omega c(t)\,\mathrm{d}x = \mathfrak{m}_0 \quad \text{for all } t \in [0,T]. \tag{3.37}$$

Now, from (1.3b) we deduce that

$$\fint_\Omega \mu \,\mathrm{d}x = \fint_\Omega \phi'(c)\,\mathrm{d}x + \fint_\Omega W_{,c}(c,\varepsilon(\mathbf{u}),z)\,\mathrm{d}x - \fint_\Omega \vartheta\,\mathrm{d}x \quad \text{a.e. in } (0,T). \tag{3.38}$$

Thanks to estimates (3.20) and (3.36), we have that

$$\|\fint_\Omega \vartheta\,\mathrm{d}x\|_{L^2(0,T)} + \left\|\fint_\Omega W_{,c}(c,\varepsilon(\mathbf{u}),z)\,\mathrm{d}x\right\|_{L^\infty(0,T)} \le C. \tag{3.39}$$

Therefore, in order to estimate $\fint_\Omega \mu\,\mathrm{d}x$ it is sufficient to gain a bound for $\fint_\Omega \phi'(c)\,\mathrm{d}x$. We shall do so by testing (1.3b) by $c - \fint_\Omega c\,\mathrm{d}x = c - \mathfrak{m}_0$. This gives for a.a. $t \in (0,T)$

$$\begin{aligned}
&\int_\Omega |\nabla c(t)|^p \,\mathrm{d}x + \int_\Omega \beta(c(t))(c(t)-\mathfrak{m}_0) + \gamma'(c(t))(c(t)-\mathfrak{m}_0)\,\mathrm{d}x \\
&= \int_\Omega \big(\vartheta(t) - W_{,c}(c(t),\varepsilon(\mathbf{u}(t)),z(t))\big)(c(t)-\mathfrak{m}_0)\,\mathrm{d}x + \int_\Omega \left(\mu(t) - \fint_\Omega \mu(t)\,\mathrm{d}x\right)(c(t)-\mathfrak{m}_0)\,\mathrm{d}x \\
&\quad - \int_\Omega c_t(t)c(t)\,\mathrm{d}x \\
&\le C\left(\|\vartheta(t)\|_{L^2(\Omega)} + \|W_{,c}(c(t),\varepsilon(\mathbf{u}(t)),z(t))\|_{L^2(\Omega)}\right)\|c(t)\|_{L^2(\Omega)} + \|\nabla\mu(t)\|_{L^2(\Omega)}\|\nabla c(t)\|_{L^2(\Omega)} \\
&\quad + \|c_t(t)\|_{L^1(\Omega)}\|c(t)\|_{L^\infty(\Omega)}
\end{aligned} \tag{3.40}$$

where for the first equality we have used that $(\fint_\Omega \mu(t)\,\mathrm{d}x)(\int_\Omega (c(t)-\mathfrak{m}_0)\,\mathrm{d}x) = 0$ and $\mathfrak{m}_0 \int_\Omega c_t(t)\,\mathrm{d}x = 0$, and for the second one the Poincaré inequality for the second integral. Now, observe that

$$\int_\Omega \gamma'(c(t))(c(t)-\mathfrak{m}_0)\,\mathrm{d}x \ge -C \tag{3.41}$$

since, by the $L^\infty(0,T;W^{1,p}(\Omega))$-estimate for c and the fact that $p > d$, we have

$$\|\gamma'(c)\|_{L^\infty(Q)} \le C\,. \tag{3.42}$$

Combining (3.40) and (3.41) with (3.1), yielding

$$\exists\, C_{\mathfrak{m}_0},\, C'_{\mathfrak{m}_0} > 0 \ \text{ for a.a. } t \in (0,T)\ :\quad \int_\Omega |\beta(c(t))|\,\mathrm{d}x \le C_{\mathfrak{m}_0}\int_\Omega \beta(c(t))(c(t)-\mathfrak{m}_0)\,\mathrm{d}x + C'_{\mathfrak{m}_0}, \tag{3.43}$$

and taking into account estimates (3.20), (3.26), (3.27), and (3.36), we conclude that $\|\beta(c)\|_{L^2(0,T;L^1(\Omega))} \le C$, whence $\|\phi'(c)\|_{L^2(0,T;L^1(\Omega))} \le C$. Then, arguing by comparison in (3.38) and taking into account (3.39) we ultimately conclude $\|\fint_\Omega \mu\,\mathrm{d}x\|_{L^2(0,T)} \le C$. Combining this with (3.27) and using the Poincaré inequality we infer that

$$\|\mu\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{3.44}$$

Sixth estimate: We now argue by comparison in (1.3b) and take into account estimates (3.20), (3.26), (3.36), and (3.44), as well as (3.42). Then we conclude that

$$\|\Delta_p(c) + \eta\|_{L^2(0,T;L^2(\Omega))} \le C.$$

Now, in view of the monotonicity of the function β, it is not difficult to deduce from the above estimate that

$$\|\Delta_p(c)\|_{L^2(0,T;L^2(\Omega))} + \|\eta\|_{L^2(0,T;L^2(\Omega))} \le C. \tag{3.45}$$

Seventh estimate: We test (1.3d) by $\frac{w}{\vartheta}$, with w a test function in $W^{1,d+\epsilon}(\Omega)$ with $\epsilon > 0$, which then ensures $w \in L^\infty(\Omega)$. Thus, using the place-holders

$$H := -c_t - z_t - \rho \mathrm{div}(\mathbf{u}_t),$$
$$J := \frac{1}{\vartheta}(g + a(c,z)\varepsilon(\mathbf{u}_t) : \mathbb{V}\varepsilon(\mathbf{u}_t) + |c_t|^2 + |z_t|^2 + m(c,z)|\nabla\mu|^2),$$

we obtain that

$$\begin{aligned}
&\left|\int_\Omega \partial_t \log(\vartheta) w \,\mathrm{d}x\right| \\
&= \left|\int_\Omega \left(Hw - \frac{\mathsf{K}(\vartheta)}{\vartheta}\nabla\vartheta\cdot\nabla w - \frac{\mathsf{K}(\vartheta)}{\vartheta^2}|\nabla\vartheta|^2 w + Jw\right)\mathrm{d}x + \int_{\partial\Omega} h\frac{w}{\vartheta}\,\mathrm{d}S\right| \\
&\le \left|\int_\Omega Hw\,\mathrm{d}x\right| + \left|\int_\Omega \frac{\mathsf{K}(\vartheta)}{\vartheta}\nabla\vartheta\cdot\nabla w\,\mathrm{d}x\right| + \left|\int_\Omega \frac{\mathsf{K}(\vartheta)}{\vartheta^2}|\nabla\vartheta|^2 w\,\mathrm{d}x\right| + \left|\int_\Omega Jw\,\mathrm{d}x\right| + \left|\int_{\partial\Omega} h\frac{w}{\vartheta}\,\mathrm{d}S\right| \\
&\doteq I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

From estimate (3.26) we deduce that $\|H\|_{L^2(0,T;L^2(\Omega))} \le C$, therefore

$$|I_1| \le \mathcal{H}(t)\|w\|_{L^2(\Omega)} \qquad \text{with } \mathcal{H}(t) = \|H(\cdot,t)\|_{L^2(\Omega)} \in L^2(0,T).$$

Analogously, also in view of (2.22c), of (3.4) and of estimate (3.26), we infer that

$$|I_4| \le \frac{1}{\underline{\vartheta}}\mathcal{J}(t)\|w\|_{L^\infty(\Omega)} \qquad \text{with } \mathcal{J}(t) := \|J(\cdot,t)\|_{L^1(\Omega)} \in L^1(0,T).$$

Moreover, $|I_5| \le \frac{1}{\underline{\vartheta}}\|h(t)\|_{L^2(\partial\Omega)}\|w\|_{L^2(\partial\Omega)}$, with $\|h(t)\|_{L^2(\partial\Omega)} \in L^1(0,T)$ thanks to (2.22d). In order to estimate I_2 and I_3 we develop the very same calculations as in the proof of [38, Sec. 3, *Sixth estimate*]. Referring to the latter paper for all details, we mention here that, exploiting the growth condition (2.17) on K, the positivity of ϑ (3.4), and the Hölder inequality, we have

$$\begin{aligned}
&|I_2| \le C\frac{C}{\underline{\vartheta}}\mathcal{O}(t)\|\nabla w\|_{L^2(\Omega;\mathbb{R}^d)} + C\widetilde{\mathcal{O}}(t)\|\nabla w\|_{L^{d+\epsilon}(\Omega;\mathbb{R}^d)} \\
&\quad\text{with } \begin{cases} \mathcal{O}(t) := \|\nabla\vartheta(t)\|_{L^2(\Omega;\mathbb{R}^d)} \in L^2(0,T) & \text{by (3.20)}, \\ \widetilde{\mathcal{O}}(t) := \|\vartheta(t)^{(\kappa+\alpha-2)/2}\nabla\vartheta(t)\|_{L^2(\Omega;\mathbb{R}^d)}\|\vartheta(t)^{(\kappa-\alpha)/2}\|_{L^{d^\star-\eta}(\Omega)} \in L^1(0,T) & \text{by (3.19), (3.20), (3.23)}, \end{cases}
\end{aligned}$$

with $\frac{1}{d+\epsilon} + \frac{1}{d^\star-\eta} + \frac{1}{2} \le 1$. With analogous arguments, we find

$$\begin{aligned}
&|I_3| \le \frac{C}{\underline{\vartheta}^2}\mathcal{O}(t)^2\|w\|_{L^\infty(\Omega)} + C\overline{\mathcal{O}}(t)\|w\|_{L^\infty(\Omega)} \\
&\text{with } \overline{\mathcal{O}}(t) = \int_\Omega \vartheta(t)^{\kappa+\alpha-2}|\nabla\vartheta(t)|^2\,\mathrm{d}x + \int_\Omega |\nabla\vartheta(t)|^2\,\mathrm{d}x \in L^1(0,T) \ \text{ by (3.19) and (3.20).}
\end{aligned}$$

All in all, we infer that there exists a positive function $\mathcal{C} \in L^1(0,T)$ such that $|\int_\Omega \partial_t \log(\vartheta(t)) w\,\mathrm{d}x| \le \mathcal{C}(t)\|w\|_{W^{1,d+\epsilon}(\Omega)}$ for a.a. $t \in (0,T)$. Hence,

$$\|\partial_t \log(\vartheta)\|_{L^1(0,T;(W^{1,d+\epsilon}(\Omega)'))} \le C. \tag{3.46}$$

Eighth estimate [$\boldsymbol{\kappa \in (1, 5/3)}$ if $\mathbf{d=3}$ and $\boldsymbol{\kappa \in (1,2)}$ if $\mathbf{d=2}$]: We multiply (1.3d) by a test function $w \in W^{1,\infty}(\Omega)$ (which e.g. holds if $w \in W^{2,d+\epsilon}(\Omega)$ for $\epsilon > 0$) and find

$$\left|\int_\Omega \vartheta_t w\,\mathrm{d}x\right| \le \left|\int_\Omega Lw\,\mathrm{d}x\right| + \left|\int_\Omega \mathsf{K}(\vartheta)\nabla\vartheta\cdot\nabla w\,\mathrm{d}x\right| + \left|\int_{\partial\Omega} hw\,\mathrm{d}S\right| \doteq I_1 + I_2 + I_3,$$

where we have set

$$L = -c_t\vartheta - z_t\vartheta - \rho\vartheta\mathrm{div}(\mathbf{u}_t) + g + a(c,z)\varepsilon(\mathbf{u}_t) : \mathbb{V}\varepsilon(\mathbf{u}_t) + |c_t|^2 + |z_t|^2 + m(c,z)|\nabla\mu|^2.$$

Therefore,

$$|I_1| \leq \mathcal{L}(t)\|w\|_{L^\infty(\Omega)} \quad \text{with } \mathcal{L}(t) := \|L(t)\|_{L^1(\Omega)} \in L^1(0,T), \quad |I_3| \leq \|h(t)\|_{L^2(\partial\Omega)}\|w\|_{L^2(\partial\Omega)} \text{ with } h \in L^1(0,T)$$

thanks to (2.22c), (3.20), and (3.26) for I_1, and (2.22d) for I_3. We estimate I_2 by proceeding exactly in the same way as for [38, Sec. 3, *Seventh estimate*]. Namely, taking into account once again the growth condition (2.17) on K, we find

$$|I_2| \leq C\|\vartheta^{(\kappa-\alpha+2)/2}\|_{L^2(\Omega)}\|\vartheta^{(\kappa+\alpha-2)/2}\nabla\vartheta\|_{L^2(\Omega;\mathbb{R}^d)}\|\nabla w\|_{L^\infty(\Omega;\mathbb{R}^d)} + C\|\nabla\vartheta\|_{L^2(\Omega;\mathbb{R}^d)}\|\nabla w\|_{L^2(\Omega;\mathbb{R}^d)}. \tag{3.47}$$

Observe that, since $\kappa < \frac{5}{3}$ if $d = 3$, and $\kappa < 2$ if $d = 2$, and α can be chosen arbitrarily close to 1, from estimate (3.21) we have that $\vartheta^{(\kappa-\alpha+2)/2}$ is bounded in $L^2(0,T;L^2(\Omega))$. Thus, also taking into account (3.20), we conclude that $|I_2| \leq C\mathcal{L}^*(t)\|\nabla w\|_{L^\infty(\Omega)}$ for some $\mathcal{L}^* \in L^1(0,T)$. Hence,

$$\|\vartheta_t\|_{L^1(0,T;W^{1,\infty}(\Omega)')} \leq C. \tag{3.48}$$

Ninth estimate: We test (1.3a) by $\Delta\mu$ and integrate in time. It follows

$$\int_0^t\int_\Omega \mathrm{div}\left(m(c,z)\nabla\mu\right)\Delta\mu\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_\Omega c_t\Delta\mu\,\mathrm{d}x\,\mathrm{d}s. \tag{3.49}$$

The left-hand side is estimated below by exploiting Hypotheses (II) and the boundedness $\|c\|_{L^\infty(Q)}+\|z\|_{L^\infty(Q)} \leq C$, viz.

$$\begin{aligned}\int_0^t\int_\Omega \mathrm{div}\left(m(c,z)\nabla\mu\right)\Delta\mu\,\mathrm{d}x\,\mathrm{d}s &\geq \int_0^t\int_\Omega\left(\nabla m(c,z)\cdot\nabla\mu\right)\Delta\mu\,\mathrm{d}x\,\mathrm{d}s + m_0\int_0^t\int_\Omega|\Delta\mu|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\geq -C\int_0^t\int_\Omega(|\nabla c|+|\nabla z|)|\nabla\mu||\Delta\mu|\,\mathrm{d}x\,\mathrm{d}s + m_0\int_0^t\int_\Omega|\Delta\mu|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

By using the interpolation inequality (2.11) and by using analogous calculations as in the *Fourth estimate*, we find by Young's inequality

$$\begin{aligned}&\int_0^t\int_\Omega(|\nabla c|+|\nabla z|)|\nabla\mu||\Delta\mu|\,\mathrm{d}x\,\mathrm{d}s\\ &\quad\leq C\int_0^t\left(\|\nabla c\|_{L^{d+\varsigma}(\Omega;\mathbb{R}^d)}+\|\nabla z\|_{L^{d+\varsigma}(\Omega;\mathbb{R}^d)}\right)\|\nabla\mu\|_{L^{d^*-\eta}(\Omega;\mathbb{R}^d)}\|\Delta\mu\|_{L^2(\Omega)}\,\mathrm{d}s\\ &\quad\leq C\left(\|\nabla c\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^d))}+\|\nabla z\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^d))}\right)\int_0^t\|\nabla\mu\|_{L^{d^*-\eta}(\Omega;\mathbb{R}^d)}\|\Delta\mu\|_{L^2(\Omega)}\,\mathrm{d}s\\ &\quad\leq C'\int_0^t\left(\varrho\|\nabla\mu\|_{H^1(\Omega;\mathbb{R}^d)}+C_\varrho\|\nabla\mu\|_{L^2(\Omega;\mathbb{R}^d)}\right)\|\Delta\mu\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\\ &\quad\leq \varrho C'C_\delta\int_0^t\|\nabla\mu\|^2_{H^1(\Omega;\mathbb{R}^d)}\,\mathrm{d}s + C'C_\varrho C_\delta\int_0^t\|\nabla\mu\|^2_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s + \delta C'\int_0^t\|\Delta\mu\|^2_{L^2(\Omega)}\,\mathrm{d}s.\end{aligned}$$

By choosing suitable $\delta > 0$ and $\varrho > 0$, we see that

$$\int_0^t\int_\Omega\left(|\nabla c|+|\nabla z|\right)|\nabla\mu||\Delta\mu|\,\mathrm{d}x\,\mathrm{d}s \leq \epsilon\int_0^t\|\mu\|^2_{H^2(\Omega)}\,\mathrm{d}s + C_\epsilon\int_0^t\|\nabla\mu\|^2_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s.$$

All in all, we find from the above estimates

$$\int_0^t\|\Delta\mu\|^2_{L^2(\Omega)}\,\mathrm{d}s \leq \epsilon\int_0^t\|\mu\|^2_{H^2(\Omega)}\,\mathrm{d}s + C_\epsilon\int_0^t\|c_t\|^2_{L^2(\Omega)}\,\mathrm{d}s + C_\epsilon\int_0^t\|\nabla\mu\|^2_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s,$$

where the second and the third term on the right-hand side are bounded by (3.26) for fixed $\epsilon > 0$. By the H^2-elliptic regularity estimate for homogeneous Neumann problems, i.e.

$$\|\mu\|^2_{H^2(\Omega)} \le C\big(\|\Delta\mu\|^2_{L^2(\Omega)} + \|\mu\|^2_{H^1(\Omega)}\big),$$

we conclude by choosing $\epsilon > 0$ sufficiently small and by using the boundedness of $\|\mu\|_{L^2(0,T;H^1(\Omega))}$ in (3.26) that

$$\|\mu\|_{L^2(0,T;H^2(\Omega))} \le C. \tag{3.50}$$

□

4 Time discretization and regularizations

In this section we will introduce and motivate a *thermodynamically consistent time-discretization scheme* for system (1.3) and devote a large part of Sec. 4.2 to the proof that it admits solutions. Next, in Sec. 4.3 we will derive the energy and entropy inequalities fulfilled by the discrete solutions, and, starting from them, we will obtain a series of a priori estimates on the approximate solutions.

4.1 Setup of the time-discrete system

We consider an equidistant partition of $[0,T]$, with time-step $\tau > 0$ and nodes

$$t_\tau^k := k\tau, \tag{4.1}$$

$k = 0, \ldots, K_\tau$, and we approximate the data $\mathbf{f}$, g, and h by local means, i.e. setting for all $k = 1, \ldots, K_\tau$

$$\mathbf{f}_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \mathbf{f}(s)\,\mathrm{d}s\,, \qquad g_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} g(s)\,\mathrm{d}s\,, \qquad h_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} h(s)\,\mathrm{d}s\,. \tag{4.2}$$

In what follows, for a given K_τ-tuple $(v_\tau^k)_{k=1}^{K_\tau}$ the time-discrete derivative is denoted by

$$D_{\tau,k}(v) = \frac{v_\tau^k - v_\tau^{k-1}}{\tau} \quad \text{so that} \quad D_{\tau,k}(D_{\tau,k}(v)) = \frac{v_\tau^k - 2v_\tau^{k-1} + v_\tau^{k-2}}{\tau^2}.$$

Before stating the complete time-discrete scheme in Problem 4.1, we are going to introduce its main ingredients in what follows.

Regularization of the coefficient functions depending on c In the following we will analyze a specially chosen time-discretization scheme for system (1.3). To ensure suitable coercivity properties in the time-discrete system needed for existence of solutions we utilize the following ω-regularizations which will eventually vanish as $\omega \downarrow 0$:

- First of all, we will replace the maximally monotone operator β (the derivative of the convex part of the potential ϕ (see Hypothesis (I)) by its Yosida regularization $\beta_\omega \in \mathrm{C}^0(\mathbb{R})$ with Yosida index $\omega \in (0,\infty)$. This will be crucial to render rigorously the *Fifth a priori estimate* on the time-discrete level, cf. the calculations in Sec. 4.4. Observe that the Yosida approximation $\widehat{\beta}_\omega \in C^1(\mathbb{R})$ of $\widehat{\beta}$, fulfilling $\widehat{\beta}'_\omega = \beta_\omega$, is still convex, and that $\beta_\omega(0) = 0$. For notational consistency we set $\phi_\omega := \widehat{\beta}_\omega + \gamma$.
- Let $\{\mathcal{R}_\omega\}_{\omega>0} \subseteq C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ be a family of functions (we can think of "smoothed truncations") such that:

$$\forall M > 0 \quad \exists \omega_0 > 0 \qquad \forall \omega \in (0,\omega_0),\ c \in (-M,M): \qquad \mathcal{R}_\omega(c) = c. \tag{4.3}$$

They have the role to somehow provide for the information that c is bounded, which is not supplied by the concentration potential ϕ, defined on all of $\mathbb{R}$. In turn, this information is crucial in order to make some of the following calculations rigorous. The limit passage as $\omega \downarrow 0$ will be possible thanks to an a priori bound for c in $L^\infty(Q)$, cf. Sec. 5 ahead.

We define the following regularizations for the elastic energy density:

$$W^\omega(c, \varepsilon, z) := W(\mathcal{R}_\omega(c), \varepsilon, z).$$

and observe that for fixed $\omega > 0$ and fixed $\varepsilon \in \mathbb{R}^{d\times d}_{sym}$ and $z \in \mathbb{R}$ (cf. also (2.21)):

$$|W^\omega(c, \varepsilon, z)| + |W^\omega_{,c}(c, \varepsilon, z)| + |W^\omega_{,cc}(c, \varepsilon, z)| \le C \qquad \text{uniformly in } c \in \mathbb{R}. \tag{4.4}$$

Throughout this section we neglect the subscript ω on the solutions c, μ, z, ϑ and $\mathbf{u}$ for the sake of readability.

Convex-concave splitting of the coefficient functions Let us mention in advance how all the various nonlinear terms in (1.3) will be coped with in the discrete system (4.9), which is in fact carefully designed in such a way as to ensure the validity of the *discrete total energy inequality*, cf. the forthcoming Lemma 4.5. To this aim, it will be crucial to employ the *convex-concave splitting* of the functions $c \mapsto W^\omega(c, \varepsilon, z)$, $z \mapsto W^\omega(c, \varepsilon, z)$, $z \mapsto \sigma(z)$, as well as the specific splitting (4.8) below (cf. also (2.14)) for ϕ_ω. Recall that, a convex-concave decomposition of some real-valued $\mathrm{C}^2(I)$-function ψ with bounded second derivative on an interval I may be canonically given by $\psi = \breve{\psi} + \hat{\psi}$, with

$$\breve{\psi}(x) := \psi(x) + \frac{1}{2}\Big(\max_{y\in I} |\psi''(y)|\Big)x^2, \qquad\qquad \hat{\psi}(x) := -\frac{1}{2}\Big(\max_{y\in I} |\psi''(y)|\Big)x^2. \tag{4.5}$$

Therefore, we will proceed as follows:

– The nonlinear contribution $\sigma'(z)$ in (1.3c) will be discretized via the convex-concave splitting (4.5) on $I = [0, 1]$:

$$\sigma'(z) \;\; \text{via} \;\; (\breve{\sigma})'(z^k_\tau) + (\hat{\sigma})'(z^{k-1}_\tau).$$

– For the time-discrete version of the term $W_{,c}(c, \varepsilon(\mathbf{u}), z)$ in (1.3b) and $W_{,z}(c, \varepsilon(\mathbf{u}), z)$ in (1.3c) we will resort to partial convex-concave splittings of W^ω. To denote them, we will use the symbols (4.5), combined with subscripts to denote the variable with respect to which the splitting is computed. Therefore, we set

$$\breve{W}^\omega_1(c, \varepsilon, z) := W^\omega(c, \varepsilon(\mathbf{u}), z) + \frac{1}{2}\Big(\sup_{\widetilde{c}\in\mathbb{R}} |W^\omega_{,cc}(\widetilde{c}, \varepsilon, z)|\Big)c^2, \tag{4.6a}$$

$$\hat{W}^\omega_1(c, \varepsilon, z) := -\frac{1}{2}\Big(\sup_{\widetilde{c}\in\mathbb{R}} |W^\omega_{,cc}(\widetilde{c}, \varepsilon, z)|\Big)c^2, \tag{4.6b}$$

$$\breve{W}^\omega_3(c, \varepsilon, z) := W^\omega(c, \varepsilon(\mathbf{u}), z) + \frac{1}{2}\Big(\sup_{\widetilde{z}\in[0,1]} |W_{,zz}{}^\omega(c, \varepsilon, \widetilde{z})|\Big)z^2, \tag{4.6c}$$

$$\hat{W}^\omega_3(c, \varepsilon, z) := -\frac{1}{2}\Big(\sup_{\widetilde{z}\in[0,1]} |W_{,zz}{}^\omega(c, \varepsilon, \widetilde{z})|\Big)z^2. \tag{4.6d}$$

Note that these functions are well-defined for fixed $\omega > 0$ due to (4.4). The splitting of W^ω with respect to $\varepsilon(\mathbf{u})$ is not needed due to the convexity of W^ω with respect to $\varepsilon(\mathbf{u})$ by the structural assumption (2.19) and the non-negativity of b in Hypothesis (V). We easily see that

$$W^\omega = \breve{W}^\omega_1 + \hat{W}^\omega_1 = \breve{W}^\omega_3 + \hat{W}^\omega_3$$

and that

$$\left.\begin{array}{ll}\breve{W}_1^\omega(\cdot,\varepsilon,z) \text{ is convex on } \mathbb{R}, & \hat{W}_1^\omega(\cdot,\varepsilon,z) \text{ is concave on } \mathbb{R}\end{array}\right\}\text{ for all fixed } \varepsilon, z,$$

$$\left.\begin{array}{l}W^\omega(c,\cdot,z) \text{ is convex on } \mathbb{R}^{n\times n}_{sym}\end{array}\right\}\text{ for all fixed } c, z,$$

$$\left.\begin{array}{ll}\breve{W}_3^\omega(c,\varepsilon,\cdot) \text{ is convex on } [0,1], & \hat{W}_3^\omega(c,\varepsilon,\cdot) \text{ is concave on } [0,1]\end{array}\right\}\text{ for all fixed } c, \varepsilon.$$

We will replace the terms $W_{,c}$, $W_{,\varepsilon}$, and $W_{,z}$ in system (1.3) by their time-discretized and regularized versions:

$$\begin{aligned}
&W_{,c}(c,\varepsilon(\mathbf{u}),z) && \text{via} && \breve{W}^\omega_{1,c}(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1})+\hat{W}^\omega_{1,c}(c_\tau^{k-1},\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1}),\\
&W_{,\varepsilon}(c,\cdot,z) && \text{via} && W^\omega_{,\varepsilon}(c_\tau^k,\varepsilon(\mathbf{u}_\tau^k),z_\tau^k),\\
&W_{,z}(c,\varepsilon(\mathbf{u}),z) && \text{via} && \breve{W}^\omega_{3,z}(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^k)+\hat{W}^\omega_{3,z}(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1}).
\end{aligned}$$

By exploiting convexity and concavity estimates this time-discretization scheme leads to the crucial estimate

$$\begin{aligned}
&\Big(\breve{W}^\omega_{1,c}(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1})+\hat{W}^\omega_{1,c}(c_\tau^{k-1},\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1})\Big)(c_\tau^k-c_\tau^{k-1})\\
&+W^\omega_{,\varepsilon}(c_\tau^k,\varepsilon(\mathbf{u}_\tau^k),z_\tau^k):\varepsilon(\mathbf{u}_\tau^k-\mathbf{u}_\tau^{k-1})\\
&+\Big(\breve{W}^\omega_{3,z}(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^k)+\hat{W}^\omega_{3,z}(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1})\Big)(z_\tau^k-z_\tau^{k-1})\\
&\quad\geq W^\omega(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1})-W^\omega(c_\tau^{k-1},\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1})\\
&\qquad+W^\omega(c_\tau^k,\varepsilon(\mathbf{u}_\tau^k),z_\tau^k)-W^\omega(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^k)\\
&\qquad+W^\omega(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^k)-W^\omega(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1})\\
&\quad\geq W^\omega(c_\tau^k,\varepsilon(\mathbf{u}_\tau^k),z_\tau^k)-W^\omega(c_\tau^{k-1},\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1}),
\end{aligned}\tag{4.7}$$

which will be used later in the proof of the discrete total energy inequality.

– We will discretize the (formally written) term $\phi'(c)=\beta(c)+\gamma'(c)$ in (1.3b) in the following way: As mentioned above the maximally monotone operator β is replaced by its Yosida regularization $\beta_\omega\in C^0(\mathbb{R})$. Hence, in view of the λ_γ-convexity of γ (cf. Remark 2.1), the functions

$$\breve{\phi}_\omega(c):=\widehat{\beta}_\omega(c)+\lambda_\gamma\frac{c^2}{2}\quad\text{and}\quad\hat{\phi}(c):=\gamma(c)-\lambda_\gamma\frac{c^2}{2}\tag{4.8}$$

provide a convex-concave decomposition of $\phi_\omega:=\widehat{\beta}_\omega+\gamma$. Thus, we will approximate

$$\phi'(c)\ \text{ via }\ (\breve{\phi}_\omega)'(c_\tau^k)+(\hat{\phi})'(c_\tau^{k-1})\qquad\text{with } \breve{\phi}_\omega,\ \hat{\phi} \text{ given by (4.8).}$$

Statement of the time-discrete problem and existence result In the following we are going to describe the time-discrete problem formally. Later on the precise spaces and a weak notion of solution will be fixed. The time-discrete problem (formally) reads as follows:

Problem 4.1. Let $\omega>0$ and $\tau>0$ be given. Find functions $\{(c_\tau^k,\mu_\tau^k,z_\tau^k,\vartheta_\tau^k)\}_{k=0}^{K_\tau}$ and $\{\mathbf{u}_\tau^k\}_{k=-1}^{K_\tau}$ which satisfy for all $k\in\{1,\ldots,K_\tau\}$ the following time-discrete version of (1.3):

(i) Cahn-Hilliard system:

$$D_{\tau,k}(c)=\operatorname{div}\big(m(c_\tau^{k-1},z_\tau^{k-1})\nabla\mu_\tau^k\big),\tag{4.9a}$$

$$\begin{aligned}\mu_\tau^k=&-\Delta_p(c_\tau^k)+(\breve{\phi}_\omega)'(c_\tau^k)+(\hat{\phi})'(c_\tau^{k-1})+\breve{W}^\omega_{1,c}(c_\tau^k,\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1})\\&+\hat{W}^\omega_{1,c}(c_\tau^{k-1},\varepsilon(\mathbf{u}_\tau^{k-1}),z_\tau^{k-1})-\vartheta_\tau^k+D_{\tau,k}(c),\end{aligned}\tag{4.9b}$$

(ii) damage equation:
$$
\begin{aligned}
&D_{\tau,k}(z) - \Delta_p(z_\tau^k) + \ell_\tau^k + \zeta_\tau^k + (\breve{\sigma})'(z_\tau^k) + (\hat{\sigma})'(z_\tau^{k-1}) \\
&= -\breve{W}^\omega_{3,z}(c_\tau^k, \varepsilon(\mathbf{u}_\tau^{k-1}), z_\tau^k) - \hat{W}^\omega_{3,z}(c_\tau^k, \varepsilon(\mathbf{u}_\tau^{k-1}), z_\tau^{k-1}) + \vartheta_\tau^k
\end{aligned}
\tag{4.9c}
$$

with

$$
\ell_\tau^k \in \partial I_{[0,\infty)}\big(z_\tau^k\big), \qquad \zeta_\tau^k \in \partial I_{(-\infty,0]}\big(D_{\tau,k}(z)\big),
$$

(iii) temperature equation:
$$
\begin{aligned}
&D_{\tau,k}(\vartheta) - \mathrm{div}(\mathsf{K}(\vartheta_\tau^k)\nabla\vartheta_\tau^k) + D_{\tau,k}(c)\vartheta_\tau^k + D_{\tau,k}(z)\vartheta_\tau^k + \rho\vartheta_\tau^k \,\mathrm{div}(D_{\tau,k}(\mathbf{u})) \\
&= g_\tau^k + |D_{\tau,k}(c)|^2 + |D_{\tau,k}(z)|^2 + m(c_\tau^{k-1}, z_\tau^{k-1})|\nabla\mu_\tau^k|^2 \\
&\qquad + a(c_\tau^{k-1}, z_\tau^{k-1})\varepsilon(D_{\tau,k}(\mathbf{u})) : \mathbb{V}\varepsilon(D_{\tau,k}(\mathbf{u})),
\end{aligned}
\tag{4.9d}
$$

(iv) balance of forces:
$$
D_{\tau,k}(D_{\tau,k}(\mathbf{u})) - \mathrm{div}\Big(a(c_\tau^{k-1}, z_\tau^{k-1})\mathbb{V}\varepsilon(D_{\tau,k}(\mathbf{u})) + W^\omega_{,\varepsilon}(c_\tau^k, \varepsilon(\mathbf{u}_\tau^k), z_\tau^k) - \rho\vartheta_\tau^k\mathbb{1}\Big) = \mathbf{f}_\tau^k,
\tag{4.9e}
$$

supplemented with the initial data

$$
\left.
\begin{array}{lll}
c_\tau^0 = c^0, & z_\tau^0 = z^0, & \vartheta_\tau^0 = \vartheta^0, \\
\mathbf{u}_\tau^0 = \mathbf{u}^0, & \mathbf{u}_\tau^{-1} = \mathbf{u}^0 - \tau\mathbf{v}^0 &
\end{array}
\right\} \quad \text{a.e. in } \Omega
\tag{4.10}
$$

and the boundary data

$$
\left.
\begin{array}{lll}
\nabla c_\tau^k \cdot \mathbf{n} = 0, & m(c_\tau^{k-1}, z_\tau^{k-1})\nabla\mu_\tau^k \cdot \mathbf{n} = 0, & \nabla z_\tau^k \cdot \mathbf{n} = 0, \\
\mathsf{K}(\vartheta_\tau^k)\nabla\vartheta_\tau^k \cdot \mathbf{n} = h_\tau^k, & \mathbf{u}_\tau^k = \mathbf{d}_\tau^k &
\end{array}
\right\} \quad \text{a.e. on } \partial\Omega.
\tag{4.11}
$$

Remark 4.2. A few comments on Problem 4.1 are in order:

(i) It will turn out that a solution of the time-discrete problem always satisfies the constraints:
$$
z_\tau^k \in [0,1], \qquad D_{\tau,k}(z) \le 0, \qquad \vartheta_\tau^k \ge \underline{\vartheta} \quad (\text{for some } \underline{\vartheta} > 0) \qquad \text{a.e. in } \Omega
\tag{4.12}
$$
as long as the initial data satisfy (2.23).

(ii) Observe that the scheme is fully implicit and, in particular, the discrete temperature equation (4.9d) is coupled with (4.9b), (4.9c), and (4.9e) via the implicit term ϑ_τ^k featuring in $D_{\tau,k}(c)\vartheta_\tau^k$, $D_{\tau,k}(z)\vartheta_\tau^k$, and $\rho\,\vartheta_\tau^k\,\mathrm{div}(D_{\tau,k}(\mathbf{u}))$. Indeed, having ϑ_τ^k implicit in these terms is crucial for the argument we will develop later on for proving the positivity of ϑ_τ^k, cf. the proof of Lemma 4.7.

(iii) The subgradients ℓ_τ^k and ζ_τ^k account for non-negativity as well as irreversibility constraints for z. In the pointwise formulation we obtain by the sum rule for $z_\tau^{k-1} \neq 0$ and by direct calculations for $z_\tau^{k-1} = 0$
$$
\partial I_{[0,\infty)}\big(z_\tau^k\big) + \partial I_{(-\infty,0]}\big(D_{\tau,k}(z)\big) = \partial I_{[0,\infty)}\big(z_\tau^k\big) + \partial I_{(-\infty,z_\tau^{k-1}]}\big(z_\tau^k\big) = \partial I_{[0,z_\tau^{k-1}]}(z_\tau^k)
$$
and, consequently, the double inclusion in (ii) may be replaced by the single inclusion
$$
\begin{aligned}
&D_{\tau,k}(z) - \Delta_p(z_\tau^k) + \xi_\tau^k + (\breve{\sigma})'(z_\tau^k) + (\hat{\sigma})'(z_\tau^{k-1}) \\
&= -\breve{W}^\omega_{3,z}(c_\tau^k, \varepsilon(\mathbf{u}_\tau^{k-1}), z_\tau^k) - \hat{W}^\omega_{3,z}(c_\tau^k, \varepsilon(\mathbf{u}_\tau^{k-1}), z_\tau^{k-1}) + \vartheta_\tau^k
\end{aligned}
$$
with
$$
\xi_\tau^k \in \partial I_{[0,z_\tau^{k-1}]}(z_\tau^k).
$$

(iv) By assuming the additional growth assumptions

$$\sigma(0) \le \sigma(z), \qquad b(c,0) \le b(c,z) \text{ for all } c \in \mathbb{R}, z \in \mathbb{R} \text{ with } z < 0,$$

it is possible the prove a maximum principle for equation (4.9c) which ensures $z_\tau^k \ge 0$ as long as $z^0 \ge 0$. In this case the subdifferential term $\partial I_{[0,\infty)}(z_\tau^k)$ in equation (4.9c) may be dropped. For details we refer to [26, Proposition 5.5].

We can now state our existence result for Problem 4.1, where we also fix the concept of weak solution to system (4.9). With this aim, let us also introduce the nonlinear operator $\mathcal{A}^k : X \to H^1(\Omega)'$, with

$$\begin{aligned} &X := \Big\{\theta \in H^1(\Omega) \ : \ \int_\Omega \mathsf{K}(\theta)\nabla\theta\cdot\nabla v \,\mathrm{d}x \text{ is well-defined for all } v \in H^1(\Omega)\Big\}, \\ &\big\langle \mathcal{A}^k(\theta), v\big\rangle_{H^1} := \int_\Omega \mathsf{K}(\theta)\nabla\theta\cdot\nabla v\,\mathrm{d}x - \int_{\partial\Omega} h_\tau^k v\,\mathrm{d}x. \end{aligned} \tag{4.13}$$

Proposition 4.3. *Assume **Hypotheses (I)–(V)**, as well as* (2.22) *on* $(\mathbf{f}, g, h)$ *and* (2.23) *on* $(c^0, z^0, \vartheta^0, \mathbf{u}^0, \mathbf{v}^0)$.

Then, for every $\omega > 0$ and $\tau > 0$ Problem 4.1 admits a weak solution

$$\{(c_\tau^k, \mu_\tau^k, z_\tau^k, \vartheta_\tau^k, \mathbf{u}_\tau^k)\}_{k=1}^{K_\tau} \subseteq W^{1,p}(\Omega) \times H^2_N(\Omega) \times W^{1,p}(\Omega) \times H^1(\Omega) \times H^2(\Omega;\mathbb{R}^d) \tag{4.14}$$

in the following sense:

- (4.9a) *and* (4.9e) *are fulfilled a.e. in* Ω, *with the boundary conditions* $\nabla c_\tau^k \cdot n = 0$ *and* $\mathbf{u}_\tau^k = \mathbf{d}_\tau^k$ *a.e. in* $\partial\Omega$,
- (4.9b) *is fulfilled in* $W^{1,p}(\Omega)'$,
- (4.9d) *is fulfilled in* $H^1(\Omega)'$, *in the form*

$$\begin{aligned} &D_{\tau,k}(\vartheta) + \mathcal{A}^k(\vartheta_\tau^k) + D_{\tau,k}(c)\vartheta_\tau^k + D_{\tau,k}(z)\vartheta_\tau^k + \rho\vartheta_\tau^k \operatorname{div}(D_{\tau,k}(\mathbf{u})) \\ &= g_\tau^k + |D_{\tau,k}(c)|^2 + |D_{\tau,k}(z)|^2 + m(c_\tau^{k-1}, z_\tau^{k-1})|\nabla\mu_\tau^k|^2 \\ &\qquad + a(c_\tau^{k-1}, z_\tau^{k-1})\varepsilon(D_{\tau,k}(\mathbf{u})) : \mathbb{V}\varepsilon(D_{\tau,k}(\mathbf{u})), \end{aligned}$$

- (4.9c) *is reformulated as (cf. Remark 4.2 (ii))*

$$\begin{aligned} &D_{\tau,k}(z) - \Delta_p(z_\tau^k) + \xi_\tau^k + (\breve{\sigma})'(z_\tau^k) + (\hat{\sigma})'(z_\tau^{k-1}) \\ &= -\breve{W}^\omega_{3,z}(c_\tau^k, \varepsilon(\mathbf{u}_\tau^{k-1}), z_\tau^k) - \hat{W}^\omega_{3,z}(c_\tau^k, \varepsilon(\mathbf{u}_\tau^{k-1}), z_\tau^{k-1}) + \vartheta_\tau^k \end{aligned} \tag{4.15}$$

  and fullfilled in $W^{1,p}(\Omega)'$ *with* $\xi_\tau^k \in \partial I_{Z_\tau^{k-1}}(z_\tau^k)$ *where*

$$Z_\tau^{k-1} := \{z \in W^{1,p}(\Omega)\,|\, 0 \le z \le z_\tau^{k-1}\}, \tag{4.16}$$

- *the initial conditions* (4.10) *and the boundary conditions* (4.11) *are satisfied,*
- *the constraints* (4.12) *are satisfied.*

We will prove Proposition 4.3 in the ensuing section by performing a double passage to the limit in a carefully devised approximation of system (4.9), depending on two additional parameters ν and ϱ.

4.2 Proof of Proposition 4.3

We will split the proof of Prop. 4.3 in several steps and obtain a series of intermediate results. Our argument is based on a double approximation procedure and two consecutive limit passages. More precisely, we approximate system (4.9) by

(i) adding the higher order terms

$$
\begin{array}{ll}
+\,\nu \operatorname{div}\left(|\nabla\mu_\tau^k|^{\varrho-2}\nabla\mu_\tau^k\right) - \nu\mu_\tau^k & \text{to the right-hand sides of the discrete Cahn-Hilliard equation (4.9a),}\\
+\,\nu|c_\tau^k|^{\varrho-2}c_\tau^k & \text{to the right-hand sides of the discrete Cahn-Hilliard equation (4.9b),}\\
+\,\nu|z_\tau^k|^{\varrho-2}z_\tau^k & \text{to the left-hand sides of the discrete damage equation (4.9c),}\\
-\,\nu \operatorname{div}\left(|\varepsilon(\mathbf{u}_\tau^k-\mathbf{d}_\tau^k)|^{\varrho-2}\varepsilon(\mathbf{u}_\tau^k-\mathbf{d}_\tau^k)\right) & \text{to the left-hand side of the discrete momentum equation (4.9e)}
\end{array}
$$

with $\nu > 0$ and $\varrho > 4$. In this way, the quadratic growth of the terms on the right-hand side of the temperature equation will be compensated and coercivity properties of the elliptic operators involved in the time-discrete scheme ensured.

(ii) Truncating the heat conduction function K and replacing it with a bounded K_M with $M \in \mathbb{N}$. In this way the elliptic operator in the discrete heat equation will be defined on $H^1(\Omega)$, with values in $H^1(\Omega)'$, but we will of course loose the enhanced estimates on the temperature variable provided by the coercivity properties of K. That is why, we will have to accordingly truncate all occurrences of ϑ in the quadratic terms.

Let us mention in advance that this double approximation, leading to system (4.21) later on, shall be devised in such a way as to allow us to prove the existence of solutions to (4.21), by resorting to a result from the theory of elliptic systems featuring pseudomonotone operators, cf. [39].

A caveat on notation: the solutions to the approximate discrete system (4.21) at the k-th time step, with given $S_\tau^{k-1} := (c_\tau^{k-1}, z_\tau^{k-1}, \mathbf{u}_\tau^{k-1}, \vartheta_\tau^{k-1})$ and $\mathbf{u}_\tau^{k-2}$, will depend on the parameters τ, ν and M (and on ω which we omit at the moment). Therefore, we should denote them by $S_{\tau,\nu,M}^k := (c_{\tau,\nu,M}^k, \mu_{\tau,\nu,M}^k, z_{\tau,\nu,M}^k, \vartheta_{\tau,\nu,M}^k, \mathbf{u}_{\tau,\nu,M}^k)$. However, to increase readability, we will simply write c^k, μ^k, z^k, ϑ^k and $\mathbf{u}^k$ and use the notation $c_M^k, \ldots, \mathbf{u}_M^k$ ($c_\nu^k, \ldots, \mathbf{u}_\nu^k$, respectively), only upon addressing the limit passage as $M \to \infty$ (as $\nu \downarrow 0$, respectively).

Outline of the proof of Proposition 4.3: For given $\tau > 0$, the construction of the solution quintuples $S_{\tau,\nu,M}^k$ and the limit passages as $M \to \infty$ and as $\nu \downarrow 0$ are performed recursively over $k = 1, \ldots, K_\tau$ in the following order:

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(S_\tau^{k-2}, \mathbf{u}_\tau^{k-3}) & \xrightarrow[\text{Step 1}]{\text{pseudo-mon. op. theory}} & S_{\tau,\nu,M}^{k-1} & \xrightarrow[\text{Step 2}]{M\to\infty} & S_{\tau,\nu}^{k-1} & \xrightarrow[\text{Step 3}]{\nu\downarrow 0} & S_\tau^{k-1} \\
(S_\tau^{k-1}, \mathbf{u}_\tau^{k-2}) & \xrightarrow[\text{Step 1}]{\text{pseudo-mon. op. theory}} & S_{\tau,\nu,M}^{k} & \xrightarrow[\text{Step 2}]{M\to\infty} & S_{\tau,\nu}^{k} & \xrightarrow[\text{Step 3}]{\nu\downarrow 0} & S_\tau^{k} \\
(S_\tau^{k}, \mathbf{u}_\tau^{k-1}) & \xrightarrow[\text{Step 1}]{\text{pseudo-mon. op. theory}} & S_{\tau,\nu,M}^{k+1} & \xrightarrow[\text{Step 2}]{M\to\infty} & S_{\tau,\nu}^{k+1} & \xrightarrow[\text{Step 3}]{\nu\downarrow 0} & S_\tau^{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

The construction of $S_{\tau,\nu,M}^k$ will be tackled in Subsection 4.2.1, the limit passage as $M \to \infty$ to $S_{\tau,\nu}^k$ in Subsection 4.2.2, and the one as $\nu \downarrow 0$ to S_τ^k in Subsection 4.2.3. Throughout all of them, we will work under the assumptions of Proposition 4.3, and omit to explicitly invoke them in the following statements.

4.2.1 Step 1: Existence and uniform estimates of the time-discrete system with ν- and M-regularization.

From now on let $\nu > 0$, $\varrho > 4$ and $M \in \mathbb{N}$. Let

$$\mathsf{K}_M(r) := \begin{cases} \mathsf{K}(0) & \text{if } r < 0, \\ \mathsf{K}(r) & \text{if } 0 \le r \le M, \\ \mathsf{K}(M) & \text{if } r > M \end{cases} \tag{4.17}$$

and accordingly we introduce the quasilinear operator $\mathcal{A}_M^k$ in analogy to (4.13):

$$\mathcal{A}_M^k : H^1(\Omega) \to H^1(\Omega)' \ \text{ defined by } \ \langle \mathcal{A}_M^k(\theta), v\rangle_{H^1(\Omega)} := \int_\Omega \mathsf{K}_M(\theta)\nabla\theta\cdot\nabla v \,\mathrm{d}x - \int_{\partial\Omega} h_\tau^k v \,\mathrm{d}S \tag{4.18}$$

Observe that, thanks to (2.17) there still holds $\mathsf{K}_M(r) \ge c_0$ for all $r \in \mathbb{R}$, and therefore by the trace theorem

$$\langle \mathcal{A}_M^k(\theta), \theta\rangle_{H^1(\Omega)} \ge \tilde{c}_0\|\nabla\theta\|^2_{L^2(\Omega)} - c_1\|\theta\|^2_{L^2(\Omega)} - c_1\|h_\tau^k\|^2_{L^2(\partial\Omega)} \qquad \text{for all } \theta \in H^1(\Omega). \tag{4.19}$$

We also introduce the truncation operator $\mathcal{T}_M : \mathbb{R} \to \mathbb{R}$

$$\mathcal{T}_M(r) := \begin{cases} 0 & \text{if } r < 0, \\ r & \text{if } 0 \le r \le M, \\ M & \text{if } r > M. \end{cases} \tag{4.20}$$

The (ν, M)-regularized time-discrete system at time step k reads as follows:

$$D_k(c) = \mathrm{div}\left(m(c^{k-1}, z^{k-1})\nabla\mu^k\right) + \nu\,\mathrm{div}\left(|\nabla\mu^k|^{\varrho-2}\nabla\mu^k\right) - \nu\mu^k, \tag{4.21a}$$

$$\begin{aligned}\mu^k = &-\Delta_p(c^k) + (\breve{\phi}_\omega)'(c^k) + (\hat{\phi})'(c^{k-1}) + \breve{W}^\omega_{1,c}(c^k, \varepsilon(\mathbf{u}^{k-1}), z^{k-1}) + \hat{W}^\omega_{1,c}(c^{k-1}, \varepsilon(\mathbf{u}^{k-1}), z^{k-1}) \\ &- \mathcal{T}_M(\vartheta^k) + D_k(c) + \nu|c^k|^{\varrho-2}c^k,\end{aligned} \tag{4.21b}$$

$$\begin{aligned}&D_k(z) - \Delta_p(z^k) + \xi^k + (\breve{\sigma})'(z^k) + (\hat{\sigma})'(z^{k-1}) + \nu|z^k|^{\varrho-2}z^k \\ &\quad = -\breve{W}^\omega_{3,z}(c^k, \varepsilon(\mathbf{u}^{k-1}), z^k) - \hat{W}^\omega_{3,z}(c^k, \varepsilon(\mathbf{u}^{k-1}), z^{k-1}) + \mathcal{T}_M(\vartheta^k) \quad \text{with } \xi^k \in \partial I_{[0,z^{k-1}]}(z^k),\end{aligned} \tag{4.21c}$$

$$\begin{aligned}&D_k(\vartheta) + \mathcal{A}_M^k(\vartheta^k) + D_k(c)\mathcal{T}_M(\vartheta^k) + D_k(z)\mathcal{T}_M(\vartheta^k) + \rho\mathcal{T}_M(\vartheta^k)\,\mathrm{div}(D_k(\mathbf{u})) \\ &\quad = g^k + |D_k(c)|^2 + |D_k(z)|^2 + a(c^{k-1}, z^{k-1})\varepsilon(D_k(\mathbf{u})) : \mathbb{V}\varepsilon(D_k(\mathbf{u})) + m(c^{k-1}, z^{k-1})|\nabla\mu^k|^2,\end{aligned} \tag{4.21d}$$

$$\begin{aligned}&D_k(D_k(\mathbf{u})) - \mathrm{div}\left(a(c^{k-1}, z^{k-1})\mathbb{V}\varepsilon(D_k(\mathbf{u})) + W^\omega_{,\varepsilon}(c^k, \varepsilon(\mathbf{u}^k), z^k) - \rho\mathcal{T}_M(\vartheta^k)\mathbb{1}\right) \\ &\quad - \nu\,\mathrm{div}\left(|\varepsilon(\mathbf{u}^k - \mathbf{d}^k)|^{\varrho-2}\varepsilon(\mathbf{u}^k - \mathbf{d}^k)\right) = \mathbf{f}^k,\end{aligned} \tag{4.21e}$$

supplemented with the previously given boundary conditions. Please note that the functions c^k, μ^k, z^k, ϑ^k and $\mathbf{u}^k$ depend on M, ν, τ and ω whereas the functions from the previous time steps c^{k-1}, μ^{k-1}, z^{k-1}, ϑ^{k-1}, $\mathbf{u}^{k-1}$ and $\mathbf{u}^{k-2}$ only depend on τ and ω and do **not** depend on M and ν.

We are now in the position to prove existence of weak solutions for system (4.21) by resorting to an existence result for pseudomonotone operators from [39], which is in turn based on a fixed point argument.

Lemma 4.4 (Existence of the time-discrete system for $\nu > 0$ and $M \in \mathbb{N}$)**.** *Let $\omega > 0$, $\tau > 0$, $k \in \{1, \ldots, K_\tau\}$, $\nu > 0$ and $M \in \mathbb{N}$ be given. We assume that*

$$(c^{k-1}, \mu^{k-1}, z^{k-1}, \vartheta^{k-1}, \mathbf{u}^{k-1}, \mathbf{u}^{k-2}) \in W^{1,p}(\Omega) \times H^2(\Omega) \times W^{1,p}(\Omega) \times H^1(\Omega) \times H^2(\Omega; \mathbb{R}^d) \times H^2(\Omega; \mathbb{R}^d).$$

Then there exists a weak solution

$$(c^k, \mu^k, z^k, \vartheta^k, \mathbf{u}^k) \in W^{1,p}(\Omega) \times W^{1,\varrho}(\Omega) \times W^{1,p}(\Omega) \times H^1(\Omega) \times W^{1,\varrho}(\Omega; \mathbb{R}^d)$$

to system (4.21) at time step k in the following sense:

– (4.21a) *is fulfilled in* $W^{1,\varrho}(\Omega)'$,

– (4.21b) *is fulfilled in* $W^{1,p}(\Omega)'$,

– (4.21c) *is fulfilled in* $W^{1,p}(\Omega)'$ *with* $\xi^k \in \partial I_{Z^{k-1}}(z^k)$,

– (4.21d) *is fulfilled in* $H^1(\Omega)'$,

– (4.21e) *is fulfilled in* $W_0^{1,\varrho}(\Omega;\mathbb{R}^d)'$,

– *the initial conditions* (4.10) *and the boundary condition* $\mathbf{u}^k = \mathbf{d}^k$ *a.e. on* $\partial\Omega$ *are satisfied,*

– *the constraints* (4.12) *are satisfied.*

Proof. Our approach for finding a solution to (4.21) for a given k is to rewrite the system as

$$0 \in \mathbf{A}(c^k,\mu^k,z^k,\vartheta^k,\mathbf{u}^k-\mathbf{d}^k) + \partial\Psi(c^k,\mu^k,z^k,\vartheta^k,\mathbf{u}^k-\mathbf{d}^k), \tag{4.22}$$

where $\mathbf{A}$ is a (to be specified) pseudomonotone and coercive operator and $\partial\Psi$ is the subdifferential of a (to be specified) proper, convex and l.s.c. potential Ψ. Note that both the operator $\mathbf{A}$ as well as Ψ will depend on the discrete functions obtained in previous time step $k-1$, but we choose not to highlight this for notational simplicity.

To be more precise, we introduce the space

$$\mathbf{X} := W^{1,p}(\Omega)\times W^{1,\varrho}(\Omega)\times W^{1,p}(\Omega)\times H^1(\Omega)\times W_0^{1,\varrho}(\Omega;\mathbb{R}^d)$$

and the announced operator

$$\mathbf{A} = \begin{bmatrix} A_1\\ A_2\\ A_3\\ A_4\\ A_5\end{bmatrix} : \mathbf{X}\to\mathbf{X}'$$

given component-wise by

$$\begin{aligned}
A_1(c,\mu,z,\vartheta,\widetilde{\mathbf{u}}) =\ & -\mu - \Delta_p(c) + \nu|c|^{\varrho-2}c + (\breve{\phi}_\omega)'(c) + (\hat{\phi})'(c^{k-1}) + \breve{W}^\omega_{1,c}(c,\varepsilon(\mathbf{u}^{k-1}),z^{k-1})\\
&+ \hat{W}^\omega_{1,c}(c^{k-1},\varepsilon(\mathbf{u}^{k-1}),z^{k-1}) - \mathcal{T}_M(\vartheta) + (c-c^{k-1})\tau^{-1},\\
A_2(c,\mu,z,\vartheta,\widetilde{\mathbf{u}}) =\ & -\operatorname{div}(m(c^{k-1},z^{k-1})\nabla\mu) - \nu\operatorname{div}(|\nabla\mu|^{\varrho-2}\nabla\mu) + \nu\mu + (c-c^{k-1})\tau^{-1},\\
A_3(c,\mu,z,\vartheta,\widetilde{\mathbf{u}}) =\ & -\Delta_p(z) + \nu|z|^{\varrho-2}z + (z-z^{k-1})\tau^{-1} + (\breve{\sigma})'(\mathcal{T}(z)) + (\hat{\sigma})'(z^{k-1})\\
&+ \breve{W}^\omega_{3,z}(c^k,\varepsilon(\mathbf{u}^{k-1}),\mathcal{T}(z)) + \hat{W}^\omega_{3,z}(c^k,\varepsilon(\mathbf{u}^{k-1}),z^{k-1}) - \mathcal{T}_M(\vartheta),
\end{aligned}$$

$$\begin{aligned}
A_4(c,\mu,z,\vartheta,\widetilde{\mathbf{u}}) =\ & \mathcal{A}^k_M(\vartheta) + (\vartheta-\vartheta^{k-1})\tau^{-1} + (c-c^{k-1})\tau^{-1}\mathcal{T}_M(\vartheta) + (z-z^{k-1})\tau^{-1}\mathcal{T}_M(\vartheta)\\
&+ \rho\mathcal{T}_M(\vartheta)\operatorname{div}(\widetilde{\mathbf{u}}+\mathbf{d}^k-\mathbf{u}^{k-1})\tau^{-1} - g^k_\tau - |(c-c^{k-1})\tau^{-1}|^2 - |(z-z^{k-1})\tau^{-1}|^2\\
&- a(c^{k-1},z^{k-1})\varepsilon((\widetilde{\mathbf{u}}+\mathbf{d}^k-\mathbf{u}^{k-1})\tau^{-1}) : \mathbb{V}\varepsilon((\widetilde{\mathbf{u}}+\mathbf{d}^k-\mathbf{u}^{k-1})\tau^{-1}) - m(c^{k-1},z^{k-1})|\nabla\mu|^2,\\
A_5(c,\mu,z,\vartheta,\widetilde{\mathbf{u}}) =\ & (\widetilde{\mathbf{u}}+\mathbf{d}^k-2\mathbf{u}^{k-1}+\mathbf{u}^{k-2})\tau^{-2} - \nu\operatorname{div}\left(|\varepsilon(\widetilde{\mathbf{u}})|^{\varrho-2}\varepsilon(\widetilde{\mathbf{u}})\right)\\
&- \operatorname{div}\left(a(c^{k-1},z^{k-1})\mathbb{V}\varepsilon((\widetilde{\mathbf{u}}+\mathbf{d}^k-\mathbf{u}^{k-1})\tau^{-1}) + W^\omega_{,\varepsilon}(c^k,\varepsilon(\widetilde{\mathbf{u}}+\mathbf{d}^k),\mathcal{T}(z)) - \rho\mathcal{T}_M(\vartheta)\mathbb{1}\right)\\
&- \mathbf{f}^k_\tau,
\end{aligned}$$

where we make use of the truncation operator $\mathcal{T}$

$$\mathcal{T}(z) := \begin{cases} 0 & \text{if } z < 0, \\ z & \text{if } 0 < z < 1, \\ 1 & \text{if } z > 1. \end{cases}$$

The potential $\Psi : \mathbf{X} \to (-\infty, +\infty]$ featuring in (4.22) is given by

$$\Psi(c, \mu, z, \vartheta, \widetilde{\mathbf{u}}) := I_{Z^{k-1}}(z) = \begin{cases} 0 & \text{if } 0 \le z \le z^{k-1} \text{ a.e. in } \Omega, \\ \infty & \text{else}, \end{cases}$$

where the set Z^{k-1} is defined in (4.16).

We remark that for solutions of (4.22) the truncation operator $\mathcal{T}$ will disappear in the resulting system since $\mathrm{dom}(\partial\Psi) \subseteq \{(c, \mu, z, \vartheta, \widetilde{\mathbf{u}}) \in \mathbf{X} \,|\, 0 \le z \le 1 \text{ a.e. in } \Omega\}$. It is merely used as an auxiliary construction to ensure coercivity of the operator $\mathbf{A}$. Furthermore, the boundary values for the displacement variable are shifted to 0 in order to obtain a vector space structure for the domain $\mathbf{X}$ of $\mathbf{A}$. As a result, we have to add $\mathbf{d}^k$ to the displacement $\widetilde{\mathbf{u}}$ of the solution afterwards.

In following we are going to verify coercivity of $\mathbf{A}$. To this end, we will estimate $\langle \mathbf{A}(\boldsymbol{x}), \boldsymbol{x} \rangle_{\mathbf{X}}$ for every $\boldsymbol{x} = (c, \mu, z, \vartheta, \widetilde{\mathbf{u}}) \in \mathbf{X}$ from below:

$$\begin{aligned} \langle \mathbf{A}(\boldsymbol{x}), \boldsymbol{x} \rangle_{\mathbf{X}} = \big\langle \mathbf{A}(c, \mu, z, \vartheta, \widetilde{\mathbf{u}}), (c, \mu, z, \vartheta, \widetilde{\mathbf{u}}) \big\rangle_{\mathbf{X}} &= \big\langle A_1(c, \mu, z, \vartheta, \widetilde{\mathbf{u}}), c \big\rangle_{W^{1,p}(\Omega)} + \big\langle A_2(c, \mu, z, \vartheta, \widetilde{\mathbf{u}}), \mu \big\rangle_{W^{1,\varrho}(\Omega)} \\ &\quad + \big\langle A_3(c, \mu, z, \vartheta, \widetilde{\mathbf{u}}), z \big\rangle_{W^{1,p}(\Omega)} + \big\langle A_4(c, \mu, z, \vartheta, \widetilde{\mathbf{u}}), \vartheta \big\rangle_{H^1(\Omega)} \\ &\quad + \big\langle A_5(c, \mu, z, \vartheta, \widetilde{\mathbf{u}}), \widetilde{\mathbf{u}} \big\rangle_{W^{1,\varrho}(\Omega;\mathbb{R}^d)} \\ &=: I_1 + I_2 + \ldots + I_5. \end{aligned} \tag{4.23}$$

We now estimate the partial derivatives $\breve{W}^\omega_{1,c}$ and $\hat{W}^\omega_{1,c}$ of $\breve{W}^\omega_1$ and $\hat{W}^\omega_1$ w.r.t. c, i.e.

$$\begin{aligned} \breve{W}^\omega_{1,c}(c, \varepsilon(\mathbf{u}), z) &= W^\omega_{,c}(c, \varepsilon(\mathbf{u}), z) + \big(\sup_{\widetilde{c} \in \mathbb{R}} |W^\omega_{,cc}(\widetilde{c}, \varepsilon(\mathbf{u}), z)|\big)\, c, \\ \hat{W}^\omega_{1,c}(c, \varepsilon(\mathbf{u}), z) &= -\big(\sup_{\widetilde{c} \in \mathbb{R}} |W^\omega_{,cc}(\widetilde{c}, \varepsilon(\mathbf{u}), z)|\big)\, c. \end{aligned}$$

Taking into account (4.6) and Hypothesis (V) (cf. also (2.21)), we obtain

$$\Big| \breve{W}^\omega_{1,c}(c, \varepsilon(\mathbf{u}), z) \Big| \le C(|c| + 1)(1 + |\varepsilon(\mathbf{u})|^2), \tag{4.24}$$

$$\Big| \hat{W}^\omega_{1,c}(c, \varepsilon(\mathbf{u}), z) \Big| \le C|c|(1 + |\varepsilon(\mathbf{u})|^2) \tag{4.25}$$

We can also verify that

$$\big| W^\omega_{,\varepsilon}(c, \varepsilon(\mathbf{u}), z) \big| \le C(1 + |\varepsilon(\mathbf{u})|), \tag{4.26}$$

and

$$\Big| \breve{W}^\omega_{3,z}(c, \varepsilon(\mathbf{u}), z) \Big| \le C(1 + |\varepsilon(\mathbf{u})|^2), \tag{4.27}$$

$$\Big| \hat{W}^\omega_{3,z}(c, \varepsilon(\mathbf{u}), z) \Big| \le C(1 + |\varepsilon(\mathbf{u})|^2)\,. \tag{4.28}$$

Estimates (4.24)–(4.28) are valid for all $c \in \mathbb{R}$, $z \in [0, 1]$ and $\mathbf{u} \in \mathbb{R}^d$, and fixed $C > 0$. Taking also the boundedness properties

$$\mathcal{T}(z),\ z^{k-1} \in [0, 1] \qquad \text{a.e. in } \Omega, \qquad \mathcal{T}_M(\vartheta) \in [0, M] \qquad \text{a.e. in } \Omega$$

into account, we obtain

$$
\begin{aligned}
\breve{W}^{\omega}_{1,c}(c,\varepsilon(\mathbf{u}^{k-1}),z^{k-1}) \geq &\ -C(|c|+1)(1+|\varepsilon(\mathbf{u}^{k-1})|^2),\\
\hat{W}^{\omega}_{1,c}(c^{k-1},\varepsilon(\mathbf{u}^{k-1}),z^{k-1}) \geq &\ -C|c^{k-1}|(1+|\varepsilon(\mathbf{u}^{k-1})|^2),\\
W^{\omega}_{,\varepsilon}(c,\varepsilon(\widetilde{\mathbf{u}}+\mathbf{d}^k),\mathcal{T}(z)) \geq &\ -C(1+|\varepsilon(\widetilde{\mathbf{u}})|^2+|\varepsilon(\mathbf{d}^k)|^2),\\
\breve{W}^{\omega}_{3,z}(c,\varepsilon(\mathbf{u}^{k-1}),\mathcal{T}(z)) \geq &\ -C(1+|\varepsilon(\mathbf{u}^{k-1})|^2),\\
\hat{W}^{\omega}_{3,z}(c,\varepsilon(\mathbf{u}^{k-1}),z^{k-1}) \geq &\ -C(1+|\varepsilon(\mathbf{u}^{k-1})|^2).
\end{aligned}
$$

Together with Young's inequality and estimates (4.24)–(4.28), a calculation reveals for the terms $I_1,\dots,I_5$ from (4.23) the following bounds (hereafter, we will write $\|\cdot\|_{L^p}$ in place of $\|\cdot\|_{L^p(\Omega)}$ for shorter notation and we will denote by δ a positive constant, to be chosen later, and by $C_\delta>0$ a constant depending on δ):

$$
\begin{aligned}
I_1 = {}& \|\nabla c\|_{L^p}^p+\nu\|c\|_{L^\varrho}^\varrho+\tau^{-1}\|c\|_{L^2}^2-\tau^{-1}\int_\Omega c^{k-1}c\,\mathrm{d}x-\int_\Omega \mu c\,\mathrm{d}x\\
&+\int_\Omega\Big(\beta_\omega(c)+\lambda_\gamma c+\gamma'(c^{k-1})-\lambda_\gamma c^{k-1}+\breve{W}^{\omega}_{1,c}(c,\varepsilon(\mathbf{u}^{k-1}),z^{k-1})\Big)c\,\mathrm{d}x\\
&+\int_\Omega\Big(\hat{W}^{\omega}_{1,c}(c^{k-1},\varepsilon(\mathbf{u}^{k-1}),z^{k-1})-\mathcal{T}_M(\vartheta)\Big)c\,\mathrm{d}x\\
\geq{}& \|\nabla c\|_{L^p}^p+\nu\|c\|_{L^\varrho}^\varrho-\delta\|\mu\|_{L^2}^2-C_\delta\|c\|_{L^2}^2-C_\delta\|\varepsilon(\mathbf{u}^{k-1})\|_{L^4}^4-C_\delta,\\
I_2 = {}& \int_\Omega m(c^{k-1},z^{k-1})|\nabla\mu|^2\,\mathrm{d}x+\nu\|\nabla\mu\|_{L^\varrho}^\varrho+\nu\|\mu\|_{L^2}^2+\tau^{-1}\int_\Omega(c-c^{k-1})\mu\,\mathrm{d}x\\
\geq{}& \nu\|\nabla\mu\|_{L^\varrho}^\varrho+\nu\|\mu\|_{L^2}^2-\delta\|\mu\|_{L^2}^2-C_\delta\|c\|_{L^2}^2-C_\delta\\
I_3 = {}& \|\nabla z\|_{L^p}^p+\nu\|z\|_{L^\varrho}^\varrho+\tau^{-1}\|z\|_{L^2}^2-\tau^{-1}\int_\Omega z^{k-1}z\,\mathrm{d}x\\
&+\int_\Omega\Big((\breve{\sigma})'(\mathcal{T}(z))+(\hat{\sigma})'(z^{k-1})+\breve{W}^{\omega}_{3,z}(c,\varepsilon(\mathbf{u}^{k-1}),\mathcal{T}(z))+\hat{W}^{\omega}_{3,z}(c,\varepsilon(\mathbf{u}^{k-1}),z^{k-1})-\mathcal{T}_M(\vartheta)\Big)z\,\mathrm{d}x\\
\geq{}& \|\nabla z\|_{L^p}^p+\nu\|z\|_{L^\varrho}^\varrho-\delta\|z\|_{L^2}^2-C_\delta\|\varepsilon(\mathbf{u}^{k-1})\|_{L^4}^4-C_\delta,\\
I_4 = {}& \int_\Omega \mathsf{K}_M(\vartheta)|\nabla\vartheta|^2\,\mathrm{d}x-\int_{\partial\Omega}h_\tau^k\vartheta\,\mathrm{d}x+\tau^{-1}\|\vartheta\|_{L^2}^2-\tau^{-1}\int_\Omega\vartheta^{k-1}\vartheta\,\mathrm{d}x\\
&+\tau^{-1}\int_\Omega\Big((c-c^{k-1})+(z-z^{k-1})+\rho\,\mathrm{div}(\widetilde{\mathbf{u}}+\mathbf{d}^k-\mathbf{u}^{k-1})\Big)\mathcal{T}_M(\vartheta)\vartheta\,\mathrm{d}x-\int_\Omega g^k\vartheta\,\mathrm{d}x\\
&-\int_\Omega\Big(|(c-c^{k-1})\tau^{-1}|^2+|(z-z^{k-1})\tau^{-1}|^2+a(c^{k-1},z^{k-1})\varepsilon\Big(\frac{\widetilde{\mathbf{u}}+\mathbf{d}^k-\mathbf{u}^{k-1}}{\tau}\Big):\mathbb{V}\varepsilon\Big(\frac{\widetilde{\mathbf{u}}+\mathbf{d}^k-\mathbf{u}^{k-1}}{\tau}\Big)\Big)\vartheta\,\mathrm{d}x\\
&-\int_\Omega m(c^{k-1},z^{k-1})|\nabla\mu|^2\vartheta\,\mathrm{d}x\\
\geq{}& c_0\|\nabla\vartheta\|_{L^2}^2+\tau^{-1}\|\vartheta\|_{L^2}^2-\delta\|\vartheta\|_{H^1}^2-C_\delta\|h^k\|_{H^{1/2}(\partial\Omega)}^2-C_\delta\|\vartheta^{k-1}\|_{L^2}^2-C_\delta\|c\|_{L^4}^4-C_\delta\|z\|_{L^4}^4-C_\delta\|\varepsilon(\widetilde{\mathbf{u}})\|_{L^4}^4\\
&-C_\delta\|\varepsilon(\mathbf{d}^k)\|_{L^4}^4-C_\delta\|c^{k-1}\|_{L^4}^4-C_\delta\|z^{k-1}\|_{L^4}^4-C_\delta\|\varepsilon(\mathbf{u}^{k-1})\|_{L^4}^4-C_\delta\|\nabla\mu\|_{L^4}^4-C_\delta\|g^k\|_{L^2}^2-C_\delta,
\end{aligned}
$$

$$
\begin{aligned}
I_5 = {}& \nu\|\varepsilon(\widetilde{\mathbf{u}})\|_{L^\varrho}^\varrho+\tau^{-2}\|\widetilde{\mathbf{u}}\|_{L^2}^2+\tau^{-2}\int_\Omega(\mathbf{d}^k-2\mathbf{u}^{k-1}+\mathbf{u}^{k-2})\cdot\widetilde{\mathbf{u}}\,\mathrm{d}x+\tau^{-1}\int_\Omega a(c^{k-1},z^{k-1})\mathbb{V}\varepsilon(\widetilde{\mathbf{u}}):\varepsilon(\widetilde{\mathbf{u}})\,\mathrm{d}x\\
&+\tau^{-1}\int_\Omega a(c^{k-1},z^{k-1})\mathbb{V}\varepsilon(\mathbf{d}^k-\mathbf{u}^{k-1}):\varepsilon(\widetilde{\mathbf{u}})\,\mathrm{d}x+\int_\Omega W^{\omega}_{,\varepsilon}(c,\varepsilon(\widetilde{\mathbf{u}}+\mathbf{d}^k),\mathcal{T}(z)):\varepsilon(\widetilde{\mathbf{u}})\,\mathrm{d}x\\
&-\int_\Omega\rho\mathcal{T}_M(\vartheta)\,\mathrm{div}(\widetilde{\mathbf{u}})\,\mathrm{d}x-\int_\Omega\mathbf{f}^k\cdot\widetilde{\mathbf{u}}\,\mathrm{d}x\\
\geq{}& \nu\|\varepsilon(\widetilde{\mathbf{u}})\|_{L^\varrho}^\varrho+\tau^{-2}\|\widetilde{\mathbf{u}}\|_{L^2}^2-\delta\|\widetilde{\mathbf{u}}\|_{H^1}^2-C_\delta\|\mathbf{u}^{k-1}\|_{H^1}^2-C_\delta\|\mathbf{d}^k\|_{H^1}^2-C_\delta\|\mathbf{u}^{k-2}\|_{L^2}^2-C_\delta\|\mathbf{f}^k\|_{L^2}^2-C_\delta.
\end{aligned}
$$

In conclusion, choosing $\delta > 0$ sufficiently small in such a way as to absorb the negative terms multiplied by δ into suitable positive contributions, we obtain constants $c', C > 0$ such that

$$\begin{aligned}\langle \mathbf{A}(\boldsymbol{x}), \boldsymbol{x} \rangle_{\mathbf{X}} \geq c' \Big(&\|\nabla c\|_{L^p}^p + \|c\|_{L^\varrho}^\varrho + \|\nabla \mu\|_{L^\varrho}^\varrho + \|\mu\|_{L^2}^2 + \|\nabla z\|_{L^p}^p + \|z\|_{L^\varrho}^\varrho + \|\nabla \vartheta\|_{L^2}^2 + \|\vartheta\|_{L^2}^2 \Big) \\ &+ c' \Big(\|\varepsilon(\widetilde{\mathbf{u}})\|_{L^\varrho}^\varrho + \|\widetilde{\mathbf{u}}\|_{L^2}^2 \Big) - C\end{aligned}$$

which leads to coercivity of $\mathbf{A}$ by using Korn's inequality. The pseudomonotonicity follows from standard arguments in the theory of quasilinear elliptic equations, cf. [39, Chapter 2.4].

By virtue of the existence theorem in [39, Theorem 5.15] together with [39, Lemma 5.17], we find an $\boldsymbol{x} \in \mathbf{X}$ solving (4.22). Thus a solution of (4.22) proves the claim. □

We now derive the incremental energy inequality satisfied by the solutions to system (4.21). This will be the starting point for the derivation of all a priori estimates allowing us to pass to the limit, first as $M \to \infty$ and then $\nu \to 0$.

Lemma 4.5 (Incremental energy inequality for the approximate discrete system)**.** *Let $(c^k, \mu^k, z^k, \vartheta^k, \mathbf{u}^k)$ be the weak solution to system (4.21) at time step k according to Lemma 4.4. Then, for every $M \in \mathbb{N}$ and $\nu > 0$ the following energy inequality holds:*

$$\begin{aligned}&\mathscr{E}_\omega(c^k, z^k, \vartheta^k, \mathbf{u}^k, \mathbf{v}^k) + \frac{\nu}{\varrho}\|c^k\|_{L^\varrho(\Omega)}^\varrho + \frac{\nu}{\varrho}\|z^k\|_{L^\varrho(\Omega)}^\varrho + \frac{\nu}{\varrho}\|\varepsilon(\mathbf{u}^k)\|_{L^\varrho(\Omega)}^\varrho + \nu\tau\Big(\|\nabla \mu^k\|_{L^\varrho(\Omega;\mathbb{R}^d)}^\varrho + \|\mu^k\|_{L^2}^2\Big) \\ &\leq \mathscr{E}_\omega(c^{k-1}, z^{k-1}, \vartheta^{k-1}, \mathbf{u}^{k-1}, \mathbf{v}^{k-1}) + \frac{\nu}{\varrho}\|c^{k-1}\|_{L^\varrho(\Omega)}^\varrho + \frac{\nu}{\varrho}\|z^{k-1}\|_{L^\varrho(\Omega)}^\varrho + \frac{\nu}{\varrho}\|\varepsilon(\mathbf{u}^{k-1})\|_{L^\varrho(\Omega)}^\varrho \\ &\quad + \tau\Big(\int_\Omega g^k \,\mathrm{d}x + \int_{\partial\Omega} h^k \,\mathrm{d}x + \int_\Omega \mathbf{f}^k \cdot \mathbf{v}^k \,\mathrm{d}x\Big) \\ &\quad + \tau\int_\Omega D_k(\mathbf{v}) \cdot D_k(\mathbf{d}) \,\mathrm{d}x + \tau\int_\Omega a(c^{k-1}, z^{k-1})\mathbb{V}\varepsilon(\mathbf{v}^k) : \varepsilon(D_k(\mathbf{d})) \,\mathrm{d}x \\ &\quad + \tau\int_\Omega W^\omega_{,\varepsilon}(c^k, \varepsilon(\mathbf{u}^k), z^k) : \varepsilon(D_k(\mathbf{d})) \,\mathrm{d}x - \tau\int_\Omega \rho\mathcal{T}_M(\vartheta^k)\,\mathrm{div}(D_k(\mathbf{d})) \,\mathrm{d}x - \tau\int_\Omega \mathbf{f}^k \cdot D_k(\mathbf{d}) \,\mathrm{d}x\end{aligned} \tag{4.29}$$

where we set $\mathbf{v}^k := D_k(\mathbf{u})$ and denote by $\mathscr{E}_\omega$ the approximation of the total energy $\mathscr{E}$ from (1.15) obtained by replacing ϕ with $\phi_\omega = \widehat{\beta}_\omega + \gamma$ and W with W^ω.

Proof. The convex-concave splitting give rise to the following crucial estimates, (cf. also (4.7)):

$$\Big((\breve{\phi}_\omega)'(c^k) + (\hat{\phi})'(c^{k-1})\Big)(c^k - c^{k-1}) \geq \phi_\omega(c^k) - \phi_\omega(c^{k-1}), \tag{4.30a}$$

$$\Big((\breve{\sigma})'(z^k) + (\hat{\sigma})'(z^{k-1})\Big)(z^k - z^{k-1}) \geq \sigma(z^k) - \sigma(z^{k-1}), \tag{4.30b}$$

$$\begin{aligned}&\Big(\breve{W}^\omega_{1,c}(c^k, \varepsilon(\mathbf{u}^{k-1}), z^{k-1}) + \hat{W}^\omega_{1,c}(c^{k-1}, \varepsilon(\mathbf{u}^{k-1}), z^{k-1})\Big)(c^k - c^{k-1}) \\ &\quad + W^\omega_{,\varepsilon}(c^k, \varepsilon(\mathbf{u}^k), z^k) : \varepsilon(\mathbf{u}^k - \mathbf{u}^{k-1}) + \Big(\breve{W}^\omega_{3,z}(c^k, \varepsilon(\mathbf{u}^{k-1}), z^k) + \hat{W}^\omega_{3,z}(c^k, \varepsilon(\mathbf{u}^{k-1}), z^{k-1})\Big)(z^k - z^{k-1}) \\ &\geq W^\omega(c^k, \varepsilon(\mathbf{u}^k), z^k) - W^\omega(c^{k-1}, \varepsilon(\mathbf{u}^{k-1}), z^{k-1})\,.\end{aligned} \tag{4.30c}$$

Moreover, we will make use of standard convexity estimates:

$$|\nabla c^k|^{p-2}\nabla c^k \cdot \nabla(c^k - c^{k-1}) \geq \frac{1}{p}|\nabla c^k|^p - \frac{1}{p}|\nabla c^{k-1}|^p, \tag{4.31a}$$

$$|c^k|^{\varrho-2}c^k(c^k - c^{k-1}) \geq \frac{1}{\varrho}|c^k|^\varrho - \frac{1}{\varrho}|c^{k-1}|^\varrho, \tag{4.31b}$$

$$|\nabla z^k|^{p-2}\nabla z^k \cdot \nabla(z^k - z^{k-1}) \geq \frac{1}{p}|\nabla z^k|^p - \frac{1}{p}|\nabla z^{k-1}|^p, \tag{4.31c}$$

$$|z^k|^{\varrho-2}z^k(z^k - z^{k-1}) \geq \frac{1}{\varrho}|z^k|^\varrho - \frac{1}{\varrho}|z^{k-1}|^\varrho, \tag{4.31d}$$

$$|\varepsilon(\mathbf{u}^k)|^{\varrho-2}\varepsilon(\mathbf{u}^k) : \varepsilon(\mathbf{u}^k - \mathbf{u}^{k-1}) \geq \frac{1}{\varrho}|\varepsilon(\mathbf{u}^k)|^\varrho - \frac{1}{\varrho}|\varepsilon(\mathbf{u}^{k-1})|^\varrho, \tag{4.31e}$$

$$\Big(\mathbf{u}^k - 2\mathbf{u}^{k-1} + \mathbf{u}^{k-2}\Big) \cdot (\mathbf{u}^k - \mathbf{u}^{k-1}) \geq \frac{1}{2}|\mathbf{u}^k - \mathbf{u}^{k-1}|^2 - \frac{1}{2}|\mathbf{u}^{k-1} - \mathbf{u}^{k-2}|^2. \tag{4.31f}$$

To obtain the energy estimate, we test the time-discrete system (4.21) as follows:

$$\begin{aligned}&(4.21\text{a}) \times (c^k - c^{k-1}) \;+\; (4.21\text{b}) \times \tau\mu^k \;+\; (4.21\text{c}) \times (z^k - z^{k-1}) \;+\; (4.21\text{d}) \times \tau \\ &+ (4.21\text{e}) \times (\mathbf{u}^k - \mathbf{u}^{k-1} - (\mathbf{d}^k - \mathbf{d}^{k-1}))\end{aligned}$$

and exploit estimates (4.30) and (4.31). $\square$

Remark 4.6. We note that in comparison with the calculations in the *First estimate* in Section 3, where we assumed spatial H^2-regularity for $\mathbf{u}$, we cannot test the weak formulation (4.21e) with $\mathbf{u}^k - \mathbf{u}^{k-1}$ because the boundary values of $\mathbf{u}^k - \mathbf{u}^{k-1}$ are not necessarily 0.

Lemma 4.7 (Positivity of ϑ^k)**.** *There exists a constant $\underline{\vartheta} > 0$, independent of ω, τ, k, M and ν, such that $\vartheta^k \geq \underline{\vartheta}$ a.e. in Ω.*

Proof. The proof is carried out in two steps: At first we show non-negativity of ϑ^k and then, in the second step, strictly positivity as claimed is shown.

Step 1: Testing the discrete heat equation (4.21d) with $-(\vartheta^k)^- := \min\{\vartheta^k, 0\}$ shows after integration over Ω:

$$\begin{aligned}&\int_\Omega \frac{1}{\tau}\underbrace{\vartheta^k(-(\vartheta^k)^-)}_{=|(\vartheta^k)^-|^2} \underbrace{-\frac{1}{\tau}\vartheta^{k-1}(-(\vartheta^k)^-)}_{\geq 0} + \Big(D_k(c) + D_k(z) + \rho\,\mathrm{div}(D_k(\mathbf{u}))\Big)\underbrace{\mathcal{T}_M(\vartheta^k)(-(\vartheta^k)^-)}_{=0}\,\mathrm{d}x \\ &= \int_\Omega \underbrace{\Big(g^k + |D_k(c)|^2 + |D_k(z)|^2 + a(c^{k-1}, z^{k-1})\varepsilon(D_k(\mathbf{u})) : \mathbb{V}\varepsilon(D_k(\mathbf{u}))\Big)}_{\geq 0}\underbrace{(-(\vartheta^k)^-)}_{\leq 0}\,\mathrm{d}x \\ &\quad + \int_\Omega \underbrace{m(c^{k-1}, z^{k-1})|\nabla\mu_\tau^k|^2}_{\geq 0}\underbrace{(-(\vartheta^k)^-)}_{\leq 0}\,\mathrm{d}x.\end{aligned}$$

Here we have merely used the information that $\vartheta^{k-1} \geq 0$ a.e. in Ω. We obtain

$$\int_\Omega |(\vartheta^k)^-|^2\,\mathrm{d}x \leq 0$$

and thus $\vartheta^k \geq 0$ a.e. in Ω.

Step 2:

The proof follows the very same lines as the argument developed in [38, Lemma 4.4 - Step 3], hence we will just outline it and refer to [38] for all details. Namely, repeating the arguments formally developed in Sec. 3 (cf. (3.2)), we deduce from (4.21d) that there exists $C > 0$ such that

$$\int_\Omega D_k(\vartheta) w \,\mathrm{d}x + \int_\Omega \mathsf{K}_M(\vartheta^k) w \,\mathrm{d}x \geq -C \int_\Omega (\vartheta^k)^2 w \,\mathrm{d}x \qquad \text{for every } w \in W^{1,2}_+(\Omega)\,.$$

Then, we compare the functions $(\vartheta^k)_{k=1}^{K_\tau}$ with the solutions $(v^k)_{k=1}^{K_\tau}$ of the finite difference equation $\frac{v_k - v_{k-1}}{\tau} = -C v_k^2$, with $v_0 = \vartheta_*$, and we conclude that $\vartheta^k \geq v_k$ a.e. in Ω. Finally, with a comparison argument we prove that

$$\vartheta^k \geq v_k \geq \frac{\vartheta_*}{1 + CT\vartheta_*} \doteq \underline{\vartheta} \quad \text{a.e. in } \Omega \qquad \text{for all } k = 1, \ldots, K_\tau.$$

□

Lemma 4.5 and Lemma 4.7 give rise to the following uniform estimates:

Lemma 4.8. *The following estimates hold uniformly in $\nu > 0$ and $M \in \mathbb{N}$:*

$$\|c^k\|_{W^{1,p}(\Omega)} + \|z^k\|_{W^{1,p}(\Omega)} + \|\mathbf{v}^k\|_{L^2(\Omega;\mathbb{R}^d)} + \|\vartheta^k\|_{L^1(\Omega)} \leq C, \tag{4.32a}$$

$$\nu^{\frac{1}{\varrho}} \|c^k\|_{L^\varrho(\Omega)} + \nu^{\frac{1}{\varrho}} \|z^k\|_{L^\varrho(\Omega)} + \nu^{\frac{1}{\varrho}} \|\varepsilon(\mathbf{u}^k)\|_{L^\varrho(\Omega;\mathbb{R}^{d\times d})} \leq C, \tag{4.32b}$$

$$\nu\tau \Big(\|\nabla\mu^k\|^\varrho_{L^\varrho(\Omega)} + \|\mu^k\|^2_{L^2(\Omega)} \Big) \leq C. \tag{4.32c}$$

Proof. In order to deduce estimates (4.32), it suffices to estimate the terms of the k-th time step on the right-hand side of the incremental energy inequality (4.29) from Lemma 4.5. The following calculations are an adaption of the calculations performed First estimate in Section 3.

- At first we observe by Young's inequality

$$\tau \int_\Omega \mathbf{f}^k \cdot \mathbf{v}^k \,\mathrm{d}x \leq \delta \|\mathbf{v}^k\|^2_{L^2(\Omega;\mathbb{R}^d)} + C_\delta \|\mathbf{f}^k\|^2_{L^2(\Omega;\mathbb{R}^d)},$$

$$\tau \int_\Omega D_k(\mathbf{v}) \cdot D_k(\mathbf{d}) \,\mathrm{d}x \leq \delta \|\mathbf{v}^k\|^2_{L^2(\Omega;\mathbb{R}^d)} + \delta \|\mathbf{v}^{k-1}\|^2_{L^2(\Omega;\mathbb{R}^d)} + C_\delta \|D_k(\mathbf{d})\|^2_{L^2(\Omega;\mathbb{R}^d)},$$

$$-\tau \int_\Omega \mathbf{f}^k \cdot D_k(\mathbf{d}) \,\mathrm{d}x \leq C \|\mathbf{f}^k\|^2_{L^2(\Omega;\mathbb{R}^d)} + C \|D_k(\mathbf{d})\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

  By choosing $\delta > 0$ sufficiently small, the term $\delta\|\mathbf{v}^k\|^2_{L^2(\Omega;\mathbb{R}^d)}$ is absorbed by the left-hand side of (4.29). The remaining terms are bounded due to (2.22).

- We continue with the next term on the right-hand side of (4.29) by using that $\mathbf{v}^k = D_k(\mathbf{d})$ a.e. on $\partial\Omega$, the trace theorem and Young's inequality

$$\begin{aligned}
&\tau\int_\Omega a(c^{k-1},z^{k-1})\mathbb{V}\varepsilon(\mathbf{v}^k):\varepsilon(D_k(\mathbf{d}))\,\mathrm{d}x\\
&=-\tau\int_\Omega \mathbf{v}^k\cdot\operatorname{div}\big(a(c^{k-1},z^{k-1})\mathbb{V}\varepsilon(D_k(\mathbf{d}))\big)\,\mathrm{d}x+\tau\int_{\partial\Omega}\mathbf{v}^k\cdot\big(a(c^{k-1},z^{k-1})\mathbb{V}\varepsilon(D_k(\mathbf{d}))n\big)\,\mathrm{d}S\\
&=-\tau\int_\Omega \mathbf{v}^k\cdot\Big(\big(a_{,c}(c^{k-1},z^{k-1})\nabla c^{k-1}+a_{,z}(c^{k-1},z^{k-1})\nabla z^{k-1}\big)\mathbb{V}\varepsilon(D_k(\mathbf{d}))\Big)\,\mathrm{d}x\\
&\quad-\tau\int_\Omega \mathbf{v}^k\cdot a(c^{k-1},z^{k-1})\operatorname{div}\big(\mathbb{V}\varepsilon(D_k(\mathbf{d}))\big)\,\mathrm{d}x+\tau\int_{\partial\Omega}D_k(\mathbf{d})\cdot\big(a(c^{k-1},z^{k-1})\mathbb{V}\varepsilon(D_k(\mathbf{d}))n\big)\,\mathrm{d}S\\
&\le\delta\|\mathbf{v}^k\|^2_{L^2(\Omega;\mathbb{R}^d)}+C_\delta\|\varepsilon(D_k(\mathbf{d}))\|^2_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\Big(\|a_{,c}(c^{k-1},z^{k-1})\|^2_{L^\infty(\Omega)}\|\nabla c^{k-1}\|^2_{L^2(\Omega;\mathbb{R}^d)}\\
&\qquad\qquad+\|a_{,z}(c^{k-1},z^{k-1})\|^2_{L^\infty(\Omega)}\|\nabla z^{k-1}\|^2_{L^2(\Omega;\mathbb{R}^d)}\Big)\\
&\quad+C_\delta\|a(c^{k-1},z^{k-1})\|^2_{L^\infty(\Omega)}\|\tau\operatorname{div}(\mathbb{V}\varepsilon(D_k(\mathbf{d})))\|^2_{L^2(\Omega;\mathbb{R}^d)}\\
&\quad+C\|D_k(\mathbf{d}^k)\|^2_{H^1(\Omega;\mathbb{R}^d)}\|a(c^{k-1},z^{k-1})\|_{L^\infty(\Omega)}\|\tau\varepsilon(D_k(\mathbf{d}))n)\|^2_{H^1(\Omega;\mathbb{R}^{d\times d})}.
\end{aligned}$$

Taking Hypothesis (IV) and (2.22a) into account, we ultimately find that

$$\begin{aligned}
&\tau\int_\Omega a(c^{k-1},z^{k-1})\mathbb{V}\varepsilon(\mathbf{v}^k):\varepsilon(D_k(\mathbf{d}))\,\mathrm{d}x\\
&\qquad\le\delta\|\mathbf{v}^k\|^2_{L^2(\Omega;\mathbb{R}^d)}+C_\delta\big(\|\nabla c^{k-1}\|^2_{L^2(\Omega;\mathbb{R}^d)}+\|\nabla z^{k-1}\|^2_{L^2(\Omega;\mathbb{R}^d)}+1\big).
\end{aligned}$$

For small $\delta>0$ the first term on the right-hand side can be absorbed into the left-hand side of (4.29).

- Moreover, we estimate the $\tau\int W^\omega_{,\varepsilon}(\ldots):\ldots$-term on the right-hand side of (4.29) as follows

$$\begin{aligned}
&\tau\int_\Omega W^\omega_{,\varepsilon}(c^k,\varepsilon(\mathbf{u}^k),z^k):\varepsilon(D_k(\mathbf{d}))\,\mathrm{d}x\\
&\le\delta\|b(c^k,z^k)\|_{L^\infty(\Omega)}\int_\Omega\underbrace{\frac12 b(c^k,z^k)\mathbb{C}(\varepsilon(\mathbf{u}^k)-\varepsilon^*(c)):(\varepsilon(\mathbf{u}^k)-\varepsilon^*(c))}_{=W(c^k,\varepsilon(\mathbf{u}^k),z^k)}\,\mathrm{d}x+C_\delta\|\varepsilon(D_k(\mathbf{d}))\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})},
\end{aligned}$$

which can be absorbed by the left-hand side of (4.29) for small $\delta>0$.

- Finally,

$$-\tau\int_\Omega\rho\mathcal{T}_M(\vartheta^k)\operatorname{div}(D_k(\mathbf{d}))\,\mathrm{d}x\le\tau\rho\|\operatorname{div}(D_k(\mathbf{d}))\|_{L^\infty(\Omega)}\int_\Omega|\vartheta^k|\,\mathrm{d}x.$$

In the end, by choosing $\tau>0$ small enough depening only on ρ and the data $\mathbf{d}$, the right-hand side can be absorbed by the left-hand side of (4.29) (recall that ϑ^k is positive). □

Remark 4.9. We see that the calculation above takes advantage of the fact that the $W_{,\varepsilon}(\ldots)$-term in the discrete force balance equation is discretized fully implicit.

4.2.2 Step 2: Limit passage $M\to\infty$.

In the following we focus on the limit passage $M\to\infty$ and keep M as a subscript in c^k_M, μ^k_M, z^k_M, $\mathbf{u}^k_M$ and ϑ^k_M. By adapting the proof in [38, Proof of Lemma 4.4 - Step 4] to our situation, we obtain enhanced estimates for $(\vartheta^k_M)_M$.

Lemma 4.10. *The following estimate holds uniformly in $M\in\mathbb{N}$:*

$$\|\vartheta^k_M\|_{H^1(\Omega)}\le C. \tag{4.33}$$

Proof. In [38, Proof of Lemma 4.4 - Step 4] estimate (4.33) is obtained in two steps which can be both applied in our case since the additional variable c enjoys the same regularity properties and estimates as z. At first (4.21d) is tested by $\mathcal{T}_M(\vartheta_M^k)$ leading to the estimates

$$\|\mathcal{T}_M(\vartheta_M^k)\|_{H^1(\Omega)} + \|\mathcal{T}_M(\vartheta_M^k)\|_{L^{3\kappa+6}(\Omega)} \leq C.$$

Secondly, (4.21d) is tested by ϑ_M^k leading to the claimed estimate (4.33). □

Lemma 4.11. *For given $\nu > 0$ there exist functions*

$$c^k \in W^{1,p}(\Omega), \qquad \mu^k \in W^{1,\varrho}(\Omega), \qquad z^k \in W^{1,p}(\Omega) \text{ with } z \in [0,1] \text{ a.e. in } \Omega,$$
$$\vartheta^k \in H^1(\Omega) \text{ with } \vartheta^k \geq \underline{\vartheta} > 0 \text{ a.e. in } \Omega, \qquad \mathbf{u}^k \in W^{1,\varrho}(\Omega;\mathbb{R}^d)$$

such that for a subsequence $M \to \infty$

$$c_M^k \to c^k \text{ strongly in } W^{1,p}(\Omega), \tag{4.34a}$$
$$\mu_M^k \to \mu^k \text{ strongly in } W^{1,\varrho}(\Omega), \tag{4.34b}$$
$$z_M^k \to z^k \text{ strongly in } W^{1,p}(\Omega), \tag{4.34c}$$
$$\vartheta_M^k \rightharpoonup \vartheta^k \text{ weakly in } H^1(\Omega), \tag{4.34d}$$
$$\mathbf{u}_M^k \to \mathbf{u}^k \text{ strongly in } W^{1,\varrho}(\Omega;\mathbb{R}^d). \tag{4.34e}$$

Proof. First of all, observe that the a priori estimates in Lemma 4.8 and Lemma 4.10 imply (4.34) with weak instead of strong topologies.

The strong convergence (4.34c) may be shown by rewriting (4.21c) as a variational inequality

$$\begin{aligned} -\int_\Omega & |\nabla z_M^k|^{p-2}\nabla z_M^k \cdot \nabla(\zeta - z_M^k)\,\mathrm{d}x \\ &\leq \int_\Omega \Big(\frac{z_M^k - z^{k-1}}{\tau} + (\breve{\sigma})'(z_M^k) + (\hat{\sigma})'(z^{k-1}) + \nu|z_M^k|^{\varrho-2}z_M^k\Big)(\zeta - z_M^k)\,\mathrm{d}x \\ &\quad + \int_\Omega \Big(\breve{W}_{3,z}^\omega(c^k,\varepsilon(\mathbf{u}^{k-1}),z_M^k) + \hat{W}_{3,z}^\omega(c^k,\varepsilon(\mathbf{u}^{k-1}),z^{k-1}) - \mathcal{T}_M(\vartheta_M^k)\Big)(\zeta - z_M^k)\,\mathrm{d}x \end{aligned} \tag{4.35}$$

holding for all $\zeta \in W^{1,p}(\Omega)$ with $0 \leq \zeta \leq z^{k-1}$ a.e. in Ω.

To proceed we can argue by recovery sequences: By now we know the following:

$$\begin{array}{ccccl} 0 \leq & z_M^k & \leq & z^{k-1} & \\ & \downarrow & & \downarrow & \text{weakly in } W^{1,p}(\Omega) \text{ as } M \to \infty. \\ 0 \leq & z^k & \leq & z^{k-1} & \end{array}$$

Due to the compact embedding $W^{1,p}(\Omega) \hookrightarrow \mathrm{C}^0(\overline{\Omega})$, we find another sequence denoted by $\widetilde{z}_M^k$ such that

$$\widetilde{z}_M^k \to z^k \text{ strongly in } W^{1,p}(\Omega) \text{ as } M \to \infty \quad \text{and} \quad 0 \leq \widetilde{z}_M^k \leq z^{k-1}.$$

We may take, for instance, $\widetilde{z}_M^k := \max\{z^k - \delta_M, 0\}$ for suitable values $\delta_M > 0$ with $\delta_M \to 0$ as $M \to \infty$.

We test (4.35) with the admissible function $\zeta = \widetilde{z}_M^k$. Taking into account the already proved weak convergences (4.34) as well as the growth properties of the functions $(\breve{\sigma})'$ and $\breve{W}_{3,z}^\omega$ (cf. (4.27)), we manage to pass to the limit on the right-hand side of (4.35) and conclude that

$$\limsup_{M\to\infty} \int_\Omega -|\nabla z_M^k|^{p-2}\nabla z_M^k \cdot \nabla(\widetilde{z}_M^k - z_M^k)\,\mathrm{d}x \leq 0. \tag{4.36}$$

By exploiting the uniform p-convexity of the $\|\cdot\|^p_{L^p(\Omega)}$-function and strong $W^{1,p}(\Omega)$-convergence of the recovery sequence, from (4.36) we deduce that $\|\nabla(\widetilde{z}^k_M - z^k_M)\|_{L^p(\Omega)} \to 0$ as $M \to \infty$. Together with $\|\nabla(\widetilde{z}^k_M - z^k)\|_{L^p(\Omega)} \to 0$, property (4.34c) is shown.

To prove the strong convergences (4.34a), (4.34b) and (4.34e), we use a lim sup–argument. We adapt the proof from [38, Proof of Lemma 4.4 - Step 4] to our situation:

– Let $\Lambda \in L^{\varrho/(\varrho-1)}(\Omega;\mathbb{R}^d)$ be a weak cluster point of $|\nabla\mu^k_M|^{\varrho-2}\nabla\mu^k_M$. Testing (4.21a) with μ^k_M yields by exploiting a lower semicontinuity argument

$$\begin{aligned}\limsup_{M\to\infty}\nu\int_\Omega|\nabla\mu^k_M|^\varrho\,\mathrm{d}x &= \limsup_{M\to\infty}\int_\Omega -\frac{c^k_M-c^{k-1}}{\tau}\mu^k_M - m(c^{k-1},z^{k-1})|\nabla\mu^k_M|^2-\nu|\mu^k_M|^2\,\mathrm{d}x\\ &\le \int_\Omega -\frac{c^k-c^{k-1}}{\tau}\mu^k - \liminf_{M\to\infty}\int_\Omega m(c^{k-1},z^{k-1})|\nabla\mu^k_M|^2\,\mathrm{d}x - \int_\Omega \nu|\mu|^2\,\mathrm{d}x\\ &\le \int_\Omega -\frac{c^k-c^{k-1}}{\tau}\mu^k - m(c^{k-1},z^{k-1})|\nabla\mu|^2-\nu|\mu|^2\,\mathrm{d}x.\end{aligned}$$

However, the right-hand side equals $\nu\int_\Omega\Lambda\cdot\nabla\mu\,\mathrm{d}x$ by passing to the limit $M\to\infty$ in (4.21b) and testing the limit equation with μ. In conclusion, taking into account the previously proved convergences we have that

$$\limsup_{M\to\infty}\int_\Omega|\nabla\mu^k_M|^\varrho\,\mathrm{d}x \le \int_\Omega\Lambda\cdot\nabla\mu\,\mathrm{d}x,$$

which results in (4.34b).

– Convergence (4.34a) can be gained with a similar argument as above, whereas (4.34e) can be shown as in [38, Proof of Lemma 4.4 - Step 4].

□

We are now in the position to carry out the limit passage as $M\to\infty$ and conclude the existence of a solution to an intermediate approximate version of the time-discrete system (4.9), only featuring the higher regularizing terms and the ω-regularizations, i.e. (4.37) below.

Lemma 4.12 (Existence of the time-discrete system for $\nu>0$ and $M\to\infty$)**.** *Let the assumption from Lemma 4.4 be fulfilled. Then for every $\nu>0$ there exists a weak solution*

$$\{(c^k,\mu^k,z^k,\vartheta^k,\mathbf{u}^k)\}_{k=1}^{K_\tau}\subseteq W^{1,p}(\Omega)\times W^{1,\varrho}(\Omega)\times W^{1,p}(\Omega)\times H^1(\Omega)\times W^{1,\varrho}(\Omega;\mathbb{R}^d)$$

to the following time-discrete PDE system:

$$D_k(c)=\mathrm{div}\Big(m(c^{k-1},z^{k-1})\nabla\mu^k\Big)+\nu\,\mathrm{div}\Big(|\nabla\mu^k|^{\varrho-2}\nabla\mu^k\Big)-\nu\mu^k \qquad \textit{in } W^{1,\varrho}(\Omega)', \tag{4.37a}$$

$$\begin{aligned}\mu^k&=-\Delta_p(c^k)+(\breve{\phi}_\omega)'(c^k)+(\hat{\phi})'(c^{k-1})+\breve{W}^\omega_{1,c}(c^k,\varepsilon(\mathbf{u}^{k-1}),z^{k-1})\\&\quad+\hat{W}^\omega_{1,c}(c^{k-1},\varepsilon(\mathbf{u}^{k-1}),z^{k-1})-\vartheta^k+D_k(c)+\nu|c^k|^{\varrho-2}c^k \qquad \textit{in } W^{1,p}(\Omega)',\end{aligned} \tag{4.37b}$$

$$\begin{aligned}&D_k(z)-\Delta_p(z^k)+\xi^k+(\breve{\sigma})'(z^k)+(\hat{\sigma})'(z^{k-1})+\nu|z^k|^{\varrho-2}z^k\\&\quad=-\breve{W}^\omega_{3,z}(c^k,\varepsilon(\mathbf{u}^{k-1}),z^k)-\hat{W}^\omega_{3,z}(c^k,\varepsilon(\mathbf{u}^{k-1}),z^{k-1})+\vartheta^k\\&\quad\textit{with } \xi^k\in\partial I_{Z^{k-1}}(z^k) \qquad \textit{in } W^{1,p}(\Omega)',\end{aligned} \tag{4.37c}$$

$$\begin{aligned}&D_k(\vartheta)+\mathcal{A}^k(\vartheta^k)+D_k(c)\vartheta^k+D_k(z)\vartheta^k+\rho\vartheta^k\,\mathrm{div}(D_k(\mathbf{u}))\\&\quad=g^k+|D_k(c)|^2+|D_k(z)|^2+a(c^{k-1},z^{k-1})\varepsilon(D_k(\mathbf{u})):\mathbb{V}\varepsilon(D_k(\mathbf{u}))\\&\quad\quad+m(c^{k-1},z^{k-1})|\nabla\mu^k|^2 \qquad \textit{in } H^1(\Omega)',\end{aligned} \tag{4.37d}$$

$$\begin{aligned}&D_k(D_k(\mathbf{u}))-\mathrm{div}\Big(a(c^{k-1},z^{k-1})\mathbb{V}\varepsilon(D_k(\mathbf{u}))+W^\omega_{,\varepsilon}(c^k,\varepsilon(\mathbf{u}^k),z^k)\Big)-\rho\vartheta^k\mathbb{1}\Big)\\&\quad-\nu\,\mathrm{div}\Big(|\varepsilon(\mathbf{u}^k-\mathbf{d}^k)|^{\varrho-2}\varepsilon(\mathbf{u}^k-\mathbf{d}^k)\Big)=\mathbf{f}^k \qquad \textit{in } W^{1,\varrho}_0(\Omega;\mathbb{R}^d)'\end{aligned} \tag{4.37e}$$

satisfying the initial conditions (4.10)*, the boundary condition* $\mathbf{u}^k = \mathbf{d}^k$ *a.e. on* $\partial\Omega$ *and the constraints* (4.12)*.*

Proof. At the beginning we notice that

$$\mathcal{T}_M(\vartheta_M^k) \to \vartheta^k \text{ strongly in } L^{p^*-\epsilon}(\Omega) \text{ for all } \epsilon \in (0, p^*-1] \text{ as } M \to \infty, \tag{4.38}$$

which follows from the pointwise convergence $\mathcal{T}_M(\vartheta_M^k) \to \vartheta^k$ as $M \to \infty$ a.e. in Ω and the uniform boundedness of $\|\mathcal{T}_M(\vartheta_M^k)\|_{L^{p^*}(\Omega)}$ with respect to M.

We see that with the help of Lemma 4.11 and (4.38), also taking into account the growth properties of $W^\omega_{,\varepsilon}$ (cf. (4.26)), we can pass to $M \to \infty$ along a subsequence in (4.21a) for c and (4.21e) for $\mathbf{u}$ and obtain (4.37a) and (4.37e), respectively. The limit passages for the remaining equations are carried out as follows:

- It follows from $\|c_M^k\|_{W^{1,p}(\Omega)} \le C$ and from the Lipschitz continuity of β_ω that $(\breve{\phi}_\omega)'(c_M^k) = \beta_\omega(c_M^k) + \lambda c_M^k$ is bounded, uniformly in M and $k = 1, \ldots, K_\tau$, in $L^\infty(\Omega)$. This and the growth properties of $\breve{W}^\omega_{1,c}$ and $\hat{W}^\omega_{1,c}$ (cf. (4.24)–(4.25)), together with Lemma 4.11 and convergence (4.38), enable us to pass to $M \to \infty$ in equation (4.21b) for μ. We find (4.37b).
- The limit passage in equation (4.21c) for z is managed via the variational formulation (4.35). To this end we pick an arbitrary test-function $\zeta \in W^{1,p}(\Omega)$ with $0 \le \zeta \le z^{k-1}$ and construct the recovery sequence

  $$\zeta_M := \max\{z^{k-1} - \delta_M, 0\}$$

  for suitable values $\delta_M > 0$ with $\delta_M \to 0$ such that $0 \le \zeta_M \le z^{k-1}$ is fulfilled for all $M \in \mathbb{N}$. Now, testing (4.35) with ζ_M and passing to $M \to \infty$ with the help of Lemma 4.11 and (4.38) yields (4.37c).
- By exploiting Lemma 4.11, property (4.38) and a comparison argument as done in [38, Lemma 4.4, Step 3] we find

  $$\mathcal{A}_M^k(\vartheta_M^k) \rightharpoonup \mathcal{A}^k(\vartheta^k) \quad \text{weakly in } H^1(\Omega)' \text{ as } M \to \infty.$$

  This allows us to pass to the limit $M \to \infty$ in equation (4.21d) for ϑ in order to obtain (4.37d).

□

4.2.3 Step 3: Limit passage $\nu\downarrow 0$.

We now address the limit passage $\nu \downarrow 0$ and denote by $(c_\nu^k, \mu_\nu^k, z_\nu^k, \mathbf{u}_\nu^k)_\nu$ the family of solutions to system (4.37) found in Lemma 4.11. By lower semicontinuity, estimates (4.32) from Lemma 4.8 are thus inherited by the functions $(c_\nu^k, \mu_\nu^k, z_\nu^k, \mathbf{u}_\nu^k)_\nu$. Furthermore, we obtain a uniform $H^1(\Omega)$-estimate for $(\vartheta_\nu^k)_\nu$. Indeed, since the higher order terms

$$\nu \operatorname{div}\left(|\nabla\mu_\nu^k|^{\varrho-2}\nabla\mu_\nu^k\right) - \nu\mu_\nu^k, \ldots\ldots, -\nu \operatorname{div}\left(|\varepsilon(\mathbf{u}_\nu^k - \mathbf{d}^k)|^{\varrho-2}\varepsilon(\mathbf{u}_\nu^k - \mathbf{d}^k)\right)$$

vanish as $\nu \downarrow 0$, we loose the $L^2(\Omega)$-estimate for the right-hand side of the discrete temperature equation (4.37d). Therefore, to prove this H^1-bound for ϑ_ν^k we have to resort to the arguments from the proof of the *Second a priori estimate* in Sec. 3, and in particular fully exploit the coercivity properties of the function K.

Lemma 4.13. *The following estimates holds uniformly in* $\nu > 0$*:*

$$\|\vartheta_\nu^k\|_{H^1(\Omega)} \le C, \qquad \|(\vartheta_\nu^k)^{(\kappa+\alpha)/2}\|_{H^1(\Omega)} \le C_\alpha \quad \textit{for all } \alpha \in (0,1). \tag{4.39}$$

Proof. We test (4.37d) by $(\vartheta_\nu^k)^{\alpha-1}$, with $\alpha \in (0,1)$. With the very same calculations as for the *Second a priori estimate* (cf. (3.9) and also the proof of Prop. 4.18 later on), we conclude

$$
\begin{aligned}
&c\int_\Omega \mathsf{K}(\vartheta_\nu^k)|\nabla(\vartheta_\nu^k)^{\alpha/2}|^2\,\mathrm{d}x + c\int_\Omega \left(\left|\varepsilon\left(D_k(u_\nu^k)\right)\right|^2 + |\nabla\mu_\nu^K|^2\right)(\vartheta_\nu^k)^{\alpha-1}\,\mathrm{d}x + c\int_\Omega \left(\left|D_k(z_\nu^k)\right|^2 + \left|D_k(c_\nu^k)\right|^2\right)(\vartheta_\nu^k)^{\alpha-1}\,\mathrm{d}x\\
&\le C + C\int_\Omega (\vartheta_\nu^k)^{\alpha+1}\,\mathrm{d}x\,.
\end{aligned}
$$

Then, with the same arguments as in Sec. 3, we arrive at $\int_\Omega |\nabla(\vartheta_\nu^k)^{(\kappa+\alpha)/2}|^2\,\mathrm{d}x \le C$ for a constant independent of ν. Ultimately, we conclude (4.39). □

By comparison arguments based on the *Third estimate* we then obtain uniform estimates for $(\mathbf{u}_\nu^k)_\nu$ and for $(\mu_\nu^k)_\nu$ with respect to ν.

Lemma 4.14. *The following estimates hold uniformly in $\nu > 0$:*

$$
\|\mathbf{u}_\nu^k\|_{H^1(\Omega;\mathbb{R}^d)} + \|\mu_\nu^k\|_{H^1(\Omega)} \le C. \tag{4.40}
$$

Proof. We proceed as in the *Third estimate* in Section 3: Testing the time-discrete heat equation (4.37d) with τ, and subtracting the resulting equation from the incremental energy inequality (4.29) (the limit version $M \to \infty$). In particular we obtain boundedness with respect to ν of

$$
\int_\Omega a(c^{k-1}, z^{k-1})\frac{\varepsilon(\mathbf{u}_\nu^k - \mathbf{u}^{k-1})}{\tau} : \mathbb{V}\frac{\varepsilon(\mathbf{u}_\nu^k - \mathbf{u}^{k-1})}{\tau}\,\mathrm{d}x + \int_\Omega m(c^{k-1}, z^{k-1})|\nabla\mu_\nu^k|^2\,\mathrm{d}x \le C.
$$

Hence $\|\varepsilon(\mathbf{u}_\nu^k)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}$ and $\|\nabla\mu_\nu^k\|_{L^2(\Omega)}$ are bounded in ν. Korn's inequality applied to $\mathbf{u}_\nu^k - \mathbf{d}^k$ shows the first part of the claim, namely boundedness of $\|\mathbf{u}_\nu^k\|_{H^1(\Omega;\mathbb{R}^d)}$.

The proof of the second part makes use of the Poincaré inequality. To this end boundedness of the spatial mean of μ_ν^k has to be shown. Testing the time-discrete equation (4.37b) with $1/|\Omega|$ shows

$$
\begin{aligned}
\fint_\Omega \mu_\nu^k\,\mathrm{d}x = &\fint_\Omega (\breve{\phi}_\omega)'(c_\nu^k) + (\hat{\phi})'(c^{k-1}) + \breve{W}_{1,c}^\omega(c_\nu^k, \varepsilon(\mathbf{u}^{k-1}), z^{k-1}) + \hat{W}_{1,c}^\omega(c^{k-1}, \varepsilon(\mathbf{u}^{k-1}), z^{k-1})\,\mathrm{d}x\\
&+ \fint_\Omega -\vartheta_\nu^k + \frac{c_\nu^k - c^{k-1}}{\tau} + \nu|c_\nu^k|^{\varrho-2}c_\nu^k\,\mathrm{d}x.
\end{aligned}
$$

By the known boundedness properties of $(c_\nu^k)_\nu$, $(\mathbf{u}_\nu^k)_\nu$, $(z_\nu^k)_\nu$ and $(\vartheta_\nu^k)_\nu$, and the growth of $\breve{W}_{1,c}^\omega$, $\hat{W}_{1,c}^\omega$ (cf. (4.24)–(4.25)), and of $(\breve{\phi}_\omega)'$ (affine-linear growth in c due to Yosida approximation with parameter τ), we then infer boundedness of $\fint_\Omega \mu_\nu^k$. Together with boundedness of $\|\nabla\mu_\nu^k\|_{L^2(\Omega)}$ we conclude the second part of the claim by the Poincaré inequality. □

We then have the following counterpart to Lemma 4.11, which reflects the lesser regularity of the solution components μ^k and $\mathbf{u}^k$ as a result of the limit passage as $\nu \downarrow 0$. Its proof is a straightforward adaptation of the argument developed for Lemma 4.11.

Lemma 4.15. *There exist $(c^k, \mu^k, z^k, \vartheta^k, \mathbf{u}^k) \in W^{1,p}(\Omega) \times H^1(\Omega) \times W^{1,p}(\Omega) \times H^1(\Omega) \times H^1(\Omega;\mathbb{R}^d)$ and a (not relabeled) subsequence $\nu \downarrow 0$ such that convergences* (4.34a), (4.34c)–(4.34d) *hold, as well as*

$$
\begin{aligned}
&\mu_\nu^k \to \mu^k \text{ strongly in } H^1(\Omega), && \text{(4.41a)}\\
&\nu|\nabla\mu_\nu^k|^{\varrho-2}\nabla\mu_\nu^k \to 0 \text{ strongly in } L^{\varrho/(\varrho-1)}(\Omega;\mathbb{R}^d), && \text{(4.41b)}\\
&\mathbf{u}_\nu^k \to \mathbf{u}^k \text{ strongly in } H^1(\Omega;\mathbb{R}^d), && \text{(4.41c)}\\
&\nu|\varepsilon(\mathbf{u}_\nu^k - \mathbf{d}^k)|^{\varrho-2}\varepsilon(\mathbf{u}_\nu^k - \mathbf{d}^k) \to 0 \text{ strongly in } L^{\varrho/(\varrho-1)}(\Omega;\mathbb{R}^{d\times d}). && \text{(4.41d)}
\end{aligned}
$$

We are now in the position to carry out the **limit passage as $\nu\downarrow 0$ in system** (4.37). The aguments for taking the limits in (4.37a), (4.37b), (4.37c), and (4.37e) are completely analogous to those developed in the proof of Lemma 4.12.

Hence we only comment on the limit passage in the discrete heat equation (4.37d). Estimate (4.39) allows us to conclude that, up to a subsequence, $(\vartheta_\nu^k)^{(\kappa+\alpha)/2} \rightharpoonup (\vartheta^k)^{(\kappa+\alpha)/2}$ in $H^1(\Omega)$, hence $(\vartheta_\nu^k)^{(\kappa+\alpha)/2} \to (\vartheta^k)^{(\kappa+\alpha)/2}$ in $L^{6-\epsilon}(\Omega)$ for all $\epsilon > 0$, whence, taking into account the growth condition on K, that

$$\mathsf{K}(\vartheta_\nu) \to \mathsf{K}(\vartheta) \qquad \text{in } L^\gamma(\Omega) \quad \text{with } \gamma = \frac{(6-\epsilon)(\kappa+\alpha)}{2\kappa} \quad \text{for all } \epsilon > 0.$$

This allows us to pass to the limit in the term $\mathsf{K}(\vartheta_\nu)\nabla\vartheta_\nu$, tested against $v \in W^{1,s}(\Omega)$ for some sufficiently large $s > 0$. All in all, we infer that $(c, \mu, z, \vartheta, \mathbf{u}, \mathcal{X})$ solves system (4.9), with (4.9b) and (4.9c) in $W^{1,p}(\Omega)'$, and with the discrete heat equation (4.9d) understood in $W^{1,s}(\Omega)'$.

In the next step, we will address enhancements of the regularities of $\mathbf{u}$ and μ.

As a by-product we will obtain the discrete heat equation (4.9d) understood in the $H^1(\Omega)'$-sense.

4.2.4 Step 4: $\mathbf{H^2}$-regularity of $\mathbf{u^k}$ and $\boldsymbol{\mu^k}$ and conclusion of the proof of Prop. 4.3

To complete the **proof of Proposition 4.3** we have to improve the regularity of $\mathbf{u}^k$ and μ^k. This is achieved by transforming the corresponding equations in a way that enables us to apply standard elliptic regularity results.

Lemma 4.16. *We get $\mu^k \in H^2_N(\Omega)$ and $\mathbf{u}^k \in H^2(\Omega;\mathbb{R}^d)$ for the functions obtained in Lemma 4.15.*

Proof. We will use an iteration argument as in [38, 24] (see also [2]) and sketch the proof for the case $d = 3$, since the calculations for $d = 2$ are completely analogous.

We already know that $\mu^k \in H^1(\Omega)$ satisfies the elliptic equation

$$\int_\Omega m(c^{k-1}, z^{k-1})\nabla\mu^k \cdot \nabla w \,\mathrm{d}x = \int_\Omega -D_k(c) w \,\mathrm{d}x \qquad \text{for all } w \in H^1(\Omega).$$

Substituting $w = \frac{\zeta}{m(c^{k-1},z^{k-1})} \in H^1(\Omega)$ for an arbitrarily chosen test-function $\zeta \in H^1(\Omega)$ yields

$$\int_\Omega \nabla\mu^k \cdot \nabla\zeta \,\mathrm{d}x = \int_\Omega \Big(\frac{-D_k(c)}{m(c^{k-1}, z^{k-1})} + \frac{m_{,c}(c^{k-1}, z^{k-1})\nabla c^{k-1} + m_{,z}(c^{k-1}, z^{k-1})\nabla z^{k-1}}{m(c^{k-1}, z^{k-1})} \cdot \nabla\mu^k\Big)\zeta \,\mathrm{d}x$$

valid for all $\zeta \in H^1(\Omega)$. Note that, due to Hypothesis (II) and the fact that $c^{k-1}, z^{k-1} \in W^{1,p}(\Omega)$ and $\nabla\mu^k \in L^2(\Omega;\mathbb{R}^d)$, the function in the bracket on the right-hand side is in $L^{2p/(2+p)}(\Omega)$. Applying a higher elliptic regularity result for homogeneous Neumann problems with $L^{2p/(2+p)}(\Omega)$-right-hand side proves $\mu^k \in W^{2,2p/(2+p)}(\Omega)$ and thus $\nabla\mu^k \in L^{6p/(6+p)}(\Omega;\mathbb{R}^d)$. Due to $p > 3$ we end up with $\mu^k \in H^2_N(\Omega)$ after repeating this procedure finitely many times (cf. [24, Proof of Lemma 4.1]).

The proof for obtaining $\mathbf{u}^k \in H^2(\Omega;\mathbb{R}^d)$ from the elliptic equation (4.21e) in $H^1_0(\Omega;\mathbb{R}^n)'$ works as in [38, Proof of Lemma 4.4 - Step 6] (cf. also [24]), with the exception that one needs to take the Dirichlet data $\mathbf{d}^k \in H^2(\Omega;\mathbb{R}^d)$ into account. This is the very point where we need to assume that $\mathbb{V} = \omega\mathbb{C}$ for some $\omega > 0$ (cf. (2.20)). □

The enhanced regularity for $\mathbf{u}^k$ yields, by a comparison argument in (4.9d), that (4.9d) not only holds in $W^{1,s}(\Omega)'$ for large $s > 1$ but even in $H^1(\Omega)'$.

Finally, we end up with a quintuple $\{(c_\tau^k, \mu_\tau^k, z_\tau^k, \vartheta_\tau^k, \mathbf{u}_\tau^k)\}_{k=1}^{K_\tau} \subseteq W^{1,p}(\Omega)\times H^2_N(\Omega)\times W^{1,p}(\Omega)\times H^1(\Omega)\times H^2(\Omega;\mathbb{R}^d)$ satisfying the assertion stated in Proposition 4.3.

This concludes the proof. □

4.3 Discrete energy and entropy inequalities

We introduce the left-continuous and right-continuous piecewise constant, and the piecewise linear interpolants for a given sequence $\{\mathfrak{h}_\tau^k\}_{k=0}^{K_\tau}$ on the nodes $\{t_\tau^k\}_{k=0}^{K_\tau}$ (see 4.1) by

$$\left.\begin{array}{lll} \overline{\mathfrak{h}}_\tau : (0,T) \to B & \text{defined by} & \overline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^k \\ \underline{\mathfrak{h}}_\tau : (0,T) \to B & \text{defined by} & \underline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^{k-1} \\ \mathfrak{h}_\tau : (0,T) \to B & \text{defined by} & \mathfrak{h}_\tau(t) := \frac{t-t_\tau^{k-1}}{\tau}\mathfrak{h}_\tau^k + \frac{t_\tau^k-t}{\tau}\mathfrak{h}_\tau^{k-1} \end{array}\right\} \qquad \text{for } t \in (t_\tau^{k-1}, t_\tau^k].$$

Furthermore, we denote by $\bar{\mathfrak{t}}_\tau$ and by $\underline{\mathfrak{t}}_\tau$ the left-continuous and right-continuous piecewise constant interpolants associated with the partition, i.e. $\bar{\mathfrak{t}}_\tau(t) := t_\tau^k$ if $t_\tau^{k-1} < t \le t_\tau^k$ and $\underline{\mathfrak{t}}_\tau(t) := t_\tau^{k-1}$ if $t_\tau^{k-1} \le t < t_\tau^k$. Clearly, for every $t \in [0,T]$ we have $\bar{\mathfrak{t}}_\tau(t) \downarrow t$ and $\underline{\mathfrak{t}}_\tau(t) \uparrow t$ as $\tau \downarrow 0$.

Proposition 4.17. *Let the assumptions of Proposition 4.3 be satisfied. Then the time-discrete solutions* $\{(c_\tau^k, \mu_\tau^k, z_\tau^k, \vartheta_\tau^k, \mathbf{u}_\tau^k)\}_{k=1}^{K_\tau}$ *to Problem 4.1 fulfill for all* $0 \le s \le t \le T$

(i) the discrete entropy inequality

$$\begin{aligned} &\int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega (\log(\underline{\vartheta}_\tau) + \underline{c}_\tau + \underline{z}_\tau)\partial_t \varphi_\tau \,\mathrm{d}x\,\mathrm{d}r - \rho \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \mathrm{div}(\partial_t \mathbf{u}_\tau)\overline{\varphi}_\tau \,\mathrm{d}x\,\mathrm{d}r \\ &\qquad - \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \mathsf{K}(\overline{\vartheta}_\tau)\nabla \log(\overline{\vartheta}_\tau) \cdot \nabla \overline{\varphi}_\tau \,\mathrm{d}x\,\mathrm{d}r \\ &\le \int_\Omega (\log(\overline{\vartheta}_\tau(t)) + \overline{c}_\tau(t) + \overline{z}_\tau(t))\overline{\varphi}_\tau(t)\,\mathrm{d}x - \int_\Omega (\log(\overline{\vartheta}_\tau(s)) + \overline{c}_\tau(s) + \overline{z}_\tau(s))\overline{\varphi}_\tau(s)\,\mathrm{d}x \\ &\qquad - \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \mathsf{K}(\overline{\vartheta}_\tau)|\nabla \log(\overline{\vartheta}_\tau)|^2 \overline{\varphi}_\tau \,\mathrm{d}x\,\mathrm{d}r \\ &\quad - \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \left(\overline{g}_\tau + |\partial_t c_\tau|^2 + |\partial_t z_\tau|^2 + a(\underline{c}_\tau, \underline{z}_\tau)\varepsilon(\partial_t \mathbf{u}_\tau) : \mathbb{V}\varepsilon(\partial_t \mathbf{u}_\tau) + m(\underline{c}_\tau, \underline{z}_\tau)|\nabla \overline{\mu}_\tau|^2\right) \frac{\overline{\varphi}_\tau}{\overline{\vartheta}_\tau}\,\mathrm{d}x\,\mathrm{d}r \\ &\quad - \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_{\partial\Omega} \overline{h}_\tau \frac{\overline{\varphi}_\tau}{\overline{\vartheta}_\tau}\,\mathrm{d}S\,\mathrm{d}r, \end{aligned} \tag{4.42}$$

for all $\varphi \in \mathrm{C}^0([0,T]; W^{1,d+\epsilon}(\Omega)) \cap H^1(0,T; L^{(d^\star)'}(\Omega))$ *for some* $\epsilon > 0$, *with* $\varphi \ge 0$;

(ii) the discrete total energy inequality

$$\begin{aligned} &\mathscr{E}_\omega(\overline{c}_\tau(t), \overline{z}_\tau(t), \overline{\vartheta}_\tau(t), \overline{\mathbf{u}}_\tau(t), \overline{\mathbf{v}}_\tau(t)) \\ &\quad \le \mathscr{E}_\omega(\overline{c}_\tau(s), \overline{z}_\tau(s), \overline{\vartheta}_\tau(s), \overline{\mathbf{u}}_\tau(s), \overline{\mathbf{v}}_\tau(s)) \\ &\qquad + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \overline{g}_\tau \,\mathrm{d}x\,\mathrm{d}r + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \overline{\mathbf{f}}_\tau \cdot \overline{\mathbf{v}}_\tau \,\mathrm{d}x\,\mathrm{d}r + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_{\partial\Omega} \overline{h}_\tau \,\mathrm{d}S\,\mathrm{d}r \\ &\qquad + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_{\partial\Omega} (\boldsymbol{\sigma}_\tau \mathbf{n}) \cdot \partial_t \mathbf{d}_\tau \,\mathrm{d}S\,\mathrm{d}r \end{aligned} \tag{4.43}$$

with the discrete stress tensor

$$\boldsymbol{\sigma}_\tau := a(\underline{c}_\tau, \underline{z}_\tau)\mathbb{V}\varepsilon(\partial_t \mathbf{u}_\tau) + W^\omega_{,\varepsilon}(\overline{c}_\tau, \varepsilon(\overline{\mathbf{u}}_\tau), \overline{z}_\tau) - \rho\overline{\vartheta}_\tau \mathbb{1}.$$

Proof.

To (i): The proof is based on [38, Proof of Proposition 4.8]. Testing the time-discrete heat equation (4.9d) for time step k with $\frac{\varphi_\tau^k}{\vartheta_\tau^k} \in H^1(\Omega)$ shows

$$
\begin{aligned}
&\int_\Omega \Big(g_\tau^k + |D_{\tau,k}(c)|^2 + |D_{\tau,k}(z)|^2 + m(c_\tau^{k-1}, z_\tau^{k-1}) |\nabla \mu_\tau^k|^2 \Big) \frac{\varphi_\tau^k}{\vartheta_\tau^k} \,\mathrm{d}x \\
&+ \int_\Omega a(c_\tau^{k-1}, z_\tau^{k-1}) \varepsilon(D_{\tau,k}(\mathbf{u})) : \mathbb{V}\varepsilon(D_{\tau,k}(\mathbf{u})) \frac{\varphi_\tau^k}{\vartheta_\tau^k} \,\mathrm{d}x + \int_{\partial\Omega} h_\tau^k \frac{\varphi_\tau^k}{\vartheta_\tau^k} \,\mathrm{d}S \\
&\leq \int_\Omega \bigg(\mathsf{K}(\vartheta_\tau^k) \nabla \vartheta_\tau^k \cdot \nabla \frac{\varphi_\tau^k}{\vartheta_\tau^k} + \Big(\frac{1}{\tau} \big(\log(\vartheta_\tau^k) - \log(\vartheta_\tau^{k-1}) \big) + D_{\tau,k}(c) + D_{\tau,k}(z) + \rho \operatorname{div}(D_{\tau,k}(\mathbf{u})) \Big) \varphi_\tau^k \,\mathrm{d}x
\end{aligned}
$$

by using the concavity estimate

$$
\frac{\vartheta_\tau^k - \vartheta_\tau^{k-1}}{\vartheta_\tau^k} \leq \log(\vartheta_\tau^k) - \log(\vartheta_\tau^{k-1}).
$$

Summing over $k = \frac{\bar{\mathrm{t}}_\tau(s)}{\tau} + 1, \ldots, \frac{\bar{\mathrm{t}}_\tau(t)}{\tau}$ and using discrete by-part-integration proves (4.42).

To (ii): The total energy inequality is inherited by the incremental energy inequality (4.29) of the (M, ν)-regularized system in Lemma 4.5. Indeed, let $0 \leq s \leq t \leq T$. Passing to the limits $M \to \infty$ and $\nu \downarrow 0$ in (4.29) by means of lower semicontinuity arguments and then summing over $j = \frac{\bar{\mathrm{t}}_\tau(s)}{\tau} + 1, \ldots, \frac{\bar{\mathrm{t}}_\tau(t)}{\tau}$ yields

$$
\begin{aligned}
&\mathscr{E}_\omega(\bar{c}_\tau(t), \bar{z}_\tau(t), \bar{\vartheta}_\tau(t), \bar{\mathbf{u}}_\tau(t), \bar{\mathbf{v}}_\tau(t)) \\
&\quad \leq \mathscr{E}_\omega(\bar{c}_\tau(s), \bar{z}_\tau(s), \bar{\vartheta}_\tau(s), \bar{\mathbf{u}}_\tau(s), \bar{\mathbf{v}}_\tau(s)) + \int_{\bar{\mathrm{t}}_\tau(s)}^{\bar{\mathrm{t}}_\tau(t)} \Big(\int_\Omega \bar{g}_\tau \,\mathrm{d}x + \int_{\partial\Omega} \bar{h}_\tau \,\mathrm{d}S + \int_\Omega \bar{\mathbf{f}}_\tau \cdot \bar{\mathbf{v}}_\tau \,\mathrm{d}x \Big) \,\mathrm{d}r \\
&\qquad \left. \begin{aligned}
&+ \int_{\bar{\mathrm{t}}_\tau(s)}^{\bar{\mathrm{t}}_\tau(t)} \int_\Omega \partial_t \mathbf{v}_\tau \cdot \partial_t \mathbf{d}_\tau \,\mathrm{d}x \,\mathrm{d}r + \int_{\bar{\mathrm{t}}_\tau(s)}^{\bar{\mathrm{t}}_\tau(t)} \int_\Omega a(\underline{c}_\tau, \underline{z}_\tau) \mathbb{V}\varepsilon(\bar{\mathbf{v}}_\tau) : \varepsilon(\partial_t \mathbf{d}_\tau) \,\mathrm{d}x \,\mathrm{d}r \\
&+ \int_{\bar{\mathrm{t}}_\tau(s)}^{\bar{\mathrm{t}}_\tau(t)} \int_\Omega W_{,\varepsilon}^\omega(\bar{c}_\tau, \varepsilon(\bar{\mathbf{u}}_\tau), \bar{z}_\tau) : \varepsilon(\partial_t \mathbf{d}_\tau) \,\mathrm{d}x \,\mathrm{d}r - \int_{\bar{\mathrm{t}}_\tau(s)}^{\bar{\mathrm{t}}_\tau(t)} \int_\Omega \rho \bar{\vartheta}_\tau \operatorname{div}(\partial_t \mathbf{d}_\tau) \,\mathrm{d}x \,\mathrm{d}r \\
&- \int_{\bar{\mathrm{t}}_\tau(s)}^{\bar{\mathrm{t}}_\tau(t)} \int_\Omega \bar{\mathbf{f}}_\tau \cdot \partial_t \mathbf{d}_\tau \,\mathrm{d}x \,\mathrm{d}r.
\end{aligned} \right\} =: I_1
\end{aligned}
$$

Finally, integration by parts in space and using (4.9e) shows

$$
\begin{aligned}
I_1 &= \int_{\bar{\mathrm{t}}_\tau(s)}^{\bar{\mathrm{t}}_\tau(t)} \int_\Omega \underbrace{\Big(\partial_t \mathbf{v}_\tau - \operatorname{div} \Big(a(\underline{c}_\tau, \underline{z}_\tau) \mathbb{V}\varepsilon(\bar{\mathbf{v}}_\tau) + W_{,\varepsilon}^\omega(\bar{c}_\tau, \varepsilon(\bar{\mathbf{u}}_\tau), \bar{z}_\tau) - \rho \bar{\vartheta}_\tau \mathbb{1} \Big) - \bar{\mathbf{f}}_\tau \Big)}_{=0} \cdot \partial_t \mathbf{d}_\tau \,\mathrm{d}x \,\mathrm{d}r \\
&\quad + \int_{\bar{\mathrm{t}}_\tau(s)}^{\bar{\mathrm{t}}_\tau(t)} \int_{\partial\Omega} \Big(a(\underline{c}_\tau, \underline{z}_\tau) \mathbb{V}\varepsilon(\bar{\mathbf{v}}_\tau) + W_{,\varepsilon}^\omega(\bar{c}_\tau, \varepsilon(\bar{\mathbf{u}}_\tau), \bar{z}_\tau) - \rho \bar{\vartheta}_\tau \mathbb{1} \Big) \mathbf{n} \cdot \partial_t \mathbf{d}_\tau \,\mathrm{d}S \,\mathrm{d}r \\
&= \int_{\bar{\mathrm{t}}_\tau(s)}^{\bar{\mathrm{t}}_\tau(t)} \int_{\partial\Omega} (\boldsymbol{\sigma}_\tau \mathbf{n}) \cdot \partial_t \mathbf{d}_\tau \,\mathrm{d}S \,\mathrm{d}r.
\end{aligned}
$$

□

4.4 A priori estimates

The aim of this section is to customize the a priori estimates which we have developed in Section 3 to the time-discrete setting described in Problem 4.1, for a time-discrete solution $(\bar{c}_\tau, \underline{c}_\tau, c_\tau, \bar{\mu}_\tau, \bar{z}_\tau, \underline{z}_\tau, z_\tau, \bar{\vartheta}_\tau, \underline{\vartheta}_\tau, \vartheta_\tau, \bar{\mathbf{u}}_\tau, \underline{\mathbf{u}}_\tau, \bar{\mathbf{v}}_\tau, \mathbf{v}_\tau)$ (recall that $\mathbf{v}^k = D_k(\mathbf{u})$ for all $k \in \{1, \ldots, K_\tau\}$. Let us mention in advance that, in this time-discrete setting we are only able to estimate (cf. (4.51) below) the supremum of the total variation $\langle \log(\overline{\vartheta}_\tau), \varphi\rangle_{W^{1,d+\epsilon}}$ over all test-functions $\varphi \in W^{1,d+\epsilon}(\Omega)$ with $\|\varphi\|_{W^{1,d+\epsilon}(\Omega)} \leq 1$, which is a slightly weaker result than the **Seventh a priori estimate** in Section 3 however strong enough to apply the compactness result proved in [38, Theorem A.5].

Proposition 4.18. *Let the assumptions of Proposition 4.3 be satisfied. Then the time-discrete solutions* $\{(c_\tau^k, \mu_\tau^k, z_\tau^k, \vartheta_\tau^k, \mathbf{u}_\tau^k)\}_{k=1}^{K_\tau}$ *to Problem 4.1 fulfill the following a priori estimates uniformly in* $\omega > 0$ *and* $\tau > 0$*:*

$$\|\overline{c}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|\underline{c}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq C, \tag{4.44}$$
$$\|c_\tau\|_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;W^{1,p}(\Omega))} \leq C, \tag{4.45}$$
$$\|\Delta_p(\overline{c}_\tau)\|_{L^2(0,T;L^2(\Omega))} \leq C, \tag{4.46}$$
$$\|\overline{\eta}_\tau\|_{L^2(0,T;L^2(\Omega))} \leq C \qquad \text{with } \overline{\eta}_\tau := \beta_\omega(\overline{c}_\tau), \tag{4.47}$$
$$\|\overline{\mu}_\tau\|_{L^2(0,T;H^2(\Omega))} \leq C, \tag{4.48}$$
$$\|\overline{z}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|\underline{z}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq C, \tag{4.49}$$
$$\|z_\tau\|_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;W^{1,p}(\Omega))} \leq C, \tag{4.50}$$
$$\|\overline{\vartheta}_\tau\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^1(\Omega))} \leq C, \tag{4.51}$$
$$\big\|(\overline{\vartheta}_\tau)^{\frac{\kappa+\alpha}{2}}\big\|_{L^2(0,T;H^1(\Omega))} \leq C_\alpha \text{ for all } \alpha \in (0,1), \tag{4.52}$$
$$\|\log(\overline{\vartheta}_\tau)\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{4.53}$$
$$\|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^d))} + \|\underline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^d))} \leq C, \tag{4.54}$$
$$\|\mathbf{u}_\tau\|_{H^1(0,T;H^2(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^d))} \leq C, \tag{4.55}$$
$$\|\mathbf{v}_\tau\|_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))\cap H^1(0,T;L^2(\Omega;\mathbb{R}^d))} \leq C \tag{4.56}$$

as well as

$$\sup_{\varphi\in W^{1,d+\epsilon}(\Omega),\|\varphi\|_{W^{1,d+\epsilon}(\Omega)}\leq 1} \operatorname{Var}\big(\langle \log(\overline{\vartheta}_\tau), \varphi\rangle_{W^{1,d+\epsilon}}; [0,T]\big) \leq C_\epsilon \quad \text{for all } \epsilon > 0. \tag{4.57}$$

Under the additional assumption (2.49) *we also have*

$$\|\vartheta_\tau\|_{\mathrm{BV}([0,T];W^{2,d+\epsilon}(\Omega)')} \leq C_\epsilon \quad \text{for all } \epsilon > 0. \tag{4.58}$$

Proof. The proof mainly follows the lines in Section 3. Besides this, the estimates for the time-discrete variables z_τ, ϑ_τ and $\mathbf{u}_\tau$ are based on [38, Proof of Proposition 4.10]. To avoid repetition we will refer to the estimates in Section 3 when necessary.

(i) The time-discrete total energy inequality from Proposition 4.17 (ii) implies the following estimates (see **First a priori estimate**):

$$\|\overline{c}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|\nabla\overline{z}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^d))} + \|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{v}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq C.$$

(ii) The **Second a priori estimate** is performed by testing the time-discrete heat equation (4.9d) with $F'(\vartheta_\tau^k) = (\vartheta_\tau^k)^{\alpha-1}$ with $\alpha \in (0,1)$ and the concave function $F(\vartheta) := \vartheta^\alpha/\alpha$, we obtain

$$\begin{aligned}
&\int_\Omega \Big(g_\tau^k + |D_{\tau,k}(c)|^2 + |D_{\tau,k}(z)|^2 + m(c_\tau^{k-1}, z_\tau^{k-1})|\nabla\mu_\tau^k|^2\Big)F'(\vartheta_\tau^k)\,\mathrm{d}x\\
&+\int_\Omega a(c_\tau^{k-1}, z_\tau^{k-1})\varepsilon(D_{\tau,k}(\mathbf{u})) : \mathbb{V}\varepsilon(D_{\tau,k}(\mathbf{u}))F'(\vartheta_\tau^k)\,\mathrm{d}x + \int_{\partial\Omega} h_\tau^k F'(\vartheta_\tau^k)\,\mathrm{d}S\\
&\leq \int_\Omega \frac{F(\vartheta_\tau^k) - F(\vartheta_\tau^{k-1})}{\tau} + \mathsf{K}(\vartheta_\tau^k)\nabla\vartheta_\tau^k\cdot\nabla(F'(\vartheta_\tau^k))\,\mathrm{d}x\\
&\quad+\int_\Omega \Big(D_{\tau,k}(c) + D_{\tau,k}(z) + \rho\operatorname{div}(D_{\tau,k}(\mathbf{u}))\Big)\vartheta_\tau^k F'(\vartheta_\tau^k)\,\mathrm{d}x
\end{aligned}$$

by using the concavity estimate $(\vartheta_\tau^k - \vartheta_\tau^{k-1})F'(\vartheta_\tau^k) \le F(\vartheta_\tau^k) - F(\vartheta_\tau^{k-1})$. Multplication by τ and summing over $k = 1, \ldots, \bar{\mathfrak{t}}_\tau(t)/\tau$ shows for every $t \in (0,T]$ the precise time-discrete analogon to (3.8). With the same calculations as in Section 3 we end up with

$$\|\overline{\vartheta}_\tau\|_{L^2(0,T;H^1(\Omega))} \le C, \qquad \|(\overline{\vartheta}_\tau)^{(\kappa+\alpha)/2}\|_{L^2(0,T;H^1(\Omega))} \le C_\alpha.$$

(iii) By testing the time-discrete heat equation (4.9d) with τ, integrating over Ω, summing over k and subtracting the result from the total energy inequality (4.43) we obtain the **Third a priori estimate**:

$$\|\partial_t c_\tau\|_{L^2(Q)} + \|\nabla\overline{\mu}_\tau\|_{L^2(Q;\mathbb{R}^d)} + \|\partial_t z_\tau\|_{L^2(Q)} + \|\partial_t \mathbf{u}_\tau\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))} \le C$$

as well as

$$\|\overline{z}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))} \le C.$$

(iv) The **Fourth a priori estimate** is obtained by testing the time-discrete force balance equation (4.9e) by $-\tau\,\mathrm{div}(\mathbb{V}\varepsilon(\mathbf{v}_\tau^k))$. The calculation in Section 3 carry over to the time-discrete setting. However, let us point out that the discrete analogue of (3.29) is given by the convexity estimate

$$\begin{aligned}-\int_0^{\bar{\mathfrak{t}}_\tau(t)}\int_\Omega \partial_t\mathbf{v}_\tau\cdot\mathrm{div}(\mathbb{V}\varepsilon(\overline{\mathbf{v}}_\tau))\,\mathrm{d}x\,\mathrm{d}s \ge\; &-\int_0^{\bar{\mathfrak{t}}_\tau(t)}\int_{\partial\Omega}\partial_t\mathbf{v}_\tau\cdot(\mathbb{V}\varepsilon(\overline{\mathbf{v}}_\tau)\mathbf{n})\,\mathrm{d}S\,\mathrm{d}s\\ &+\int_\Omega \frac{1}{2}\varepsilon(\overline{\mathbf{v}}_\tau(t)) : \mathbb{V}\varepsilon(\overline{\mathbf{v}}_\tau(t))\,\mathrm{d}x - \int_\Omega\frac{1}{2}\varepsilon(\mathbf{v}^0) : \mathbb{V}\varepsilon(\mathbf{v}^0)\,\mathrm{d}x.\end{aligned}$$

With analogous calculations we arrive at

$$\begin{aligned}&\|\mathbf{u}_\tau\|_{H^1(0,T;H^2(\Omega;\mathbb{R}^d))} + \|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^d))} \le C,\\ &\|\mathbf{v}_\tau\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^d))\cap L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))\cap L^2(0,T;H^2(\Omega;\mathbb{R}^d))} + \|\overline{\mathbf{v}}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))\cap L^2(0,T;H^2(\Omega;\mathbb{R}^d))} \le C.\end{aligned}$$

(v) For the **Fifth a priori estimate** we test (4.9b) with $c_\tau^k - \mathfrak{m}_0$ where $\mathfrak{m}_0 := \fint_\Omega c^0\,\mathrm{d}x$. With exactly the same calculations as in Section 3 we find

$$\|\overline{\mu}_\tau\|_{L^2(0,T;H^1(\Omega))} \le C.$$

(vi) A comparison in (4.9b) as done in the **Sixth a priori estimate** gives

$$\|\Delta_p(\overline{c}_\tau)\|_{L^2(0,T;L^2(\Omega))} + \|\overline{\eta}_\tau\|_{L^2(0,T;L^2(\Omega))} \le C.$$

(vii) Estimate (4.51) can be shown by utilizing the calculations in [38, Proof of Proposition 4.10 - Sixth estimate] and additionally noticing that $\{\overline{c}_\tau\}_{\tau>0}$ is bounded in $\mathrm{BV}([0,T];L^2(\Omega))$ due to the *Third estimate*. We thus obtain (4.57).

(viii) The **Eighth a priori estimate** works as in Section 3 and yields (4.58).

(ix) The **Ninth a priori estimate** works as in Section 3 and yields (4.48).

□

Remark 4.19. We observe that (4.57) implies the uniform bound

$$\|\log(\overline{\vartheta}_\tau)\|_{L^\infty(0,T;W^{1,d+\epsilon}(\Omega))} \le C_\epsilon.$$

Moreover, by interpolation we infer from (4.51) that (see (3.21))

$$\|\overline{\vartheta}_\tau\|_{L^h(Q)} \le C$$

with $h = 8/3$ for $d = 3$ and $h = 3$ for $d = 2$.

5 Proof of Theorem 2.6

In this last section we are going to perform the limit passages as $\tau \downarrow 0$ and $\omega \downarrow 0$ in the time-discrete system (4.9), for which the existence of solutions was proved in Proposition 4.3. This will lead us to prove Theorem 2.6.

5.1 Compactness

We shall adopt the notation from the previous section. In particular for fixed $\omega > 0$ we let $(\overline{c}_\tau, \underline{c}_\tau, c_\tau, \overline{\mu}_\tau, \overline{z}_\tau, \underline{z}_\tau, z_\tau, \overline{\vartheta}_\tau, \underline{\vartheta}_\tau, \vartheta_\tau, \overline{\mathbf{u}}_\tau, \underline{\mathbf{u}}_\tau, \overline{\mathbf{v}}_\tau, \mathbf{v}_\tau)$ be a time-discrete solution on an equi-distant partition of $[0,T]$ with fineness $\tau > 0$ according to Proposition 4.3.

Lemma 5.1. *Let the assumptions from Proposition 4.3 be satisfied and $\omega > 0$ be fixed. Then there exists a quintuple $(c, \mu, z, \vartheta, \mathbf{u})$ satisfying (2.26)–(2.30) such that along a (not relabeled) subsequence, as $\tau \downarrow 0$, the following convergences hold:*

$$\overline{c}_\tau,\ \underline{c}_\tau \overset{\star}{\rightharpoonup} c \qquad \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)), \tag{5.1}$$

$$c_\tau \overset{\star}{\rightharpoonup} c \qquad \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{5.2}$$

$$\Delta_p(\overline{c}_\tau) \rightharpoonup \Delta_p(c) \qquad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{5.3}$$

$$\overline{c}_\tau,\ \underline{c}_\tau \to c \qquad \text{strongly in } L^s(0,T;W^{1,p}(\Omega)) \text{ for all } s \in [1,\infty), \tag{5.4}$$

$$\overline{\eta}_\tau \rightharpoonup \eta \qquad \text{weakly in } L^2(0,T;L^2(\Omega)) \text{ with } \eta = \beta_\omega(c) \text{ a.e. in } Q, \tag{5.5}$$

$$\overline{\mu}_\tau \rightharpoonup \mu \qquad \text{weakly in } L^2(0,T;H^2_N(\Omega)), \tag{5.6}$$

$$\overline{z}_\tau,\ \underline{z}_\tau \overset{\star}{\rightharpoonup} z \qquad \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)), \tag{5.7}$$

$$z_\tau \overset{\star}{\rightharpoonup} z \qquad \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{5.8}$$

$$\overline{z}_\tau,\ \underline{z}_\tau \to z \qquad \text{strongly in } L^\infty(0,T;X) \text{ for all } X \text{ such that } W^{1,p}(\Omega) \Subset X \subseteq L^2(\Omega), \tag{5.9}$$

$$z_\tau \to z \qquad \text{strongly in } \mathrm{C}^0([0,T];X) \text{ for all } X \text{ such that } W^{1,p}(\Omega) \Subset X \subseteq L^2(\Omega), \tag{5.10}$$

$$\overline{\vartheta}_\tau \rightharpoonup \vartheta \qquad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{5.11}$$

$$\log(\overline{\vartheta}_\tau) \overset{\star}{\rightharpoonup} \log(\vartheta) \qquad \text{weakly-star in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,d+\epsilon}(\Omega)') \quad \text{for every } \epsilon > 0, \tag{5.12}$$

$$\log(\overline{\vartheta}_\tau) \to \log(\vartheta) \qquad \text{strongly in } L^2(0,T;L^s(\Omega)) \text{ for all } s \in [1,6) \text{ if } d=3, \text{ and all } s \in [1,\infty) \text{ if } d=2, \tag{5.13}$$

$$\log(\overline{\vartheta}_\tau(t)) \rightharpoonup \log(\vartheta(t)) \qquad \text{weakly in } H^1(\Omega) \text{ for almost all } t \in (0,T) \text{ (the chosen subsequence for } \tau \downarrow 0 \text{ does not depend on } t), \tag{5.14}$$

$$\overline{\vartheta}_\tau \to \vartheta \qquad \text{strongly in } L^q(\Omega \times (0,T)) \text{ for all } q \in [1,8/3) \text{ for } d=3 \text{ and all } q \in [1,3) \text{ if } d=2, \tag{5.15}$$

$$\overline{\mathbf{u}}_\tau, \underline{\mathbf{u}}_\tau \overset{\star}{\rightharpoonup} \mathbf{u} \qquad \text{weakly-star in } L^\infty(0,T;H^2(\Omega;\mathbb{R}^d)), \tag{5.16}$$

$$\mathbf{u}_\tau \overset{\star}{\rightharpoonup} \mathbf{u} \qquad \text{weakly-star in } H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^d)), \tag{5.17}$$

$$\overline{\mathbf{u}}_\tau,\ \underline{\mathbf{u}}_\tau \to \mathbf{u} \qquad \text{strongly in } L^\infty(0,T;X) \text{ for all } X \text{ such that } H^2(\Omega;\mathbb{R}^d) \Subset X \subseteq L^2(\Omega;\mathbb{R}^d), \tag{5.18}$$

$$\mathbf{u}_\tau \to \mathbf{u} \qquad \text{strongly in } \mathrm{C}^0([0,T];X) \text{ for all } X \text{ such that } H^2(\Omega;\mathbb{R}^d) \Subset X \subseteq L^2(\Omega;\mathbb{R}^d), \tag{5.19}$$

$$\overline{\mathbf{v}}_\tau \rightharpoonup \mathbf{u}_t \qquad \text{weakly in } L^2(0,T;H^2(\Omega;\mathbb{R}^d)), \tag{5.20}$$

$$\mathbf{v}_\tau \rightharpoonup \mathbf{u}_t \qquad \text{weakly in } H^1(0,T;L^2(\Omega;\mathbb{R}^d)) \cap L^2(0,T;H^2(\Omega;\mathbb{R}^d)). \tag{5.21}$$

Under the additional assumption (2.49) we also have for all $\epsilon > 0$ that $\vartheta \in \mathrm{BV}([0,T];W^{2,d+\epsilon}(\Omega)')$ and

$$\overline{\vartheta}_\tau \to \vartheta \qquad \text{strongly in } L^2(0,T;Y) \text{ for all } Y \text{ such that } H^1(\Omega) \Subset Y \subset W^{2,d+\epsilon}(\Omega)', \tag{5.22}$$

$$\overline{\vartheta}_\tau(t) \to \vartheta(t) \qquad \text{strongly in } W^{2,d+\epsilon}(\Omega)' \text{ for all } t \in [0,T]. \tag{5.23}$$

Proof. We immediately obtain (5.1), (5.2), (5.6), (5.7), (5.8), (5.11), (5.16), (5.17), (5.20) and (5.21) from the estimates (4.44)–(4.56) in Proposition 4.18 by standard weak compactness arguments.

From the regularity result [41, Thm. 2, Rmk. 3.5], we infer for every $1 \le \delta < \frac{1}{p}$ the enhanced regularity $\overline{c}_\tau, \underline{c}_\tau \in L^2(0,T;W^{1+\delta,p}(\Omega))$ together with the estimate

$$\|\overline{c}_\tau\|_{L^2(0,T;W^{1+\delta,p}(\Omega))} + \|\underline{c}_\tau\|_{L^2(0,T;W^{1+\delta,p}(\Omega))} \le C_\delta.$$

In combination with (4.44) and (4.45), the application of the Aubin-Lions compactness theorem yields (5.4). Now we choose a subsequence $\tau \downarrow 0$ such that $\Delta_p(\overline{c}_\tau) \rightharpoonup S$ in $L^2(Q)$ for an element $S \in L^2(Q)$ possible due to (4.46). Taking $\overline{c}_\tau \to c$ in $L^2(Q)$ into account, we may identify $S = \Delta_p(c)$ by the strong-weak closedness of the maximal monotone graph of $\Delta_p : L^2(Q) \to L^2(Q)$. We then conclude (5.3). Analogously, (5.5) ensues from the strong-weak closedness of the graph of the maximal monotone operator (induced by β_ω) $\beta_\omega : L^2(Q) \to L^2(Q)$. In addition, (5.9), (5.10), (5.18) and (5.19) follow from (4.49), (4.50), (4.54) and (4.55) via Aubin-Lions compactness results (see [42]).

It remains to show the convergences for $\overline{\vartheta}_\tau$ and $\log(\overline{\vartheta}_\tau)$. Here we proceed as in [38, Proof of Lemma 5.1]. We use the boundedness properties (4.53) and (4.57), and apply the compactness result [38, Theorem A.5] which is based on Helly's selection principle. We obtain a function $\lambda \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,d+\epsilon}(\Omega)')$ for all $\epsilon > 0$ and a further, (again not relabeled), subsequence such that

$$\begin{aligned}&\log(\overline{\vartheta}_\tau) \overset{\star}{\rightharpoonup} \lambda \text{ weakly-star in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,d+\epsilon}(\Omega)'),\\&\log(\overline{\vartheta}_\tau(t)) \rightharpoonup \lambda(t) \text{ weakly in } H^1(\Omega) \quad \text{for a.a. } t \in (0,T).\end{aligned}$$

Here the chosen subsequence for $\tau \downarrow 0$ does not depend on t in the latter convergence. We also infer from above that

$$\log(\overline{\vartheta}_\tau(t)) \to \lambda(t) \text{ strongly in } L^s(\Omega) \quad \text{for a.a. } t \in (0,T) \text{ and all } s \text{ from (5.13)}.$$

By also exploiting the boundedness of $\|\log(\overline{\vartheta}_\tau)\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;W^{1,d+\epsilon}(\Omega)')}$ and the interpolation inequality (2.8) with $X = H^1(\Omega)$, $Y = L^s(\Omega)$ and $Z = W^{1,d+\epsilon}(\Omega)$ we infer that the sequence $\{\log(\overline{\vartheta}_\tau)\}_\tau$ is uniformly integrable in $L^2(0,T;L^s(\Omega))$. Application of Vitali convergence theorem proves

$$\log(\overline{\vartheta}_\tau) \to \lambda \text{ strongly in } L^2(0,T;L^s(\Omega)) \quad \text{for all } s \text{ from (5.13)}.$$

Comparison with (5.11) yields $\lambda = \log(\vartheta)$ and hence (5.12), (5.13) and (5.14). The uniform bound (4.51) shows uniform integrability of $\{\overline{\vartheta}_\tau\}_\tau$ in $L^q(Q)$ with $q \in [1,8/3)$ for $d = 3$ and $q \in [1,3)$ for $d = 2$ (cf. (3.21)). Vitali's convergence theorem proves the strong convergence (5.15).

In particular we find $\overline{\vartheta}_\tau(t) \to \vartheta(t)$ strongly in $L^1(\Omega)$ for a.e. $t \in (0,T)$ (where the subsequence of $\tau \downarrow 0$ is independent of t). By the boundedness $\|\overline{\vartheta}_\tau(t)\|_{L^1(\Omega)} \le C$ uniformly in t and τ (see (4.51)) we infer by lower semicontinuity that $\vartheta \in L^\infty(0,T;L^1(\Omega))$. Furthermore, by considering a weak cluster point $(\overline{\vartheta}_\tau)^{\frac{\kappa+\alpha}{2}} \rightharpoonup S$ in $L^2(0,T;H^1(\Omega))$ and identifying $S = (\overline{\vartheta}_\tau)^{\frac{\kappa+\alpha}{2}}$ via a.e. limits from above we also obtain $\vartheta^{\frac{\kappa+\alpha}{2}} \in L^2(0,T;H^1(\Omega))$.

Finally, under the additional assumption (2.49) convergences (5.22) and (5.23) follow from an Aubin-Lions type compactness result for BV-functions (cf. e.g. [39, Chap. 7, Cor. 4.9]), combining estimate (4.51) together with the BV-bound (4.58). For further details we refer to [38, Proof of Lemma 5.1]. □

5.2 Conclusion of the proof of Theorem 2.6

Here is the outline of the proof:

1 First, for fixed $\omega > 0$ we will pass to the limit as $\tau \downarrow 0$, (along the same subsequence for which the convergences in Lemma 5.1 hold), in the time-discrete system (4.9). We will thus obtain an *entropic weak solution* (in the sense of Definition 2.4), to the (initial-boundary value problem for the) PDE system (1.3), where the maximal monotone operator β and the elastic energy density W are replaced by their regularized versions β_ω and W^ω.

2 Secondly, we will tackle the limit passage as $\omega \downarrow 0$.

Observe that the limit passages $\tau \downarrow 0$ and $\omega \downarrow 0$ cannot be performed simultaneously, because in the time-discrete system from Problem 4.1 the (partial) derivatives of the convex- and the concave-decompositions (4.6) may "explode" as $\omega \downarrow 0$. However, the convex-concave splitting shall disappear in the limit $\tau \downarrow 0$ for fixed $\omega > 0$ in the corresponding PDE system.

Limit passage $\tau \downarrow 0$ First of all, we mention that from the time-discrete damage equation (4.9c) we derive the following inequalities (for details we refer to [21, Section 5.2]; see also [38, Proof of Theorem 1] and [37, Proof of Theorem 4]):

– *damage energy-dissipation inequality:* for all $t \in (0,T]$, for $s = 0$, and for almost all $0 < s \le t$:

$$\begin{aligned}
&\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\partial_t z_\tau|^2 \,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \Big(\frac{1}{p}|\nabla \overline{z}_\tau(t)|^p + (\breve{\sigma})'(\overline{z}_\tau(t)) + (\hat{\sigma})'(\underline{z}_\tau(t))\Big)\,\mathrm{d}x \\
&\quad\le \int_\Omega \Big(\frac{1}{p}|\nabla \overline{z}_\tau(s)|^p + (\breve{\sigma})'(\overline{z}_\tau(s)) + (\hat{\sigma})'(\underline{z}_\tau(s))\Big)\,\mathrm{d}x \\
&\qquad + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \partial_t z_\tau \Big(-\breve{W}^\omega_{3,z}(\overline{c}_\tau, \varepsilon(\underline{\mathbf{u}}_\tau), \overline{z}_\tau) - \hat{W}^\omega_{3,z}(\overline{c}_\tau, \varepsilon(\underline{\mathbf{u}}_\tau), \underline{z}_\tau) + \overline{\vartheta}_\tau\Big)\,\mathrm{d}x\,\mathrm{d}r;
\end{aligned} \tag{5.24}$$

– *damage variational inequality:* for all $\zeta \in L^\infty(0,T;W^{1,p}(\Omega))$ with $0 \le \zeta \le \underline{z}_\tau$:

$$\begin{aligned}
&\int_0^T \int_\Omega \Big(|\nabla \overline{z}_\tau|^{p-2}\nabla \overline{z}_\tau \cdot \nabla(\zeta - \overline{z}_\tau)\big((\partial_t z_\tau) + (\breve{\sigma})'(\overline{z}_\tau) + (\hat{\sigma})'(\underline{z}_\tau)\big)(\zeta - \overline{z}_\tau)\Big)\,\mathrm{d}x\,\mathrm{d}r \\
&+ \int_0^T \int_\Omega \Big(\breve{W}^\omega_{3,z}(\overline{c}_\tau, \varepsilon(\underline{\mathbf{u}}_\tau), \overline{z}_\tau) + \hat{W}^\omega_{3,z}(\overline{c}_\tau, \varepsilon(\underline{\mathbf{u}}_\tau), \underline{z}_\tau) - \overline{\vartheta}_\tau\Big)(\zeta - \overline{z}_\tau)\,\mathrm{d}x\,\mathrm{d}r \ge 0.
\end{aligned} \tag{5.25}$$

The limit passage $\tau \downarrow 0$ in the damage energy-dissipation inequality (5.24), in the damage variational inequality (5.25), in the entropy inequality (4.42), in the total energy inequality (4.43) and in the equation for the balance of forces (4.9e) works exactly as outlined in [38, Proof of Theorem 1] by taking the growth properties (4.24)–(4.28) into account (for **fixed** $\omega > 0$) and needs no repetition here.

We end up with properties (ii), (iii), (iv) and (v) of Definition 2.4, keeping in mind that $W(c, \varepsilon(\mathbf{u}), z)$, $W_{,z}(c, \varepsilon(\mathbf{u}), z)$ and $W_{,\varepsilon}(c, \varepsilon(\mathbf{u}), z)$ are replaced by their ω-regularized versions $W^\omega(c, \varepsilon(\mathbf{u}), z)$, $W^\omega_{,z}(c, \varepsilon(\mathbf{u}), z)$ and $W^\omega_{,\varepsilon}(c, \varepsilon(\mathbf{u}), z)$, respectively. Let us comment that in the limit $\tau \downarrow 0$ of (5.25) we are only able to obtain a "one-sided variational inequality" which still suffices to obtain a weak solution in the sense of Definition 2.4 (see (2.41)). Furthermore, following the approach from [21, Proof of Theorem 4.4], the subgradient $\xi \in L^2(0,T;L^2(\Omega))$ fulfilling $\xi \in \partial I_{[0,+\infty)}(z)$ a.e. in Q can be specified precisely as

$$\xi = -\mathbf{1}_{\{z=0\}}\Big(\sigma'(z) + W_{,z}(c, \varepsilon(\mathbf{u}), z) - \vartheta\Big)^+ \qquad \text{a.e. in } Q,$$

where $\mathbf{1}_{\{z=0\}} : Q \to \{0,1\}$ denotes the characteristic function of the set $\{z = 0\} \subseteq Q$ and $(\cdot)^+ := \max\{0, \cdot\}$.

It remains to show the limit passage as $\tau \downarrow 0$ in the Cahn-Hilliard system (4.9a)–(4.9b). This can be achieved via standard convergence methods by exploiting the convergences shown in Lemma 5.1 and noticing the growth properties (4.24)–(4.28). This leads to property (i) from Definition 2.4 where $W_{,c}(c, \varepsilon(\mathbf{u}), z)$ and β should be replaced by $W^\omega_{,c}(c, \varepsilon(\mathbf{u}), z)$ and β_ω, respectively.

Limit passage $\omega\downarrow 0$ In the subsequent argumentation we let $S_\omega = (c_\omega, \mu_\omega, z_\omega, \vartheta_\omega, \mathbf{u}_\omega)$ be an ω-regularized weak solution, i.e. an entropic weak solution in the sense of Definition 2.4 where the W, $W_{,c}$, $W_{,\varepsilon}$, $W_{,z}$ and β-terms are replaced by W^ω, $W^\omega_{,c}$, $W^\omega_{,\varepsilon}$, $W^\omega_{,z}$ and β_ω, respectively. We observe that the a priori estimates in

Proposition 4.18 are inherited by the weak solutions S_ω via lower semicontinuity arguments. Hence we obtain the same convergence properties for $\omega \downarrow 0$ as for $\tau \downarrow 0$ in Lemma 5.1 where (5.5) should be replaced by

$$\overline{\eta}_\omega \rightharpoonup \eta \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \text{ as } \omega \downarrow 0 \text{ with } \eta \in \beta(c) \text{ a.e. in } Q. \tag{5.26}$$

Indeed, to prove (5.26), let $\eta_\omega = \beta_\omega(c_\omega) \rightharpoonup S$ in $L^2(Q)$ for $\omega \downarrow 0$ for some element $S \in L^2(Q)$. By convexity of the operator $\widehat{\beta}_\omega : L^2(Q) \to \mathbb{R}$ we find

$$\forall w \in L^2(Q): \quad \widehat{\beta}_\omega(c_\omega) + \langle \beta_\omega(c_\omega), w - c_\omega \rangle_{L^2(Q)} \le \widehat{\beta}_\omega(w). \tag{5.27}$$

Since $\{\beta_\omega\}$ is the Yosida-approximation of β we conclude that (cf. [39, Lemma 5.17])

$$\forall w \in L^2(Q): \quad \widehat{\beta}_\omega(w) \to \widehat{\beta}(w) \text{ strongly in } L^2(Q) \text{ as } \omega \downarrow 0 \qquad \text{and} \qquad \liminf_{\omega \downarrow 0} \widehat{\beta}_\omega(c_\omega) \ge \widehat{\beta}(c). \tag{5.28}$$

Thus by (5.26) and (5.28) we can pass to the limit $\omega \downarrow 0$ for a subsequence in (5.27) and obtain $\eta \in \partial\beta(c)$.

The main feature for the passage $\omega \downarrow 0$ in the PDE system is the following observation: From (4.44) and (4.45) we infer via the compact embedding $W^{1,p}(\Omega) \Subset L^\infty(\Omega)$ that for all $\omega > 0$

$$\|c_\omega\|_{L^\infty(Q)} \le C.$$

An important consequence is that in combination with the definition of $\mathcal{R}_\omega$ in (4.3) we find for all sufficiently small $\omega > 0$ that $\mathcal{R}_\omega(c_\omega) = c_\omega$ a.e. in Q and thus

$$\left.\begin{array}{ll} W(c_\omega, \varepsilon(\mathbf{u}_\omega), z_\omega) = W^\omega(c_\omega, \varepsilon(\mathbf{u}_\omega), z_\omega), & W_{,c}(c_\omega, \varepsilon(\mathbf{u}_\omega), z_\omega) = W^\omega_{,c}(c_\omega, \varepsilon(\mathbf{u}_\omega), z_\omega), \\ W_{,\varepsilon}(c_\omega, \varepsilon(\mathbf{u}_\omega), z_\omega) = W^\omega_{,\varepsilon}(c_\omega, \varepsilon(\mathbf{u}_\omega), z_\omega), & W_{,z}(c_\omega, \varepsilon(\mathbf{u}_\omega), z_\omega) = W^\omega_{,z}(c_\omega, \varepsilon(\mathbf{u}_\omega), z_\omega). \end{array}\right\} \text{ a.e. in } Q.$$

Then, the limit passage $\omega \downarrow 0$ in the ω-regularized versions of (i)-(v) in Definition 2.4 works as for $\tau \downarrow 0$.

This concludes the proof of Theorem 2.6. $\square$

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