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Global-in-time existence of weak solutions to Kolmogorov's two-equation model of turbulence

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ABSTRACT

We consider Kolmogorov's model for the turbulent motion of an incompressible fluid in $\mathbb{R}^3$. This model consists in a Navier-Stokes type system for the mean flow $\boldsymbol{u}$ and two further partial differential equations: an equation for the frequency ω and for the kinetic energy k each. We investigate this system of partial differential equations in a cylinder $\Omega\times\,]0,T\,[$ ($\Omega \subset \mathbb{R}^3$ cube, $0 < T < +\infty$) under spatial periodic boundary conditions on $\partial\Omega\times\,]0,T\,[$ and initial conditions in $\Omega \times \{0\}$. We present an existence result for a weak solution $\{\boldsymbol{u},\omega,k\}$ to the problem under consideration, with ω, k obeying the inequalities $c_1+t \le \frac{1}{\omega} \le t+c_2$ and $\frac{k^{1/2}}{\omega} \ge c_3 t^{1/2}$ ($c_1,c_2,c_3 = \text{const} > 0$).

1. Introduction In [7], Kolmogorov postulated the following system of partial differential equations as a model for the turbulent motion of an incompressible fluid in $\mathbb{R}^3$:

$$\operatorname{div}\boldsymbol{u} = 0, \quad \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \operatorname{div}\Big(\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u})\Big) - \nabla p + \boldsymbol{f}\,, \tag{1}$$

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega = \operatorname{div}\Big(\frac{k}{\omega}\nabla\omega\Big) - \omega^2, \quad \frac{\partial k}{\partial t} + \boldsymbol{u}\cdot\nabla k = \operatorname{div}\Big(\frac{k}{\omega}\nabla k\Big) + \frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - k\omega\,. \tag{2}$$

Here, $\boldsymbol{u} = (u_1,u_2,u_3)$ denotes the mean velocity, p the mean pressure, $k = \frac{1}{3}\overline{|\,\widetilde{\boldsymbol{u}}\,|^2}$ the mean turbulent kinetic energy ($\widetilde{\boldsymbol{u}}$ = fluctuation velocity) and $\omega > 0$ denotes a frequency associated with the dissipation of turbulent kinetic energy ($\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top)$ mean strain-rate). $\boldsymbol{f}$ represents a given external force. The paper [7] originated from Kolmogorov's theory of turbulence published in 1941. A detailed presentation of this theory is given, e.g., in [5] (see also the article by Yaglom [15, pp. 488–503]). A discussion of (1), (2) and other two-equation models of turbulence can be found in [14], [16, Chap. 4.3].

Instead of studying (1), (2) in the whole $\mathbb{R}^3$, we consider this system in a cube $\Omega = (]0,a[\,)^3$ ($0 < a < +\infty$ fixed) and complete it by spatial periodic boundary conditions with respect to Ω. Let $\partial\Omega$ denote the boundary of Ω. We define $\Gamma_i = \partial\Omega \cap \{x_i = 0\}$, $\Gamma_{i+3} = \partial\Omega\cap\{x_i = a\}$ $(i = 1,2,3)$.

Let $0 < T < +\infty$. We study system (1), (2) in the cylinder $Q_T = \Omega\times\,]0,T\,[$ with the following conditions:

$$\left.\begin{aligned} &\boldsymbol{u}\Big|_{\Gamma_i\times]0,T[} = \boldsymbol{u}\Big|_{\Gamma_{i+3}\times]0,T[}, \quad \text{analogously for } p,\omega,k\,, \\ &\boldsymbol{D}(\boldsymbol{u})\Big|_{\Gamma_i\times]0,T[} = \boldsymbol{D}(\boldsymbol{u})\Big|_{\Gamma_{i+3}\times]0,T[}, \quad \text{analogously for } \nabla\omega,\nabla k\,, \end{aligned}\right\} \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{u}_0, \quad \omega = \omega_0, \quad k = k_0 \;\text{ in }\; \Omega\times\{0\}\,. \tag{4}$$

The aim of this Note is to present an existence result for a weak solution $\{\boldsymbol{u},\omega,k\}$ to (1)–(4).

2. Statement of the main result Let X denote a real normed space with norm $|\cdot|_X$, let X^* be its dual and let $\langle x^*,x\rangle_X$ denote the dual pairing of $x^*\in X^*$ and $x\in X$. The symbol $C_w([0,T];X)$ stands for the vector space of all mappings $u : [0,T] \to X$ such that, for every $x^* \in X^*$, the function $t \mapsto \big\langle x^*, u(t)\big\rangle_X$ is continuous on $[0,T]$. Next, by $L^p(0,T;X)$ $(1 \le p \le +\infty)$ we denote the vector space of all equivalence classes of Bochner measurable mappings $u : [0,T] \to X$ such that the function $t \mapsto \big|u(t)\big|_X$ is in $L^p(0,T)$ (cf. [2, Chap. III, §3, Chap. IV, §3], [3, App.], [4]).

For bounded domains $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ with Lipschitz boundary we denote by $W^{1,p}(\Omega)$ $(1 \le p < +\infty)$ the usual Sobolev space.

In what follows, let $\Omega = (\,]0,a\,[)^3$ be the cube introduced above. We define

$$W^{1,p}_{\text{per}}(\Omega) = \Big\{u \in W^{1,p}(\Omega);\; u\Big|_{\Gamma_i} = u\Big|_{\Gamma_{i+3}} \;\; (i = 1,2,3)\Big\}\,,$$

$$\boldsymbol{W}^{1,p}_{\text{per,div}}(\Omega) = \big\{\boldsymbol{u}\in\boldsymbol{W}^{1,p}_{\text{per}}(\Omega);\; \operatorname{div}\boldsymbol{u} = 0 \;\text{ a. e. in }\; \Omega\big\}$$

(bold-faced letters refer to vector valued mappings as well as to Banach spaces of such mappings). The conditions on the data are:

$$\boldsymbol{f}\in\boldsymbol{L}^2(Q_T)\,; \quad \boldsymbol{u}_0 \in \overline{\boldsymbol{C}^\infty_{\text{per,div}}(\overline{\Omega})}^{\|\cdot\|_{\boldsymbol{L}^2(\Omega)}}\,, \tag{5}$$

$$\left.\begin{aligned}&\omega_0 \text{ measurable in } \Omega\,,\ \omega_* \le \omega_0(x) \le \omega^* \text{ for a.e. } x\in\Omega \ (\omega_*,\omega^* = \text{ const} > 0)\,,\\ &k_0 \in L^1(\Omega)\,,\ k_0(x) \ge k_* = \text{ const} > 0 \text{ for a.e. } x\in\Omega\,.\end{aligned}\right\}\tag{6}$$

The following theorem is the main result of our paper.

Theorem *Assume (5) and (6). Then there exists a triple of measurable functions $\{\boldsymbol{u},\omega,k\}$ in Q_T such that*

$$\frac{\omega_*}{1+t\omega_*} \le \omega(x,t) \le \frac{\omega^*}{1+t\omega^*}\,,\quad k(x,t) \ge \frac{k_*}{1+t\omega_*} \text{ for a.e. } (x,t)\in Q_T\,,\tag{7}$$

$$\frac{1}{\omega^*}+t \le \frac{1}{\omega(x,t)} \le t+\frac{1}{\omega_*}\,,\quad \frac{k^{1/2}(x,t)}{\omega(x,t)} \ge \Big(\frac{k_*}{\omega^*}\,t\Big)^{1/2} \text{ for a.e. } (x,t)\in Q_T\,,\tag{8}$$

$$\left.\begin{aligned}&\boldsymbol{u}\in C_w\big([0,T];\boldsymbol{L}^2(\Omega)\big)\cap L^2\big(0,T;\boldsymbol{W}^{1,2}_{\text{per,div}}(\Omega)\big)\,,\\ &\omega\in C_w\big([0,T];L^2(\Omega)\big)\cap L^2\big(0,T;W^{1,2}_{\text{per}}(\Omega)\big)\,,\quad k\in L^\infty\big(0,T;L^1(\Omega)\big)\,,\end{aligned}\right\}\tag{9}$$

$$\left.\begin{aligned}&\int_{Q_T}\big(k^{4p/3}+|\nabla k|^p\big) < +\infty \quad \forall\, 1\le p<2\,,\quad \int_{Q_T}\big(k\,|\nabla k|\big)^q < +\infty \quad \forall\, 1\le q<\frac{8}{7}\,,\\ &\int_{Q_T}\frac{|\nabla k|^2}{(1+k)^{1+\delta}} < +\infty \quad \forall\, 0<\delta<1\,;\quad \int_{Q_T}\frac{k}{\omega}\big(|\boldsymbol{D}(\boldsymbol{u})|^2+|\nabla\omega|^2\big) < +\infty\,,\end{aligned}\right\}\tag{10}$$

$$\boldsymbol{u}'\in\bigcup_{1\le p<2} L^{8p/(4p+3)}\big(0,T;\big(\boldsymbol{W}^{1,8p/(4p-3)}_{\text{per,div}}(\Omega)\big)^*\big)\,,\quad \omega'\in\bigcup_{1\le p<2} L^{8p/(4p+3)}\big(0,T;\big(W^{1,8p/(4p-3)}_{\text{per}}(\Omega)\big)^*\big)\,,\tag{11}$$

$$\left.\begin{aligned}&\int_0^T\big\langle \boldsymbol{u}'(t),\boldsymbol{v}(t)\big\rangle_{\boldsymbol{W}^{1,r}_{\text{per,div}}}dt - \int_{Q_T}(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla\boldsymbol{v} + \int_{Q_T}\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v}) =\\ &= \int_{Q_T}\boldsymbol{f}\cdot\boldsymbol{v} \quad \forall\,\boldsymbol{v}\in\bigcup_{r>16/5} L^r\big(0,T;W^{1,r}_{\text{per}}(\Omega)\big)\,,\quad \boldsymbol{u}(\cdot,0)=\boldsymbol{u}_0 \text{ a.e. in } \Omega\,,\end{aligned}\right\}\tag{12}$$

$$\left.\begin{aligned}&\int_0^T\big\langle \omega'(t),\varphi(t)\big\rangle_{W^{1,r}_{\text{per}}}dt - \int_{Q_T}\omega\boldsymbol{u}\cdot\nabla\varphi + \int_{Q_T}\frac{k}{\omega}\nabla\omega\cdot\nabla\varphi =\\ &= -\int_{Q_T}\omega^2\varphi \quad \forall\,\varphi\in\bigcup_{r>16/5} L^r\big(0,T;W^{1,r}_{\text{per}}(\Omega)\big)\,,\quad \omega(\cdot,0)=\omega_0 \text{ a.e. in } Q_T\,,\end{aligned}\right\}\tag{13}$$

$$\left.\begin{aligned}&\exists \text{ bounded Radon measure } \mu \text{ on the Borel } \sigma\text{-algebra of } \overline{Q}_T \text{ such that}\\ &-\int_{Q_T}k\frac{\partial z}{\partial t} - \int_{Q_T}k\boldsymbol{u}\cdot\nabla z + \int_{Q_T}\frac{k}{\omega}\nabla k\cdot\nabla z = \int_\Omega k_0(x)z(x,0)dx + \int_{Q_T}\Big(\frac{k}{\omega}\left|\boldsymbol{D}(\boldsymbol{u})\right|^2 - k\omega\Big)z + \int_{\overline{Q}_T}z\,d\mu\\ &\forall\, z\in C^1(\overline{Q}_T),\ z\Big|_{\Gamma_i\times\,]0,T[} = z\Big|_{\Gamma_{i+3}\times\,]0,T[} \quad (i=1,2,3)\,,\quad z(\cdot,T)=0\,.\end{aligned}\right\}\tag{14}$$

In addition, the following inequalities hold for a.e. $t\in\,]0,T[$:

$$\frac12\int_\Omega\left|\boldsymbol{u}(x,t)\right|^2dx + \int_0^t\!\!\int_\Omega\frac{k}{\omega}\left|\boldsymbol{D}(\boldsymbol{u})\right|^2 \le \frac12\int_\Omega\left|\boldsymbol{u}_0\right|^2 + \int_0^t\!\!\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}\,,\tag{15}$$

$$\int_\Omega\Big(\frac12\left|\boldsymbol{u}(x,t)\right|^2 + k(x,t)\Big)dx + \int_0^t\!\!\int_\Omega k\omega \le \int_\Omega\Big(\frac12\left|\boldsymbol{u}_0\right|^2 + k_0\Big) + \int_0^t\!\!\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}\,,\tag{16}$$

Remarks 1. Obviously, (8) follows from (7). Except for the additive constants $\frac{1}{\omega^*}$ and $\frac{1}{\omega_*}$ in (8), the estimates for $\frac{1}{\omega}$ are in coincidence with Kolmogorov's theory of turbulence (cf. [5, pp. 100–103], [8, Chap. 33]).
The function $L := \frac{k^{1/2}}{\omega}$ characterizes the "external length scale" of the turbulent motion (see [5, Chap. 7], [8, Chap. 33], [16, Chap. 8.1]). Instead of the growth of L in (8), Kolmogorov [7] claims the weaker growth $L \geq c_0 t^{2/7}$ ($c_0 =$ const > 0).

2. The integral relations in (11) and (12) represent a weak formulation of the $\boldsymbol{u}$-equation and the ω-equation, respectively, with spatial periodic boundary condition (cf. (1), (2), (3)). The derivatives $\boldsymbol{u}'$ and ω' in (11) (and (12), (13)) have to be understood in the sense of distributions from $]0,T[$ into the spaces $\big(\boldsymbol{W}^{1,8p/(4p-3)}_{\text{per,div}}(\Omega)\big)^*$ and $\big(W^{1,8p/(4p-3)}_{\text{per}}(\Omega)\big)^*$, respectively. An analogous remark refers to $\{\boldsymbol{u}'_\varepsilon, \omega'_\varepsilon, k'_\varepsilon\}$ below.

3. The defect measure μ in (14) arises from our approximation method for the proof the existence of a weak solution to (1)–(4). The measure μ vanishes, provided the weak solution under consideration satisfies appropriate regularity properties. More precisely:

Let the triple $\{\boldsymbol{u}, \omega, k\}$ *satisfy* $\omega > 0$, $k > 0$ *a.e. in* Q_T *and let* (9)–(16) *be fulfilled. If equality holds in both* (15) *and* (16), *then* (i) $\mu = 0$ and (ii) $\exists\, k' \in \bigcap_{1<s<8/7} L^1\big(0,T;\big(W^{1,s}_{\text{per}}(\Omega)\big)^*\big)$.

To prove (i), let α be any Lipschitz function on $[0,T]$, $\alpha(t) = 0$ for all $t \in [t_\alpha, T]$ $(0 < t_\alpha < T)$. Then (14) continues to hold for functions $z = z(x,t) = \zeta(x)\alpha(t)$, where $\zeta \in W^{1,s}_{\text{per}}(\Omega)$ ($s > 8$, observe (10)). Given $t \in]0,T[$ and $m > \frac{1}{T-t}$ ($m \in \mathbb{N}$), define

$$\alpha_m(\tau) = \begin{cases} 1 & \text{if } \ 0 \leq \tau \leq t\,, \\ m\Big(t + \dfrac{1}{m} - \tau\Big) & \text{if } \ t < \tau < t + \dfrac{1}{m}\,, \\ 0 & \text{if } \ t + \dfrac{1}{m} \leq \tau \leq T\,. \end{cases}$$

We insert $z = 1 \cdot \alpha_m$ into (14) and obtain

$$m \int_t^{t+1/m} \int_\Omega k(x,\tau)\,dx\,d\tau \geq \int_\Omega k_0 + \int_0^{t+1/m} \int_\Omega \Big(\frac{k}{\omega}\left|\boldsymbol{D}(\boldsymbol{u})\right|^2 - k\omega\Big)\alpha_m + \mu\big(\overline{\Omega} \times [0,t]\big)\,.$$

Letting $m \to +\infty$ it follows that

$$\int_\Omega k(x,t)\,dx \geq \int_\Omega k_0 + \int_0^t\int_\Omega \Big(\frac{k}{\omega}\left|\boldsymbol{D}(\boldsymbol{u})\right|^2 - k\omega\Big) + \mu\big(\overline{\Omega} \times [0,t]\big)$$

for all Lebesgue points t of the function $\tau \mapsto \displaystyle\int_\Omega k(x,\tau)\,dx$. Adding this inequality to (15) [with equality therein] one finds

$$\int_\Omega \Big(\frac{1}{2}\left|\boldsymbol{u}(x,t)\right|^2 + k(x,t)\Big)dx + \int_0^t\int_\Omega k\omega - \int_\Omega \Big(\frac{1}{2}\left|\boldsymbol{u}_0\right|^2 + k_0\Big) - \int_0^t\int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \ \geq \ \mu\big(\overline{\Omega} \times [0,t]\big)\,.$$

From (16) [with equality therein] it follows $\mu(\overline{\Omega} \times [0,t]) = 0$. Thus, $\mu(\overline{Q_T}) = 0$.

The claim (ii) can be easily established by routine arguments (cf. [3, Appendice, Prop. A6], [4]).

With (i) and (ii) in hand we obtain $k \in C([0,T]; (W^{1,s}_{\text{per}}(\Omega))^*)$ and $k(\cdot,0) = k_0$ in the sense of $(W^{1,s}_{\text{per}}(\Omega))^*$. Now, (14) turns into the weak formulation of the k-equation.

4. The defect measure μ in (14) reflects the deep problem to establish an energy equality for weak solutions to the Navier-Stokes equations (see also [12], [13]). In [9], the author studies a simplified one-equation model of turbulence, where a defect measure appears on p. 397 and 416. We notice that defect measures also occur for other types of nonlinear partial differential equations (cf., e.g., [1], [6], [10]).

3. Sketch of proof Let $\Phi \in C([0,+\infty[\,)$ be a fixed, non-increasing function fulfilling the conditions $0 \le \Phi \le 1$ in $[0,+\infty[\,$, $\Phi = 1$ in $[0,1]$ and $\Phi = 0$ in $[2,+\infty[\,$. For $0 < \varepsilon \le 1$, define $\Phi_\varepsilon(\xi) = \Phi(\varepsilon\xi)$, $0 \le \xi < +\infty$.

1° *Existence of an approximate solution.* Fix any $6 < \rho < +\infty$, $3 < \sigma < \frac{11}{3}$. For every $0 < \varepsilon < +\infty$ there exist measurable functions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ in Q_T such that $\omega_\varepsilon \ge 0$, $k_\varepsilon \ge 0$ a.e. in Q_T,

$$\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\} \in L^\rho\big(0,T;\boldsymbol{W}^{1,\rho}_{\text{per,div}}(\Omega)\big) \times L^4\big(0,T;W^{1,4}_{\text{per}}(\Omega)\big) \times L^\sigma\big(0,T;W^{1,\sigma}_{\text{per}}(\Omega)\big)\,,$$

$$\{\boldsymbol{u}'_\varepsilon, \omega'_\varepsilon, k'_\varepsilon\} \in L^{\rho'}\big(0,T;\big(\boldsymbol{W}^{1,\rho}_{\text{per,div}}(\Omega)\big)^*\big) \times L^{4/3}\big(0,T;\big(W^{1,4}_{\text{per}}(\Omega)\big)^*\big) \times L^{\sigma'}\big(0,T;\big(W^{1,\sigma}_{\text{per}}(\Omega)\big)^*\big)\,,$$

$$\left.\begin{aligned} &\big\langle \boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}\big\rangle_{W^{1,\rho}_{\text{per,div}}} - \int_\Omega \Phi_\varepsilon\big(|\boldsymbol{u}_\varepsilon|^2\big)(\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \nabla \boldsymbol{v} + \int_\Omega \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon\,\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^{\rho-2}\Big)\boldsymbol{D}(\boldsymbol{u}_\varepsilon) : \boldsymbol{D}(\boldsymbol{v}) = \\ &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \;\text{ for a.e. }\; t \in\,]0,T[\,,\;\; \forall\, \boldsymbol{v} \in \boldsymbol{W}^{1,\rho}_{\text{per}}(\Omega)\,;\;\; \boldsymbol{u}_\varepsilon(\cdot,0) = \boldsymbol{u}_0\,, \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} &\big\langle \omega'_\varepsilon(t), \varphi\big\rangle_{W^{1,4}_{\text{per}}} - \int_\Omega \omega_\varepsilon \boldsymbol{u}_\varepsilon\cdot\nabla\varphi + \int_\Omega \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon\,|\nabla\omega_\varepsilon|^2\Big)\nabla\omega_\varepsilon\cdot\nabla\varphi = \\ &= -\int_\Omega \omega_\varepsilon^2\varphi \;\text{ for a.e. }\; t \in\,]0,T[\,,\;\; \forall\, \varphi \in W^{1,4}_{\text{per}}(\Omega)\,;\;\; \omega_\varepsilon(\cdot,0) = \omega_0\,, \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned} &\big\langle k'_\varepsilon(t), z\big\rangle_{W^{1,\sigma}_{\text{per}}} - \int_\Omega k_\varepsilon \boldsymbol{u}_\varepsilon\cdot\nabla z + \int_\Omega \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon\,|\nabla k_\varepsilon|^{\sigma-2}\Big)\nabla k_\varepsilon\cdot\nabla z = \\ &= \int_\Omega \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon}\,\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 - k_\varepsilon\omega_\varepsilon\Big) z \;\text{ for a.e. }\; t \in\,]0,T[\,,\;\; \forall\, z \in W^{1,\sigma}_{\text{per}}(\Omega)\,;\;\; k_\varepsilon(\cdot,0) = k_0\,. \end{aligned}\right\} \tag{19}$$

This result can be proved by reformulating (17)–(19) in terms of an abstract operator equation and applying [11, Chap. 3.1.4, Théorème 1.2]. For this we have to pass from the data $\{\boldsymbol{u}_0, \omega_0, k_0\}$ to zero initial data. With regard to $\boldsymbol{u}_0$, this is easily done by (5) and with regard to ω_0, k_0 by routine arguments.

2° *A-priori estimates a.e. in Q_T for ω_ε and k_ε. For every $\varepsilon > 0$ there holds*

$$\frac{\omega_*}{1 + t\omega_*} \le \omega_\varepsilon(x,t) \le \frac{\omega^*}{1+t\omega^*}\,, \quad k_\varepsilon(x,t) \ge \frac{k_*}{1+t\omega^*} \;\; \textit{for a.e.}\;\; (x,t) \in Q_T \;\; (\omega_*, \omega^* \textit{ and } k_* \textit{ as in } (6)). \tag{20}$$

We establish the estimate from below for ω_ε. Set $\underline{\omega}(t) := \dfrac{\omega_*}{1+t\omega_*}$, $0 \le t \le T$. Then

$$\big(\omega_\varepsilon(\cdot,t) - \underline{\omega}(t)\big) \in W^{1,4}_{\text{per}}(\Omega) \;\text{ for a.e. }\; t \in\,]0,T[\,, \quad \big(\omega_\varepsilon(x,0) - \underline{\omega}(0)\big)^- = 0 \;\text{ for a.e. }\; x \in \Omega\,.$$

We take $\varphi = \big(\omega_\varepsilon(\cdot,t) - \underline{\omega}(t)\big)^-$ in (18), add the term $-\dot{\underline{\omega}}(t)\int_\Omega \big(\omega_\varepsilon(x,t) - \underline{\omega}(t)\big)^- dx$ ($\dot{\underline{\omega}}$ = derivative of ω) to both sides and integrate over the interval $[0,t]$. It follows that

$$\frac{1}{2}\int_\Omega \big((\omega_\varepsilon(x,t) - \underline{\omega}(t))^-\big)^2 dx - \int_0^t\!\!\int_\Omega \omega_\varepsilon \boldsymbol{u}_\varepsilon\cdot\nabla(\omega_\varepsilon - \underline{\omega})^- \le \int_0^t\!\!\int_\Omega \big(\underline{\omega}^2 - \omega_\varepsilon^2\big)(\omega_\varepsilon - \underline{\omega})^- \le 0$$

for a.e. $t \in\,]0,T[$ (notice that $\dot{\underline{\omega}} = -\underline{\omega}^2$). Since

$$\begin{aligned} &\int_\Omega \omega_\varepsilon(x,s)\boldsymbol{u}_\varepsilon(x,s)\cdot\nabla\big(\omega_\varepsilon(x,s) - \underline{\omega}(s)\big)^- dx = \\ &= \int_\Omega \big(\omega_\varepsilon(x,s) - \underline{\omega}(s)\big)^- \boldsymbol{u}_\varepsilon(x,s)\cdot\nabla\big(\omega_\varepsilon(x,s) - \underline{\omega}(s)\big)^- dx + \underline{\omega}(s)\int_\Omega \boldsymbol{u}_\varepsilon(x,s)\cdot\nabla\big(\omega_\varepsilon(x,s) - \underline{\omega}(s)\big)^- dx = 0 \end{aligned}$$

for a.e. $s \in]0,t[$, the estimate from below for ω_ε follows.

The estimate for ω_ε from above by $\overline{\omega}(t) := \dfrac{\omega^*}{1+t\omega^*}$ $(0 \le t \le T)$ can be proved by testing (18) with $\varphi = \big(\omega_\varepsilon(\cdot,t) - \overline{\omega}(t)\big)^+$. To prove the estimate from below for k_ε, set $\kappa(t) := \dfrac{k_*}{1+t\omega_*}$, $0 \le t \le T$. We insert $z = \big(k_\varepsilon(\cdot,t) - \kappa(t)\big)^-$ into (19), make use of $\dot{\kappa} = -\kappa\overline{\omega}$ and obtain

$$\frac{1}{2}\int\limits_\Omega \Big(\big(k_\varepsilon(x,t) - \kappa(t)\big)^-\Big)^2 dx - \int\limits_0^t\int\limits_\Omega k_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla(k_\varepsilon - \kappa)^- \le \int\limits_0^t\int\limits_\Omega (\kappa\overline{\omega} - k_\varepsilon\omega_\varepsilon)(k_\varepsilon - \kappa)^- \le 0$$

for a.e. $t \in]0,T[$. By an analogous reasoning as above, $\displaystyle\int_0^t\int_\Omega k_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla(k_\varepsilon - \kappa)^- = 0$. Thus, $k_\varepsilon \ge \kappa$ a.e. in Q_T.

3° *Integral estimates* We insert $\boldsymbol{v} = \boldsymbol{u}(\cdot,t)$ into (17) and $z = 1$ into (19). This gives

$$\|\boldsymbol{u}_\varepsilon\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \int\limits_{Q_T} \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon\,\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^{\rho-2}\Big)\,\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 \le c\,, \quad \|k_\varepsilon\|_{L^\infty(0,T;L^1)} \le c \tag{21}$$

(by c we denote different positive constants which do not depend on ε).

Next, define $\Psi(\xi) = \displaystyle\int_0^\xi \Big(1 - \frac{1}{(1+s)^\delta}\Big) ds$ $(0 \le \xi < +\infty,\ 0 < \delta < 1)$. We take $z = \Psi'\big(k_\varepsilon(\cdot,t)\big)$ in (19). With the help of (21) we obtain

$$\delta \int\limits_{Q_T} \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon\,|\nabla k_\varepsilon|^{\sigma-2}\Big)\frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^{1+\delta}} \le c\,.$$

From this estimate it follows that

$$\int\limits_{Q_T} \big(k_\varepsilon^{4p/3} + |\nabla k_\varepsilon|^p\big) \le c \quad \forall\, 1 \le p < 2\,, \qquad \int\limits_{Q_T} \big(k_\varepsilon\,|\nabla k_\varepsilon|\big)^q \le c \quad \forall\, 1 \le q < \frac{8}{7}\,,$$

$$\varepsilon \int\limits_{Q_T} |\nabla k_\varepsilon|^{\sigma-1}|\nabla z| \le \varepsilon^{1/\sigma}\|\nabla z\|_{\boldsymbol{L}^r(Q_T)} \quad \forall\, z \in L^r\big(0,T;W^{1,r}_{\mathrm{per}}(\Omega)\big)\,,$$

where $r = \dfrac{\sigma\kappa}{\kappa - 1}$, $\kappa = \dfrac{4p}{3}\cdot\dfrac{1}{(1+\delta)(\sigma-1)}$ $\Big(0 < \delta < \dfrac{11 - 3\sigma}{3(\sigma-1)}\Big)$. The integral estimates for ω_ε are straightforward.

Estimates for $\boldsymbol{u}'_\varepsilon$ and ω'_ε with respect to appropriate dual norms are easily derived. Finally, given $8 < s < +\infty$, there exists a constant $c(s)$ such that $\|k'_\varepsilon\|_{L^1(0,T;(W^{1,s}_{\mathrm{per}})^*)} \le c(s)$.

4° *Passage to the limit* $\varepsilon \to 0$ From (21), (22), the estimates for ω_ε and the estimates for $\boldsymbol{u}'_\varepsilon$, ω'_ε and k'_ε we obtain the existence of a subsequence of $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ which converges weakly [or weakly*] to a triple $\{\boldsymbol{u}, \omega, k\}$ in the respective spaces as well as a.e. in Q_T. Then (8)–(13), (15) and (16) are readily seen.

Finally, there exists a bounded Radon measure μ on the Borel σ-algebra of $\overline{Q}_T$ such that, for all $z \in C(\overline{Q}_T)$,

$$\int\limits_{Q_T} \frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon}\,\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 z \longrightarrow \int\limits_{Q_T} \frac{k}{\omega}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 z + \int\limits_{\overline{Q}_T} z\, d\mu \quad \text{as } \varepsilon \to 0\,.$$

The passage to the limit $\varepsilon \to 0$ in (19) is now easily done by routine arguments.

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