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Considering copositivity locally

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Abstract

Let A be an element of the copositive cone $\mathcal{COP}^n$. A zero $\mathbf{u}$ of A is a nonnegative vector whose elements sum up to one and such that $\mathbf{u}^T A\mathbf{u} = 0$. The support of $\mathbf{u}$ is the index set $\operatorname{supp}\mathbf{u} \subset \{1,\dots,n\}$ corresponding to the nonzero entries of $\mathbf{u}$. A zero $\mathbf{u}$ of A is called minimal if there does not exist another zero $\mathbf{v}$ of A such that its support $\operatorname{supp}\mathbf{v}$ is a strict subset of $\operatorname{supp}\mathbf{u}$. Our main result is a characterization of the cone of feasible directions at A, i.e., the convex cone $\mathcal{K}^A$ of real symmetric $n \times n$ matrices B such that there exists $\delta > 0$ satisfying $A + \delta B \in \mathcal{COP}^n$. This cone is described by a set of linear inequalities on the elements of B constructed from the set of zeros of A and their supports. This characterization furnishes descriptions of the minimal face of A in $\mathcal{COP}^n$, and of the minimal exposed face of A in $\mathcal{COP}^n$, by sets of linear equalities and inequalities constructed from the set of minimal zeros of A and their supports. In particular, we can check whether A lies on an extreme ray of $\mathcal{COP}^n$ by examining the solution set of a system of linear equations. In addition, we deduce a simple necessary and sufficient condition on the irreducibility of A with respect to a copositive matrix C. Here A is called irreducible with respect to C if for all $\delta > 0$ we have $A - \delta C \notin \mathcal{COP}^n$.

1 Introduction

Let $\mathcal{S}^n$ be the vector space of real symmetric $n \times n$ matrices. A matrix $A \in \mathcal{S}^n$ is called *copositive* if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n_+$. The set of copositive matrices forms a convex cone, the *copositive cone*, $\mathcal{COP}^n$. This matrix cone is of interest for combinatorial optimization, for surveys see [4, 7, 9, 14]. It is a classical result by Diananda [5, Theorem 2] that for $n \leq 4$ the copositive cone can be described as the sum of the cone of positive semi-definite matrices $\mathcal{S}^n_+$ and the cone of element-wise nonnegative symmetric matrices $\mathcal{N}^n$. In general, this sum is a subset of the copositive cone, $\mathcal{S}^n_+ + \mathcal{N}^n \subset \mathcal{COP}^n$. Horn showed that for $n \geq 5$ the inclusion is strict [5, p.25].

In this contribution, we investigate different properties of copositive matrices, in particular their minimal faces and irreducibility with respect to other copositive matrices, and relate them to the set of its zeros or its minimal zeros.

A vector $\mathbf{u} \in \mathbb{R}^n_+$ whose elements sum up to one is called a *zero* of a copositive matrix A if $\mathbf{u}^T A\mathbf{u} = 0$. Note that in the literature, instead of limiting the sum of the elements to be equal to one, sometimes there are no restrictions and sometimes there is only the restriction that $\mathbf{u} \neq \mathbf{0}$. As for all $\lambda > 0$ we have $\mathbf{u}^T A\mathbf{u} = 0$ if and only if $(\lambda\mathbf{u})^T A(\lambda\mathbf{u}) = 0$, it is a trivial matter to transfer between these definitions.

A zero $\mathbf{u}$ of A is called *minimal* if for no other zero $\mathbf{v}$ of A, the index set of positive entries of $\mathbf{v}$ is a strict subset of the index set of positive entries of $\mathbf{u}$.

A copositive matrix A is called *irreducible* with respect to another copositive matrix C if for every $\delta > 0$, we have $A - \delta C \notin \mathcal{COP}^n$, and it is called irreducible with respect to a subset $\mathcal{M} \subset \mathcal{COP}^n$ if it is irreducible with respect to all nonzero elements $C \in \mathcal{M}$.

It has been recognised early that the zero set of a copositive matrix is a useful tool in the study of the structure of the cone $\mathcal{COP}^n$ [5, 11]. In [3] Baumert considered the possible zero sets of matrices in $\mathcal{COP}^5$. He provided a partial classification of the zero sets of matrices $A \in \mathcal{COP}^5$ which are irreducible with respect to the cone $\mathcal{N}^5$. In [8] this classification was completed, and a necessary and sufficient condition for irreducibility of a copositive matrix $A \in \mathcal{COP}^n$ with respect to the cone $\mathcal{N}^n$ was given in terms of its zero set. In [12], a similar condition in terms of the minimal zero set was given for irreducibility of a copositive matrix $A \in \mathcal{COP}^n$ with respect to the cone $\mathcal{S}_+^n$.

In [6], the facial structure and the extreme rays of the copositive cone $\mathcal{COP}^n$ and its dual, the completely positive cone, has been investigated. It has been shown that not every extreme ray of $\mathcal{COP}^n$ is exposed.

Our main result in this paper is a necessary and sufficient condition on a pair (A, B), where $A \in \mathcal{COP}^n$ and $B \in \mathcal{S}^n$, for the existence of a scalar $\delta > 0$ such that $A + \delta B \in \mathcal{COP}^n$. For fixed A, the set of all such matrices $B \in \mathcal{S}^n$ forms a convex cone $\mathcal{K}^A$, which is referred to as the *cone of feasible directions* [15]. We express this cone in terms of the zeros of A and their supports.

The obtained description of the cone $\mathcal{K}^A$ is a powerful tool. It will allow us to compute the minimal face and the minimal exposed face of A. In particular, we obtain a simple test of extremality of A, which amounts to checking the rank of a certain matrix constructed from the minimal zeros of A. The necessary and sufficient conditions for the irreducibility of A with respect to a nonnegative matrix $C \in \mathcal{N}^n$ or a positive semi-definite matrix $C \in \mathcal{S}_+^n$, which have been given in [8] and [12], respectively, are generalized to the case of arbitrary matrices $C \in \mathcal{COP}^n$. The conditions in [8] and [12] follow as particular cases.

The remainder of the paper is structured as follows. In the next section we provide necessary definitions and notations, and in the following section we collect some results from the literature and provide some preliminary results. In Section 4 we provide our main result, the description of the cone $\mathcal{K}^A$ of feasible directions of $\mathcal{COP}^n$ at A. We also compute its closure, the *tangent cone* $\mathrm{cl}(\mathcal{K}^A)$, and the *tangent space* $\mathrm{cl}(\mathcal{K}^A) \cap -\mathrm{cl}(\mathcal{K}^A)$ [15]. In Section 5 we deduce the descriptions of the minimal face and the minimal exposed face of a copositive matrix. In Section 6 we consider irreducibility of a copositive matrix with respect to another arbitrary copositive matrix. Finally, we give a summary in the last section.

2 Notations

We shall denote vectors with bold lower-case letters and matrices with upper-case letters. Individual entries of a vector $\mathbf{u}$ and a matrix A will be denoted by u_i and a_{ij} respectively. For a matrix A and a vector $\mathbf{u}$ of compatible size, the i-th element of the matrix-vector product $A\mathbf{u}$ will be denoted by $(A\mathbf{u})_i$. Inequalities $\mathbf{u} \geq \mathbf{0}$ on vectors will be meant element-wise, where we denote by $\mathbf{0} = (0, \ldots, 0)^T$ the all-zeros vector. Similarly we denote by $\mathbf{1} = (1, \ldots, 1)^T$ the all-ones vector. We further let $\mathbf{e}_i$ be the unit vector with i-th entry equal to one and all other entries equal to zero. Let $\Delta^n = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{1}^T\mathbf{x} = 1\}$ be the standard simplex in $\mathbb{R}^n$. For a subset $\mathcal{I} \subset \{1, \ldots, n\}$ we denote by $A_\mathcal{I}$ the principal submatrix of A whose elements have row and column indices in $\mathcal{I}$, i.e. $A_\mathcal{I} = (a_{ij})_{i,j\in\mathcal{I}} \in \mathcal{S}^{|\mathcal{I}|}$. Similarly for a vector $\mathbf{u} \in \mathbb{R}^n$ we define the subvector $\mathbf{u}_\mathcal{I} = (u_i)_{i\in\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$.

We call a vector $\mathbf{u} \in \Delta^n$ a *zero* of a matrix $A \in \mathcal{S}^n$ if $\mathbf{u}^T A\mathbf{u} = 0$, and we denote the set of zeros of A by $\mathcal{V}^A = \{\mathbf{u} \in \Delta^n \mid \mathbf{u}^T A\mathbf{u} = 0\}$. For a vector $\mathbf{u} \in \mathbb{R}^n$ we define its *support* as $\operatorname{supp}\mathbf{u} = \{i \in \{1, \ldots, n\} \mid u_i \neq 0\}$. For a matrix $A \in \mathcal{COP}^n$ and a zero $\mathbf{v}$ of A we define the set $\mathcal{J}(\mathbf{v}, A)$ of indices i such that there exists $\mathbf{u} \in \mathcal{V}^A$ satisfying $\{i\} \cup \operatorname{supp}(\mathbf{v}) \subset \operatorname{supp}(\mathbf{u})$. In other words, $\mathcal{J}(\mathbf{v}, A)$ is the union of $\operatorname{supp}(\mathbf{u})$ over all $\mathbf{u} \in \mathcal{V}^A$ with $\operatorname{supp}(\mathbf{v}) \subset \operatorname{supp}(\mathbf{u})$.

A zero $\mathbf{u}$ of a copositive matrix A is called *minimal* if there exists no zero $\mathbf{v}$ of A such that the inclusion $\operatorname{supp}\mathbf{v} \subset \operatorname{supp}\mathbf{u}$ holds strictly. We shall denote the set of minimal zeros of a copositive matrix A by $\mathcal{V}^A_{\min}$. From [12], for all $A \in \mathcal{COP}^n$, the set $\mathcal{V}^A_{\min}$ is a finite set, and there are algorithmic methods for finding this set.

3 Preliminary Results

We start by considering the following preliminary results on sets of zeros.

Lemma 1 ([2, p.200])**.** *Let $A \in \mathcal{COP}^n$ and $\mathbf{u} \in \mathcal{V}^A$. Then $A\mathbf{u} \geq 0$.*

Lemma 2 ([8, Lemma 2.5])**.** *Let $A \in \mathcal{COP}^n$ and $\mathbf{u} \in \mathcal{V}^A$. Then $(A\mathbf{u})_i = 0$ for all $i \in \operatorname{supp}(\mathbf{u})$.*

Lemma 3 ([12, Corollary 3.4])**.** *Let $A \in \mathcal{COP}^n$ and $\mathbf{u} \in \mathcal{V}^A$. Then $\mathbf{u}$ can be represented as a convex combination of minimal zeros of A.*

We now consider the following corollary.

Corollary 4. *For $A \in \mathcal{COP}^n$ and $\mathbf{u}, \mathbf{v} \in \mathcal{V}^A$ such that $\operatorname{supp}(\mathbf{u}) \subset \operatorname{supp}(\mathbf{v})$, we have*

$$\emptyset = \operatorname{supp}(\mathbf{u}) \cap \operatorname{supp}(A\mathbf{v}) = \operatorname{supp}(A\mathbf{u}) \cap \operatorname{supp}(\mathbf{v}),$$

and thus $(\theta\mathbf{u} + \lambda\mathbf{v})^T A(\theta\mathbf{u} + \lambda\mathbf{v}) = 0$ for all $\theta, \lambda \in \mathbb{R}$.

Proof. From Lemma 2 we have $\operatorname{supp}(\mathbf{v}) \subset \{1, \ldots, n\} \setminus \operatorname{supp}(A\mathbf{v})$, and as $\operatorname{supp}(\mathbf{u}) \subset \operatorname{supp}(\mathbf{v})$ this implies that $\emptyset = \operatorname{supp}(\mathbf{u}) \cap \operatorname{supp}(A\mathbf{v})$. This in turn implies that

$$0 = \mathbf{u}^T A\mathbf{v} = \sum_{i=1}^n \underbrace{v_i}_{\geq 0} \underbrace{(A\mathbf{u})_i}_{\geq 0},$$

where we used the fact that $A\mathbf{u} \geq \mathbf{0}$ from Lemma 1. From this observation we then get $\emptyset = \operatorname{supp}(\mathbf{v}) \cap \operatorname{supp}(A\mathbf{u})$, which completes the proof. $\square$

The set $\mathcal{J}(\mathbf{v}, A)$ will be considered later in the paper. Due to its applications we would like to have an algorithmic method to find it. Based on the fact that the finite set $\mathcal{V}^A_{\min}$ can be found algorithmically [12], such a method is provided by the following lemma.

Lemma 5. *For $A \in \mathcal{COP}^n$, $\mathbf{v} \in \mathcal{V}^A$ and $i \in \{1, \dots, n\}$, we have $i \in \mathcal{J}(\mathbf{v}, A)$ if and only if there exists $\mathbf{w} \in \mathcal{V}^A_{\min}$ such that $i \in \operatorname{supp}(\mathbf{w}) \subset \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{v})$.*

Proof. We begin by considering the forward implication. If $i \in \mathcal{J}(\mathbf{v}, A)$ then there exists $\mathbf{u} \in \mathcal{V}^A$ such that $\{i\} \cup \operatorname{supp}(\mathbf{v}) \subset \operatorname{supp}(\mathbf{u})$, and from Corollary 4 we have $\operatorname{supp}(\mathbf{u}) \subset \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{v})$. By Lemma 3 there exists $\mathbf{w} \in \mathcal{V}^A_{\min}$ such that $i \in \operatorname{supp}(\mathbf{w}) \subset \operatorname{supp}(\mathbf{u})$, which completes the proof of the forward implication.

For the reverse implication we let $\mathbf{u} = \frac{1}{2}(\mathbf{v} + \mathbf{w})$, and note that $\{i\} \cup \operatorname{supp}(\mathbf{v}) \subset \operatorname{supp}(\mathbf{u})$. We have $0 = \mathbf{v}^T A\mathbf{v} = \mathbf{w}^T A\mathbf{w} = \mathbf{w}^T A\mathbf{v}$ and thus $\mathbf{u}^T A\mathbf{u} = 0$. Therefore $\mathbf{u} \in \mathcal{V}^A$ and $i \in \mathcal{J}(\mathbf{v}, A)$. $\square$

We shall also consider the concept of irreducibility, which is defined as follows:

Definition 6 ([8, Definition 1.1])**.** For a matrix $A \in \mathcal{COP}^n$ and a subset $\mathcal{M} \subset \mathcal{COP}^n$, we say that A is *irreducible with respect to* $\mathcal{M}$ if there do not exist $\delta > 0$ and $M \in \mathcal{M} \setminus \{0\}$ such that $A - \delta M \in \mathcal{COP}^n$.

Note that this definition differs from the concept of an irreducible matrix that is normally used in matrix theory. For simplicity we speak about irreducibility with respect to M when $\mathcal{M} = \{M\}$. We also note that as $\mathcal{COP}^n$ is a convex cone, the following result holds.

Lemma 7 ([12, Lemma 2.2])**.** *Let $A \in \mathcal{COP}^n$ and $\mathcal{M} \subset \mathcal{COP}^n$. Then the following are equivalent:*

(a) A is irreducible with respect to $\mathcal{M}$,

(b) A is irreducible with respect to M for all $M \in \mathcal{M}$,

(c) A is irreducible with respect to $\mathbb{R}_+\mathcal{M}$,

(d) A is irreducible with respect to the convex conic hull of $\mathcal{M}$.

4 Main result

For a matrix $A \in \mathcal{COP}^n$ we consider the set $\mathcal{K}^A = \{B \in \mathcal{S}^n \mid \exists \delta > 0 \text{ s.t. } A + \delta B \in \mathcal{COP}^n\}$. This set is trivially a cone. As $\mathcal{COP}^n$ is a convex cone, we have $\mathcal{COP}^n \subset \mathcal{K}^A$ and we have that $\mathcal{K}^A$ is a convex set. The convex cone $\mathcal{K}^A$ is not pointed, unless $A = 0$, as we always have $\pm A \in \mathcal{K}^A$. It is also in general not closed, as we shall see at the end of this section.

We now present the following main result.

Theorem 8. *For $A \in \mathcal{COP}^n$ we have:*

$$\mathcal{K}^A = \left\{ B \in \mathcal{S}^n \;\middle|\; \begin{array}{l} \mathbf{v}^T B\mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A, \\ (B\mathbf{v})_i \geq 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A \cap \mathcal{V}^B,\ i \in \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{v}) \end{array} \right\}$$

Proof. We shall prove the forward and reverse inclusions separately.

$\subset$: Suppose that $A + \delta B \in \mathcal{COP}^n$ for some $\delta > 0$.

Then for all $\mathbf{v} \in \mathcal{V}^A$ we have $0 \leq \frac{1}{\delta}\mathbf{v}^T(A + \delta B)\mathbf{v} = \mathbf{v}^T B\mathbf{v}$.

Also, for all $\mathbf{v} \in \mathcal{V}^A \cap \mathcal{V}^B$, $i \in \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{v})$ and $\varepsilon > 0$ we have $(\mathbf{v} + \varepsilon\mathbf{e}_i) \in \mathbb{R}^n_+$ and thus $0 \leq \frac{1}{2\varepsilon\delta}(\mathbf{v} + \varepsilon\mathbf{e}_i)^T(A + \delta B)(\mathbf{v} + \varepsilon\mathbf{e}_i) = (B\mathbf{v})_i + \frac{\varepsilon}{2\delta}(a_{ii} + \delta b_{ii})$. Letting $\varepsilon \to 0$ this implies that $(B\mathbf{v})_i \geq 0$, which completes this part of the proof.

$\supset$: Suppose for the sake of contradiction that the conditions on the right hand side hold for a given $B \in \mathcal{S}^n \setminus \mathcal{K}^A$.

For $\delta > 0$ let

$$\mathbf{v}_\delta \in \arg\min_{\mathbf{v}} \left\{ \mathbf{v}^T(A + \delta B)\mathbf{v} \;\middle|\; \mathbf{v} \in \Delta^n \right\}, \tag{1}$$

and note that for all $\delta > 0$ we would have $\mathbf{v}_\delta^T(A + \delta B)\mathbf{v}_\delta < 0$ and thus $\mathbf{v}_\delta^T B\mathbf{v}_\delta < -\frac{1}{\delta}\mathbf{v}_\delta^T A\mathbf{v}_\delta \leq 0$.

As Δ^n is a compact set, there exists a sequence $\{\delta_k \mid k \in \mathbb{N}\} \subset \mathbb{R}_{++}$ and $\mathbf{v}^* \in \Delta^n$ such that $\lim_{k\to\infty} \delta_k = 0$ and $\lim_{k\to\infty} \mathbf{v}_{\delta_k} = \mathbf{v}^*$. Furthermore, without loss of generality (by possibly throwing away leading δ_k's), for all $k \in \mathbb{N}$ we have

$$\operatorname{supp}(\mathbf{v}^*) \subset \operatorname{supp}(\mathbf{v}_{\delta_k}), \qquad \operatorname{supp}_+(A\mathbf{v}^*) \subset \operatorname{supp}_+((A + \delta_k B)\mathbf{v}_{\delta_k}),$$

where $\operatorname{supp}_+(\mathbf{u}) := \{i \in \{1, \dots, n\} \mid u_i > 0\}$.

We have $0 \leq \mathbf{v}^{*T} A\mathbf{v}^* = \lim_{k\to\infty} \mathbf{v}_{\delta_k}^T(A + \delta_k B)\mathbf{v}_{\delta_k} \leq 0$ and thus $\mathbf{v}^* \in \mathcal{V}^A$ and $\operatorname{supp}(A\mathbf{v}^*) = \operatorname{supp}_+(A\mathbf{v}^*)$ by Lemma 1. Therefore, by the assumptions, we have

$$0 \leq \mathbf{v}^{*T} B\mathbf{v}^* = \lim_{k\to\infty} \mathbf{v}_{\delta_k}^T B\mathbf{v}_{\delta_k} \leq 0.$$

This implies that $\mathbf{v}^* \in \mathcal{V}^A \cap \mathcal{V}^B$ and thus again by the assumptions we have $(B\mathbf{v}^*)_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{v}^*)$.

From now on we consider an arbitrary fixed $k \in \mathbb{N}$.

Having only linear constraints, problem (1) fulfills a constraint qualification [1],[10, p.52], and by the Karush-Kuhn-Tucker optimality conditions there exist $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ and $\mu \in \mathbb{R}$ such that $\operatorname{supp}(\mathbf{v}_{\delta_k}) \subset \{1, \dots, n\} \setminus \operatorname{supp}(\boldsymbol{\lambda})$ and $(A + \delta_k B)\mathbf{v}_{\delta_k} = \boldsymbol{\lambda} - \mu\mathbf{1}$. We then have

$$\mu = -\mathbf{v}_{\delta_k}^T(A + \delta_k B)\mathbf{v}_{\delta_k} + \mathbf{v}_{\delta_k}^T\boldsymbol{\lambda} = -\mathbf{v}_{\delta_k}^T(A + \delta_k B)\mathbf{v}_{\delta_k} > 0$$

and thus

$$\operatorname{supp}(\boldsymbol{\lambda}) \supset \operatorname{supp}_+((A + \delta_k B)\mathbf{v}_{\delta_k}) \supset \operatorname{supp}_+(A\mathbf{v}^*) = \operatorname{supp}(A\mathbf{v}^*).$$

Therefore $(B\mathbf{v}^*)_i \geq 0$ for all $i \in \{1,\dots,n\}\setminus\operatorname{supp}(\boldsymbol{\lambda})$. Furthermore, from $\operatorname{supp}(\mathbf{v}_{\delta_k}) \subset \{1,\dots,n\} \setminus \operatorname{supp}(\boldsymbol{\lambda})$, we have $(B\mathbf{v}^*)_i \geq 0$ for all $i \in \operatorname{supp}(\mathbf{v}_{\delta_k})$. This implies ${\mathbf{v}^*}^T B\mathbf{v}_{\delta_k} \geq 0$ and we get the contradiction

$$\begin{aligned}
0 &= {\mathbf{v}^*}^T\mathbf{0} \\
&= {\mathbf{v}^*}^T\left((A+\delta_k B)\mathbf{v}_{\delta_k} - \boldsymbol{\lambda} + \mu\mathbf{1}\right) \\
&= \underbrace{{\mathbf{v}^*}^T A}_{\geq\mathbf{0}}\underbrace{\mathbf{v}_{\delta_k}}_{\geq\mathbf{0}} + \underbrace{\delta_k {\mathbf{v}^*}^T B\mathbf{v}_{\delta_k}}_{\geq 0} - \underbrace{{\mathbf{v}^*}^T\boldsymbol{\lambda}}_{=0} + \underbrace{\mu}_{>0}\underbrace{{\mathbf{v}^*}^T\mathbf{1}}_{=1} \\
&> 0.
\end{aligned}$$

□

Noting that for $A \in \mathcal{COP}^n$ and $\mathbf{v} \in \mathcal{V}^A$ we have $\mathbf{v} \in \mathbb{R}^n_+$ and $\operatorname{supp}(\mathbf{v}) \subset \{1,\dots,n\} \setminus \operatorname{supp}(A\mathbf{v})$, an alternative characterization is as follows:

$$\mathcal{K}^A = \left\{ B \in \mathcal{S}^n \;\middle|\; \begin{array}{l} \mathbf{v}^T B\mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A, \\ (B\mathbf{v})_i = 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A \cap \mathcal{V}^B,\ i \in \operatorname{supp}(\mathbf{v}), \\ (B\mathbf{v})_i \geq 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A \cap \mathcal{V}^B,\ i \in \{1,\dots,n\} \setminus (\operatorname{supp}(\mathbf{v}) \cup \operatorname{supp}(A\mathbf{v})) \end{array} \right\}$$

We now consider a quick example, from which we see that $\mathcal{K}^A$ is not in general closed.

Example 9. Let $A = \sum_{i=2}^n \mathbf{e}_i\mathbf{e}_i^T$, then we have $\mathcal{V}^A = \{\mathbf{e}_1\}$ and $\operatorname{supp}(A\mathbf{e}_1) = \emptyset$. Therefore

$$\mathcal{K}^A = \{B \in \mathcal{S}^n \mid b_{11} > 0\} \cup \{B \in \mathcal{S}^n \mid b_{11} = 0,\ b_{1i} \geq 0 \text{ for all } i\},$$

which is not closed.

Using the characterizations of $\mathcal{K}^A$ so far presented, it is in general difficult to give an explicit expression for $\mathcal{K}^A$. Later in the paper, in Corollary 19, we shall present yet another characterization, which would allow for an explicit expression (using Theorem 18). For the moment, however, we shall content ourselves by considering a characterization for the somewhat simpler set of its closure, denoted $\operatorname{cl}(\mathcal{K}^A)$. In the literature this cone is generally referred to as the tangent cone [13, 15].

Theorem 10. *For $A \in \mathcal{COP}^n$ the tangent cone of $\mathcal{COP}^n$ at A is given by*

$$\operatorname{cl}(\mathcal{K}^A) = \{B \in \mathcal{S}^n \mid \mathbf{v}^T B\mathbf{v} \geq 0 \textit{ for all } \mathbf{v} \in \mathcal{V}^A\}.$$

Proof. We let $\mathcal{M}$ be the set on the right-hand side of the equation above. From Theorem 8 we have $\mathcal{K}^A \subset \mathcal{M}$. Furthermore, the set $\mathcal{M}$ is the intersection of closed sets, and thus is itself closed. From this we then get $\operatorname{cl}(\mathcal{K}^A) \subset \mathcal{M}$.

We now consider an arbitrary $B \in \mathcal{M}$. Letting I be the identity matrix we have

$$\mathbf{v}^T(B+\varepsilon\mathrm{I})\mathbf{v} = \underbrace{\mathbf{v}^T B\mathbf{v}}_{\geq 0} + \underbrace{\varepsilon\mathbf{v}^T\mathbf{v}}_{>0} > 0 \qquad \text{for all } \mathbf{v} \in \mathcal{V}^A,\ \varepsilon > 0.$$

Therefore $B + \varepsilon\mathrm{I} \in \mathcal{K}^A$ for all $\varepsilon > 0$, and thus $B \in \operatorname{cl}(\mathcal{K}^A)$, which completes the proof. □

From the work of [6, 12] it can be seen that for $A \in \mathcal{COP}^n$ we have that $\mathcal{V}^A$ is the union of finitely many polyhedra contained in Δ^n. We can then characterise $\mathrm{cl}(\mathcal{K}^A)$ by noting the following two trivial results.

Lemma 11. *For a polyhedron $\mathcal{M} = \mathrm{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ (where 'conv' denotes the convex hull), letting $V = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_m\end{bmatrix}$, we have*

$$\{B \in \mathcal{S}^n \mid \mathbf{v}^T B\mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathcal{M}\} = \{B \in \mathcal{S}^n \mid V^T BV \in \mathcal{COP}^m\}.$$

Lemma 12. *For sets $\mathcal{M}_1, \ldots, \mathcal{M}_p \subset \mathbb{R}^n$ we have*

$$\left\{B \in \mathcal{S}^n \;\middle|\; \mathbf{v}^T B\mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \bigcup_{i=1}^{p} \mathcal{M}_i\right\} = \bigcap_{i=1}^{p} \left\{B \in \mathcal{S}^n \;\middle|\; \mathbf{v}^T B\mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathcal{M}_i\right\}.$$

In fact, for $A \in \mathcal{COP}^n \setminus \{0\}$, all the polyhedra composing $\mathcal{V}^A$ are of dimension less than or equal to $n-2$, which we could partition into simplices, each with at most $n-1$ vertices. This then means that $\mathcal{K}^A$ can be characterised using cones $\mathcal{COP}^m$ where $m \leq n-1$. Noting that for $m \leq 4$ we have $\mathcal{COP}^m = \mathcal{S}_+^m + \mathcal{N}^m$, we then get the following result:

Lemma 13. *For $A \in \mathcal{COP}^5 \setminus \{0\}$ we have that $\mathrm{cl}(\mathcal{K}^A)$ is a semi-definite representable set.*

This is of interest as it is still an open question whether $\mathcal{COP}^5$ itself is a semi-definite representable set.

From Theorem 10 we can also obtain an expression for the tangent space, which is defined as $\mathrm{cl}(\mathcal{K}^A) \cap -\mathrm{cl}(\mathcal{K}^A)$ [15].

Theorem 14. *For $A \in \mathcal{COP}^n$ the tangent space of $\mathcal{COP}^n$ at A is given by*

$$\begin{aligned}\mathrm{cl}(\mathcal{K}^A) \cap -\mathrm{cl}(\mathcal{K}^A) &= \{B \in \mathcal{S}^n \mid \mathcal{V}^A \subset \mathcal{V}^B\} \\ &= \{B \in \mathcal{S}^n \mid \mathbf{u}^T B\mathbf{v} = 0 \text{ for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}^A_{\min} \text{ s.t. } \mathbf{u}^T A\mathbf{v} = 0\}.\end{aligned}$$

Proof. The expression "$\mathrm{cl}(\mathcal{K}^A) \cap -\mathrm{cl}(\mathcal{K}^A) = \{B \in \mathcal{S}^n \mid \mathcal{V}^A \subset \mathcal{V}^B\}$" follows directly from Theorem 10.

We now consider an arbitrary $B \in \mathcal{S}^n$ such that $\mathcal{V}^A \subset \mathcal{V}^B$. For all $\mathbf{u}, \mathbf{v} \in \mathcal{V}^A_{\min} \subset \mathcal{V}^B$ such that $\mathbf{u}^T A\mathbf{v} = 0$ we have $\frac{1}{2}(\mathbf{u}+\mathbf{v}) \in \mathcal{V}^A \subset \mathcal{V}^B$ and thus $0 = (\mathbf{u}+\mathbf{v})^T B(\mathbf{u}+\mathbf{v}) = 2\mathbf{u}^T B\mathbf{v}$. Therefore

$$\{B \in \mathcal{S}^n \mid \mathcal{V}^A \subset \mathcal{V}^B\} \subset \{B \in \mathcal{S}^n \mid \mathbf{u}^T B\mathbf{v} = 0 \text{ for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}^A_{\min} \text{ s.t. } \mathbf{u}^T A\mathbf{v} = 0\}.$$

To prove that the reverse inclusion relation also holds, we consider an arbitrary $\mathbf{w} \in \mathcal{V}^A$ and $B \in \mathcal{S}^n$ such that $\mathbf{u}^T B\mathbf{v} = 0$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}^A_{\min}$ with $\mathbf{u}^T A\mathbf{v} = 0$. By Lemma 3, there exist $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathcal{V}^A_{\min}$ and $\theta_1, \ldots, \theta_m > 0$ such that $\mathbf{w} = \sum_{i=1}^m \theta_i \mathbf{v}_i$. Using Lemma 1 we note that

$$0 = \mathbf{w}^T A\mathbf{w} = \sum_{i,j=1}^{m} \underbrace{\theta_i\,\theta_j}_{>0}\,\underbrace{\mathbf{v}_i^T}_{\geq \mathbf{0}}\,\underbrace{A\mathbf{v}_j}_{\geq \mathbf{0}},$$

and thus $\mathbf{v}_i^T A\mathbf{v}_j = 0$ for all i, j. By the assumptions, this implies that $\mathbf{v}_i^T B\mathbf{v}_j = 0$ for all i, j, and thus $\mathbf{w} \in \mathcal{V}^B$. $\square$

Note that the latter characterization of the tangent space is as a linear space described by finitely many linear equality relations.

5 Minimal Faces

In this section we apply Theorem 8 to determine the minimal face and the minimal exposed face of a copositive matrix.

Definition 15. A convex subset $\mathcal{F} \subset \mathcal{COP}^n$ is a *face* of $\mathcal{COP}^n$ if every closed line segment in $\mathcal{COP}^n$ with a relative interior point in $\mathcal{F}$ must have both end points in $\mathcal{F}$. For $A \in \mathcal{COP}^n$ we let $\mathcal{F}^A$ equal the intersection of all faces of $\mathcal{COP}^n$ containing A. This is itself a face, and is referred to as the *minimal face* of $\mathcal{COP}^n$ containing A.

We say that $\widehat{\mathcal{F}}$ is an exposed face of a convex set $\mathcal{C} \subset \mathcal{S}^n$ if there exists $\beta \in \mathbb{R}$ and $B \in \mathcal{S}^n$ such that $\widehat{\mathcal{F}} = \{X \in \mathcal{C} \mid \langle X, B\rangle = \beta\}$ and $\mathcal{C} \subset \{X \in \mathcal{S}^n \mid \langle X, B\rangle \geq \beta\}$. Here $\langle A, B\rangle = \mathrm{trace}(AB)$ is the Frobenius scalar product on $\mathcal{S}^n$. For $\mathcal{C}$ being the copositive cone it is trivial to see that this is equivalent to the following definition.

Definition 16. A set $\widehat{\mathcal{F}} \subset \mathcal{COP}^n$ is an *exposed face* of $\mathcal{COP}^n$ if there exists $\mathcal{W} \subset \mathbb{R}^n_+$ such that $\widehat{\mathcal{F}} = \{X \in \mathcal{COP}^n \mid \mathcal{W} \subset \mathcal{V}^X\}$. For $A \in \mathcal{COP}^n$ we let $\widehat{\mathcal{F}^A} = \{X \in \mathcal{COP}^n \mid \mathcal{V}^A \subset \mathcal{V}^X\}$. This is an exposed face which is the intersection of all exposed faces of $\mathcal{COP}^n$ containing A, and it is referred to as the *minimal exposed face* of $\mathcal{COP}^n$ containing A.

Note that for $A \in \mathcal{COP}^n$ we always have $\{\lambda A \mid \lambda \geq 0\} \subset \mathcal{F}^A \subset \widehat{\mathcal{F}^A} \subset \mathcal{COP}^n$.

If $A \in \mathcal{COP}^n \setminus \{0\}$ and $\mathcal{F}^A$ is of dimension equal to one then we say that A gives an *extreme ray* of the copositive cone. We in fact then have $\mathcal{F}^A = \{\lambda A \mid \lambda \geq 0\}$.

If $A \in \mathcal{COP}^n \setminus \{0\}$ and $\widehat{\mathcal{F}^A}$ is of dimension equal to one then we say that A gives an *exposed ray* of the copositive cone, which is a special type of extreme ray. Similarly to before we then have $\widehat{\mathcal{F}^A} = \{\lambda A \mid \lambda \geq 0\}$.

For $A \in \mathcal{COP}^n$ we now let

$$\mathcal{L}^A = \{B \in \mathcal{S}^n \mid \exists \delta > 0 \text{ s.t. } A + \delta B \in \mathcal{F}^A\}$$
$$\widehat{\mathcal{L}^A} = \{B \in \mathcal{S}^n \mid \exists \delta > 0 \text{ s.t. } A + \delta B \in \widehat{\mathcal{F}^A}\}.$$

We then have

$$\dim(\mathcal{F}^A) = \dim(\mathcal{L}^A), \qquad \dim(\widehat{\mathcal{F}^A}) = \dim(\widehat{\mathcal{L}^A}).$$

Therefore A gives an extreme (resp. exposed) ray of the copositive cone if and only if $\mathcal{L}^A$ (resp. $\widehat{\mathcal{L}^A}$) is of dimension equal to one.

The advantage of using the sets $\mathcal{L}^A$ and $\widehat{\mathcal{L}^A}$ comes from Theorems 18 and 20 below, in which we see that the characterizations of $\mathcal{L}^A$ and $\widehat{\mathcal{L}^A}$ are relatively simple. This then gives us a method for checking whether a copositive matrix gives an extreme/exposed ray.

Before presenting these theorems, we first recall the following result relating $\mathcal{F}^A$ and $\mathcal{K}^A$:

Lemma 17 ([15, Lemma 3.2.1]). *For $A \in \mathcal{COP}^n$ we have $\mathcal{K}^A = \mathcal{COP}^n + \operatorname{span}(\mathcal{F}^A)$.*

We are now ready to present the first of our results on minimal faces.

Theorem 18. *For $A \in \mathcal{COP}^n$ and $B \in \mathcal{S}^n$ the following are equivalent:*

1 $B \in \mathcal{L}^A$,

2 $B \in \operatorname{span}(\mathcal{F}^A)$,

3 $\exists \delta > 0$ *such that* $A \pm \delta B \in \mathcal{COP}^n$ *(equivalently $B \in \mathcal{K}^A \cap (-\mathcal{K}^A)$),*

4 $(B\mathbf{v})_i = 0$ *for all* $\mathbf{v} \in \mathcal{V}^A$, $i \in \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{v})$,

5 $(B\mathbf{v})_i = 0$ *for all* $\mathbf{v} \in \mathcal{V}^A_{\min}$, $i \in \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{v})$.

Proof. We shall split this proof into the following parts:

(1)⇒(2): This follows trivially from the definitions.

(2)⇒(3): This follows directly from Lemma 17.

(3)⇒(1): Suppose that (3) holds and consider the set $\mathcal{M} = \{A + \theta B \mid -\delta \leq \theta \leq \delta\}$. This is a closed line segment in $\mathcal{COP}^n$ with A in its relative interior. Therefore, from the definition of a face, we have $A + \delta B \in \mathcal{F}^A$, and thus $B \in \mathcal{L}^A$.

(3)⇔(4): This follows from Theorem 8 and the fact that $\operatorname{supp}(\mathbf{v}) \subset \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{v})$.

(4)⇒(5): This follows trivially from the fact that $\mathcal{V}^A_{\min} \subset \mathcal{V}^A$.

(5)⇒(4): Suppose that statement (5) holds and consider an arbitrary $\mathbf{u} \in \mathcal{V}^A$. By Lemma 3, there exist $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathcal{V}^A_{\min}$ and $\theta_1, \dots, \theta_m > 0$ such that $\mathbf{u} = \sum_{j=1}^m \theta_j \mathbf{v}_j$. For all j we have $A\mathbf{v}_j \geq \mathbf{0}$ and thus $\operatorname{supp}(A\mathbf{u}) \supset \operatorname{supp}(A\mathbf{v}_j)$. Therefore for all $i \in \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{u})$ we have $(B\mathbf{v}_j)_i = 0$ for all j, and thus $(B\mathbf{u})_i = (B\mathbf{v}_1)_i + \dots + (B\mathbf{v}_m)_i = 0$. □

Note that from this theorem $\mathcal{L}^A = \operatorname{span}(\mathcal{F}^A)$ is a linear subspace of $\mathcal{S}^n$, and as $\mathcal{V}^A_{\min}$ is a finite set, the system of linear equations in Theorem 18 (5) is finite. We can thus algorithmically compute the dimension of $\mathcal{L}^A$ by finding the rank of the coefficient matrix of this system of linear equations. This then allows us to determine if the copositive matrix A gives an extreme ray.

Also note that using Lemma 17 we get the following alternative characterization for $\mathcal{K}^A$. In comparison to the previous characterizations in Section 4, in general it is easier to give this one explicitly.

Corollary 19. *For $A \in \mathcal{COP}^n$ we have*

$$\mathcal{K}^A = \mathcal{COP}^n + \{B \in \mathcal{S}^n \mid (B\mathbf{v})_i = 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A_{\min},\ i \in \{1, \dots, n\} \setminus \operatorname{supp}(A\mathbf{v})\}.$$

In comparison to the characterization of $\mathrm{cl}(\mathcal{K}^A)$ in Section 4, our latest characterization of $\mathcal{K}^A$ involves one copositive cone of order n, whereas the characterization of $\mathrm{cl}(\mathcal{K}^A)$ involves multiple copositive cones of order strictly less than n.

We now present the following result for minimal exposed faces, which is closely related to Theorem 18.

Theorem 20. *For $A \in \mathcal{COP}^n$ and $B \in \mathcal{S}^n$ the following are equivalent:*

1 $B \in \widehat{\mathcal{L}^A}$,

2 $\exists\delta > 0$ *such that* $A + \delta B \in \mathcal{COP}^n$ *and* $\mathcal{V}^A \subset \mathcal{V}^{A+\delta B}$,

3 *For all* $\mathbf{v} \in \mathcal{V}^A$ *we have* $\mathbf{v}^T B\mathbf{v} = 0$ *and* $(B\mathbf{v})_i \geq 0$ *for all* $i \in \{1, \dots, n\} \setminus \mathrm{supp}(A\mathbf{v})$,

4 *For all* $\mathbf{v} \in \mathcal{V}^A_{\min}$ *we have*

$$\begin{aligned} (B\mathbf{v})_i &= 0 && \text{for all } i \in \mathcal{J}(\mathbf{v}, A), \\ (B\mathbf{v})_i &\geq 0 && \text{for all } i \in \{1, \dots, n\} \setminus \mathrm{supp}(A\mathbf{v}). \end{aligned}$$

Proof. We shall split this proof into the following parts:

(1)⇔(2): This equivalence follows directly from the definition of $\widehat{\mathcal{L}^A}$.

(2)⇒(3): Let $\mathbf{v} \in \mathcal{V}^A$. Then $\mathbf{v} \in \mathcal{V}^{A+\delta B}$ and hence $\mathbf{v} \in \mathcal{V}^B$. Moreover, $B \in \mathcal{K}^A$ and hence by Theorem 8 we have $(B\mathbf{v})_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \mathrm{supp}(A\mathbf{v})$.

(3)⇒(2): By assumption $\mathcal{V}^A \subset \mathcal{V}^B$ and hence also $\mathcal{V}^A \subset \mathcal{V}^{A+\delta B}$ for all δ. Moreover, from Theorem 8 we have $B \in \mathcal{K}^A$ and hence there exists $\delta > 0$ such that $A + \delta B \in \mathcal{COP}^n$.

(3)⇒(4): By assumption we have $\mathcal{V}^A \subset \mathcal{V}^B$. Since $\mathcal{V}^A_{\min} \subset \mathcal{V}^A$, the inequality constraints in (4) hold. Consider an arbitrary $\mathbf{v} \in \mathcal{V}^A_{\min} \subset \mathcal{V}^B$ and $i \in \mathcal{J}(\mathbf{v}, A)$. By definition of $\mathcal{J}(\mathbf{v}, A)$ there exists $\mathbf{u} \in \mathcal{V}^A \subset \mathcal{V}^B$ such that $\{i\} \cup \mathrm{supp}(\mathbf{v}) \subset \mathrm{supp}(\mathbf{u})$. From Corollary 4 we have $\mathrm{supp}(\mathbf{u}) \subset \{1, \dots, n\} \setminus \mathrm{supp}(A\mathbf{v})$ and $\frac{1}{2}(\mathbf{u} + \mathbf{v}) \in \mathcal{V}^A \subset \mathcal{V}^B$. Hence by assumption $(B\mathbf{v})_j \geq 0$ for all $j \in \mathrm{supp}(\mathbf{u})$ and

$$0 = (\mathbf{u} + \mathbf{v})^T B(\mathbf{u} + \mathbf{v}) = 2\mathbf{u}^T B\mathbf{v} = 2 \sum_{j \in \mathrm{supp}(\mathbf{u})} \underbrace{u_j}_{>0} \underbrace{(B\mathbf{v})_j}_{\geq 0} \geq 0.$$

It follows that $(B\mathbf{v})_j = 0$ for all $j \in \mathrm{supp}(\mathbf{u})$, and in particular $(B\mathbf{v})_i = 0$, which yields (4).

(4)⇒(3): Consider an arbitrary $\mathbf{u} \in \mathcal{V}^A$. By Lemma 3, there exist $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathcal{V}^A_{\min}$ and $\theta_1, \dots, \theta_m > 0$ such that $\mathbf{u} = \sum_{j=1}^m \theta_j \mathbf{v}_j$. For all $j = 1, \dots, m$ we have $\mathrm{supp}(\mathbf{v}_j) \subset \mathrm{supp}(\mathbf{u})$ and hence $\mathrm{supp}(\mathbf{u}) \subset \mathcal{J}(\mathbf{v}_j, A)$. Moreover, we have $A\mathbf{u} = \sum_{j=1}^m \theta_j A\mathbf{v}_j$ and hence $\mathrm{supp}(A\mathbf{v}_j) \subset \mathrm{supp}(A\mathbf{u})$ by Lemma 1. Therefore by assumption for all $j = 1, \dots, m$ we have $(B\mathbf{v}_j)_i = 0$ for all $i \in \mathrm{supp}(\mathbf{u})$ and $(B\mathbf{v}_j)_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \mathrm{supp}(A\mathbf{u})$. This implies that $(B\mathbf{u})_i = \sum_{j=1}^m \theta_j (B\mathbf{v}_j)_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \mathrm{supp}(A\mathbf{u})$ and

$$\mathbf{u}^T B\mathbf{u} = \sum_{i \in \mathrm{supp}(\mathbf{u})} u_i (B\mathbf{u})_i = \sum_{i \in \mathrm{supp}(\mathbf{u})} \sum_{j=1}^m \theta_j u_i (B\mathbf{v}_j)_i = 0. \qquad \square$$

Example 21. Consider $A = \mathbf{e}_1\mathbf{e}_1^T$. We have

$$\begin{aligned} \mathcal{V}^A &= \operatorname{conv}\{\mathbf{e}_j \mid j \in \{2,\dots,n\}\}, \\ \mathcal{V}^A_{\min} &= \{\mathbf{e}_j \mid j \in \{2,\dots,n\}\}, \\ \operatorname{supp}(A\mathbf{e}_j) &= \emptyset \qquad\qquad \text{for all } j \in \{2,\dots,n\}, \\ \mathcal{J}(\mathbf{e}_j, A) &= \{2,\dots,n\} \qquad \text{for all } j \in \{2,\dots,n\}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}^A &= \{B \in \mathcal{S}^n \mid (B)_{ij} = 0 \text{ for all } (i,j) \in \{1,\dots,n\} \times \{2,\dots,n\}\} = \operatorname{span}(\{A\}), \\ \widehat{\mathcal{L}^A} &= \left\{ B \in \mathcal{S}^n \;\middle|\; \begin{array}{l} (B)_{ij} = 0 \text{ for all } (i,j) \in \{2,\dots,n\} \times \{2,\dots,n\}, \\ (B)_{1j} \geq 0 \text{ for all } j \in \{2,\dots,n\} \end{array} \right\} \end{aligned}$$

We thus have $\dim(\mathcal{L}^A) = 1$ and $\dim(\widehat{\mathcal{L}^A}) = n$. This implies that A gives an extreme but not exposed ray of the copositive cone.

This example also shows that the index set $\mathcal{J}(\mathbf{v}, A)$ in Theorem 20 cannot be replaced by $\{1,\dots,n\} \setminus \operatorname{supp}(A\mathbf{v})$.

6 Irreducibility

In this subsection we describe irreducibility of a copositive matrix A with respect to another copositive matrix C. This allows us to recover the results on irreducibility from [8] and [12] as special cases.

Theorem 22. *For $A, C \in \mathcal{COP}^n$ the following are equivalent:*

1 A is not irreducible with respect to C,

2 For all $\mathbf{v} \in \mathcal{V}^A$ we have $\operatorname{supp}(C\mathbf{v}) \subset \operatorname{supp}(A\mathbf{v})$ (and thus $\mathbf{v} \in \mathcal{V}^C$),

3 For all $\mathbf{v} \in \mathcal{V}^A_{\min}$ we have $\operatorname{supp}(C\mathbf{v}) \subset \operatorname{supp}(A\mathbf{v})$ (and thus $\mathbf{v} \in \mathcal{V}^C$).

Proof. Let (1) hold, then there exists $\delta > 0$ such that $A - \delta C \in \mathcal{COP}^n$. For every $\mathbf{v} \in \mathcal{V}^A$ we have $0 \leq \mathbf{v}^T(A - \delta C)\mathbf{v} = -\delta \mathbf{v}^T C \mathbf{v} \leq 0$. Hence $\mathbf{v} \in \mathcal{V}^C$ and $\mathbf{v} \in \mathcal{V}^{A-\delta C}$. It follows that (2) of Theorem 20 holds with $B = -C$. On the other hand, (2) of Theorem 20 with $B = -C$ implies (1).

Theorem 23 now follows directly from Theorem 20 with $B = -C$, by noting that if $C \in \mathcal{COP}^n$ and $\mathbf{v} \in \mathcal{V}^C$ then $(C\mathbf{v}) \geq \mathbf{0}$, and hence that $(B\mathbf{v})_i \geq 0$ implies $(B\mathbf{v})_i = 0$. □

Alternatively we could have stated this theorem as follows:

Theorem 23. *For $A, C \in \mathcal{COP}^n$ the following are equivalent:*

1 A is irreducible with respect to C,

2 There exists $\mathbf{v} \in \mathcal{V}^A$, $i \in \{1, \dots, n\}$ such that $(A\mathbf{v})_i = 0 \neq (C\mathbf{v})_i$.

3 There exists $\mathbf{v} \in \mathcal{V}^A_{\min}$, $i \in \{1, \dots, n\}$ such that $(A\mathbf{v})_i = 0 \neq (C\mathbf{v})_i$.

We will now recover the results from [8] and [12].

Corollary 24 ([8, Theorem 2.6]). *Let $A \in \mathcal{COP}^n$ and $C = \mathbf{e}_k\mathbf{e}_l^T + \mathbf{e}_l\mathbf{e}_k^T$ where $k, l \in \{1, \dots, n\}$. Then A is irreducible with respect to C if and only if there exists $\mathbf{v} \in \mathcal{V}^A$ such that $(A\mathbf{v})_k = (A\mathbf{v})_l = 0 < v_k + v_l$.*

Proof. By Theorem 23, A is irreducible with respect to C if and only if there exists $\mathbf{v} \in \mathcal{V}^A$ and $i \in \{1, \dots, n\}$ such that $(A\mathbf{v})_i = 0 \neq (C\mathbf{v})_i = \delta_{ik}v_l + \delta_{il}v_k$. We shall now show that this is equivalent to the condition in the corollary.

To show the reverse implication we suppose that $\mathbf{v} \in \mathcal{V}^A$ such that $(A\mathbf{v})_k = (A\mathbf{v})_l = 0 < v_k + v_l$. Without loss of generality we have $v_l > 0$, and taking $i = k$ we get $(A\mathbf{v})_i = 0 < v_l + \delta_{il}v_k = (C\mathbf{v})_i$.

To prove the forward implication we suppose that $\mathbf{v} \in \mathcal{V}^A$ such that $(A\mathbf{v})_i = 0 \neq \delta_{ik}v_l + \delta_{il}v_k$. Without loss of generality we have $i = k$, and thus $(A\mathbf{v})_k = 0 \neq (1 + \delta_{kl})v_l$. Therefore $v_l > 0$ and thus $(A\mathbf{v})_l = 0 < v_k + v_l$. □

From this we get that A is irreducible with respect to $\mathcal{N}^n$ if and only if for all $k, l \in \{1, \dots, n\}$ there exists $\mathbf{v} \in \mathcal{V}^A$ such that $(A\mathbf{v})_k = (A\mathbf{v})_l = 0 < v_k + v_l$.

Corollary 24 still holds if we replace $\mathcal{V}^A$ by $\mathcal{V}^A_{\min}$ [12, Corollary 4.2]. The short proof of Corollary 24 provided above remains valid for this strengthened version by replacing $\mathcal{V}^A$ by $\mathcal{V}^A_{\min}$.

Corollary 25 ([12, Corollary 4.4]). *Let $A \in \mathcal{COP}^n$, $\mathbf{c} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, and $C = \mathbf{c}\mathbf{c}^T$. Then A is irreducible with respect to C if and only if there exists $\mathbf{v} \in \mathcal{V}^A_{\min}$ such that $\mathbf{v}^T\mathbf{c} \neq 0$.*

Proof. Suppose that A is irreducible with respect to C. Then from Theorem 23, there exist $\mathbf{v} \in \mathcal{V}^A_{\min}$, $i \in \{1, \dots, n\}$ such that $(A\mathbf{v})_i = 0 \neq (C\mathbf{v})_i = c_i\mathbf{c}^T\mathbf{v}$, and thus $\mathbf{v}^T\mathbf{c} \neq 0$.

Now suppose that there exists $\mathbf{v} \in \mathcal{V}^A_{\min}$ such that $\mathbf{v}^T\mathbf{c} \neq 0$. Then there exists $i \in \operatorname{supp}(\mathbf{v}) \cap \operatorname{supp}(\mathbf{c})$ and by Lemma 2 we have $(A\mathbf{v})_i = 0 \neq c_i\mathbf{c}^T\mathbf{v} = (C\mathbf{v})_i$. Therefore, by Theorem 23, A is irreducible with respect to C. □

From this we get that A is irreducible with respect $\mathcal{S}^n_+$ if and only if $\operatorname{span}\left(\mathcal{V}^A_{\min}\right) = \mathbb{R}^n$.

7 Conclusions

In this paper we have given necessary and sufficient conditions on a pair $(A, B) \in \mathcal{COP}^n \times \mathcal{S}^n$ for the existence of $\delta > 0$ such that $A + \delta B$ is copositive. For fixed $A \in \mathcal{COP}^n$, the set of matrices B satisfying this condition forms a convex cone $\mathcal{K}^A$. We have described this cone by a set of linear inequalities constructed from the set of zeros of A. This description allowed us to compute the linear span of the minimal face of A in $\mathcal{COP}^n$. In particular, we devised a simple test for the extremality of A. The result can also be applied for checking irreducibility of A with respect to an arbitrary matrix $C \in \mathcal{COP}^n$. This result covers previous results from [8] and [12] on irreducibility as special cases.

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