Weierstraß-Institut

für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Existence of solutions to a two–dimensional model for nonisothermal two–phase flows of incompressible fluids

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submitted: June 6, 2014

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No. 1966

Berlin 2014

2010 *Mathematics Subject Classification.* 35Q35, 35K25, 76D05, 35D30.

Key words and phrases. Cahn-Hilliard, Navier-Stokes, incompressible non-isothermal binary fluid, global-in-time existence, a-priori estimates.

The research of E.R. has been supported by the FP7-IDEAS-ERC-StG Grant #256872 (EntroPhase). The work of G.S. and M.E. was partially supported by the FP7-IDEAS-ERC-StG #256872 (EntroPhase). The work of M.E., E.R. and G.S. was partially supported by GNAMPA (Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica).

Edited by
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Abstract

We consider a thermodynamically consistent diffuse interface model describing two-phase flows of incompressible fluids in a non-isothermal setting. The model was recently introduced in [12] where existence of weak solutions was proved in three space dimensions. Here, we aim at studying the properties of solutions in the two-dimensional case. In particular, we can show existence of global in time solutions satisfying a stronger formulation of the model with respect to the one considered in [12]. Moreover, we can admit slightly more general conditions on some material coefficients of the system.

1 Introduction

We consider here a mathematical model for two-phase flows of non-isothermal incompressible fluids in a bounded container Ω in $\mathbb{R}^2$ during a finite time interval $(0,T)$. The model consists in a PDE system describing the evolution of the unknown variables $\boldsymbol{u}$ (macroscopic velocity), φ (order parameter), μ (chemical potential), ϑ (absolute temperature), and it takes the form

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.1}$$

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \Delta \boldsymbol{u} - \operatorname{div}(\nabla \varphi \otimes \nabla \varphi), \tag{1.2}$$

$$\varphi_t + \boldsymbol{u} \cdot \nabla \varphi = \Delta \mu, \tag{1.3}$$

$$\mu = -\Delta \varphi + F'(\varphi) - \vartheta, \tag{1.4}$$

$$\vartheta_t + \boldsymbol{u} \cdot \nabla \vartheta + \vartheta\big(\varphi_t + \boldsymbol{u} \cdot \nabla \varphi\big) - \operatorname{div}(\kappa(\vartheta) \nabla \vartheta) = |\nabla \boldsymbol{u}|^2 + |\nabla \mu|^2. \tag{1.5}$$

Relation (1.2), with the incompressibility constraint (1.1), represents a variant of the Navier-Stokes system; (1.3)-(1.4) correspond to a form of the Cahn-Hilliard system [11] for phase separation, while (1.5) is the internal energy equation describing the evolution of temperature. Note that transport effects are admitted for all variables in view of the occurrence of material derivatives in (1.2), (1.3), and (1.5). As usual, the variable p in the Navier-Stokes system (1.2) represents the (unknown) pressure. The function F whose derivative appears in (1.4) is a possibly non-convex potential whose minima represent the least energy configurations of the phase variable. Here we will assume that F is smooth and has at most a power-like growth at infinity. Indeed, it is not clear whether our result could be extended to other classes of physically significant potential, having nonsmooth or singular character (like, e.g., the so-called logarithmic potential $F(r) = (1+r)\log(1+r) + (1-r)\log(1-r) - r^2$ which typically appears in Cahn-Hilliard-based models, see, e.g., [31]). Finally, the coefficient $\kappa(\vartheta)$ in (1.5) stands for the heat conductivity of the fluid. Here we shall assume that κ grows at infinity like a sufficiently high power of ϑ (see (2.11) below).

The PDE system (1.1)-(1.4) in the case of a constant temperature ϑ, refereed in the literature as Model H, is a diffuse interface model for incompressible isothermal two-phase flows which consists of the Navier-Stokes equations for the (averaged) velocity $\boldsymbol{u}$ nonlinearly coupled with a convective Cahn-Hilliard equation for the phase variable φ (cf., for instance, [3, 19, 20, 21, 22, 23, 26] and the references [1, 2, 4, 5, 16, 17, 18, 28, 36, 42, 43] for the study of the resulting evolution system). However,

even if many authors considered the isothermal Model H, up to our knowledge no contributions are so far present in the literature in the non-isothermal case, except for [37] where a linearization of (1.5) is considered and our previous paper [12] where we introduced system (1.1)-(1.5) in the 3D case.

The above model was indeed proposed in our recent work [12] starting from the balance laws for internal energy and entropy; in particular, thermodynamical consistence was shown to hold for any (positive) value of the absolute temperature ϑ. Moreover, existence of solutions for a *weak formulation* of (1.1)-(1.5) was proved when the system is settled in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ and complemented with no-flux conditions for φ, μ and ϑ and with *slip* conditions for $\boldsymbol{u}$. Mathematically speaking, the main source of difficulty in system (1.1)-(1.5) comes from the quadratic terms on the right hand side of (1.5). Their occurrence is physically motivated as one considers the derivation of the model in terms of the energy and entropy balances (cf. [12, Sec. 2]). Roughly speaking, one can say that these terms represent a source of thermal energy coming from the dissipation of kinetic energy due to viscosity (cf. (1.2)) and of configuration energy due to action of micro-forces (cf. (1.3)-(1.4)). This energy dissipation, as expected, happens in such a way to increase the entropy of the system.

From the analytical viewpoint, the quadratic terms in (1.5) can be controlled only in the L^1-norm, at least in the 3D-case. For this reason, proving existence for the formulation (1.1)-(1.5) appears to be out of reach. Actually, the notion of weak solution considered in [12] is based on a suitable reformulation of the model along the lines of an idea originally developed in [8, 13] for dealing with heat conduction in fluids, in [14] for solid-liquid phase transitions, and more recently in [34] for damage phenomena. In such a setting, the "heat" equation (1.5) is replaced with a relation describing the balance of *total energy* (i.e., not only of thermal energy), which does no longer contain quadratic terms. This is complemented with a distributional version of the *entropy inequality*. It is worth observing that the weak formulation considered in [12] is consistent with the standard (strong) one (1.1)-(1.5). Indeed, it is not difficult to prove that, at least for *sufficiently smooth* weak solutions, the total energy balance together with the entropy inequality imply the original form of the heat (or, more precisely, internal energy balance) equation (1.5). However, as noted above, the required regularity in the 3D case is not at all known.

Looking at the 2D model, whose analysis is the aim of this paper, it is well-known that, for the Navier-Stokes system (1.2), additional regularity is available provided that the forcing term (here given by $-\operatorname{div}(\nabla\varphi \otimes \nabla\varphi)$) lies in L^2 (cf., e.g., [33]). Fortunately, this seems to happen in our case, as one can readily check starting from the available energy and entropy estimates; hence, there is hope to get additional summability for the quadratic term $|\nabla \boldsymbol{u}|^2$ in (1.5). This was the motivation which led us to investigate whether it is possible to prove existence of a solution to the *original* (strong) system (1.1)-(1.5) in two space dimensions. Indeed, we may give a positive answer to this question, but the argument we use for arriving at this conclusion is far from being a straighforward one. So, let us try to give some ideas of the mathematical difficulties we met.

To make things clear, we start by introducing some basic assumptions. First of all, in order to avoid technical complications related with the choice of boundary data, we ask the system to be settled in the unit torus $\Omega := [0,1] \times [0,1]$ and complemented with periodic boundary conditions for all unknowns. It is worth noting that, at the price of some notational change and of limited technical complications, other types of boundary conditions could be assumed. For instance, we may take no-flux (i.e., homogeneous Neumann) conditions for φ, μ and ϑ (as it is physically reasonable if one assumes the container Ω to be insulated from the exterior), whereas for $\boldsymbol{u}$ we may consider any conditions that could allow the transport terms to have zero spatial mean and are compatible with the existence of smooth solutions to the 2D Navier-Stokes system (cf. estimate (3.29) below). This is the case, for instance, of homogeneous Dirichlet conditions (cf., e.g., [38, Thm. 3.10, p. 314]). The

evolution is assumed to take place on a given reference interval $(0, T)$, with no restrictions on the magnitude of the final time $T > 0$.

Coming to our mathematical argument, once additional regularity for $\boldsymbol{u}$ has been obtained, we need to get further bounds for the other variables, with the aim of proving an estimate for the remaining nonlinear terms in the system (and, particularly, for the quadratic term $|\nabla\mu|^2$ on the right hand side of (1.5)). Actually, due to the strong coupling between the energy balance equation (1.5) and the Cahn-Hilliard system (1.3)-(1.4), getting a regularity estimate for the variables φ, μ and ϑ requires to manage all equations simultaneously in a non-straighforward way. This further regularity estimate represents, in our view, the main novelty of the present paper.

Referring to Section 3 for more details, we just give here some brief explanation of the procedure. The first thing one can naturally do is to differentiate in time the Cahn-Hilliard relation (1.4), and to test the result by φ_t. However, this trick works only if one is able to control the product $\vartheta_t\varphi_t$. In view of the highly nonlinear structure of (1.5), getting a bound for ϑ_t by working directly on the "heat" equation seems difficult. Hence, the only possibility seems that of testing (1.5) by φ_t in order to cancel the bad term. However, we then need to control the quadratic terms $|\nabla\boldsymbol{u}|^2$ and $|\nabla\mu|^2$ on the right hand side, and particularly the latter one, for which only an L^1-estimate is available at this level. Actually, in order to provide a bound for $|\nabla\mu|^2$, we have to use some duality technique, and, more precisely, we need to rely on a sharp two-dimensional interpolation-embedding inequality which is proved in the Appendix. This property follows from well-known two-dimensional embedding theorems; however we were not able to find it anywhere in the literature. The underlying idea stands in optimizing with respect to q the embedding constant of the immersion $\|v\|_{H^1(\Omega)'} \leq c_q\|v\|_{L^q(\Omega)}$ which holds true in 2D for every $q \in (1, +\infty]$ and $v \in L^q(\Omega)$ (cf. [39, (17), p. 479]). Then, applying the embedding inequality to the term $|\nabla\mu|^2$, and performing a notable amount of technical work, we can actually prove the desired enhanced a-priori bounds. These estimates permit us to pass to the limit in a suitable approximation scheme (which is just sketched, for brevity), obtaining in this way a solution to the original (strong) system (1.1)-(1.5), in a proper regularity class, coupled with periodic boundary conditions and with the initial conditions.

It is finally worth noting that, while in [12] we needed to assume non-constant specific heat and heat conductivity, both having a suitable growth at 0 and at ∞ (the reasons were mainly of mathematical type; namely, we needed to get a sufficient summability for ϑ), here we just need a sufficiently fast growing heat conductivity, but we can allow for a constant specific heat. Even if this choice is mainly motivated by mathematical reasons, a physical justification for it can be found, e.g., in [41].

The arguments given in this paper may also be adapted to deal with other interesting related models. For instance, we could consider the case when the "Cahn-Hilliard" relations (1.3)-(1.4) are replaced by their "Allen-Cahn" equivalent (cf. [10])

$$\varphi_t + \boldsymbol{u} \cdot \nabla\varphi - \Delta\varphi + F'(\varphi) - \vartheta = 0. \tag{1.6}$$

Moreover, as a byproduct of our results, one can deduce the existence of solutions in 2D for the so-called Frémond's model of phase transitions with microscopic effects introduced in [15], at least in the case of power-like heat conductivity. The Frémond model basically corresponds to system (1.1)-(1.5) where the velocity $\boldsymbol{u}$ is assumed to be identically equal to 0. Indeed, for this model, in the case of Neumann boundary conditions and standard Fourier heat flux law, existence of global in time "strong" solutions was known only in the one-dimensional setting (cf. [29, 30]), while *weak solutions* were proved to exist in 3D (cf. [14]) when (1.5) is replaced by the *total energy balance* and an *entropy inequality*. Hence, this paper covers the missing 2D case, at least for the case of power-like heat conductivity and periodic boundary conditions.

Let us finally note that uniqueness of solutions, as well as their long-time behavior (both in terms of trajectories and of attractors) for the whole system (1.1)-(1.5), are still open issues, which will be the subject of further investigations.

Here is the plan of the paper: in the next Section 2 we specify our assumptions on coefficients and data and state the precise mathematical formulation of our problem together with the related existence theorem. The remainder of the paper is devoted to the proof of the theorem. In particular, the core of our argument is given in Section 3, where we provide the a-priori estimates and the compactness argument necessary to pass to the limit in the approximation scheme. Indeed, in order to avoid technicalities, the estimates are obtained in a formal way leaving the details of a possible regularization in the subsequent Section 4. Finally, the Appendix contains the proof of the mentioned two-dimensional interpolation-embedding inequality, which plays a key role in the derivation of the a-priori bounds.

2 Assumptions and main results

In order to state the precise mathematical formulation of our problem we first need to introduce some functional spaces. Recalling that $\Omega = [0,1] \times [0,1]$, we note as $H := L^2_{\rm per}(\Omega)$ the space of functions in $L^2(\mathbb{R}^2)$ which are Ω-periodic (i.e., 1-periodic both in x_1 and in x_2). Analogously, we set $V := H^1_{\rm per}(\Omega)$. The spaces H and V are endowed with the norms of $L^2(\Omega)$ and $H^1(\Omega)$, respectively. For brevity, the norm in H will be simply indicated by $\| \cdot \|$. We will note by $\| \cdot \|_X$ the norm in the generic Banach space X. The symbol $\langle \cdot, \cdot \rangle$ will indicate the duality between V' and V and $(\cdot, \cdot)$ will stand for the scalar product of H. We also write $L^p(\Omega)$ in place of $L^p_{\rm per}(\Omega)$, and the same for other spaces; indeed, no confusion should arise since periodic boundary conditions are assumed to hold for all unknowns. Still for brevity, we use the same notation for indicating vector-valued (or tensor-valued) function spaces and related norms. For instance, writing $\boldsymbol{u} \in H$, we will in fact mean $\boldsymbol{u} \in L^2_{\rm per}(\Omega)^2$. Also the incompressibility constraint (1.1) will not be emphasized in the notation for functional spaces (hence, the notation $\boldsymbol{u} \in H$ will also implicitly subsume that $\operatorname{div} \boldsymbol{u} = 0$ in the sense of distributions). These simplifications will allow us to shorten a bit some formulas.

For any function $v \in H$, we will note as

$$v_\Omega := \frac{1}{|\Omega|} \int_\Omega v = \int_\Omega v \tag{2.1}$$

the spatial mean of v. Replacing the integral with a duality pairing, the same notation will be used in case $v \in V'$. The symbols V_0, H_0 and V_0' denote the subspaces of V, H and, respectively, V' containing the function(al)s having zero spatial mean. We notice that the distributional operator $(-\Delta)$ is invertible if seen as a mapping from V_0 to V_0'. In the sequel we shall denote by $\mathcal{N}$ its inverse operator.

Moreover, in the following we will frequently use the following 2D interpolation inequalities:

$$\|v\|_{L^4(\Omega)} \le c \|v\|_V^{1/2} \|v\|^{1/2}, \tag{2.2}$$

$$\|v\|_{L^\infty(\Omega)} \le c \|v\|_{H^2(\Omega)}^{1/2} \|v\|^{1/2}, \tag{2.3}$$

$$\|v\|_{L^r(\Omega)} \le c \|v\|_{L^s(\Omega)}^{1-\alpha} \|v\|_{L^\infty(\Omega)}^{\alpha}, \qquad \alpha = 1 - \frac{s}{r}, \tag{2.4}$$

holding for any sufficiently smooth function v and for suitable embedding constants, all denoted by the same symbol $c > 0$ for brevity.

We will also use the following nonlinear version of the Poincaré inequality

$$\|v^{p/2}\|_V^2 \le c_p\big(\|v\|_{L^1(\Omega)}^p + \|\nabla v^{p/2}\|^2\big), \tag{2.5}$$

holding for all nonnegative $v \in L^1(\Omega)$ such that $\nabla v^{p/2} \in L^2(\Omega)$, and for all $p \in [2,\infty)$. We also note that

$$\|v\| \le c\|\nabla v\|^{1/2}\|v\|_{V'}^{1/2} \quad \text{for all } v \in V_0. \tag{2.6}$$

This property can be proved by combining the standard interpolation inequality $\|v\| \le c\|v\|_V^{1/2}\|v\|_{V'}^{1/2}$ with the Poincaré-Wirtinger inequality.

In the sequel we will frequently use the continuous embedding $V \subset L^p(\Omega)$, holding for all $p \in [1,\infty)$. Actually, using interpolation of L^p-spaces and Young's inequality, it is not difficult to see that (2.6) implies

$$\|v\|_{L^p(\Omega)}^2 \le \epsilon\|\nabla v\|^2 + c_\epsilon\|v\|_{V'}^2 \quad \text{for all } v \in V_0, \tag{2.7}$$

for all (small) $\epsilon > 0$ and correspondingly large $c_\epsilon > 0$ whose value additionally depends on $p \in [1,\infty)$.

With the above notation at disposal, we can present our main assumptions. First of all, we ask the configuration potential F to satisfy:

$$F \in C^2(\mathbb{R};\mathbb{R}), \quad \liminf_{|r|\to\infty} \frac{F(r)}{|r|} > 0, \tag{2.8}$$

$$F''(r) \ge -\lambda \quad \text{for some } \lambda \ge 0 \quad \text{and all } r \in \mathbb{R}, \tag{2.9}$$

$$|F''(r)| \le c_F\big(1+|r|^{p_F}\big) \quad \text{for some } c_F \ge 0,\ p_F \ge 0, \quad \text{and all } r \in \mathbb{R}. \tag{2.10}$$

In other words, we ask for F to be a smooth, coercive (in view of (2.8)), λ-convex (cf. (2.9)) function, with at most polynomial growth at infinity (cf. (2.10)). These conditions may probably be relaxed (admitting, for instance, functions with exponential growth at infinity), at the price, however, of technical complications. On the other hand, it is not clear whether it would be possible to admit *singular potentials* like the logarithmic function mentioned in the Introduction.

Next, we assume the heat conductivity to be given by

$$\kappa(r) = 1 + r^q, \quad q \in [2,\infty), \quad r \ge 0. \tag{2.11}$$

Correspondingly, we define

$$K(r) := \int_0^r \kappa(s)\,\mathrm{d}s = r + \frac{1}{q+1}r^{q+1}, \quad r \ge 0. \tag{2.12}$$

In the sequel we will often need to estimate the value $\|K(\vartheta)\|_V^2$. To this aim, we first observe that, for some $k_q > 0$,

$$\int_\Omega \kappa(\vartheta)^2|\nabla\vartheta|^2 = \|\nabla K(\vartheta)\|^2 \ge \|\nabla\vartheta\|^2 + k_q\|\nabla\vartheta^{q+1}\|^2. \tag{2.13}$$

Then, exploiting (2.5) with the choice $p = 2$ we obtain

$$\|K(\vartheta)\|_V^2 \le c_q\left(\int_\Omega(\vartheta+\vartheta^{q+1})\right)^2 + c_q\left(\int_\Omega\left|\nabla\vartheta + \nabla\left(\frac{\vartheta^{q+1}}{q+1}\right)\right|^2\right) =: I + II \tag{2.14}$$

for some $c_q > 0$. Now, using again (2.5), this time with the choice $p = 2(q+1)$, we deduce

$$I \le c_q\|\vartheta\|_{L^1(\Omega)}^2 + c_q\left(\|\vartheta\|_{L^1(\Omega)}^{2(q+1)} + \|\nabla\vartheta^{q+1}\|^2\right) \le c_q\left(1 + \|\vartheta\|_{L^1(\Omega)}^{2(q+1)} + \|\nabla\vartheta^{q+1}\|^2\right). \tag{2.15}$$

Estimating II with the help of (2.13) we then conclude

$$\|K(\vartheta)\|_V^2 \le c_q \left(1 + \|\vartheta\|_{L^1(\Omega)}^{2(q+1)} + \int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2\right). \tag{2.16}$$

Finally we come to our assumptions on the initial data:

$$\boldsymbol{u}_0 \in V, \quad \operatorname{div} \boldsymbol{u}_0 = 0, \tag{2.17}$$
$$\varphi_0 \in H^3_{\rm per}(\Omega), \tag{2.18}$$
$$\vartheta_0 \in V, \quad \vartheta_0 \ge \underline{\vartheta} > 0 \text{ a.e. in } \Omega, \tag{2.19}$$

where $\underline{\vartheta}$ is some positive constant (actually the last condition in (2.19) could be relaxed by asking $\vartheta_0 > 0$ almost everywhere with $\log\vartheta \in L^1(\Omega)$).

With the above machinery at disposal, we can conclude this section by stating our main existence theorem, whose proof will occupy the remainder of the paper:

Theorem 2.1. *Let us assume* (2.8)-(2.10)*,* (2.11)*, and* (2.17)-(2.19)*. Let also* $T > 0$*. Then, there exists at least one* strong solution *to the non-isothermal model for two-phase fluid flows, namely, one quadruple* $(\boldsymbol{u}, \varphi, \mu, \vartheta)$ *with*

$$\boldsymbol{u} \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{2.20}$$
$$\varphi \in W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{2.21}$$
$$\mu \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{2.22}$$
$$\vartheta \in H^1(0,T;V') \cap L^\infty(0,T;L^{q+2}(\Omega)) \cap L^2(0,T;V), \quad \vartheta > 0 \quad \text{a.e. in } (0,T)\times\Omega, \tag{2.23}$$
$$K(\vartheta) \in L^2(0,T;V), \tag{2.24}$$

such that the equations of the system (1.1)-(1.4) *hold in the sense of distributions as well as almost everywhere in* $(0,T)\times\Omega$*, while* (1.5) *holds, for a.e.* $t \in (0,T)$*, as the following relation in* V'*:*

$$\vartheta_t + \boldsymbol{u}\cdot\nabla\vartheta + \vartheta\big(\varphi_t + \boldsymbol{u}\cdot\nabla\varphi\big) - \Delta K(\vartheta) = |\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2, \tag{2.25}$$

where Δ *is a weak form of the Laplace operator with periodic boundary conditions. Moreover, the quadruple* $(\boldsymbol{u}, \varphi, \mu, \vartheta)$ *complies with the initial condition*

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \qquad \varphi|_{t=0} = \varphi_0, \qquad \vartheta|_{t=0} = \vartheta_0, \tag{2.26}$$

almost everywhere in Ω*.*

3 Global existence

We start by deriving the a-priori estimates leading to existence of weak solutions. As noted in the Introduction, we shall work directly, though formally, on the original system (1.1)-(1.5) without referring to any approximation or regularization. Indeed, this permits us to make the argument more readable and to avoid technical complications. The details of a possible regularization scheme are posponed to Section 4 below. In the following, the letter c will denote a generic positive constant depending only on the data of the problem, whose value is allowed to vary on occurrence. In particular, c is intended to be independent of all regularization parameters.

Energy estimate. This basic property corresponds, in the physical derivation of the model, to the energy conservation principle. To deduce it from the equations, we test (1.2) by $\boldsymbol{u}$, (1.3) by μ, (1.4) by $-\varphi_t$, (1.5) by 1, integrate over Ω, and sum all the obtained relations together. Then, using the fact that

$$(\boldsymbol{u} \cdot \nabla \varphi)\mu = (\boldsymbol{u} \cdot \nabla \varphi)(-\Delta \varphi + F'(\varphi) - \vartheta) \tag{3.1}$$

thanks to (1.4), and performing standard integration by parts (cf. [12, Sec. 2] for more details), it is not difficult to arrive at the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\boldsymbol{u}, \varphi, \vartheta) = 0, \qquad \text{where } \mathcal{E}(\boldsymbol{u}, \varphi, \vartheta) := \int_\Omega \left(\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla \varphi|^2 + F(\varphi) + \vartheta \right) \tag{3.2}$$

is the *total energy* of the system, given by the sum of the *kinetic*, *interfacial*, *configuration*, and *thermal* energies (the four summands in $\mathcal{E}$).

Relation (3.2), in turn, yields the following *a priori* estimates:

$$\|\boldsymbol{u}\|_{L^\infty(0,T;H)} \le c, \tag{3.3}$$

$$\|\varphi\|_{L^\infty(0,T;V)} \le c, \tag{3.4}$$

$$\|\vartheta\|_{L^\infty(0,T;L^1(\Omega))} \le c, \tag{3.5}$$

where the control of the full V-norm of φ (and not only of the L^2-norm of the gradient) is reached thanks to the superlinear growth of F at infinity (cf. (2.8)). Note that, for getting (3.5) from (3.2), the nonnegativity of ϑ is exploited (which holds as a consequence of the approximation scheme, cf. Lemma 4.3 below). We also observe that, thanks to (3.4) and Sobolev's embeddings, there follows

$$\|\varphi\|_{L^\infty(0,T;L^p(\Omega))} \le c_p \quad \text{for all } p \in [1, \infty). \tag{3.6}$$

Conservation properties. Integrating (1.2) and (1.3) over Ω, and using (1.1) together with the periodic boundary conditions, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \boldsymbol{u} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi = 0 \quad \text{a.e. in } (0,T). \tag{3.7}$$

In other words, the spatial mean values of the velocity and of the phase variable are constant in time. This basically corresponds to the physical principles of conservation of momentum and of mass. Of course, (3.7) can be equivalently rewritten as

$$(\boldsymbol{u}_t(t))_\Omega = (\varphi_t(t))_\Omega = 0 \quad \text{for a.e. } t \in (0,T). \tag{3.8}$$

Entropy estimate. The following estimate corresponds to the entropy production principle. It is simply obtained by testing (1.5) by $-\vartheta^{-1}$ and integrating over Ω. As before, in order for the procedure to be rigorous, we need that ϑ is strictly positive in the approximation. Recalling also (2.11), we then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (-\log \vartheta - \varphi) + \int_\Omega \frac{1}{\vartheta}\big(|\nabla \boldsymbol{u}|^2 + |\nabla \mu|^2\big) + \int_\Omega \big(|\nabla \log \vartheta|^2 + k_q |\nabla \vartheta^{q/2}|^2\big) = 0, \tag{3.9}$$

where $k_q > 0$ only depends on the exponent q (cf. (2.11)). Integrating in time and recalling (3.4)-(3.5), we get the a priori bounds

$$\|\log \vartheta\|_{L^\infty(0,T;L^1(\Omega))} + \|\log \vartheta\|_{L^2(0,T;V)} \le c, \tag{3.10}$$

$$\|\nabla \vartheta^{q/2}\|_{L^2(0,T;H)} \le c. \tag{3.11}$$

In particular, (3.10) entails that the strict positivity (almost everywhere in $(0,T)\times\Omega$) of ϑ is preserved also in the limit. Note also that, from (3.11), (3.5) and inequality (2.5), there follows

$$\|\vartheta^{q/2}\|_{L^2(0,T;V)} \leq c. \tag{3.12}$$

Now, (3.12) entails in particular

$$\|\vartheta\|_{L^2(0,T;H)} \leq c. \tag{3.13}$$

On the other hand, arguing in a similar way as in [12, Sec. 4.2], by the elementary inequality

$$1 \leq c\left(\frac{1}{x^2} + x^{q-2}\right) \quad \text{for all } x \geq 0, \tag{3.14}$$

holding for $q \geq 2$, we obtain from (3.10)-(3.11) the additional bound

$$\|\nabla\vartheta\|_{L^2(0,T;H)} \leq c. \tag{3.15}$$

Putting together (3.13) and (3.15), we deduce

$$\|\vartheta\|_{L^2(0,T;V)} \leq c. \tag{3.16}$$

Temperature estimate. While the bounds obtained above basically correspond to physical principles, in order to get weak stability of families of solutions we need to derive more refined a-priori estimates by properly managing the equations of the system. To this aim, we first integrate (1.5) over Ω with the purpose of obtaining some information from the quadratic terms on the right hand side. Using the periodic boundary conditions, we actually infer

$$\int_\Omega \left(|\nabla \boldsymbol{u}|^2 + |\nabla\mu|^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta + \int_\Omega \vartheta(\varphi_t + \boldsymbol{u}\cdot\nabla\varphi) \tag{3.17}$$

and we aim at controlling the terms on the right hand side. Actually, the first one, after integration in time, is estimated simply recalling (3.5). Using (1.3) and Hölder's and Young's inequalities, we control the second integral as follows:

$$\int_\Omega \vartheta(\varphi_t + \boldsymbol{u}\cdot\nabla\varphi) = \int_\Omega \vartheta\Delta\mu = -\int_\Omega \nabla\vartheta\cdot\nabla\mu \leq \frac{1}{2}\left(\|\nabla\mu\|^2 + \|\nabla\vartheta\|^2\right). \tag{3.18}$$

The first term on the right hand side is absorbed by the corresponding one on the left hand side of (3.17), while the latter is estimated thanks to (3.16). Hence, we get

$$\|\boldsymbol{u}\|_{L^2(0,T;V)} \leq c, \tag{3.19}$$

$$\|\nabla\mu\|_{L^2(0,T;H)} \leq c. \tag{3.20}$$

Now, integrating (1.4) in space, using (3.5), (3.6) and (2.10), and taking the (essential) supremum with respect to time, we readily infer

$$\|\mu_\Omega\|_{L^\infty(0,T)} \leq c. \tag{3.21}$$

This property, combined with (3.20), yields

$$\|\mu\|_{L^2(0,T;V)} \leq c. \tag{3.22}$$

The estimates proved up to this point basically correspond (up to the slightly different assumptions on coefficients and data) to the procedure used in [12] to get existence of at least one solution to a

suitable reformulation of the problem. In particular, we recall that, in the approach of [12], the "heat" equation (1.5) was restated in the form of a *total energy balance* complemented with an *energy production inequality*. Indeed, in that formulation the troublesome quadratic terms characterizing the right hand side of (1.5) do no longer appear.

It is worth noting that, in spite of the better embeddings we have at disposal, the above estimates do not seem sufficient to prove anything better in our 2D case, because, still, the right hand side of (1.5) is controlled only in L^1 (thanks to (3.19)-(3.20)). In view of these considerations, it is natural to investigate whether higher order a-priori estimates could hold in the 2D case. As we will see in a while, such estimates can, indeed, be obtained, but the argument is not at all straighforward. For clarity, the procedure will be split into several steps.

Remark 3.1. To be more precise, a further difference in our assumptions stands in the choice of a *linear* latent heat (namely, we have no coefficient in front of ϑ_t in (1.5)), while in [12] we required to have a power-like function multiplying ϑ_t, which yielded some additional summability of ϑ. Actually, even in 2D, it is not clear whether one could prove existence of weak solutions (i.e., solutions complying with the formulation given in [12]) in the case of a linear latent heat. Nevertheless, this assumption is allowed as one looks for ***strong*** solutions, as we are doing now.

Second estimate for φ. We test (1.4) by $\Delta^2\varphi$ and integrate over Ω. Recalling (2.10), we get

$$\begin{aligned}\|\nabla\Delta\varphi\|^2 &= \int_\Omega F''(\varphi)\nabla\varphi\cdot\nabla\Delta\varphi - \int_\Omega \nabla(\vartheta+\mu)\cdot\nabla\Delta\varphi\\ &\le c\|\nabla\Delta\varphi\|\big(\|\nabla\varphi\| + \|\varphi\|^{p_F}_{L^{2p_F}(\Omega)}\|\nabla\varphi\|_{L^\infty(\Omega)} + \|\nabla\vartheta\| + \|\nabla\mu\|\big)\\ &\le c\|\nabla\Delta\varphi\|\big(1 + \|\varphi\|_V^{1/2}\|\varphi\|^{1/2}_{H^3(\Omega)} + \|\nabla\vartheta\| + \|\nabla\mu\|\big)\\ &\le c\|\nabla\Delta\varphi\|\big(1 + \|\varphi\|_V + \|\varphi\|_V^{1/2}\|\nabla\Delta\varphi\|^{1/2} + \|\nabla\vartheta\| + \|\nabla\mu\|\big)\\ &\le \frac12\|\nabla\Delta\varphi\|^2 + c\big(1 + \|\nabla\vartheta\|^2 + \|\nabla\mu\|^2\big).\end{aligned} \tag{3.23}$$

Note that (2.3) (applied to $v=\nabla\varphi$) and estimates (3.4), (3.6) have been used. Integrating (3.23) in time and using (3.16) and (3.20), we then obtain

$$\|\varphi\|_{L^2(0,T;H^3(\Omega))} \le c. \tag{3.24}$$

This property has some notable consequences. Firstly, testing (1.3) by nonzero $v\in V$, recalling also (3.3), we can notice that

$$\begin{aligned}\langle\varphi_t, v\rangle &= -\int_\Omega \nabla\mu\cdot\nabla v - (\boldsymbol{u}\cdot\nabla\varphi, v) \le \|\nabla\mu\|\|\nabla v\| + \|\boldsymbol{u}\|\|\nabla\varphi\|_{L^\infty(\Omega)}\|v\|\\ &\le c\big(\|\nabla\mu\| + \|\varphi\|_{H^3(\Omega)}\big)\|v\|_V.\end{aligned} \tag{3.25}$$

Hence, dividing by $\|v\|_V$, passing to the supremum with respect to $v\in V\setminus\{0\}$, squaring, integrating in time, and using (3.22) and (3.24), we obtain

$$\|\varphi_t\|_{L^2(0,T;V')} \le c. \tag{3.26}$$

Moreover, (3.24) permits us to get a useful bound for the last term in (1.2). Indeed, using twice (2.2), we have

$$\begin{aligned}\big\|\operatorname{div}(\nabla\varphi\otimes\nabla\varphi)\big\| &\le c\|D^2\varphi\|_{L^4(\Omega)}\|\nabla\varphi\|_{L^4(\Omega)} \le c\|\varphi\|^{1/2}_{H^3(\Omega)}\|\varphi\|_{H^2(\Omega)}\|\varphi\|_V^{1/2}\\ &\le c\|\varphi\|_{H^3(\Omega)}\|\varphi\|_V \le c\|\varphi\|_{H^3(\Omega)},\end{aligned} \tag{3.27}$$

where the last inequality follows from (3.4). Then, squaring and integrating over $(0,T)$, thanks to (3.24) we infer

$$\big\| \operatorname{div}(\nabla\varphi \otimes \nabla\varphi)\big\|_{L^2(0,T;H)} \le c. \tag{3.28}$$

Second estimate for $\boldsymbol{u}$. Property (3.28) allows us to apply standard regularity results to the 2D Navier-Stokes system (1.1)-(1.2), (see for instance [33, Sec. 9.6]), basically corresponding to testing (1.2) by $-\Delta\boldsymbol{u}$. As a consequence, we infer

$$\|\boldsymbol{u}\|_{H^1(0,T;H)} + \|\boldsymbol{u}\|_{L^\infty(0,T;V)} + \|\boldsymbol{u}\|_{L^2(0,T;H^2(\Omega))} \le c. \tag{3.29}$$

Key estimate: φ. We start now with the key regularity estimate, which is obtained by combining in a suitable way equations (1.3), (1.4) and (1.5). At first, we deal with the Cahn-Hilliard system. Namely, we take (1.3), differentiate it with respect to time, and test the result by $\mathcal{N}\varphi_t$. Correspondingly, we differentiate (1.4) in time and test by $-\varphi_t$. Summing the obtained relations, noting that a couple of terms cancel in view of

$$(\Delta\mu_t, \mathcal{N}\varphi_t) = -\big((-\Delta)(\mu_t - (\mu_t)_\Omega), (-\Delta)^{-1}\varphi_t\big) = -\big(\mu_t - (\mu_t)_\Omega, \varphi_t\big) = -(\mu_t, \varphi_t), \tag{3.30}$$

where we have used also (3.8), we then get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|_{V'}^2 + \|\nabla\varphi_t\|^2 + \int_\Omega (F''(\varphi)+\lambda)|\varphi_t|^2 \\ &\quad = \lambda\|\varphi_t\|^2 - \langle \boldsymbol{u}_t\cdot\nabla\varphi, \mathcal{N}\varphi_t\rangle - \langle \boldsymbol{u}\cdot\nabla\varphi_t, \mathcal{N}\varphi_t\rangle + (\vartheta_t, \varphi_t).\end{aligned} \tag{3.31}$$

Thanks to (2.9), the last term on the left hand side is nonnegative. On the other hand, we need to control the right hand side. To this aim, we first notice that, by (2.6) and (3.8), we can estimate the first term as

$$\lambda\|\varphi_t\|^2 \le \frac{1}{8}\|\nabla\varphi_t\|^2 + c\|\varphi_t\|_{V'}^2. \tag{3.32}$$

Next, using (2.3) and standard embeddings, we infer

$$\begin{aligned}-\langle \boldsymbol{u}_t\cdot\nabla\varphi, \mathcal{N}\varphi_t\rangle &= (\!(\boldsymbol{u}_t\cdot\nabla\varphi, \varphi_t)\!)_{V_0'} \le \|\boldsymbol{u}_t\cdot\nabla\varphi\|_{V'}\|\varphi_t\|_{V'} \le c\|\boldsymbol{u}_t\cdot\nabla\varphi\|\|\varphi_t\|_{V'} \\ &\le c\|\boldsymbol{u}_t\|\|\nabla\varphi\|_{L^\infty(\Omega)}\|\varphi_t\|_{V'} \le c\|\boldsymbol{u}_t\|^2 + c\|\varphi\|_{H^3(\Omega)}^2\|\varphi_t\|_{V'}^2\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}-\langle \boldsymbol{u}\cdot\nabla\varphi_t, \mathcal{N}\varphi_t\rangle &= (\!(\boldsymbol{u}\cdot\nabla\varphi_t, \varphi_t)\!)_{V_0'} \le \|\boldsymbol{u}\cdot\nabla\varphi_t\|_{V'}\|\varphi_t\|_{V'} \le c\|\boldsymbol{u}\cdot\nabla\varphi_t\|\|\varphi_t\|_{V'} \\ &\le c\|\boldsymbol{u}\|_{L^\infty(\Omega)}\|\nabla\varphi_t\|\|\varphi_t\|_{V'} \le \frac{1}{8}\|\nabla\varphi_t\|^2 + c\|\boldsymbol{u}\|_{H^2(\Omega)}^2\|\varphi_t\|_{V'}^2.\end{aligned} \tag{3.34}$$

In the above formulas we noted by $(\!(\cdot,\cdot)\!)_{V_0'}$ the scalar product of V_0' and used the fact that $(-\Delta)$ corresponds to the Riesz operator from V_0 to V_0'.

Hence, on account of (3.32)-(3.34), relation (3.31) takes the form

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|_{V'}^2 + \frac{3}{4}\|\nabla\varphi_t\|^2 + \int_\Omega (F''(\varphi)+\lambda)|\varphi_t|^2 \le M_1(t)\|\varphi_t\|_{V'}^2 + M_2(t) + (\vartheta_t, \varphi_t), \tag{3.35}$$

where the functions

$$M_1(t) = c\big(1 + \|\varphi\|_{H^3(\Omega)}^2 + \|\boldsymbol{u}\|_{H^2(\Omega)}^2\big), \qquad M_2(t) = c\big(1 + \|\boldsymbol{u}_t\|^2\big) \tag{3.36}$$

lie (or, more precisely, are uniformly bounded w.r.t. all approximation parameters) in $L^1(0,T)$ thanks to (3.24) and (3.29). It remains to control the last term on the right hand side. This, however, requires to work with the energy equation (1.5), which is our next task.

Key estimate: ϑ. We start testing (1.5) by φ_t in order to compute the last term in (3.35). We get

$$(\vartheta_t,\varphi_t)+\int_\Omega \vartheta\varphi_t^2=\int_\Omega \vartheta\,\boldsymbol{u}\cdot\nabla\varphi_t-\int_\Omega \vartheta\,\varphi_t\,\boldsymbol{u}\cdot\nabla\varphi \\ -\int_\Omega \kappa(\vartheta)\nabla\vartheta\cdot\nabla\varphi_t+\int_\Omega \big(|\nabla\boldsymbol{u}|^2+|\nabla\mu|^2\big)\varphi_t, \tag{3.37}$$

where we have used (1.1). Let us provide an estimate for the terms on the right hand side. Firstly, we have

$$\int_\Omega \vartheta\,\boldsymbol{u}\cdot\nabla\varphi_t\le \|\vartheta\|_{L^4(\Omega)}\|\boldsymbol{u}\|_{L^4(\Omega)}\|\nabla\varphi_t\|\le\frac{1}{16}\|\nabla\varphi_t\|^2+c\|\vartheta\|_V^2, \tag{3.38}$$

where we have also used estimate (3.29).

Next, by (2.7) and Hölder's and Young's inequalities, thanks also to (3.4) and (3.29), we have

$$-\int_\Omega \vartheta\,\varphi_t\,\boldsymbol{u}\cdot\nabla\varphi\le\|\vartheta\|_{L^{\infty-}(\Omega)}\|\varphi_t\|_{L^{2+}(\Omega)}\|\boldsymbol{u}\|_{L^{\infty-}(\Omega)}\|\nabla\varphi\|\le c\|\vartheta\|_V\|\varphi_t\|_{L^{2+}(\Omega)} \\ \le c\|\vartheta\|_V^2+c\|\varphi_t\|_{L^{2+}(\Omega)}^2\le c\|\vartheta\|_V^2+\frac{1}{16}\|\nabla\varphi_t\|^2+c\|\varphi_t\|_{V'}^2. \tag{3.39}$$

Here (and below) the notation $\infty-$ stands for an exponent which is chosen as large (i.e. as close as infinity) as we need (in view of the fact that $V\subset L^p(\Omega)$ for all $p\in[1,\infty)$). Correspondingly, $2+$ turns out to be larger than, but close to 2. Next,

$$-\int_\Omega \kappa(\vartheta)\nabla\vartheta\cdot\nabla\varphi_t\le 4\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2+\frac{1}{16}\|\nabla\varphi_t\|^2. \tag{3.40}$$

Finally, using interpolation (cf. (2.2) and (2.6)), Young's inequality, and (3.29),

$$\int_\Omega |\nabla\boldsymbol{u}|^2\varphi_t\le c\|\varphi_t\|_{L^4(\Omega)}\|\nabla\boldsymbol{u}\|\|\nabla\boldsymbol{u}\|_{L^4(\Omega)}\le c\|\varphi_t\|^{1/2}\|\nabla\varphi_t\|^{1/2}\|\boldsymbol{u}\|_{H^2(\Omega)} \\ \le c\|\varphi_t\|_{V'}^{1/4}\|\nabla\varphi_t\|^{3/4}\|\boldsymbol{u}\|_{H^2(\Omega)}\le c+c\|\boldsymbol{u}\|_{H^2(\Omega)}^2+c\|\varphi_t\|_{V'}^2+\frac{1}{16}\|\nabla\varphi_t\|^2. \tag{3.41}$$

Thanks to (3.38)-(3.41), (3.37) gives

$$(\vartheta_t,\varphi_t)+\int_\Omega \vartheta\varphi_t^2\le c\big(1+\|\boldsymbol{u}\|_{H^2(\Omega)}^2+\|\vartheta\|_V^2+\|\varphi_t\|_{V'}^2\big) \\ +\frac{1}{4}\|\nabla\varphi_t\|^2+4\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2+\int_\Omega |\nabla\mu|^2\varphi_t. \tag{3.42}$$

The last two terms on the right hand side are still to be controlled. The quadratic term in $\nabla\mu$, which is the most difficult one, will be dealt with at the end. In order to treat the term in $\nabla\vartheta$, we need another estimate. Namely, we test (1.5) by $8K(\vartheta)$ (cf. (2.12)). Let us set

$$\mathcal{J}(r):=\int_0^r K(s)\,\mathrm{d}s=\frac{r^2}{2}+\frac{1}{(q+1)(q+2)}r^{q+2},\quad r\ge 0. \tag{3.43}$$

Then, we obtain

$$8\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\vartheta) + 8\int_\Omega \boldsymbol{u}\cdot\nabla\mathcal{J}(\vartheta) + 8\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2 \\ = -8\int_\Omega \vartheta K(\vartheta)(\varphi_t + \boldsymbol{u}\cdot\nabla\varphi) + 8\int_\Omega K(\vartheta)\big(|\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2\big), \tag{3.44}$$

where, in fact, the second integral on the left hand side is zero in view of (1.1) and the periodic boundary conditions. Again, we need to provide an estimate for the terms on the right hand side. At first, recalling (2.12) and using (2.6), we have

$$-8\int_\Omega \vartheta K(\vartheta)\varphi_t \le c\int_\Omega \big(\vartheta^2 + \vartheta^{q+2}\big)|\varphi_t| \le c\int_\Omega \big(1+\vartheta^{q+2}\big)|\varphi_t| \\ \le c\|\varphi_t\| + \int_\Omega \vartheta^{q+2}|\varphi_t| \le c + \frac{1}{16}\|\nabla\varphi_t\|^2 + c\|\varphi_t\|_{V'}^2 + c\int_\Omega \vartheta^{q+2}|\varphi_t|. \tag{3.45}$$

The last term needs to be managed accurately. We start by noting that

$$c\int_\Omega \vartheta^{q+2}|\varphi_t| \le c\|\vartheta^{q+2}\|\|\varphi_t\| \le c\|\vartheta\|_{L^{2q+4}(\Omega)}^{q+2}\|\varphi_t\|_{V'}^{1/2}\|\nabla\varphi_t\|^{1/2}. \tag{3.46}$$

Next, we observe that, using (2.4) with the choices $r = 2q+4$ and $s = 1$, we would obtain

$$\|\vartheta\|_{L^{2q+4}(\Omega)} \le \|\vartheta\|_{L^1(\Omega)}^{\frac{1}{2q+4}}\|\vartheta\|_{L^\infty(\Omega)}^{\frac{2q+3}{2q+4}}. \tag{3.47}$$

However the above interpolation exponents do not work in our case and need to be modified slightly. Actually, recalling also (3.5), we can continuate (3.46) as follows:

$$c\int_\Omega \vartheta^{q+2}|\varphi_t| \le c\Big(\|\vartheta\|_{L^1(\Omega)}^{\frac{1}{2q+4}-}\|\vartheta\|_{L^{\infty-}(\Omega)}^{\frac{2q+3}{2q+4}+}\Big)^{q+2}\|\varphi_t\|_{V'}^{1/2}\|\nabla\varphi_t\|^{1/2} \\ \le c\|\vartheta\|_{L^{\infty-}(\Omega)}^{\frac{2q+3}{2}+}\|\varphi_t\|_{V'}^{1/2}\|\nabla\varphi_t\|^{1/2} \le c\|\vartheta^{q+1}\|_{L^{\infty-}(\Omega)}^{\frac{2q+3}{2q+2}+}\|\varphi_t\|_{V'}^{1/2}\|\nabla\varphi_t\|^{1/2}. \tag{3.48}$$

As above, $\infty-$ stands for an exponent $P\in[1,\infty)$ (which we can choose as large as we need) and the number $\frac{2q+3}{2}+$ depends on the choice of P and will be closer to $\frac{2q+3}{2}$ as larger is taken P. The same applies to $\frac{2q+3}{2q+2}+$ and other exponents below. Of course, also the constants c will depend on the choice of P (and will be larger for larger P). Then, recalling (2.5) and subsequently using Young's inequality with exponents 8, 4, and $8/5$, computation (3.48) can be continued this way:

$$c\int_\Omega \vartheta^{q+2}|\varphi_t| \le c\|\varphi_t\|_{V'}^{1/2}\|\nabla\varphi_t\|^{1/2}\big(1+\|\nabla\vartheta^{q+1}\|^{\frac{2q+3}{2q+2}+}\big) \\ \le c + c\|\varphi_t\|_{V'}^4 + \frac{1}{16}\|\nabla\varphi_t\|^2 + c\|\nabla\vartheta^{q+1}\|^{\frac{4(2q+3)}{5(q+1)}+}, \tag{3.49}$$

and, thanks to (2.11) (actually, $q>1$ would be enough at this level), we can take P so large that $\frac{4(2q+3)}{5(q+1)}+$ is *strictly* smaller than 2. Hence, using Young's inequality again, and recalling (2.13), we conclude that

$$-8\int_\Omega \vartheta K(\vartheta)\varphi_t \le c + \frac{1}{8}\|\nabla\varphi_t\|^2 + c\|\varphi_t\|_{V'}^2 + c\|\varphi_t\|_{V'}^4 + c\|\nabla\vartheta^{q+1}\|^{\frac{4(2q+3)}{5(q+1)}+} \\ \le c + \frac{1}{8}\|\nabla\varphi_t\|^2 + c\|\varphi_t\|_{V'}^4 + \int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2. \tag{3.50}$$

The estimation of the subsequent summand in (3.44) is simpler. Actually, recalling also (3.4), (3.5) and (3.29), and using once more (2.5), (2.13) and Young's inequality, we get

$$-8\int_\Omega \vartheta K(\vartheta)\boldsymbol{u}\cdot\nabla\varphi \le c\int_\Omega (1+\vartheta^{q+2})|\boldsymbol{u}|\,|\nabla\varphi| \le c\big(1+\|\vartheta^{q+2}\|_{L^{2+}(\Omega)}\big)\|\boldsymbol{u}\|_{L^{\infty-}(\Omega)}\|\nabla\varphi\|$$
$$\le c\Big(1+\|\vartheta^{q+1}\|^{\frac{q+2}{q+1}}_{L^{\frac{2(q+2)}{q+1}+}(\Omega)}\Big) \le c\big(1+\|\nabla\vartheta^{q+1}\|^{\frac{q+2}{q+1}}\big) \le c+\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2, \tag{3.51}$$

since, clearly, $\frac{q+2}{q+1}<2$. Next, recalling (3.29) and (2.16) and arguing analogously to (3.41), we have

$$8\int_\Omega |\nabla\boldsymbol{u}|^2K(\vartheta) \le c\|K(\vartheta)\|_{L^4(\Omega)}\|\nabla\boldsymbol{u}\|\|\nabla\boldsymbol{u}\|_{L^4(\Omega)} \le c+c\|\boldsymbol{u}\|^2_{H^2(\Omega)}+\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2. \tag{3.52}$$

Collecting (3.45)-(3.52), (3.44) gives

$$8\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\vartheta)+5\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2$$
$$\le c+c\|\boldsymbol{u}\|^2_{H^2(\Omega)}+\frac{1}{8}\|\nabla\varphi_t\|^2+c\|\varphi_t\|^4_{V'}+8\int_\Omega K(\vartheta)|\nabla\mu|^2. \tag{3.53}$$

Summing (3.42) and (3.53) we then get

$$(\vartheta_t,\varphi_t)+\int_\Omega \vartheta\varphi_t^2+8\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\vartheta)+\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2$$
$$\le c\big(1+\|\boldsymbol{u}\|^2_{H^2(\Omega)}+\|\vartheta\|^2_V+\|\varphi_t\|^4_{V'}\big)+\frac{3}{8}\|\nabla\varphi_t\|^2+\int_\Omega (8K(\vartheta)+\varphi_t)|\nabla\mu|^2. \tag{3.54}$$

Hence, summing (3.35) and (3.54) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|^2_{V'}+8\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\vartheta)+\frac{3}{8}\|\nabla\varphi_t\|^2+\int_\Omega (F''(\varphi)+\lambda)|\varphi_t|^2$$
$$+\int_\Omega \vartheta\varphi_t^2+\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2 \le M_1(t)\|\varphi_t\|^2_{V'}+M_2(t)$$
$$+c\big(1+\|\boldsymbol{u}\|^2_{H^2(\Omega)}+\|\vartheta\|^2_V+\|\varphi_t\|^4_{V'}\big)+\int_\Omega (8K(\vartheta)+\varphi_t)|\nabla\mu|^2. \tag{3.55}$$

Neglecting some positive terms in the left hand side and rearranging, we then arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|^2_{V'}+8\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\vartheta)+\frac{3}{8}\|\nabla\varphi_t\|^2+\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2$$
$$\le M_3(t)\|\varphi_t\|^2_{V'}+M_4(t)+\int_\Omega (8K(\vartheta)+\varphi_t)|\nabla\mu|^2, \tag{3.56}$$

where we have set

$$M_3(t)=c\big(1+\|\varphi\|^2_{H^3(\Omega)}+\|\boldsymbol{u}\|^2_{H^2(\Omega)}+\|\varphi_t\|^2_{V'}\big), \qquad M_4(t)=c\big(1+\|\boldsymbol{u}_t\|^2+\|\boldsymbol{u}\|^2_{H^2(\Omega)}+\|\vartheta\|^2_V\big). \tag{3.57}$$

Key estimate: quadratic terms. The most difficult part of our argument concerns the control of the last term in the right hand side of (3.56). This is based on the embedding inequality (A.1) proved in

Lemma A.1 below. Indeed, applying (A.1) to $\xi = |\nabla\mu|^2$ and using once more (2.16) together with (3.8) and the Poincaré-Wirtinger inequality, we get

$$\begin{aligned}\int_\Omega (8K(\vartheta)+\varphi_t)|\nabla\mu|^2 &\le c\big(\|K(\vartheta)\|_V + \|\nabla\varphi_t\|\big)\big\||\nabla\mu|^2\|_{V'}\\ &\le c + \frac12\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2 + \frac18\|\nabla\varphi_t\|^2 + c\big\||\nabla\mu|^2\big\|_{V'}^2\\ &\le c + \frac12\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2 + \frac18\|\nabla\varphi_t\|^2 + c\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2 \log\big(e+\big\||\nabla\mu|^2\big\|_{L^2(\Omega)}\big). \end{aligned} \tag{3.58}$$

Now, let us notice that $\psi(r) = e^r$, $r\in\mathbb{R}$ and $\psi^*(s) = s(\log s - 1)$, $s>0$ (extended by continuity to $s=0$ by setting $\psi^*(0)=0$) are *convex conjugate* functions. Consequently, for any $r\in\mathbb{R}$, $s\ge 0$, we have (cf., e.g., [6, Sec. 1.4]) $rs \le \psi(r)+\psi^*(s)$. Applying this property to $r = \log\big(e+\||\nabla\mu|^2\|_{L^2(\Omega)}\big)$ and $s = c\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2$, the last term (let us note it as $\mathcal{I}$) in (3.58) can be controlled as follows:

$$\begin{aligned}\mathcal{I} &\le c\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2\Big(\log\big(c\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2\big)-1\Big) + e + \big\||\nabla\mu|^2\big\|_{L^2(\Omega)}\\ &\le c + c\|\nabla\mu\|^4\log\big(e+\|\nabla\mu\|^2\big) + \|\nabla\mu\|_{L^4(\Omega)}^2, \end{aligned} \tag{3.59}$$

where, observing that $\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2 = \|\nabla\mu\|^4$, we used the fact that

$$\begin{aligned}\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2\log\big(c\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2\big) &= \|\nabla\mu\|^4\log\big(c\|\nabla\mu\|^4\big)\\ &= \|\nabla\mu\|^4\big(\log c + 2\log\|\nabla\mu\|^2\big)\\ &\le c\|\nabla\mu\|^4\big(1+\log\|\nabla\mu\|^2\big)\\ &\le c\|\nabla\mu\|^4\log\big(e+\|\nabla\mu\|^2\big); \end{aligned} \tag{3.60}$$

the last line is based on the elementary inequality $1+\log\lambda \le c\log(e+\lambda)$, holding for $\lambda>0$. Now, to manage the last term of (3.59), we use equation (1.3), estimate (3.4), and inequalities (2.2) and (2.6):

$$\begin{aligned}\|\nabla\mu\|_{L^4(\Omega)}^2 &\le c\|\nabla\mu\|\|\mu\|_{H^2(\Omega)} \le c\|\nabla\mu\|\big(\|\mu\|_V + \|\Delta\mu\|\big) \le c\|\mu\|_V^2 + c\|\varphi_t\|^2 + c\|\boldsymbol{u}\cdot\nabla\varphi\|^2\\ &\le c\|\mu\|_V^2 + \frac18\|\nabla\varphi_t\|^2 + c\|\varphi_t\|_{V'}^2 + c\|\boldsymbol{u}\|_{L^\infty(\Omega)}^2\|\nabla\varphi\|^2\\ &\le c\|\mu\|_V^2 + \frac18\|\nabla\varphi_t\|^2 + c\|\varphi_t\|_{V'}^2 + c\|\boldsymbol{u}\|_{H^2(\Omega)}^2. \end{aligned} \tag{3.61}$$

On the other hand, the first nonconstant term on the right hand side of (3.59) needs a further manipulation. Namely, we have to test equation (1.3) by $-\mu$. Then, noting that both terms on the left hand side of (1.3) have zero spatial mean and using the Poincaré-Wirtinger inequality, we get

$$\begin{aligned}\|\nabla\mu\|^2 &= -\int_\Omega(\varphi_t+\boldsymbol{u}\cdot\nabla\varphi)\mu = -\int_\Omega(\varphi_t+\boldsymbol{u}\cdot\nabla\varphi)(\mu-\mu_\Omega) \le \|\mu-\mu_\Omega\|_V\|\varphi_t+\boldsymbol{u}\cdot\nabla\varphi\|_{V'}\\ &\le \frac12\|\nabla\mu\|^2 + c\|\varphi_t\|_{V'}^2 + c\|\boldsymbol{u}\cdot\nabla\varphi\|_{V'}^2 \le \frac12\|\nabla\mu\|^2 + c\|\varphi_t\|_{V'}^2 + c\|\boldsymbol{u}\cdot\nabla\varphi\|_{L^{4/3}(\Omega)}^2\\ &\le \frac12\|\nabla\mu\|^2 + c\|\varphi_t\|_{V'}^2 + c\|\boldsymbol{u}\|_{L^4(\Omega)}^2\|\nabla\varphi\|^2 \le \frac12\|\nabla\mu\|^2 + c\|\varphi_t\|_{V'}^2 + c, \end{aligned} \tag{3.62}$$

where in the last line we also used (3.29). Consequently,

$$\begin{aligned}c\|\nabla\mu\|^4\log\big(e+\|\nabla\mu\|^2\big) &\le c\big(1+\|\varphi_t\|_{V'}^4\big)\log\Big(e+c\big(1+\|\varphi_t\|_{V'}^2\big)\Big)\\ &\le c + c\|\varphi_t\|_{V'}^4\log\big(e+\|\varphi_t\|_{V'}^2\big). \end{aligned} \tag{3.63}$$

Hence, collecting (3.59)-(3.63), (3.58) gives

$$\int_\Omega (8K(\vartheta)+\varphi_t)|\nabla\mu|^2 \le c+\frac{1}{2}\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2+\frac{1}{4}\|\nabla\varphi_t\|^2+c\|\varphi_t\|_{V'}^2+c\|\mu\|_V^2 \\ +c\|\boldsymbol{u}\|_{H^2(\Omega)}^2+c\|\varphi_t\|_{V'}^4\log\big(e+\|\varphi_t\|_{V'}^2\big). \tag{3.64}$$

Plugging (3.64) into (3.56) we then get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|_{V'}^2+8\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\vartheta)+\frac{1}{8}\|\nabla\varphi_t\|^2+\frac{1}{2}\int_\Omega \kappa^2(\vartheta)|\nabla\vartheta|^2 \\ \le M_5(t)\|\varphi_t\|_{V'}^2+M_6(t)+c\|\varphi_t\|_{V'}^4\log\big(e+\|\varphi_t\|_{V'}^2\big). \tag{3.65}$$

where

$$M_5(t)=c\big(1+\|\varphi\|_{H^3(\Omega)}^2+\|\boldsymbol{u}\|_{H^2(\Omega)}^2+\|\varphi_t\|_{V'}^2\big), \tag{3.66}$$

$$M_6(t)=c\big(1+\|\boldsymbol{u}_t\|^2+\|\boldsymbol{u}\|_{H^2(\Omega)}^2+\|\vartheta\|_V^2+\|\mu\|_V^2\big). \tag{3.67}$$

Let us now set

$$Y_1(t):=\frac{1}{2}\|\varphi_t(t)\|_{V'}^2, \qquad Y_2(t):=8\int_\Omega \mathcal{J}(\vartheta(t)). \tag{3.68}$$

Then, from (3.64) we obtain the following differential inequality:

$$Y_1'(t)+Y_2'(t)\le M_5(t)Y_1(t)+M_6(t)+cY_1^2(t)\log\big(e+Y_1(t)\big). \tag{3.69}$$

Setting $Z(t):=e+Y_1(t)+Y_2(t)$, and dividing both hand sides of (3.69) by $Z\log Z$, it is then easy to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\log Z(t)=\frac{Z'(t)}{Z(t)\log Z(t)}\le M_5(t)+\frac{M_6(t)}{Z(t)\log Z(t)}+cY_1(t). \tag{3.70}$$

Now, let us notice that, in view of the a-priori estimates (3.12), (3.16), (3.22), (3.24), (3.26), and (3.29), we have

$$\|Y_1\|_{L^1(0,T)}+\|M_5\|_{L^1(0,T)}+\|M_6\|_{L^1(0,T)}\le c. \tag{3.71}$$

Moreover, we can notice that, at least formally,

$$\begin{aligned} Z(0)&=e+\frac{1}{2}\|\varphi_t(0)\|_{V'}^2+8\int_\Omega \mathcal{J}(\vartheta_0)\\ &\le c+c\|\Delta\mu(0)\|_{V'}^2+c\|\boldsymbol{u}_0\cdot\nabla\varphi_0\|_{V'}^2+c\int_\Omega \mathcal{J}(\vartheta_0)\\ &\le c+\frac{1}{2}\|\mu(0)\|_V^2+c\|\boldsymbol{u}_0\|_V^2\|\varphi_0\|_{H^3(\Omega)}^2+c\|\vartheta_0\|^2+c\|\vartheta_0\|_{L^{q+2}(\Omega)}^{q+2}\\ &\le c+c\|\Delta\varphi_0\|_V^2+c\|F'(\varphi_0)\|_V^2+c\|\vartheta_0\|_V^{q+2}<\infty, \end{aligned} \tag{3.72}$$

where in the second row we used equation (1.3), in the third we used standard interpolation and embeddings and the definition of $\mathcal{J}$, and in the fourth we used equation (1.4) and the assumptions (2.10) on the potential and (2.26) on the initial data.

Hence, thanks to (3.71) and to (3.72), we can integrate (3.70) over $(0,T)$ to obtain

$$\|Z\|_{L^\infty(0,T)}\le c. \tag{3.73}$$

In particular, as often happens in 2D models, Z turns out to grow with respect to time (at most) as fast as a double exponential; however it does not explode in finite times. This is the key point in our existence proof.

Key estimate: consequences. Recalling also (3.43), (3.73) gives

$$\|\varphi_t\|_{L^\infty(0,T;V')} \le c, \tag{3.74}$$
$$\|\vartheta\|_{L^\infty(0,T;L^{q+2}(\Omega))} \le c. \tag{3.75}$$

Using (1.3) we can estimate, as in (3.62), the H-norm of $\nabla\mu$ in terms of the V'-norm of φ_t. Hence, recalling also (3.21), we get

$$\|\mu\|_{L^\infty(0,T;V)} \le c. \tag{3.76}$$

With these properties at disposal, we can go back to relation (3.65). Integrating it over $(0,T)$, we infer (cf. also (2.12), (3.8))

$$\|\varphi_t\|_{L^2(0,T;V)} \le c, \tag{3.77}$$
$$\|K(\vartheta)\|_{L^2(0,T;V)} \le c. \tag{3.78}$$

In addition to that, an easy interpolation argument and estimates (3.24), (3.29) permit us to check that

$$\|\boldsymbol{u}\cdot\nabla\varphi\|_{L^2(0,T;V)} \le c\|\boldsymbol{u}\|_{L^\infty(0,T;V)}\|\varphi\|_{L^2(0,T;H^3(\Omega))} \le c. \tag{3.79}$$

Hence, viewing (1.3) as a time-dependent family of elliptic problems and using standard regularity results with (3.79) and (3.77), we infer

$$\|\mu\|_{L^2(0,T;H^3(\Omega))} \le c. \tag{3.80}$$

Remark 3.2. The argument given in (3.72) in order to estimate $\varphi_t(0)$ in terms of $\boldsymbol{u}_0$, ϑ_0 and μ_0 is just formal since, at least in principle, it is not clear whether equation (1.3) holds pointwise for *all* values of the time variable. However, as will be clear from the approximation scheme detailed in the next section, at least locally in time we can get an approximate solution which is essentially as smooth as we need. Hence, at the price of some additional technicalities, the argument could be made fully rigorous with a limited effort.

Estimate of μ_t and ϑ_t. In order to pass to the limit in the nonlinear terms involving ϑ and μ we will apply the Aubin-Lions lemma. To this aim, we need to deduce some a-priori estimates on μ_t and ϑ_t. Hence, we (formally) differentiate (1.4) with respect to time and use (1.5) to get

$$\mu_t = -\Delta\varphi_t + F''(\varphi)\varphi_t + \boldsymbol{u}\cdot\nabla\vartheta + \vartheta(\varphi_t + \boldsymbol{u}\cdot\nabla\varphi) - \Delta K(\vartheta) - |\nabla\boldsymbol{u}|^2 - |\nabla\mu|^2. \tag{3.81}$$

Here, to justify the procedure, one could make similar observations as in Remark 3.2 (see also Remark 4.7 below). Let us test the above relation by nonzero $v \in V$. We get, using our choice of boundary conditions,

$$\begin{aligned}\langle\mu_t, v\rangle &= \int_\Omega \nabla(\varphi_t + K(\vartheta))\cdot\nabla v + \big(F''(\varphi)\varphi_t + \boldsymbol{u}\cdot\nabla\vartheta + \vartheta(\varphi_t + \boldsymbol{u}\cdot\nabla\varphi) - |\nabla\boldsymbol{u}|^2 - |\nabla\mu|^2, v\big)\\ &\le \|v\|_V\Big(\big\|\nabla(\varphi_t + K(\vartheta))\big\|\\ &\qquad + \big\|F''(\varphi)\varphi_t + \boldsymbol{u}\cdot\nabla\vartheta + \vartheta(\varphi_t + \boldsymbol{u}\cdot\nabla\varphi) - |\nabla\boldsymbol{u}|^2 - |\nabla\mu|^2\big\|_{L^{3/2}(\Omega)}\Big).\end{aligned} \tag{3.82}$$

Dividing by $\|v\|$, passing to the supremum with respect to $v \in V\setminus\{0\}$, squaring, and integrating in time, we would then obtain

$$\|\mu\|_{H^1(0,T;V')} \le c, \tag{3.83}$$

provided we could prove that

$$\big\|\nabla(\varphi_t + K(\vartheta))\big\|_{L^2(0,T;H)} \le c, \tag{3.84}$$

$$\big\|F''(\varphi)\varphi_t + \boldsymbol{u}\cdot\nabla\vartheta + \vartheta(\varphi_t + \boldsymbol{u}\cdot\nabla\varphi) - |\nabla\boldsymbol{u}|^2 - |\nabla\mu|^2\big\|_{L^2(0,T;L^{3/2}(\Omega))} \le c, \tag{3.85}$$

where the exponent $3/2$ is chosen just for simplicity (any number strictly greater than 1 would be allowed, indeed). Now, (3.84) is an immediate consequence of (3.77)-(3.78), whereas (3.85) follows by appropriately combining all the above a-priori estimates and using standard inequalities. The details are lengthy but straighforward; hence, they are left to the reader.

To conclude, we test equation (1.5) by $v \in V \setminus \{0\}$. Performing the very same computations as above we then get

$$\|\vartheta\|_{H^1(0,T;V')} \le c. \tag{3.86}$$

This is the last estimate we need.

Weak sequential stability. We assume to have a sequence of solutions $(\boldsymbol{u}_n, \varphi_n, \mu_n, \vartheta_n)$ satisfying the a-priori estimates obtained above uniformly with respect to n. This could be, for instance, a sequence of approximate solutions provided by the fixed-point argument performed in the next section. Then, we aim at proving that, up to the extraction of a subsequence, we can find a limit quadruple $(\boldsymbol{u}, \varphi, \mu, \vartheta)$ satisfying (1.1)-(1.5) in the sense of Theorem 2.1. Note that, in view of the uniform character of the estimates, even though the approximate solutions are defined only locally in time, we will have global solutions in the limit (cf. also Remark 4.5 below). For this reason, and also for the sake of simplicity, we shall directly work on the original reference time interval $(0,T)$.

That said, the above a-priori estimates (cf., in particular, (3.16), (3.24), (3.29), (3.74)-(3.80), (3.83), and (3.86)), together with standard weak compactness results, entail

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{3.87}$$

$$\varphi_n \to \varphi \quad \text{weakly star in } W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{3.88}$$

$$\mu_n \to \mu \quad \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{3.89}$$

$$\vartheta_n \to \vartheta \quad \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;L^{q+2}(\Omega)) \cap L^2(0,T;V). \tag{3.90}$$

Here and below, all convergence relations are intended to hold up to the extraction of non-relabelled subsequences. Note also that (3.88), by interpolation (cf., e.g., [7, formula (2.5.38)]), yields

$$\varphi_n \to \varphi \quad \text{weakly in } H^s(0,T;H^{3-2s}(\Omega)) \quad \text{for all } s \in [0,1]. \tag{3.91}$$

Then, the Aubin-Lions compactness theorem (cf. [27, p. 58] or [35, Cor. 4, Sec. 8]) entails (actually something more is true)

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V), \tag{3.92}$$

$$\varphi_n \to \varphi \quad \text{strongly in } C^0([0,T];V) \cap L^2(0,T;H^2(\Omega)), \tag{3.93}$$

$$\mu_n \to \mu \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;H^2(\Omega)), \tag{3.94}$$

$$\vartheta_n \to \vartheta \quad \text{strongly in } C^0([0,T];V') \cap L^2(0,T;H). \tag{3.95}$$

Notice that, to deduce the first (3.93), also (3.91) has been used together with the compact embedding $H^s(0,T) \subset\subset C^0([0,T])$ (or, more precisely, its vector-valued analogue), holding for $s > 1/2$.

We now claim that the above relations suffice to take the limit $n \nearrow \infty$ in all equations of our system (1.1)-(1.5). To see this, we limit ourselves to consider the most troublesome nonlinear terms, the other ones being in fact almost straighforward to treat. To start with, we note that

$$\nabla\varphi_n \otimes \nabla\varphi_n \to \nabla\varphi \otimes \nabla\varphi \quad \text{strongly in } C^0([0,T];L^1(\Omega)) \tag{3.96}$$

thanks to (3.93). Actually, proceeding as in (3.27)-(3.28), we can check that $\operatorname{div}(\nabla\varphi \otimes \nabla\varphi)$ lies in $L^2(0,T;H)$, which allows for (1.2) to hold pointwise (almost everywhere). Next, we notice that (3.95) entails $\vartheta_n \to \vartheta$ a.e. in $(0,T)\times\Omega$. Such a property, combined with (3.75), (3.78), and a generalized version of Lebesgue's theorem, gives

$$\vartheta_n \to \vartheta \quad \text{strongly in } L^p(0,T;L^{(q+2)-}(\Omega)) \cap L^{(2q+2)-}(0,T;L^p(\Omega)) \quad \text{for all } p \in [1,\infty). \tag{3.97}$$

In particular, recalling (2.12), we get

$$K(\vartheta_n) \to K(\vartheta) \quad \text{strongly in } L^p((0,T)\times\Omega) \quad \text{for a suitable } p > 1. \tag{3.98}$$

Hence,

$$-\operatorname{div}(\kappa(\vartheta_n)\nabla\vartheta_n) = -\Delta K(\vartheta_n) \to -\Delta K(\vartheta) \tag{3.99}$$

at least in the sense of distributions. More precisely, the limit function $K(\vartheta)$ lies in $L^2(0,T;V)$ in view of (3.78), which allows the limit of (1.5) to hold as a relation in $L^2(0,T;V')$ as specified by (2.25).

Moreover, recalling that $q \geq 2$ and using (3.88), (3.92), (3.93) and (3.97), we can easily check that

$$\vartheta_n(\varphi_{n,t}+\boldsymbol{u}_n\cdot\nabla\varphi_n) \to \vartheta(\varphi_t+\boldsymbol{u}\cdot\nabla\varphi) \quad \text{weakly in } L^p((0,T)\times\Omega) \quad \text{for a suitable } p > 1. \tag{3.100}$$

Next, thanks to (3.92) and (3.94), we get (actually, something more is true)

$$|\nabla\boldsymbol{u}_n|^2 + |\nabla\mu_n|^2 \to |\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2 \quad \text{strongly in } L^1((0,T)\times\Omega). \tag{3.101}$$

Relations (3.96)-(3.99) permit us to let $n \searrow \infty$ in all equations of the system (with (1.5) replaced by (2.25) in the limit, as noted above). In particular, the regularity properties (3.92)-(3.95) are a direct consequence of our argument. Finally, it is worth noting that the limit functions $\boldsymbol{u}$, φ, ϑ also satisfy the initial conditions (2.26). Indeed, they are continuous with respect to time with values in suitable Banach spaces (and the corresponding uniform estimates (3.92)-(3.95) hold for the approximating sequences). Hence, we may conclude that the limit quadruple $(\boldsymbol{u},\varphi,\mu,\vartheta)$ solves our system in the sense of Theorem 2.1, as desired.

4 Approximation and local existence

In this section we give some highlights regarding a possible approximation of system (1.1)-(1.5) and provide a proof of local existence by means of a fixed point argument of Schauder type. In order to reduce the length of the exposition we leave most technical details to the reader, just limiting ourselves to outline the main steps of the procedure.

Regularized system. For (small) $\varepsilon \in (0,1)$ we consider the following regularized statement:

$$\operatorname{div}\boldsymbol{u} = 0, \tag{4.1}$$

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \Delta\boldsymbol{u} - \operatorname{div}(\nabla\varphi\otimes\nabla\varphi), \tag{4.2}$$

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \Delta\mu, \tag{4.3}$$

$$\mu = -\varepsilon\Delta\varphi_t - \Delta\varphi + F_\varepsilon'(\varphi) - \vartheta, \tag{4.4}$$

$$\vartheta_t + \boldsymbol{u}\cdot\nabla\vartheta + \vartheta\big(\varphi_t + \boldsymbol{u}\cdot\nabla\varphi\big) - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) = |\nabla\boldsymbol{u}|^2 + T_\varepsilon\big(|\nabla\mu|^2\big). \tag{4.5}$$

The above system differs from the original one due in view of the additional term $-\varepsilon\Delta\varphi_t$ in (4.3), which provides further parabolic regularity to φ, and of the *truncation operator* T_ε in (4.5), where

$$T_\varepsilon(v) := \min\Big\{\varepsilon^{-1}, \max\big\{-\varepsilon^{-1}, v\big\}\Big\}, \quad \text{for } v : (0,T)\times\Omega \to \mathbb{R}, \tag{4.6}$$

which yields boundedness of the last term in the right hand side of (4.5). Moreover F_ε is a smooth regularization of F such that F_ε' is Lipschitz continuous. We assume that F_ε still enjoys the coercivity property (2.8). We also truncate the initial temperature in such a way that

$$\vartheta_{0,\varepsilon} \in H^1(\Omega)\cap L^\infty(\Omega), \qquad \varepsilon \le \vartheta_{0,\varepsilon} \le \varepsilon^{-1} \text{ a.e. in } \Omega. \tag{4.7}$$

Then, local existence for (4.1)-(4.5), complemented with the initial data $\boldsymbol{u}_0$, φ_0 and $\vartheta_{0,\varepsilon}$, and with periodic boundary conditions, is proved via a fixed point argument detailed below. This is essentially divided into three separate Lemmas. At first, we fix ϑ and $\boldsymbol{u}$ in the Cahn-Hilliard system (4.3)-(4.4).

Lemma 4.1. *Let $\varepsilon \in (0,1)$ and let $R > 0$ be a number, depending on the initial data and on ε, such that*

$$\|\varphi_0\|_{H^3(\Omega)} + \|\boldsymbol{u}_0\|_V + \|\vartheta_{0,\varepsilon}\|_{H^1(\Omega)} + \|\vartheta_{0,\varepsilon}\|_{L^\infty(\Omega)} \le R. \tag{4.8}$$

Let also

$$\overline{\vartheta} \in L^2(0,T;V), \quad \overline{\boldsymbol{u}} \in L^2(0,T;L^4(\Omega)), \quad \text{with } \|\overline{\vartheta}\|_{L^2(0,T;V)} + \|\overline{\boldsymbol{u}}\|_{L^2(0,T;L^4(\Omega))} \le R. \tag{4.9}$$

Then there exist unique functions φ and μ satisfying, a.e in $(0,T)\times\Omega$, the system

$$\varphi_t + \overline{\boldsymbol{u}}\cdot\nabla\varphi = \Delta\mu, \tag{4.10}$$

$$\mu = -\varepsilon\Delta\varphi_t - \Delta\varphi + F_\varepsilon'(\varphi) - \overline{\vartheta}, \tag{4.11}$$

with the initial condition $\varphi|_{t=0} = \varphi_0$. Moreover, the following regularity properties hold:

$$\varphi \in H^1(0,T;H^3(\Omega)), \tag{4.12}$$

$$\mu \in L^2(0,T;H^2(\Omega)), \tag{4.13}$$

$$\|\varphi\|_{H^1(0,T;H^3(\Omega))} + \|\mu\|_{L^2(0,T;H^2(\Omega))} \le Q_1(R,T). \tag{4.14}$$

Here and below, $Q_i : (\mathbb{R}^+)^2 \to \mathbb{R}^+$, $i = 1, 2, \ldots$, are computable functions, increasingly monotone in each of their arguments, whose expression may additionally depend on ε.

PROOF. We just give the highlights. Actually, once $\overline{\boldsymbol{u}}$ and $\overline{\vartheta}$ are assigned, (4.10)-(4.11) is a semilinear pseudo-parabolic system with Lipschitz nonlinearity. Hence, existence is standard. For example, it could be proved by relying on a Faedo-Galerkin scheme, or on a time-discretization argument. The a-priori estimates corresponding to the regularity conditions (4.12)-(4.13) are the following ones: first, one reproduces the energy estimate for the complete system by testing (4.10) by μ and (4.11) by φ_t. Noticing that

$$\int_\Omega \mu\,\overline{\boldsymbol{u}}\cdot\nabla\varphi = -\int_\Omega \varphi\,\overline{\boldsymbol{u}}\cdot\nabla\mu \le \|\varphi\|_{L^4(\Omega)}\|\overline{\boldsymbol{u}}\|_{L^4(\Omega)}\|\nabla\mu\| \le \frac{1}{2}\|\nabla\mu\|^2 + c\|\varphi\|_V^2\|\overline{\boldsymbol{u}}\|_{L^4(\Omega)}^2, \tag{4.15}$$

an estimate follows by using (4.9) and applying the Gronwall Lemma. With this estimate at disposal we can test (4.11) by $\Delta^2\varphi_t$. Thanks to the fact that $\mu, \overline{\vartheta} \in L^2(0,T;V)$ due to the energy estimate and to (4.9), and using the Lipschitz continuity of F_ε', it is then not difficult to obtain (4.12).

Subsequently, noting that the left hand side of (4.10) lies in $L^2(0,T;H)$, by elliptic regularity we obtain (4.13). Then, relation (4.14) is also a direct consequence of the a-priori estimates, as one can see by writing them in a quantitative way. Finally, uniqueness in the regularity class specified by (4.12)-(4.13) can be proved by a standard contractive argument. Namely, one may test (the difference of) (4.10) by $\varphi_1-\varphi_2$ (where (φ_1,μ_1) and (φ_2,μ_2) are two solutions) and (the difference of) (4.11) by $\Delta(\varphi_1-\varphi_2)$ and perform standard calculations. ∎

Lemma 4.2. *Let us assume that the hypotheses of* Lemma 4.1 *are satisfied, and let φ, μ be the functions provided by* Lemma 4.1*. Then, there exists a unique function $\boldsymbol{u}$ such that*

$$\boldsymbol{u}\in H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega)), \tag{4.16}$$

$$\|\boldsymbol{u}\|_{H^1(0,T;H)}+\|\boldsymbol{u}\|_{L^\infty(0,T;V)}+\|\boldsymbol{u}\|_{L^2(0,T;H^2(\Omega))}\le Q_2(R,T). \tag{4.17}$$

Moreover, $\boldsymbol{u}$ satisfies, a.e in $(0,T)\times\Omega$, the system

$$\operatorname{div}\boldsymbol{u}=0, \tag{4.18}$$

$$\boldsymbol{u}_t+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p-\Delta\boldsymbol{u}=-\operatorname{div}(\nabla\varphi\otimes\nabla\varphi), \tag{4.19}$$

with the initial condition $\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0$.

PROOF. Also in this case we just give the highlights. Actually, as φ is given satisfying (4.12) and (4.14), it is clear that the right hand side of (4.19) lies in $L^2(0,T;H)$. Hence, existence and uniqueness of a solution satisfying (4.16) follow from the general theory of Navier-Stokes systems (cf., e.g., [33] or [38]). Moreover, writing explicitly the a-priori bounds, one also immediately gets (4.17), where, in principle, the expression of Q_2 may also depend on a suitable norm of φ. However, thanks to (4.14), Q_2 can in fact be written as a (computable and monotone) function of R and T. ∎

Finally, we come to the "heat" equation, which is a little bit more involved to deal with:

Lemma 4.3. *Let the assumptions of* Lemma 4.1 *hold and let φ, μ, $\boldsymbol{u}$ be the functions provided by* Lemmas 4.1, 4.2*. Then, there exists a unique function ϑ such that*

$$\vartheta\in H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))\cap L^\infty((0,T)\times\Omega),$$
$$\vartheta>0 \ \text{ a.e. in } (0,T)\times\Omega, \tag{4.20}$$

$$\|\vartheta\|_{H^1(0,T;H)}+\|\vartheta\|_{L^\infty(0,T;V)}+\|\vartheta\|_{L^2(0,T;H^2(\Omega))}\le Q_3(R,T). \tag{4.21}$$

Moreover, ϑ satisfies, a.e in $(0,T)\times\Omega$,

$$\vartheta_t+\boldsymbol{u}\cdot\nabla\vartheta+\vartheta\big(\varphi_t+\boldsymbol{u}\cdot\nabla\varphi\big)-\operatorname{div}(\kappa(\vartheta)\nabla\vartheta)=|\nabla\boldsymbol{u}|^2+T_\varepsilon\big(|\nabla\mu|^2\big), \tag{4.22}$$

with the initial condition $\vartheta|_{t=0}=\vartheta_{0,\varepsilon}$.

PROOF. Equation (4.22) enjoys the quasilinear structure

$$\vartheta_t+\boldsymbol{u}\cdot\nabla\vartheta+m_1\vartheta-\Delta K(\vartheta)=f, \tag{4.23}$$

where K was defined in (2.12), and, in view of estimates (4.17), (4.14) and of interpolation, it is not difficult to infer

$$f:=|\nabla\boldsymbol{u}|^2+T_\varepsilon\big(|\nabla\mu|^2\big)\in L^p((0,T)\times\Omega)\quad\text{for all } p\in(1,\infty), \tag{4.24}$$

$$m_1:=\varphi_t+\boldsymbol{u}\cdot\nabla\varphi\in L^1(0,T;L^\infty(\Omega))\cap L^2(0,T;L^p(\Omega))\quad\text{for all } p\in(1,\infty). \tag{4.25}$$

Hence, existence of solutions to the initial-boundary value problem for (4.23) follows, as before, from standard techniques (as before, one could use time discretization or Faedo-Galerkin approximation).

Moreover, testing (4.23) by ϑ and performing simple calculations, we obtain the a-priori estimates leading to the regularity $\vartheta \in L^\infty(0,T;H) \cap L^2(0,T;V)$. By virtue of the high summability of f and m_1, a standard application of Moser's iteration technique (see, e.g., [25, Chap. 5]) yields $\vartheta \in L^\infty((0,T)\times\Omega)$.

Strict positivity of ϑ follows from the maximum principle. Next, to deduce (4.20)-(4.21) one tests (4.22) by $K(\vartheta)_t$. Recalling (2.11), and using the L^∞-bound for ϑ coming from Moser's iterations, we then get the estimate

$$\begin{aligned}\int_\Omega (1+\vartheta^q)|\vartheta_t|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla K(\vartheta)\|^2 &= \int_\Omega (f - \boldsymbol{u}\cdot\nabla\vartheta - m_1\vartheta)(1+\vartheta^q)\vartheta_t \\ &\le \frac{1}{2}\|\vartheta_t\|^2 + c\big\|f - \boldsymbol{u}\cdot\nabla\vartheta - m_1\vartheta\big\|^2\big(1+\|\vartheta\|^{2q}_{L^\infty(\Omega)}\big) \\ &\le \frac{1}{2}\|\vartheta_t\|^2 + c\|f\|^2 + c\big(\|\boldsymbol{u}\|^2_{L^\infty(\Omega)} + \|m_1\|^2_{L^4(\Omega)}\big)\|\vartheta\|^2_V. \end{aligned} \tag{4.26}$$

Hence, noting that $\|\vartheta\|_V \le c\|K(\vartheta)\|_V$ by (2.11), conditions $\vartheta \in H^1(0,T;H)$ and $K(\vartheta) \in L^\infty(0,T;V)$ follow from Gronwall's lemma. Consequently we also have $\vartheta \in L^\infty(0,T;V)$. In addition to that, viewing (4.23) as a time-dependent family of elliptic problems (for the variable $K(\vartheta)$) with L^2-data and applying standard regularity results, we obtain that $K(\vartheta) \in L^2(0,T;H^2(\Omega))$.

Let now $k := K(\vartheta)$ and denote as η_q the inverse function of K over $[0,+\infty)$. Observe that η_q' and η_q'' are uniformly bounded. Hence

$$\Delta\vartheta = \Delta\eta_q(k) = \eta_q'(k)\Delta k + \eta_q''(k)|\nabla k|^2 \tag{4.27}$$

belongs to $L^2(0,T;H)$ thanks to the fact that $k \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;V)$. This entails (4.20). As before, as one writes explicitly the estimates leading to (4.20), also (4.21) follows.

Finally, to show uniqueness, we take two solutions ϑ_1 and ϑ_2 to (4.22) with the same initial datum and the same $\varphi, \mu, \boldsymbol{u}$. Then, the difference $\tilde\vartheta := \vartheta_1 - \vartheta_2$ solves

$$\tilde\vartheta_t + \boldsymbol{u}\cdot\nabla\tilde\vartheta + m_1\tilde\vartheta - \Delta(K(\vartheta_1) - K(\vartheta_2)) = 0. \tag{4.28}$$

Then, testing the above by $\operatorname{sign}\tilde\vartheta$ (more precisely, one should first take an approximation of the sign function and then pass to the limit), using monotonicity of K, the incompressibility (4.18), and the periodic boundary conditions, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde\vartheta\|_{L^1(\Omega)} \le \|m_1\|_{L^\infty(\Omega)}\|\tilde\vartheta\|_{L^1(\Omega)}, \tag{4.29}$$

whence uniqueness follows from Gronwall's Lemma recalling the first condition in (4.25). ∎

With the three lemmas at disposal, we can make explicit our fixed-point argument

Theorem 4.4. *Let $\varepsilon \in (0,1)$ and let us assume* (4.6) *and* (4.7)*. Then there exist a time T_0 (depending on ε and on the initial data) and at least one quadruple $(\boldsymbol{u},\varphi,\mu,\vartheta)$ such that*

$$\boldsymbol{u} \in H^1(0,T_0;H) \cap L^\infty(0,T_0;V) \cap L^2(0,T_0;H^2(\Omega)), \tag{4.30}$$

$$\varphi \in H^1(0,T_0;H^3(\Omega)), \tag{4.31}$$

$$\mu \in L^2(0,T_0;H^2(\Omega)), \tag{4.32}$$

$$\vartheta \in H^1(0,T_0;H) \cap L^\infty(0,T_0;V) \cap L^2(0,T_0;H^2(\Omega)) \cap L^\infty((0,T_0)\times\Omega), \tag{4.33}$$

with $\vartheta > 0$ a.e. in $(0,T_0)\times\Omega$, satisfying system (4.1)-(4.5) *a.e. in $(0,T_0)\times\Omega$ and complying with the initial conditions*

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \qquad \varphi|_{t=0} = \varphi_0, \qquad \vartheta|_{t=0} = \vartheta_{0,\varepsilon}. \tag{4.34}$$

PROOF. Given $\varepsilon > 0$, we truncate the initial temperature as specified in (4.7). Then, we choose $R > 0$ correspondingly (cf. (4.8)). Hence, we can consider the closed ball (cf. (4.9))

$$\mathcal{B} := \big\{(\overline{\vartheta},\overline{u}) : \|\overline{\vartheta}\|_{L^2(0,T_0;V)} + \|\overline{\boldsymbol{u}}\|_{L^2(0,T_0;L^4(\Omega))} \le R\big\}, \tag{4.35}$$

where $T_0 \in (0,T]$ will be chosen later on. Notice that the chosen radius R depends only on the initial data (and on ε by the truncation applied to ϑ_0). Let us consider the fixed point map (also depending on ε, of course)

$$\mathcal{T} : \mathcal{B} \to L^2(0,T_0;V)\times L^2(0,T_0;L^4(\Omega)), \qquad \mathcal{T} : (\overline{\vartheta},\overline{u}) \mapsto (\vartheta,\boldsymbol{u}). \tag{4.36}$$

We aim at applying the Schauder fixed point theorem to the above map, for a sufficiently small choice of the final time $T_0 > 0$. To this aim, we can observe the following:

(a) The map $\mathcal{T}$ is continuous: this follows from the fact that the fixed point equations (4.10)-(4.11), (4.19) and (4.22) only contain Lipschitz or locally Lipschitz nonlinearities. To give a formal proof (which is omitted for brevity), one could just put together (and refine a bit) the contractive arguments used to prove uniqueness in the three fixed point Lemmas.

(b) The map $\mathcal{T}$ is compact: this follows immediately from (4.20), (4.16) and the Aubin-Lions lemma. Indeed, the output of the map $\mathcal{T}$ lies in a bounded set of a space which is compactly embedded into $L^2(0,T_0;V)\times L^2(0,T_0;L^4(\Omega))$.

(c) The map $\mathcal{T}$ takes values into $\mathcal{B}$. Indeed, thanks to (4.21), (4.17), and the continuous embedding $V \subset L^4(\Omega)$, we get

$$\|\boldsymbol{u}\|_{L^\infty(0,T_0;L^4(\Omega))} + \|\vartheta\|_{L^\infty(0,T_0;V)} \le Q_4(R,T_0), \tag{4.37}$$

for some function $Q_4(R,T_0)$, whence

$$\|\boldsymbol{u}\|_{L^2(0,T_0;L^4(\Omega))} + \|\vartheta\|_{L^2(0,T_0;V)} \le T_0^{1/2} Q_4(R,T_0) \le R, \tag{4.38}$$

provided T_0 is small enough.

In view of the above conditions (a)-(c) the assumptions of the Schauder fixed point argument are satisfied, whence (at least) one solution to (4.1)-(4.5) exists. The regularity conditions (4.30)-(4.33) follow immediately from the above three Lemmas. Theorem 4.4 is proved. ∎

We conclude this section with three observations aimed at clarifying why the present approximation-fixed point argument is compatible with the a-priori estimates of the previous section.

Remark 4.5. As usual, the solution provided by the fixed point argument is local in time and the final time T_0 may depend on ε and be smaller as smaller is ε. However, the a-priori estimates performed in the previous section are uniform with respect to time. Hence, thanks to standard extension arguments, the (approximate) solution turns out to be defined, in fact, on the whole reference interval $(0,T)$, and the same will hold for the limit solution.

Remark 4.6. It is also worth noting that neither the regularizing term $-\varepsilon\Delta\varphi_t$ added on the right hand side of (4.4) nor the truncation operator on the right hand side of (4.5) really interfer with the a-priori estimates of the previous section, which turn out to be independent of the approximation parameter ε. Actually, $-\varepsilon\Delta\varphi_t$ just gives some more information, vanishing as ε goes to 0, in the energy and subsequent bounds. On the other hand, the truncation operator yields some (positive) remainder term on the left hand side of the energy estimate, coming from the fact that, as one tests (4.3) by μ and (4.5) by 1, the contribution of μ does not vanish completely. It is immediate to see that this additional term compensates exactly the lack of information one gets in the subsequent temperature estimate (due to the fact that now only the truncated $|\nabla\mu|^2$ appears on the left hand side of (3.17)). Hence, (3.20) can still be obtained.

Remark 4.7. Finally, we notice that the regularity class (4.30)-(4.33) is not really sufficient in order for the estimates of the previous section to be rigorous. For instance, in principle we have no information on the term μ_t, which would be needed as we perform the "Key estimate". However, it is easy to realize that the regularity given for the local approximate solution is just the outcome of the fixed point argument and, hence, is not at all optimal. Further regularity properties can be standardly proved by working separately on the equations of the approximate system and performing simple bootstrap arguments. This procedure may involve some boring technical details (and may also require some further regularization of the initial data) and is omitted for brevity. However, it is worth noting at least that, for what concerns the "heat" equation (which contains the more delicate nonlinearities), the regularity obtained in the fixed-point argument is sufficient. So, it would be sufficient to bootstrap regularity for $\boldsymbol{u}$ and φ.

A Appendix

We prove here the two-dimensional interpolation-embedding inequality used in the proof of existence. It is worth noting that the result is independent of the use of periodic boundary conditions and holds in any smooth bounded subset of $\mathbb{R}^2$.

Lemma A.1. *Let $\mathcal{O}$ a smooth bounded domain in $\mathbb{R}^2$. Then, there exists $c > 0$ depending only on $\mathcal{O}$ such that*

$$\|\xi\|_{H^1(\mathcal{O})'} \le c\Big(1 + \|\xi\|_{L^1(\mathcal{O})} \log^{1/2}\big(e + \|\xi\|_{L^2(\mathcal{O})}\big)\Big) \tag{A.1}$$

for any $\xi \in L^2(\mathcal{O})$.

PROOF. We start by recalling (see, e.g., [39, (17), p. 479]) that, for all $p \in [1, \infty)$,

$$\|v\|_{L^p(\mathcal{O})} \le cp^{1/2}\|v\|_{H^1(\mathcal{O})} \quad \text{for all } v \in H^1(\mathcal{O}). \tag{A.2}$$

where the constant $c > 0$ can be taken *independent* of p. The above inequality makes precise the rate of explosion of the embedding constant of H^1 into L^p as p becomes large. As before, the value of c can vary in the computations below; in any case, c will always be intended to be independent of p.

Let us identify, as before, $L^2(\mathcal{O})$ with its dual in such a way that $L^2(\mathcal{O})$ can be (compactly and continuously) embedded into $H^1(\mathcal{O})'$. Then, given $\eta \in L^2(\mathcal{O})$, we have, for $c > 0$ as above,

$$\frac{\langle \eta, v\rangle}{\|v\|_{H^1(\mathcal{O})}} \le cp^{1/2}\frac{\langle \eta, v\rangle}{\|v\|_{L^p(\mathcal{O})}} \le cp^{1/2}\|\eta\|_{L^{p'}(\mathcal{O})}, \tag{A.3}$$

for all nonzero $v \in H^1(\mathcal{O})$ and where p' is the conjugate exponent to p. Hence,

$$\|\eta\|_{H^1(\mathcal{O})'} \leq cp^{1/2}\|\eta\|_{L^{p'}(\mathcal{O})} = cp^{1/2}\|\eta\|_{L^{\frac{p}{p-1}}(\mathcal{O})}. \tag{A.4}$$

Taking $p \geq 2$ and using interpolation, it then follows

$$\|\eta\|_{H^1(\mathcal{O})'} \leq cp^{1/2}\|\eta\|_{L^1(\mathcal{O})}^{\frac{p-2}{p}}\|\eta\|_{L^2(\mathcal{O})}^{\frac{2}{p}}. \tag{A.5}$$

Let us temporarily assume that $\|\eta\|_{L^2(\mathcal{O})} = 1$. Then, squaring, we obtain

$$\|\eta\|_{H^1(\mathcal{O})'}^2 \leq cp\|\eta\|_{L^1(\mathcal{O})}^{\frac{2(p-2)}{p}} = cp\|\eta\|_{L^1(\mathcal{O})}^2\|\eta\|_{L^1(\mathcal{O})}^{-\frac{4}{p}}. \tag{A.6}$$

Now we use the so-called Yudovich' trick (see, e.g., [40]), namely we optimize the above right hand side with respect to $p \in [2, \infty)$. To this aim, let us consider the function

$$f : [2, \infty) \to (0, \infty), \qquad f(p) := pA^{-\frac{4}{p}}, \tag{A.7}$$

where $A > 0$ is given. Then, clearly,

$$f'(p) = A^{-\frac{4}{p}}\Big(1 + \frac{4}{p}\log A\Big). \tag{A.8}$$

Now, we have to distinguish between two cases. Firstly, if $A \geq e^{-1/2}$, then f is increasing over $[2, +\infty)$, whence its minimum is achieved for $p = 2$:

$$\min f = f(2) = 2A^{-2} \leq 2e. \tag{A.9}$$

On the other hand, if $A \in (0, e^{-1/2})$, then $-4\log A$ (the zero of f') is strictly larger than 2, whence

$$\min f = f(-4\log A) = -4A^{\frac{1}{\log A}}\log A = -4e\log A. \tag{A.10}$$

Let us now choose $A = \|\eta\|_{L^1(\mathcal{O})}$ for $\eta \in L^2(\mathcal{O})$ with $\|\eta\|_{L^2(\mathcal{O})} = 1$. Then,

$$\|\eta\|_{H^1(\mathcal{O})'}^2 \leq \begin{cases} 2ec\|\eta\|_{L^1(\mathcal{O})}^2 & \text{if } \|\eta\|_{L^1(\mathcal{O})} \geq e^{-1/2} \\ -4ec\|\eta\|_{L^1(\mathcal{O})}^2 \log\big(\|\eta\|_{L^1(\mathcal{O})}\big) & \text{if } \|\eta\|_{L^1(\mathcal{O})} < e^{-1/2}. \end{cases} \tag{A.11}$$

Let us now take any nonzero $\xi \in L^2(\mathcal{O})$ and apply the above to $\eta = \xi/\|\xi\|_{L^2(\mathcal{O})}$. If

$$\|\eta\|_{L^1(\mathcal{O})} \geq e^{-1/2}, \quad \text{i.e. } \|\xi\|_{L^2(\mathcal{O})} \leq e^{1/2}\|\xi\|_{L^1(\mathcal{O})}, \tag{A.12}$$

then it follows from the first (A.11) that

$$\|\xi\|_{H^1(\mathcal{O})'}^2 \leq 2ec\|\xi\|_{L^1(\mathcal{O})}^2, \tag{A.13}$$

and, in particular, (A.1) holds. On the other hand, if

$$\|\eta\|_{L^1(\mathcal{O})} < e^{-1/2}, \quad \text{i.e. } \|\xi\|_{L^2(\mathcal{O})} > e^{1/2}\|\xi\|_{L^1(\mathcal{O})}, \tag{A.14}$$

then from the second (A.11) we obtain

$$\|\xi\|_{H^1(\mathcal{O})'}^2 \leq 4ec\|\xi\|_{L^1(\mathcal{O})}^2\big(\log\|\xi\|_{L^2(\mathcal{O})} - \log\|\xi\|_{L^1(\mathcal{O})}\big). \tag{A.15}$$

Here, we have to distinguish again some cases. First, if it is both $\|\xi\|_{L^2(\mathcal{O})} \geq 1$ and $\|\xi\|_{L^1(\mathcal{O})} \geq 1$, then (A.15) is continuated as

$$\|\xi\|^2_{H^1(\mathcal{O})'} \leq 4ec\|\xi\|^2_{L^1(\mathcal{O})} \log \|\xi\|_{L^2(\mathcal{O})}, \tag{A.16}$$

whence (A.1) follows. Second, if $\|\xi\|_{L^2(\mathcal{O})} \geq 1$ and $\|\xi\|_{L^1(\mathcal{O})} < 1$, then, observing that $0 \leq -r \log r \leq c$ for all $r \in (0,1)$, we get

$$\|\xi\|^2_{H^1(\mathcal{O})'} \leq 4ec\|\xi\|^2_{L^1(\mathcal{O})} \log \|\xi\|_{L^2(\mathcal{O})} - 4ec\|\xi\|^2_{L^1(\mathcal{O})} \log \|\xi\|_{L^1(\mathcal{O})} \leq 4ec\|\xi\|^2_{L^1(\mathcal{O})} \log \|\xi\|_{L^2(\mathcal{O})} + c, \tag{A.17}$$

and we still have (A.1). Finally, if it is both $\|\xi\|_{L^2(\mathcal{O})} < 1$ and $\|\xi\|_{L^1(\mathcal{O})} < 1$ then, we simply observe that $\|\xi\|^2_{H^1(\mathcal{O})'} \leq c\|\xi\|^2_{L^2(\mathcal{O})} \leq c$, which concludes the proof. ∎

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