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Unique determination of balls and polyhedral scatterers with a single point source wave

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Abstract

In this paper, we prove uniqueness in determining a sound-soft ball or polyhedral scatterer in the inverse acoustic scattering problem with a single incident point source wave in $\mathbb{R}^N$ $(N = 2, 3)$. Our proofs rely on the reflection principle for the Helmholtz equation with respect to a Dirichlet hyperplane or sphere, which is essentially a 'point-to-point' extension formula. The method has been adapted to proving uniqueness in inverse scattering from sound-soft cavities with interior measurement data incited by a single point source. The corresponding uniqueness for sound-hard balls or polyhedral scatterers has also been discussed.

1 Introduction

This paper is concerned with the inverse time-harmonic acoustic scattering by an impenetrable scatterer D in $\mathbb{R}^N$ $(N \geq 2)$. The incident field is given by the time-harmonic point source wave

$$u^{in}(x; y) = \begin{cases} \frac{i}{4}H_0^{(1)}(k|x-y|), & N = 2, \\ \frac{e^{ik|x-y|}}{4\pi|x-y|}, & N = 3, \end{cases} \qquad x \neq y, \tag{1.1}$$

where $k > 0$ is the wave number, $y \in \mathbb{R}^N$ the position of the point source and $H_0^{(1)}$ the Hankel function of the first kind of order zero. We consider both the classical exterior scattering problems and the fairly new interior scattering problems with near-field measurement data (see Figure 1) . To describe the scattering system, we shall use u^{sc} and u to represent the scattered and total fields, respectively, where $u = u^{in} + u^{sc}$.

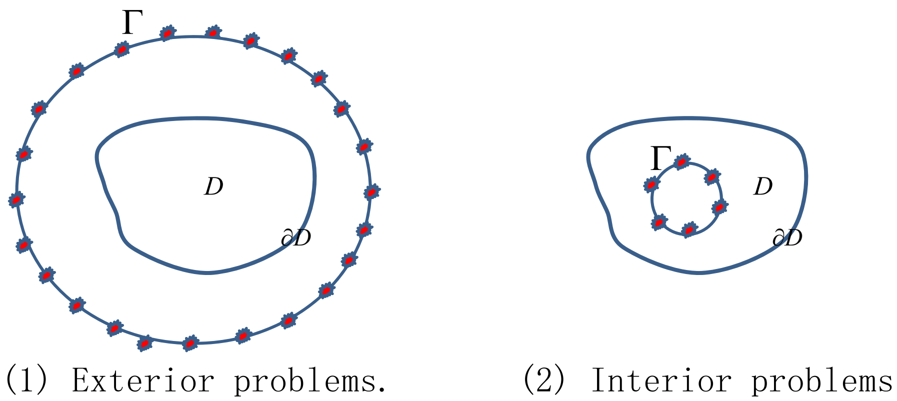

Figure 1: Object D and the curve Γ $(N = 2)$ where measurements are taken. (1): The typical exterior scattering system; (2) The fairly new interior scattering system.

Exterior scattering problems. The typical inverse scattering problems are exterior problems where the measurements (far-field or near-field data) are taken outside of the scatterer, i.e., in $D^e := \mathbb{R}^N \backslash \overline{D}$ [5]. Such problems arise in diverse areas such as medical imaging, ultrasound tomography, material science, radar, remote sensing and seismic exploration. The direct exterior scattering problem is to find the scattered field $u^{sc} \in H^1_{loc}(D^e)$ such that u^{sc} solves the Helmholtz equation

$$\Delta u^{sc} + k^2 u^{sc} \;=\; 0 \qquad \text{in} \quad D^e, \tag{1.2}$$

and satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{(N-1)/2}\left(\frac{\partial u^{sc}}{\partial r} - iku^{sc}\right) = 0, \quad r = |x|, \tag{1.3}$$

uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. If D is an impenetrable scatterer, the total field u fulfills a boundary condition of the form

$$\mathscr{B}(u) = 0 \qquad \text{on} \quad \partial D, \tag{1.4}$$

where $\mathscr{B}(u) = u$ for a sound-soft scatterer and $\mathscr{B}(u) = \partial u/\partial\nu$ for a sound-hard scatterer. In the later case, ν denotes the unit outward normal vector at $x \in \partial D$. We emphasis that D may contain several (finitely many) connected components but its exterior $D^e := \mathbb{R}^N \backslash \overline{D}$ is always connected. It is well known that there exists a unique solution $u^{sc} \in H^1_{loc}(D^e)$ to the scattering problem (1.2)-(1.4) if ∂D is Lipschitz (see e.g., [2, 5]).

Interior scattering problems. The interior scattering problems are fairly new research topics (see the recent papers [15, 16] and the references therein). These problems occur in many industrial applications of non-destructive testing where both the sources (incident waves) and measurements (scattered waves) are put inside the object D. In this case, we suppose that D is bounded and simply connected. The direct scattering is to find the scattered field $u^{sc} \in H^1(D)$ such that u^{sc} solves the Helmholtz equation

$$\Delta u^{sc} + k^2 u^{sc} = 0 \quad \text{in} \quad D, \tag{1.5}$$

and the total field u satisfies the boundary condition (1.4). The well-posedness of the direct problem has been established in [2] provided that k^2 is not an interior eigenvalue of $-\Delta$ in D with respect to the boundary condition under consideration.

Let Γ be a closed Lipschitz surface in $\mathbb{R}^N$ where the near-field measurement data are taken, and let D_0 be the bounded domain enclosed by Γ. As shown in Figure 1, we assume that

$$\Gamma \subset \begin{cases} D^e & \text{for the exterior problems;} \\ D & \text{for the interior problems.} \end{cases}$$

The inverse problem we consider is to reconstruct ∂D from the available measurements taken on Γ due to one point source wave at one single wave number. To the best of our knowledge, uniqueness is still an open problem for general scatterers without any a priori information. The aim of this paper is to provide a confirmative answer to the uniqueness in inverse scattering from balls or polyhedral scatterers. The concept of a *polyhedral scatterer* for the exterior problems is defined as follows.

Definition 1.1. A compact set $D \subset \mathbb{R}^N$ is called a *polyhedral scatterer* if ∂D is the union of finitely many cells and its exterior D^e is connected. Here a cell is defined as the closure of an open connected subset of an $(N-1)$-dimensional hyperplane.

Note that the definition of a *polyhedral scatterer* is more general than the terminology *polyhedral obstacle* used in the literature (see e.g., [1,8,9,14]). A *polyhedral obstacle* is defined as the union of finitely many convex polyhedra, which always coincides with the closure of its interior. Hence, a *polyhedral scatterer* can be equivalently defined as the union of a *polyhedral obstacle* and finitely many cells.

The uniqueness for sound-soft and sound-hard balls with a single plane wave was proved in [13,20]. In the case of an incident point source wave, the total field turns out to be singular at the source position, giving rise to essential difficulties in justifying the analytical extension of the solution from D^e into D (this plays a central role in plane wave incidence case, see [13,20] or [5, Chapter 5.1]). To overcome this difficulty, we prove that the singularity of solutions to the Helmholtz equation can be 'propagated', if the solution vanishes on a sphere; see Lemma 2.1. This property leads to uniqueness in determining sound-soft balls with an incoming point source wave. Our mathematical analysis is based on the Schwartz reflection principle for harmonic functions [12,19] combined with the constructive method for solving the exterior Dirichlet boundary value problem of the Helmholtz equation for balls in [4,6].

Recently, uniqueness results with a minimal number of incident plane waves have been obtained within the class of polyhedral scatterers; see e.g., [1,3,8,9,14]. The key ingredients in carrying out the proof include the reflection principle for the Helmholtz equation with respect to a Dirichlet or Neumann hyperplane and the essential properties of a plane wave (for instance, there is no decaying of a plane wave at infinity). It has been shown that a sound-hard (resp. sound-soft) *polyhedral scatterer* in $\mathbb{R}^N$ can be uniquely identified using N (resp. 1) incident plane waves (see [1,14]) and this number cannot be reduced in general. Since a point source wave admits the same asymptotic behavior as Sommerfeld radiating waves, the existing argument for plane waves cannot straightforwardly apply to point source waves. Motivated by the idea used in [11] and the path argument first developed in [1] and later simplified in [14], we prove that a sound-soft or sound-hard *polyhedral scatterer* can be uniquely identified from the near-field data of one point source wave at a fixed wave number; see Section 3 for the details.

The main results of our paper for the exterior scattering problems are summarized as follows:

Theorem 1.1. *Let D be either a sound-soft ball or a sound-soft polyhedral scatterer. Then, for a fixed wave number $k > 0$ and source point $y \in D^e$, the boundary ∂D can be uniquely determined by the near-field data $u^{sc}(\cdot; y)$ on Γ generated by a single incident point source wave $u^{in}(\cdot; y)$.*

The proof of Theorem 1.1 will be presented in Sections 2 and 3 for balls and polyhedral scatterers, respectively. For the interior problems of reconstructing the boundary of a simply connected domain, we restrict our discussions to sound-soft balls and *polyhedral obstacles*. In Section 4, the following analogous results of Theorem 1.1 will be proved.

Theorem 1.2. *Let D be either a sound-soft ball or a sound-soft polyhedral obstacle. Assume further that D is simply connected and k^2 is neither a Dirichlet eigenvalue of $-\Delta$ in D nor in D_0. Then, for a fixed wave number $k > 0$ and any source point $y \in D$, the boundary ∂D can be uniquely determined by the scattered field $u^{sc}(\cdot; y)$ on Γ generated by a single incident point source wave $u^{in}(\cdot; y)$.*

2 Proof of Theorem 1.1 for balls

Let $B_r(z)$ be the ball centered at $z \in \mathbb{R}^N$ with the radius $r > 0$. For simplicity, we denote by B_r the ball centered at the origin with the radius $r > 0$. A main ingredient in our proof is the following reflection principle for the Helmholtz equation with respect to a sphere. We formulate this principle only for a ball centered at the origin in Lemma 2.1 below. However, in our subsequent applications, we will mostly use the result corresponding to the ball $B_{r_0}(z)$ centered at $z \in \mathbb{R}^N$ which can be stated analogously.

Lemma 2.1. ***(Reflection principle for the Helmholtz equation W.R.T. spheres)*** *Let $u(\cdot; y)$ be a solution to the boundary value problem*

$$\begin{aligned} \Delta u(x) + k^2 u(x) &= -\delta(x-y) \qquad \text{in} \quad \mathbb{R}^N \backslash \overline{B_{r_0}}, \\ u &= 0 \qquad\qquad\qquad \text{on} \quad \partial B_{r_0}, \end{aligned} \tag{2.1}$$

where y is a fixed point in $\mathbb{R}^N \backslash \overline{B_{r_0}}$. Then $u(\cdot; y)$ can be analytically extended into the interior of B_{r_0} except for the point $y^* = (r_0/|y|)^2\, y$. Furthermore, the extension of $u(\cdot; y)$ in B_{r_0} solves the following interior boundary value problem

$$\begin{aligned} \Delta u(x) + k^2 u(x) &= (r_0/|y^*|)^{N+2} \delta(x-y^*) \quad \text{in} \quad B_{r_0}, \\ u &= 0 \qquad\qquad\qquad\qquad\qquad \text{on} \quad \partial B_{r_0}. \end{aligned} \tag{2.2}$$

The point y^* is the inversion of y with respect to the sphere $|x| = r_0$. For $r > 0$, set $r^* := r_0^2/r$. Then by the definition of y^*, it holds that $|y^*| = r_1^*$, where $r_1 = |y|$. Arguing the same as for plane waves shown in [5], we believe that Lemma 2.1 can be justified by analyzing the explicit representation of u in terms of special functions in a subtle way. However, in what follows we prefer to provide a more general argument based on the celebrated Schwartz reflection principle for harmonic functions. The following property of harmonic functions is well-known, but will be presented only for the readers' convenience.

Lemma 2.2. *Let $r_2 > r_1 = |y| > r_0$, and let $v(x; y)$ be a solution to the boundary value problem*

$$\Delta v(x) = -\delta(x-y) \quad \text{in} \quad B_{r_2} \backslash \overline{B_{r_0}}, \qquad v = 0 \quad \text{on} \quad |x| = r_0. \tag{2.3}$$

Then, $v(x; y)$ can be harmonically extended into the domain $B_{r_0} \backslash \overline{B_{r_2^*}}$ from $B_{r_2} \backslash \overline{B_{r_0}}$ except for the point y^* given as in Lemma 2.1. Furthermore, the harmonic extension of v satisfies

$$\begin{aligned} \Delta v(x) &= (r_0/|y^*|)^{N+2} \delta(x-y^*) \quad \text{in} \quad B_{r_0} \backslash \overline{B_{r_2^*}}, \\ v &= 0 \qquad\qquad\qquad\qquad\qquad \text{on} \quad \partial B_{r_0}. \end{aligned} \tag{2.4}$$

Proof. Define

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if} \quad r_0 < |x| < r_2, \quad x \neq y \\ -(r_0/|x|)^{N-2}\, v(x^*) & \text{if} \quad r_2^* < |x| < r_0, \quad x \neq y^*, \end{cases} \tag{2.5}$$

where $x^* = (r_0/|x|)^2\, x$. The definition of $\tilde{v}$ in $r_2^* < |x| < r_0$ is nothing else but the Kelvin transform of v with respect to the sphere $|x| = r_0$. One can straightforwardly check that

$$\Delta \tilde{v}(x) = -(r_0/|x|)^{N+2}\, \Delta v(x^*) = (r_0/|y^*|)^{N+2} \delta(x-y^*) \quad \text{in} \quad B_{r_0} \backslash \overline{B_{r_2^*}}, \tag{2.6}$$

and

$$v^+ = v^- = 0, \qquad \frac{\partial v^+}{\partial \nu} = \frac{\partial v^-}{\partial \nu} \qquad \text{on} \quad |x| = r_0,$$

where $\nu \in \mathbb{S}^N$ denotes the unit normal on ∂B_{r_0} pointing into $|x| > r_0$ and the superscripts $(\cdot)^{\pm}$ stand for the limits taken from outside and inside of B_{r_0}, respectively. This implies that $\tilde{v}$ is also harmonic in a small neighborhood of the interface $|x| = r_0$. Hence, $\tilde{v}$ is harmonic in $B_{r_2} \backslash \{\overline{B_{r_2^*}} \cup \{y^*, y\}\}$. Moreover, one can conclude from (2.6) that $\tilde{v}$ is singular at $x = y^*$ and $x = y$. □

A novelty in the proof of Lemma 2.2 is derivation of the singularity of the harmonic extension $\tilde{v}$ based on the singularity of v. To prove Lemma 2.1, we shall follow the spirit of Colton [6] by constructing solutions to the Helmholtz equation in terms of harmonic functions. The singularity of u at y has to be appropriately treated. The calculations in the proof of Lemma 2.1 below provide us inspirations how to deal with Neumann and impedance boundary conditions (see Remark $2.1(i)$).

Proof of Lemma 2.1 Employing spherical coordinates $(r,\theta)=(r,\theta_1,\cdots,\theta_{N-1})$ allows us to rewrite $u(x;y)=u(r,\theta)$ with $r=|x|$, where the dependence on y has been omitted. For a fixed $r_2>r_1$, set

$$y=(r_1,\theta_y),\quad I=\{(r,\theta_y): r_1\leq r\leq r_2\},\quad \Omega=B_{r_2}\backslash\{\overline{B}_{r_0}\cup I\}.$$

Clearly, I is the one-dimensional line-segment in $\mathbb{R}^N$ connecting y and the point (r_2,θ_y). Analogously to [6], we make an ansatz on the solution $u(x;y)$ in Ω by

$$u(r,\theta)=v(r,\theta)+w(r,\theta),\quad w(r,\theta)=\int_{r_0}^{r} s^{N-3}K(r,s)v(s,\theta)\,ds,\quad (r,\theta)\in\Omega, \tag{2.7}$$

where v is a harmonic solution with vanishing boundary data on $|x|=r_0$ and $K(r,s)$ is an unknown continuous function to be determined later. In order for u to be a solution of (2.1), the function w has to satisfy

$$r^2\Big(\frac{\partial^2 w}{\partial r^2}+\frac{N-1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\hat{\Delta}_\theta w\Big)+k^2r^2(w+v)=0\quad\text{in}\quad\Omega, \tag{2.8}$$

with $\hat{\Delta}_\theta$ being the Laplace-Beltrami operator on the unit sphere in $\mathbb{R}^N$. Straightforward calculations lead to the following identities in Ω :

$$\begin{aligned} & r^2\Big(\frac{\partial^2 w}{\partial r^2}+\frac{N-1}{r}\frac{\partial w}{\partial r}+k^2 w\Big)\\ = & \int_{r_0}^{r} s^{N-3}\left[K_{rr}(r,s)+\frac{N-1}{r}K_r(r,s)+k^2K(r,s)\right]r^2v(s,\theta)ds\\ & +r^{N-1}K_r(r,r)v(r,\theta)+r^2\frac{\partial}{\partial r}[r^{N-3}K(r,r)v(r,\theta)]\\ & +r^{N-2}K(r,r)v(r,\theta)(N-1), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \hat{\Delta}_\theta w &= \int_{r_0}^{r} s^{N-3}K(r,s)\hat{\Delta}_\theta v(s,\theta)\,ds\\ &= -\int_{r_0}^{r} s^{N-1}K(r,s)\left(\frac{\partial^2}{\partial s^2}+\frac{N-1}{s}\frac{\partial}{\partial s}\right)v(s,\theta)\,ds. \end{aligned} \tag{2.10}$$

Note that we have used the relation

$$\left(\frac{\partial^2}{\partial s^2}+\frac{N-1}{s}\frac{\partial}{\partial s}\right)v(s,\theta)+\frac{1}{s^2}\hat{\Delta}_\theta v(s,\theta)=0\quad\text{in}\quad\Omega$$

in deriving (2.10). Further, integration by parts in (2.10) yields

$$\begin{aligned} \hat{\Delta}_\theta w &= -r^{N-1}[K(r,r)v_s(r,\theta)-K_s(r,r)v(r,\theta)]\\ &\quad +r_0^{N-1}[K(r,r_0)v_s(r_0,\theta)-K_s(r,r_0)v(r_0,\theta)]\\ &\quad -\int_{r_0}^{r} s^{N-3}\Big[K_{ss}(r,s)+\frac{N-1}{s}K_s(r,s)\Big]s^2\,v(s,\theta)ds. \end{aligned} \tag{2.11}$$

Inserting (2.11) and (2.9) into (2.8) and using the boundary condition $v(r_0, \theta) = 0$, it follows that w is a solution of (2.8) provided the kernel $K(r, s)$ satisfies the identities

$$\begin{aligned} &\left[K_{rr}(r,s) + \frac{N-1}{r}K_r(r,s) + k^2K(r,s)\right] r^2 = \left[K_{ss}(r,s) + \frac{N-1}{s}K_s(r,s)\right] s^2, \\ &K(r, r_0) = 0, \end{aligned} \tag{2.12}$$

and the function $\tilde{K}(r) := K(r, r)$ subjects to the ordinary differential equation

$$2r\tilde{K}'(r) = -[(2N-4)\tilde{K}(r) + k^2 r^{4-N}], \quad \forall\, r \in [r_0, r_2]. \tag{2.13}$$

The unique solution to (2.13) with the compatibility condition $\tilde{K}(r_0) = K(r_0, r_0) = 0$ is given by

$$K(r, r) = -k^2/4\, r^{2-N}(r^2 - r_0^2). \tag{2.14}$$

In [6], the problem (2.12) and (2.14) was transformed into a Goursat problem for a hyperbolic equation in a cone. Consequently, the well-posedness of analytic solutions of $K(r, s)$ in $\{(s, r) : s < r, r > r_0\}$ and $\{(s, r) : s > r, r < r_0\}$ follows from the technique of successive approximations (see [10, pp. 118-119]). It is worth noting that the kernel $K(r, s)$ is independent of v.

Having determined the kernel $K(r, s)$, we now want to represent $u(r, \theta)$ in the form $u(r, \theta) = Tv(r, \theta)$ in Ω, where v will be proved to be some harmonic function in Ω and

$$Tv(r, \theta) := v(r, \theta) + \int_{r_0}^{r} s^{N-3} K(r, s) v(s, \theta)\, ds, \quad (r, \theta) \in \Omega.$$

Since T is a Volterra integral equation of the second kind and the integral kernel $K(r, s)$ is analytic, there always exists a unique solution v to the equation $Tv = u$ in Ω. Moreover, v has the same singularity as u and satisfies $v(r_0, \theta) = 0$. Hence the values of $v(r, \theta_y)$ for $r_1 < r < r_2$ can be defined by taking the limit. Applying properties of $K(r, s)$, it is easy to verify that $\Delta v = -\delta(x - y)$ for all $x \in B_{r_2} \backslash \{\overline{B_{r_0}} \cup \{y\}\}$.

To proceed with the proof, we need to extend the total field u from $B_{r_2} \backslash \{\overline{B_{r_0}} \cup \{y\}\}$ into $B_{r_0} \backslash \{\overline{B_{r_2^*}} \cup \{y^*\}\}$. Introduce the function

$$\tilde{u}(r, \theta) := T\tilde{v}(r, \theta) \quad \text{in} \quad B_{r_2} \backslash \{\overline{B_{r_2^*}} \cup \{y^*, y\}\},$$

where $\tilde{v}$ is the extension of v into $B_{r_0} \backslash \{\overline{B_{r_2^*}} \cup \{y^*\}\}$ given by (2.5). Now it can be seen that $\tilde{u}$ is the extension of u as a solution of the Helmholtz equation to $B_{r_0} \backslash \{\overline{B_{r_2^*}} \cup \{y^*\}\}$. Since $\tilde{v}$ satisfies (2.4), the extension of u in B_{r_0} satisfies the interior boundary value problem (2.2). The proof of Lemma 2.1 is finished. □

Remark 2.1. (i) A more explicit extension formula was constructed in [18] for general analytic (Dirichlet) curves, from which the result of Lemma 2.1 for a disk also follows. The investigation of the Neumann and Robin boundary conditions would lead to a Goursat problem in a cone with the Dirichlet data (2.14) and certain Robin boundary condition on $s = r_0$, which is beyond the scope of this paper.

(ii) When $|y| \to \infty$, the incident point source wave (1.1) behaves like a plane wave with the direction $-y/|y|$ and the inversion point y^* tends to the origin. Hence, the scattered field for plane waves can be analytically extended into the interior of a sound-soft ball except for its center. This fact has been used to prove uniqueness for balls by sending an incident plane wave (see [5, 13]).

We are now ready to present the proof of Theorem 1.1 for balls in $\mathbb{R}^N$.

Proof of Theorem 1.1 for balls. Assume that there are two sound-soft balls $D_1 = B_{r_1}(z_1)$, $D_2 = B_{r_2}(z_2)$ producing the same near field data $u_1^{sc}(\cdot; y) = u_2^{sc}(\cdot; y)$ on Γ for the incident point source $u^{in}(\cdot; y)$ with $y \in D_1^e \cap D_2^e$. In the following, we shall prove the coincidence of the centers and radii, i.e., $z_1 = z_2$, $r_1 = r_2$.

By Lemma 2.1, the total field $u_l(\cdot; y) = u^{in}(\cdot; y) + u_l^{sc}(\cdot; y)$ can be analytically extended into D_l except for the point

$$y_l^* := z_l + \frac{r_l^2}{|y - z_l|^2}\big(y - z_l\big), \quad l = 1, 2.$$

Denote by $\tilde{u}_l^{sc}(\cdot; y)$ the extension of $u_l^{sc}(\cdot; y)$ into the domain $\mathbb{R}^N \backslash \{y_l^*\}$. By the uniqueness of the exterior Dirichlet boundary value problem [5], the coincidence of $\tilde{u}_1^{sc}(\cdot; y)$ and $\tilde{u}_2^{sc}(\cdot; y)$ on Γ implies that $\tilde{u}_1^{sc}(x; y) = \tilde{u}_2^{sc}(x; y)$ for all x outside of Γ. Furthermore, by analytic continuation we conclude that $\tilde{u}_1^{sc}(\cdot; y)$ and $\tilde{u}_2^{sc}(\cdot; y)$ coincide in $\mathbb{R}^N \backslash \{y_1^*, y_2^*\}$. If $y_1^* \neq y_2^*$, we can construct a non-trivial radiating solution to the Helmholtz equation in the whole space $\mathbb{R}^N$, which is impossible. Hence, we get $y_1^* = y_2^* = y^*$, from which it follows the relation

$$|z_l - y|\,|z_l - y^*| = r_l^2, \quad l = 1, 2. \tag{2.15}$$

In addition, one can readily conclude that z_1, z_2, y and y^* are collinear points and y^* is located between z_j $(j = 1, 2)$ and y.

Denote by $u_l^{\infty}(\cdot; y)$ the far-field pattern of the scattered field $u_l^{sc}(\cdot; y)$. In view of the mixed reciprocity relation (see e.g., [17, Theorem 2.1.4])

$$u_l^{\infty}(\hat{x}; y) = \eta\, u_l^{sc}(y; -\hat{x}), \quad l = 1, 2, \quad \eta = \begin{cases} e^{i\pi/4}/\sqrt{8\pi k} & \text{if} \quad N = 2, \\ 1/(4\pi) & \text{if} \quad N = 3, \end{cases} \tag{2.16}$$

where $u_l^{sc}(\cdot; -\hat{x})$ denotes the scattered field generated by the incident plane wave onto D_l with the direction $-\hat{x}$. Since $u_1^{sc}(x; y) = u_2^{sc}(x; y)$ on Γ, we know $u_1^{\infty}(\hat{x}; y) = u_2^{\infty}(\hat{x}; y)$ and thus by (2.16) the relation $u_1^{sc}(y; -\hat{x}) = u_2^{sc}(y; -\hat{x})$ for all $\hat{x} \in \mathbb{S}^{N-1}$. The explicit representation of u_l^{sc} in three dimensions is given by (see e.g., [5, Chapter 3.2] when z_l coincides with the origin)

$$u_l^{sc}(y; -\hat{x}) = -\sum_{n=0}^{\infty} i^n\,(2n+1)\frac{j_n(kr_l)}{h_n^{(1)}(kr_l)}\,h_n^{(1)}(k|z_l - y|)P_n(\cos\varphi_l), \quad l = 1, 2.$$

Here, j_n denotes the spherical Bessel function of order n; $h_n^{(1)}$ the spherical Hankel function of the first kind of order n; P_n the Legendre polynomial of order n; and φ_l the angle between $(y - z_l)/|y - z_l|$ and $-\hat{x}$. By the asymptotic behavior of the spherical Bessel and Hankel functions for large n, we see

$$\frac{j_n(kr_l)}{h_n^{(1)}(kr_l)}\,h_n^{(1)}(k|z_l - y|) \sim \frac{k^n}{(2n+1)!!}\frac{r_l^{2n+1}}{|z_l - y|^{n+1}} \quad \text{as} \quad n \to \infty,$$

where $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)$. Hence, it follows from $u_1^{sc}(y; -\hat{x}) = u_2^{sc}(y; -\hat{x})$ and $\varphi_1 = \varphi_2$ for all $\hat{x} \in \mathbb{S}^{N-1}$ that

$$\frac{r_1^{2n+1}}{|z_1 - y|^{n+1}} = \frac{r_2^{2n+1}}{|z_2 - y|^{n+1}} \quad \text{for sufficiently large} \quad n \in \mathbb{N},$$

from which we conclude that

$$\frac{r_1^2}{|z_1 - y|} = \frac{r_2^2}{|z_2 - y|}. \tag{2.17}$$

Making use of (2.15), we obtain $|y^* - z_1| = |y^* - z_2|$, implying that $z_1 = z_2$. Finally, the relation $r_1 = r_2$ follows immediately from (2.15) and (2.17). This finishes the proof of Theorem 1.1 for balls in $\mathbb{R}^3$. The two dimensional case can be treated in the same manner. □

3 Proof of Theorem 1.1 for polyhedral scatterers

We first state the reflection principle for the Helmholtz equation with respect to a Dirichlet hyperplane.

Lemma 3.1. (Reflection principle for the Helmholtz equation w.r.t. planes) *Suppose that $\Omega \subset \mathbb{R}^N$ is a symmetric connected domain with respect to an $(N-1)$-dimensional hyperplane Π and that $\Lambda = \Omega \cap \Pi \neq \emptyset$. Denote by Ω^+ and Ω^- the two connected subdomains of Ω separated by Λ. If $\Delta u + k^2 u = 0$ in Ω^+ and $u = 0$ on Λ, then u can be analytically extended into Ω^- by the formula*

$$u(x) = -u(\mathcal{R}_\Pi(x)), \quad x \in \Omega^-, \tag{3.1}$$

where $\mathcal{R}_\Pi$ stands for the reflection with respect to Π.

Lemma 3.1 arises naturally from the Schwarz reflection principle for harmonic solutions which vanish on a flat surface. The corresponding principle for the Helmholtz equation was first studied in [7], in which one may also find the extension formulae under the Neumann and impedance boundary conditions. Recently, it has been used to prove uniqueness in inverse acoustic scattering from polyhedral scatterers with one or several incident plane waves (see e.g., [1, 3, 8, 14]). To prove the uniqueness for polyhedral scatterers with a single point source wave, we introduce the concept of a Dirichlet set.

Definition 3.1. Let Π be a $(N-1)$-dimensional hyperplane in $\mathbb{R}^N$. A non-void open connected component $\Lambda \subset \Pi$ will be called a Dirichlet set of u if $u = 0$ on Π.

The reflection principle described in Lemma 3.1 immediately gives us the following properties of the Helmholtz equation.

Corollary 3.1. *With the notations used in Lemma 3.1, we suppose that u is a solution to the Helmholtz equation in Ω vanishing on Λ.*

(i) *If Λ_0 is a Dirichlet set of u in Ω^+, then $\mathcal{R}_\Pi(\Lambda_0) \subset \Omega^-$ is also a Dirichlet set of u.*

(ii) *If u is singular at $y \in \Omega$, then u is also singular at $\mathcal{R}_\Pi(y)$.*

From the second assertion of Corollary 3.1, we see the number of singularities of u in Ω cannot be one. This fact will be utilized to justify the uniqueness within the *polyhedral scatterers*, since the total field has exactly one singular point (i.e., the position of the incidence point source) in the exterior of the scatterer under investigation. Our proof will be carried out by using the path and reflection arguments first developed in [1, 3] and later modified in [9, 14]. Note that in contrast to the boundedness of the Dirichlet set for incident plane waves, a Dirichlet set caused by point source waves is allowed to be unbounded. The final contradiction in our proof also differs from that using plane wave incidence.

Proof of Theorem 1.1 for polyhedral scatterers. Assume that two sound-soft polyhedral scatterers D_1 and D_2 generate the same total fields $u_1(\cdot;y) = u_2(\cdot;y)$ on Γ due to the point source located at $y \in \Omega_0$, where Ω_0 denotes the unbounded connected component of $D_1^e \cap D_2^e$. We are aimed at proving $\partial D_1 = \partial D_2$. By well-posedness of the acoustic scattering problem in $\mathbb{R}^N \backslash \overline{D_0}$ and the unique continuation of solutions to the Helmholtz equation, we see

$$u_1(x;y) = u_2(x;y) \quad \text{in} \quad \Omega_0 \backslash \{y\}. \tag{3.2}$$

If $\partial D_1 \neq \partial D_2$, without loss of generality we may always assume there exists a Dirichlet set Λ of u_1 in D_1^e. This follows from the relation (3.2) together with the fact that D_1 and D_2 are both polyhedral scatterers in the sense of Definition 1.1 and that D_j^e for $j = 1, 2$ are connected; see e.g., [1, 14]. Since u_1 is real analytic in the exterior of D_1, a Dirichlet set of u_1 in our proof always means its maximum extension in D_1^e. It might happen that Λ extends to infinity in D_1^e or Λ is identical with some Dirichlet hyperplane. Next, we shall carry out the proof by deriving a contraction. For clarity we divide our proof into three steps.

Step 1: Path argument. Introduce the set of all Dirichlet sets of u_1 by

$$\mathcal{D} = \{\tilde{\Lambda} : \tilde{\Lambda} \text{ is a Dirichlet set of } u_1 \text{ in } D_1^e\}.$$

The set $\mathcal{D} \neq \emptyset$, because $\Lambda \in \mathcal{D}$. Choose a point $y_0 \in \Lambda$ and a continuous injective curve $\gamma(t)$ for $t \geq 0$ connecting y_0 and the position y of the incident point source. Without loss of generality, we assume that $\gamma(0) = y_0$ and $\gamma(T) = y$ for some $T > 0$. Let $\mathcal{M}$ be the set of intersection points of γ with all Dirichlet sets of u_1, i.e.,

$$\mathcal{M} = \{y_n : \text{there exist } \Lambda_n \in \mathcal{D} \text{ and } t_n \geq 0 \text{ and such that } \Lambda_n \cap \gamma(t_n) = y_n \}.$$

It is clear that the points contained in $\mathcal{M}$ are uniformly bounded, since $\gamma(t)$ is a bounded curve with finite length. Moreover, it was shown in [14] that $\mathcal{M}$ is closed in the sense that if $y_n \to y'$, there exist $t' > 0$ and a Dirichlet set $\Lambda' \in \mathcal{D}$ such that $\Lambda' \cap \gamma(t') = y'$. Hence, $\mathcal{M}$ is compact, and we can find some $t^* > 0$ such that there exists $\Lambda^* \in \mathcal{D}$ intersecting with $\gamma(t)$ at $t = t^*$ and that

$$\gamma(t) \cap \mathcal{M} = \emptyset, \qquad \forall \quad T > t > t^*.$$

Note that $t^* < T$, because u_1 vanishes at $\gamma(t^*)$ but is singular at $\gamma(T) = y$.

Step 2: Reflection argument and the final contradiction. Let Π^* be the hyperplane containing Λ^*, and let $\mathcal{R}_{\Pi^*}$ denote the reflection with respect to the plane Π^*. We now apply Corollary 3.1 to prove the existence of a symmetric open set $\Omega \supset \Lambda^*$ with respect to Π^* such that $\Omega \subset D_1^e$ and $y \in \Omega$. This will be done in the following paragraph.

Choose $x^+ = \gamma(t^* + \epsilon)$ for $\epsilon > 0$ sufficiently small such that $t^* + \epsilon < t < T$ and define $x^- := \mathcal{R}_{\Pi^*}(x^+)$. Let $G^\pm$ be the connected component of $\mathbb{R}^N \backslash \{\overline{D}_1 \cup \Lambda^*\}$ containing $x^\pm$, and denote by $\Omega^\pm$ the connected component of $G^\pm \cap \mathcal{R}_{\Pi^*}(G^\mp)$ containing $x^\pm$. Setting $\Omega := \Omega^+ \cup \Lambda^* \cup \Omega^-$, we observe that $\Omega \subset D_1^e$ is a connected symmetric domain with respect to Π^* whose boundary is a subset of $(\mathcal{D} \cup \partial D_1) \cup \mathcal{R}_{\Pi^*}(\mathcal{D} \cup \partial D_1)$. Thus, by Corollary 3.1 (i), u_1 vanishes on $\partial\Omega$. What differs from the plane wave incidence case is that the domain Ω in the current situation might be unbounded. It now remains to prove $y \in \Omega$. Assume to the contrary that $y \notin \Omega$. Since $x^+ = \gamma(t^* + \epsilon) \in \Omega^+$ and the continuous curve $\gamma(t)$ for $t^* + \epsilon < t < T$ lies in D_1^e, $\gamma(t)$ must intersect $\partial\Omega \cup \mathcal{R}_{\Pi^*}(\partial D_1)$ at some $t^{**} > t^* + \epsilon$. This implies the existence of a new Dirichlet set intersecting $\gamma(t)$ at $t^{**} > t^*$, contradicting the obtained Dirichlet set Λ^* at $t = t^*$. Hence $y \in \Omega$.

Step 3: End of the proof. Let Ω and Π^* be given as in Step 2. We observe that $y^* := \mathcal{R}_{\Pi^*}(y) \in \Omega$, since $y \in \Omega$ and Ω is a connected symmetric domain with respect to Π^*. By Corollary 3.1 (ii), u_1 is also singular at y^* $(\neq y)$. However, this is a contradiction to the analyticity of u_1 in $\Omega \subset D_1^e \backslash \{y\}$. The proof of Theorem 1.1 for polyhedral scatterers is thus complete. □

We finally remark that, for sound-hard polyhedral scatterers, one should apply the even extension formula $u(x) = u(\mathcal{R}_\Pi(x))$ in place of (3.1), which is still the 'point-to-point'-kind extension. Hence, the proof for sound-soft polyhedral scatterers carries over to the sound-hard ones. However, it was shown in [7] that the extension formula with the impedance boundary condition is no longer the 'point-to-point'-kind. To the best of the authors' knowledge, it is still unknown how to prove the uniqueness with one incident plane or point source wave within the class of non-convex polyhedral obstacles of impedance-type.

4 Uniqueness with interior measurement data

This section is devoted to extending the previous results for the exterior problems to the interior problems. In contrast to the exterior problems, we have to assume that k^2 is not an eigenvalue of $-\Delta$ in D with respect to the boundary conditions under consideration to ensure the direct problem is uniquely solvable and thus the measurements make sense. Such an assumption can be removed by adding an artificial obstacle $\mathcal{B}$ $(\overline{\mathcal{B}} \subset D)$ with impedance boundary condition to the underlying scattering system [15]. Furthermore, we assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_0. However, this is not essential since we have the freedom to choose D_0.

We emphasize that the object we want to reconstruct in the interior inverse problems is a simply connected domain. If the center of the ball is given in advance, it has been proved in [16] that the radius can be uniquely determined by a single interior measurement. In the following, we remove this additional assumption and show that both the center and the radius can be uniquely determined by a single interior measurement.

Proof of Theorem 1.2. The unique determination of a *polyhedra obstacle* follows from similar arguments as in Section 3. Hence, it remains to consider the case of a sound-soft ball. We shall carry out the proof by applying Theorem 1.1 for the exterior scattering problems.

Assume that there are two sound-soft balls $D_1 = B_{r_1}(z_1)$, $D_2 = B_{r_2}(z_2)$ generating the same near-field data $u_1^{sc}(\cdot; y) = u_2^{sc}(\cdot; y)$ on Γ due to the incident point source $u^{in}(\cdot; y)$ with $y \in D_1 \cap D_2$. Set $u_l(\cdot; y) = u_l^{sc}(\cdot; y) + u^{in}(\cdot; y)$ $(l = 1, 2)$ to be the total fields for the interior scattering problems associated with D_l. Define

$$y_l^* := z_l + \frac{r_l^2}{|y - z_l|^2}\big(y - z_l\big), \quad l = 1, 2.$$

Clearly, $y_l^* \in \mathbb{R}^N \backslash \overline{D_l}$ is the inversion of y with respect to the sphere $\partial D_l,\ l = 1, 2$.

Let $w_l(\cdot; y_l^*)$ be a radiating solution to the exterior Dirichlet problem

$$\begin{aligned} \Delta w_l + k^2 w_l = -\delta(\cdot - y_l^*) \quad &\text{in} \quad \mathbb{R}^N \backslash \overline{D_l}, \\ w_l = 0 \quad &\text{on} \quad \partial D_l. \end{aligned} \tag{4.1}$$

The solution $w_l(\cdot; y_l^*)$ can be regarded as the total field generated by the incident point source wave $u^{in}(\cdot; y_l^*)$, i.e., $w_l(\cdot; y_l^*) = u^{in}(\cdot; y_l^*) + w_l^{sc}(\cdot; y_l^*)$, where $w_l^{sc}(\cdot; y_l^*)$ denotes the associated scattered field. By Lemma 2.1, $w_l(\cdot; y_l^*)$ can be analytically extended into D_l except for the point $y \in D_1 \cap D_2$. Furthermore, w_l is a solution of the following Dirichlet boundary value problem in D_l (see (2.2)):

$$\begin{aligned} \Delta w_l + k^2 w_l = -\frac{1}{\alpha_l}\delta(\cdot - y) \quad &\text{in} \quad D_l, \\ w_l = 0 \quad &\text{on} \quad \partial D_l. \end{aligned} \tag{4.2}$$

Here, $\alpha_l := -(|y - z_l|/r_l)^{N+2}$ is a constant. Since u_l/α_l is also a solution of (4.2), we get $\alpha_l w_l = u_l$ in D_l by the assumption that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_l. Define

$$v_l(x) = \alpha_l w_l(x; y_l^*) - u^{in}(x; y), \quad x \in \mathbb{R}^N \backslash \{y_l^*\}, \quad l = 1, 2.$$

Then, v_l is a radiating solution of the Helmholtz equation in $\mathbb{R}^N \backslash \{y_l^*\}$. The assumption that $u_1^{sc}(\cdot; y) = u_2^{sc}(\cdot; y)$ on Γ and the relation $\alpha_l w_l = u_l$ in D_l imply that $v_1 = v_2$ on Γ. Since k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_0, we obtain $v_1 = v_2$ in D_0 and therefore in $\mathbb{R}^N \backslash \{y_1^*, y_2^*\}$ by analytic continuation. If $y_1^* \neq y_2^*$, we can construct a non-trivial radiating solution to the Helmholtz equation in the whole space $\mathbb{R}^N$, which is impossible. Hence, we get $y_1^* = y_2^* = y^*$. The relation $v_1 = v_2$ in $\mathbb{R}^N \backslash \{y^*\}$ implies that

$$\alpha_1 w_1 = \alpha_2 w_2 \quad \text{in} \quad \mathbb{R}^N \backslash \{y^*, y\}. \tag{4.3}$$

We claim that $\alpha_1 = \alpha_2$. Actually, if $\alpha_1 \neq \alpha_2$, it follows from (4.3) and the relation $w_l(\cdot; y_l^*) = u^{in}(\cdot; y_l^*) + w_l^{sc}(\cdot; y_l^*)$ that

$$(\alpha_1 - \alpha_2) u^{in}(\cdot; y^*) = \alpha_2 w_2^{sc}(\cdot; y^*) - \alpha_1 w_1^{sc}(\cdot; y^*) \quad \text{in} \quad \mathbb{R}^N \backslash \{y^*, y\}.$$

However, this leads to an obvious contradiction since, the left hand of the above equality is singular at $x = y^*$ due to the point source wave u^{in}, while the scattered fields w_l^{sc} $(l = 1, 2)$ on the right hand are both smooth at $y^* \in \mathbb{R}^N \backslash \overline{D_1 \cup D_2}$. Hence, $\alpha_1 = \alpha_2$.

To finish the proof, we deduce from (4.3) and $\alpha_1 = \alpha_2$ that $w_1^{sc}(\cdot; y^*) = w_2^{sc}(\cdot; y^*)$ on ∂B_R, where B_R is a large ball containing D_1, D_2 and y. Finally, we obtain $z_1 = z_2$ and $r_1 = r_2$ as a sequence of Theorem 1.1. □

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