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Connection times in large ad hoc mobile networks

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ABSTRACT. We study connectivity properties in a probabilistic model for a large mobile ad-hoc network. We consider a large number of participants of the system moving randomly, independently and identically distributed in a large domain, with a space-dependent population density of finite, positive order and with a fixed time horizon. Messages are instantly transmitted according to a relay principle, i.e., they are iteratedly forwarded from participant to participant over distances $\leq 2R$, with $2R$ the communication radius, until they reach the recipient. In mathematical terms, this is a dynamic continuum percolation model.

We consider the connection time of two sample participants, the amount of time over which these two are connected with each other. In the above thermodynamic limit, we find that the connectivity induced by the system can be described in terms of the counterplay of a local, random, and a global, deterministic mechanism, and we give a formula for the limiting behaviour.

A prime example of the movement schemes that we consider is the well-known random waypoint model (RWP). Here we describe the decay rate, in the limit of large time horizons, of the probability that the portion of the connection time is less than the expectation.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Background and goals.** *Ad-hoc networks* consist of individuals in a given domain that communicate with each other via a relay principle: messages are forwarded from individual to individual as long as this transmission is local, until the message finally arrives at the recipient. This requires of course that the sender is *connected* with the recipient, i.e., that there is a chain of individuals connecting them such that all links are not larger than a given radius, the *transmission radius* or *communication radius*. This principle of message transmission within the system of participants, rather than via antennas or fixed wires, has a number of advantages over a firmly installed communication system; e.g., its installation is cheap, it does not require much maintaining, it can accommodate more information etc. A disadvantage is of course that the connectivity is not always fulfilled, i.e., it may be that two given individuals are not connected with each other and are therefore not able to exchange messages.

The advantages of such a type of system increase if the ad-hoc network becomes *mobile*, i.e., if all the individuals independently move around in the given region and transmit the messages at their present location, since in this case a fixed system of wires would be useless, and firmly located antennas would be necessary, and this may easily lead to situations of overloads in peak times. This is why mobile ad-hoc networks are increasingly in the discussion for various applications, like telecommunication, car-to-car applications for the distribution of information about the traffic situation, downloading of large data packages, and more. However, before one can seriously think about an introduction of a mobile ad-hoc system, one needs to know how reliable it is and how much information it can reliably transmit and how well the participants of the system are connected.

The mathematical analysis of the connectivity properties of a *mobile ad-hoc network (MANET)*, is the purpose of the present paper. We discuss a natural probabilistic model and derive rigorous results about the quality of the connection in this system. Roughly speaking, in our model, a large number N of participants randomly and independently move around in a given domain $D \subset \mathbb{R}^d$ with $d \geq 2$. The movement scheme considered is quite general, but later we will discuss the prime example, the *random waypoint model (RWP)*, in detail. Each of the participants carries a device that possesses a fixed *communication radius* $2R$ (the same for everybody). The domain is so large that the individuals are distributed according to a spatial density that is of finite order, but may depend on the details of the domain (this models subareas with more or less frequent visits, like forests, lakes or public places). We assume that messages are transmitted instantly, i.e., without loss of time. Then we ask, for two fixed given participants, how large, during a given time interval $[0, T]$, the amount of time is during which they are connected, their *connection time* $\tau_T^{(N)}$. This is one of the most decisive quantities in such a system, since it measures the quality of the entire system by means of two sample participants.

The regime in which we will be working is the limit of a large number of participants, coupled with the limit of a large region such that the population density (number of participants per area unit) is of finite positive order. In the language of statistical mechanics, this is the *thermodynamic limit*. We will condition on the two sample trajectories. The connection time is obviously a complex function of the entire system, but we will be able to quantify the influence of the large number of the participants on the connectivity of the two sample participants in terms of a simple function. This function is known from the theory of *continuum percolation*, which studies connectivity through a union of randomly distributed balls. It turns out that the limiting connection time has a global, deterministic part and a local, probabilistic part, the latter of which is described in terms of the mentioned function. Furthermore, it also turns out that this limit is deterministic, given the two sample participants. This is due to one of our assumptions on the movement scheme, which requires that knowledge about the walker's location at a later time point does not fix the current location with positive probability. This assumption implies a certain independence of the locations of the totality of the walkers at any two given times and leads to a deterministic limit. This is presented in Sections 1.2 and 1.3.

From the practical point of view, a very large value of the connection time is highly desirable. This can be guaranteed by a large value of the communication radius $2R$. However, one also would like to have rules at hand that tell how large this radius must be picked in order that the connection time exceeds a certain threshold. Some general answer to this question is given in Section 1.3. We explain there that, under natural conditions, the main effect that may damage the connection are time lags that any of the two sample participants spend close to the boundary of the domain D, while, in the interior of D, the local connection quality of the system super-exponentially fast tends to the optimum for $R \to \infty$, depending on the local user density only.

Furthermore, another important question is about the long-time behaviour of the connection time. More precisely, we identify the limiting fraction of the connection time by means of an ergodic theorem and estimate the probability of the unwanted event that the connection time covers only a subnormal portion of the time interval. This is an event of a downward *large deviation*, and we will show that its probability decays exponentially fast as $T \to \infty$, and we quantify an upper bound of the decay rate. For this question, we restrict to the RWP and derive some recurrence properties that may be useful also for further investigations; see Section 1.4.

1.2. **Connection time of two participants in the thermodynamic limit.** Let us introduce the model; our main result here is Theorem 1.2.

We consider a system of N particles (the participants of the mobile ad-hoc network), which randomly move within a given bounded domain in $\mathbb{R}^d$ with time horizon $[0, T]$. The N movements do not have to be Markovian, but they are independent and identically distributed; more precise assumptions follow below. The underlying probability measure and expectation are denoted by $\mathbb{P}$ and $\mathbb{E}$.

We equip every walker with a fixed communication radius $2R \in (0, \infty)$. That is, there is a direct connection between any two of them if their distance is $< 2R$. Two of the N participants, located at x and y, say, are (indirectly) connected if and only if there is a sequence $x_1, \dots, x_m$ of m other participants such that all the distances between x_i and x_{i-1} are $< 2R$ for any $i = 1, \dots, m+1$, where we put $x_0 = x$ and $x_{m+1} = y$. In other words, the $m+2$ balls around $x_0, \dots, x_{m+1}$ with radius R have pairwise a nontrivial intersection along the chain $x_0, \dots, x_{m+1}$; in particular, there is a continuous path from x to y within their union. This is fulfilled if and only if x and y lie in the same connected component of the union of the balls of radius

R centered at the N participants. In this way, we see that our model is a dynamic continuum percolation process.

It is our goal to study the thermodynamic limit of this system, i.e., we assume that the volume of the domain is of order N, and we assume that the trajectories are coupled with N in an accordingly rescaled way. That is, the length scale is $N^{1/d}$. Then it is clear that a rescaled version of this picture is better suitable for a mathematical analysis. Hence, we consider the equivalent situation of a fixed domain D and a fixed movement scheme (both not depending on N), and we put the communication radius equal to $2RN^{-1/d}$. We do not rescale the time interval $[0,T]$ by $N^{1/d}$, as this is a trivial change.

By $X^{(i)} = (X_s^{(i)})_{s\in[0,T]}$ we denote the (random) trajectory of the i-th participant, and $B(x,r)$ denotes the open ball around x with radius $r>0$. Then the set

$$D_s^{(N)} = D \cap \bigcup_{i=1}^N B\big(X_s^{(i)}, RN^{-1/d}\big)$$

is the *communication zone* at time s. We introduce the notion of connectivity at time s: for $x,y\in D$ we write

$$x \overset{N}{\underset{s}{\longleftrightarrow}} y \qquad \Longleftrightarrow \qquad x \text{ and } y \text{ lie in the same component of } D_s^{(N)}. \tag{1.1}$$

We will use this notion only for $x = X_s^{(1)}$ and $y = X_s^{(2)}$. Hence, the two participants $X^{(1)}, X^{(2)}$ are connected at time s if there is a polygon line from $X_s^{(1)}$ to $X_s^{(2)}$ with line segments of lengths $< 2RN^{-1/d}$ with the vertices being the locations of other participants at time s. Hence, $X_s^{(1)} \overset{N}{\underset{s}{\longleftrightarrow}} X_s^{(2)}$ if and only if these two can exchange a message at time s.

The main object is the *connection time*

$$\tau_T^{(N)} := \Big|\big\{s\in[0,T]\colon X_s^{(1)} \overset{N}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\big\}\Big| = \int_0^T \mathrm{d}s\, \mathbb{1}\{X_s^{(1)} \overset{N}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}, \tag{1.2}$$

the amount of time during which these two participants are connected up to T. We will analyze the connection time in the limit $N\to\infty$.

Let us state our assumptions on the random movement of the N walkers. We write $\{f>r\}$ for short for the set $\{x\in D\colon f(x)>r\}$ and use analogous notation for similarly defined sets.

Assumption 1.1 (The movement scheme)**.** The distribution of the random path $X = (X_s)_{s\in[0,T]}$ in D satisfies the following.

(i) For any $s\in(0,T]$, the location X_s possesses a continuous Lebesgue density $f_s\colon D\to[0,\infty)$.
(ii) For any $x,y\in D$ and $s,\widetilde{s}\in(0,T]$ satisfying $s<\widetilde{s}$, we have $\mathbb{P}(X_s = x \mid X_{\widetilde{s}} = y) = 0$.

Sufficient for Assumption 1.1 is the existence of a jointly continuous Lebesgue density of X_s and $X_{\widetilde{s}}$ for any $0<s<\widetilde{s}\le T$. Condition (ii) is needed for the asymptotic independence of the clusters at time s from the clusters at time $\widetilde{s}$; it allows us to neglect those walkers that define both clusters and to deal only with disjoint sets of participants that form the two clusters. Note that we do not require the continuity of the trajectories; regularity is only required for the distributions at fixed times. Assumption 1.1 is satisfied for many diffusions in D and also for many continuous-time random walks in D. For practical reasons, we are mainly interested in the random waypoint model, see below.

We need to introduce some standard objects from (static, homogeneous) continuum percolation; see [MR96] and Section 2.1 below for general background. Let $(Z_i)_{i\in\mathbb{N}}$ be a standard Poisson point process on $\mathbb{R}^d$ with intensity $\lambda\in(0,\infty)$. We define the *percolation probability* $\overline{\Theta}(\lambda,R)$ as the probability that there is a path from $B(0,R)$ to infinity that never leaves the set $U_R = \bigcup_{i\in\mathbb{N}} B(Z_i,R)$. In other words,

$\overline{\Theta}(\lambda, R)$ is the probability that U_R has a connected component with infinite Lebesgue measure that intersects $B(0, R)$. Connected components will be also called *clusters* in the sequel. By rescaling, it is easy to see that $\overline{\Theta}(\lambda, R) = \overline{\Theta}(\lambda R^d, 1)$. Furthermore, it is known that the map $\lambda \mapsto \overline{\Theta}(\lambda, R)$ is increasing and that there is a $\lambda^{(\mathrm{cr})}(R) > 0$ such that $\overline{\Theta}(\lambda, R) = 0$ for $\lambda < \lambda^{(\mathrm{cr})}(R)$ and $\overline{\Theta}(\lambda, R) > 0$ for $\lambda > \lambda^{(\mathrm{cr})}(R)$. It is known that $\overline{\Theta}(\cdot, R)$ is continuous outside the critical point $\lambda^{(\mathrm{cr})}(R)$; the continuity in this point is not known, but strongly expected. Again by rescaling, $\lambda^{(\mathrm{cr})}(R) = R^{-d}\lambda^{(\mathrm{cr})}(1)$.

The function $\overline{\Theta}$ will play a crucial rôle in the asymptotic description of our model. As we will see below, the number $\overline{\Theta}(\lambda, R)$ describes, in our spatially rescaled picture, the probability that, locally, a given participant belongs to the infinitely large cluster and has therefore connection over a macroscopic part of the space.

We introduce two notions of (non-random) connectedness in the domain D as follows. By 'path' we mean a continuous curve in D. For $\diamond \in \{\geq, >\}$ and $x, y \in D$, we write

$$x \overset{\diamond}{\underset{s}{\longleftrightarrow}} y \qquad \Longleftrightarrow \qquad \text{there exists a path from } x \text{ to } y \text{ within } \{f_s \diamond \lambda^{(\mathrm{cr})}(R)\}\,.$$

Furthermore, we introduce two versions of a limiting value of $\tau_T^{(N)}$. For $\diamond \in \{\geq, >\}$, define

$$\tau_T^{(\diamond)}(X^{(1)}, X^{(2)}) = \int_0^T \mathrm{d}s\, \mathbb{1}\{X_s^{(1)} \overset{\diamond}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}\overline{\Theta}^{(\diamond)}\big(f_s(X_s^{(1)}), R\big)\overline{\Theta}^{(\diamond)}\big(f_s(X_s^{(2)}), R\big), \tag{1.3}$$

where $\overline{\Theta}^{(>)}(\lambda, R) = \overline{\Theta}(\lambda-, R) = \lim_{s\uparrow\lambda}\overline{\Theta}(s, R)$ and $\overline{\Theta}^{(\geq)}(\lambda, R) = \overline{\Theta}(\lambda+, R) = \lim_{s\downarrow\lambda}\overline{\Theta}(s, R)$ are the left- and right-continuous versions of $\overline{\Theta}$. Recall that these two functions coincide at least everywhere outside the critical value $\lambda^{(\mathrm{cr})}(R)$.

Our main result is the following.

Theorem 1.2. *Fix $T > 0$ and $R > 0$, and assume that the distributions of the N i.i.d. random movements $X^{(1)}, \dots, X^{(N)}$ satisfy Assumption 1.1. Then, for almost every paths $X^{(1)}, X^{(2)}$, we have, in probability with respect to $\mathbb{P}(\cdot \mid X^{(1)}, X^{(2)})$,*

$$\tau_T^{(>)}(X^{(1)}, X^{(2)}) \leq \liminf_{N\to\infty} \tau_T^{(N)} \leq \limsup_{N\to\infty} \tau_T^{(N)} \leq \tau_T^{(\geq)}(X^{(1)}, X^{(2)}). \tag{1.4}$$

For a proof of Theorem 1.2 see Section 2; for a discussion about whether or not the limit in (1.4) exists and how it behaves for large R, see Section 1.3.

The assertion in (1.4) shows that the connectivity of the medium that is built out of $X^{(1)}, X^{(2)}, \dots, X^{(N)}$ may be replaced, asymptotically, by some explicit deterministic term that describes two effects: a global, deterministic one and a local, stochastic one. Indeed, the two walkers at time s are connected if and only if

- their positions $x = X_s^{(1)}$ and $y = X_s^{(2)}$ are connected by a deterministic path within the set $\{f_s \diamond \lambda^{(\mathrm{cr})}(R)\}$ (with $\diamond = \geq$ for an upper bound and $\diamond = >$ for a lower bound) and
- x and y, respectively, belong locally to the giant component of the static continuum percolation process with density $f_s(x)$ and $f_s(y)$, respectively, and ball radius $RN^{-1/d}$ (note that these two events are asymptotically independent).

We will see in the proof that the first condition implies that, with probability tending to one, certain neighbourhoods of x and y are connected through the communication zone D_N, and the probabilities of the latter condition are expressed by the two $\overline{\Theta}$-terms. This argument is based on [P95], after some modifications. Our regularity assumptions on the density f_s enables to integrate over $s \in [0, T]$. From this argument, the convergence of the expectation of certain lower, respectively upper, bounds for $\tau_T^{(N)}$ towards $\tau_T^{(>)}$, respectively to $\tau_T^{(\geq)}$, follows. Afterwards, we use the second-moment method to deduce the convergence in probability. On the way, we show that the connectivity at any two given distinct times is asymptotically independent. Here the fact crucially enters, contained in Assumption 1.1(ii), that knowledge about the walker's location at a later time does not imply any deterministicity of its location at any earlier time.

Note that the same limit in the almost-sure sense has no physical relevance, since this requires the presence of all infinitely many participants of the systems on one probability space, and this limit would depend on the order that we would attach to them, i.e., on the way in which we increase the system ('add people').

1.3. **Discussion.**

1.3.1. *Does the limit in* (1.4) *exist?* Certainly, one expects that, in many cases, $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ should coincide almost surely and in (1.4) one should have a limit. This is certainly true under many additional abstract conditions, like requiring that connection within $\{f_s \geq \lambda^{(\mathrm{cr})}(R)\}$ implies connection within $\{f_s > \lambda^{(\mathrm{cr})}(R)\}$ for almost all pairs of points, e.g. Sufficient conditions like that are certainly easy to check in many explicit situations, where the structure of the connectivity landscape given by the density f_s is easy to control. However, it is difficult to give a satisfactory sufficient condition that is both reasonably general and reasonably explicit, and therefore we abstained from that.

Part of our difficulties to decide whether or not $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ coincide come also from the fact that the continuity of $\overline{\Theta}(\cdot, R)$ in the critical point is unknown: If the set $\{f_s = \lambda^{(\mathrm{cr})}(R)\}$ has a positive Lebesgue measure (which can happen only for countably many values of R), there is a positive probability that one of the two walkers belongs to its interior for a positive portion of the time, but we do not know whether $\overline{\Theta}(\lambda^{(\mathrm{cr})}(R), R)$ is positive and appears in the limiting description of $\tau_T^{(N)}$ or not. This problem is not present if $\overline{\Theta}(\cdot, R)$ is continuous in the critical point.

Furthermore, besides questions of continuity of $\overline{\Theta}(\cdot, R)$, the coincidence of $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ is also open if, for s in some set with positive Lebesgue measure, some components of $\{f_s > \lambda^{(\mathrm{cr})}(R)\}$ are separated from each other by a component of $\{f_s = \lambda^{(\mathrm{cr})}(R)\}$ that has a complicated local structure. In dimension $d = 2$, e.g., a line with some fractal structure would pose such a question. In this case, it is unclear what local properties of the separation set would imply what connectivity probabilities of the corresponding percolation process. Finding clear criteria seems to be an open problem in the study of continuum percolation. We believe that, for related reasons, one can construct situations in which $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ do not coincide, the limit in (1.4) does not exist or is random.

1.3.2. *Behaviour of the limit in* (1.4) *for* $R \to \infty$. From a practical point of view, installing a MANET makes sense only if the degree of connectivity in the system can be guaranteed to be extremely high, at least with high probability. Hence, it is a major goal to find sufficient conditions for a large value (i.e., close to T) of the communication time. Making the communication radius R large is certainly such a criterion, but it is also important to know how strongly this parameter influences the connectivity. Based on Theorem 1.2, we want to illustrate some partial answer to this question, i.e., we want to comment on the behaviour of the asymptotic lower bound for the connection time, $\tau_T^{(>)}$.

This lower bound consists, for any time $s \in [0, T]$, of two components: The values of $\overline{\Theta}$ in the two locations of the sample trajectories, and the decision whether or not they are globally connected through the super-critical area $\{f_s > R^{-d}\lambda^{(\mathrm{cr})}(1)\}$. An important fact (see Theorem 2.2 below) is that $\overline{\Theta}(\lambda, R)$ converges super-exponentially quickly towards one for $R \to \infty$, more precisely, for any $\varepsilon > 0$ and some $C_\varepsilon > 0$,

$$\overline{\Theta}(\lambda, R) \geq 1 - \mathrm{e}^{-\lambda R^d |B(0,2)|(1-\varepsilon)}, \qquad \lambda R^d \geq C_\varepsilon.$$

This shows that the 'bad' event of being not connected at a given time s does predominantly not come from the $\overline{\Theta}$-term, but from the non-connectivity, i.e., from the indicator on the counterevent of $\{X_s^{(1)} \overset{>}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}$. It is a natural assumption that the density f_s is, for every $s \in [0,T]$, bounded away from zero in most of the domain D, except possibly close to the boundary of D and that f_s decays polynomially towards the boundary of D. Then the difference $T - \tau_T^{(>)}$ can be upper bounded by some polynomially decaying term, which depends on the time that at least one of the two walkers spends polynomially close to the boundary, and some term of the form e^{-CR^d} for the remaining time. But the time that one of the walkers spends close to the boundary of D is polynomially small in R in probability, since the density is small there. The conclusion is that bad connectivity properties of the system predominantly come from the time that the users spend close to the boundary of D, at least if the domain is homogeneously filled with users.

1.4. **Further investigations for the random waypoint model.** Let us discuss the *random waypoint model (RWP)*, which we are most interested in. The RWP is a motion dynamic in a convex compact domain D that is considered in information science as a realistic model for the random movement of a human being, e.g., a participant of a telecommunication system, see for example [R11, L04, BHPC04, LV06]. Below we show that the RWP is amenable to Theorem 1.2, and we study the large-T average of the connection time and long-time deviations from the mean in terms of large-deviation estimates.

We now introduce the RWP. We assume that the compact domain D is convex. Let $(W_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. points in D, drawn from a distribution μ on D, the *waypoint measure*. Furthermore, let $(V_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of velocities drawn from some distribution $\mathcal{V}$ on $(0,\infty)$, the *velocity measure*. The walker starts from an initial location $X_0 \in D$, heading with constant initial velocity V_1 towards the waypoint W_1 on a straight line. Having arrived at W_1, the walker immediately moves along the straight line from W_1 to W_2 with velocity V_2 and so on.

This is an extension of the classical RWP, as we admit D as any convex compact domain, μ as any distribution on D, and $\mathcal{V}$ as any distribution on $(0,\infty)$. On the other hand, we do not admit pause times that the walker spends at waypoints, as this would destroy the validity of Assumption 1.1(ii); in fact, also the statement of Theorem 1.2 would have to be altered.

We denote by $U_n = |W_{n+1} - W_n|/V_{n+1}$ the time that it takes the walker to go from the n-th to the $(n+1)$-th waypoint. Then $R_n = U_0 + U_1 + \cdots + U_{n-1}$ is the time at which the walker arrives at the n-th waypoint, W_n. We put $R_0 = 0$. Introduce the time-change $N(t) = \inf\{n \in \mathbb{N} \colon R_n > t\}$, then $W_{N(t)}$ is the waypoint that the walker is heading to at time t, $V_{N(t)}$ is his current velocity, and $R_{N(t)} - t$ is the time difference after which she/he arrives there. The position of the walker at time t is denoted by X_t. Then

$$X_t = W_{N(t)} + \frac{W_{N(t)-1} - W_{N(t)}}{|W_{N(t)-1} - W_{N(t)}|} V_{N(t)}(R_{N(t)} - t). \tag{1.5}$$

We define all these processes as right-continuous. Note that the location process $X = (X_t)_{t\in[0,\infty)}$ is not Markov, but the process

$$Y = (Y_t)_{t\in[0,\infty)} = \Big(X_t, W_{N(t)}, V_{N(t)}\Big)_{t\in[0,\infty)} \tag{1.6}$$

is a continuous-time Markov process on the state space $\mathcal{D} = D \times D \times [v_-, v_+]$.

We need to assume some regularity. Throughout the paper, we assume that the waypoint measure μ and the velocity measure $\mathcal{V}$ possess continuous Lebesgue densities on D and on some interval $[v_-, v_+] \subset (0,\infty)$, respectively. In particular, the velocities are bounded away from 0 and from ∞.

First we show that we can apply Theorem 1.2 to this movement scheme; indeed, in Section 3.1 we prove the following.

Lemma 1.3 (The RWP satisfies Assumption 1.1)**.** *We initialise the process by drawing $W_0 \in D$ and a velocity V_0 from some distributions on D respectively on $[v_-, v_+]$ having continuous densities, such that all the random variables W_0, W_1, V_0 are independent, and put $X_0 = W_0$ and X_t as in* (1.5)*. Then the RWP satisfies Assumption 1.1.*

1.4.1. *Long-time limit.* Let us consider the long-time behaviour of $\tau_T^{(\diamond)} = \tau_T^{(\diamond)}(X^{(1)}, X^{(2)})$ defined in (1.3) for $\diamond \in \{>, \geq\}$ for the RWP. We will show in Section 3.2 that the RWP is Harris ergodic and in particular possesses an invariant distribution, towards which it converges as the time grows to infinitity. In particular, the distribution of the location of the RWP, X_t, converges in total variation sense towards a probability measure μ_* on D, and it has a continuous Lebesgue density $f_*\colon D \to [0, \infty)$. However, it is not so easy to deduce convergence of $\frac{1}{T}\tau_T^{(\diamond)}$ from this, and we are not able to do so in all cases. For $\diamond \in \{>, \geq\}$, introduce

$$p_*^{(\diamond)} = \int_D \mu_*(\mathrm{d}x) \int_D \mu_*(\mathrm{d}y)\, \mathbb{1}\{x \overset{\diamond}{\underset{*}{\longleftrightarrow}} y\} \overline{\Theta}^{(\diamond)}(f_*(x), R) \overline{\Theta}^{(\diamond)}(f_*(y), R) \in [0, 1], \tag{1.7}$$

where $\overset{\diamond}{\underset{*}{\longleftrightarrow}}$ denotes connectedness within the set $\{f^* \diamond \lambda^{(\mathrm{cr})}(R)\}$. Then $p_*^{(\diamond)}$ is a measure for connectedness of two independent sites in D drawn from the limiting distribution of X_t. Furthermore, introduce

$$\tau_T^{(\diamond,*)} = \int_0^T \mathrm{d}s\, \mathbb{1}\{X_s^{(1)} \overset{\diamond}{\underset{*}{\longleftrightarrow}} X_s^{(2)}\} \overline{\Theta}^{(\diamond)}\big(f_*(X_s^{(1)}), R\big) \overline{\Theta}^{(\diamond)}\big(f_*(X_s^{(2)}), R\big), \tag{1.8}$$

the special case of $\tau_T^{(\diamond)}$ for all the random waypoint walkers starting in the invariant distribution.

Lemma 1.4 (Ergodic limit)**.** *Let $X^{(1)}$ and $X^{(2)}$ be two independent copies of X. Then*

(i) *for $\diamond \in \{>, \geq\}$,*

$$\lim_{T \to \infty} \frac{1}{T} \tau_T^{(\diamond,*)}(X^{(1)}, X^{(2)}) = p_*^{(\diamond)}, \qquad \textit{almost surely and in } L^1(\mathbb{P}), \tag{1.9}$$

(ii) *If $f_s \to f_*$ as $s \to \infty$ uniformly in D, then, in probability,*

$$\limsup_{T \to \infty} \frac{1}{T} \tau_T^{(\geq)}(X^{(1)}, X^{(2)}) \leq p_*^{(\geq)}, \qquad \textit{and} \qquad \liminf_{T \to \infty} \frac{1}{T} \tau_T^{(>)}(X^{(1)}, X^{(2)}) \geq p_*^{(>)}. \tag{1.10}$$

We will give two proofs of Lemma 1.4(i); the first one is in Section 3.3 and is based on the above mentioned ergodic theorem. The second one is in Section 3.5; it is based on a time-discrete Markov chain that is introduced in Section 1.4.2. The proof of Lemma 1.4(ii) is in Section 3.3.

In general, it is not clear if $\frac{1}{T}\tau_T^{(\diamond)}$ converges towards $p_*^{(\diamond)}$. Indeed, the critical point is the convergence of $\mathbb{1}\{x \overset{\diamond}{\underset{s}{\longleftrightarrow}} y\}$ towards $\mathbb{1}\{x \overset{\diamond}{\underset{*}{\longleftrightarrow}} y\}$ for $x, y \in D$ as $s \to \infty$, which is not true in many counterexamples, as one can easily find.

We remark here that, in cases where the limit in (1.4) exists, we expect that the limits $T \to \infty$ and $N \to \infty$ can also be interchanged without changing the value, i.e.,

$$p_*^{(>)} = \lim_{N \to \infty} \lim_{T \to \infty} \frac{1}{T} \tau_T^{(N)}.$$

Indeed, in the limit $T \to \infty$, the ergodic theorem leads to the average connection probability for two out of N i.i.d. sites drawn from the invariant distribution, and then the identification of the limit $N \to \infty$ follows from Theorem 1.2, applied to the RWP starting in the invariant distribution. We decided to leave the details of the proof to the reader.

1.4.2. *Large-T deviations.* In our next result, we describe the downward deviations of $\tau_T^{(>,*)}(X^{(1)}, X^{(2)})$, more precisely, the probability of the event $\{\tau_T^{(>,*)} \le Tp\}$ for $p \in (0, p_*^{(>)})$, in the limit $T \to \infty$. This is certainly an interesting question, since one would like to effectively bound the probability of the unwanted event of being connected over less than the average portion in the long-time limit. We show that this probability decays even exponentially fast, and we give an explicit bound for the decay rate. Because of (1.4), such a bound for $\tau_T^{(>,*)}$ (rather than for $\tau_T^{(\ge,*)}$) gives a useful upper deviation bound for $\tau_T^{(N)}$. We write $\mathbb{P}_*$ for the probability measure of the RWP if both copies $Y^{(1)}$ and $Y^{(2)}$ start from the invariant distribution.

Theorem 1.5. *For any* $p \in (0, p_*^{(>)})$,

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{P}_*(\tau_T^{(>,*)} \le Tp) < 0. \tag{1.11}$$

The proof of Theorem 1.5 is in Section 3.6. It describes an explicit upper bound for the left-hand side of (1.11) in terms of a variational problem. The main novelty lies in the proof, which describes the probability in question in terms of an interesting Markov chain with nice properties, such that the theory of large deviations may be applied in a standard way. This Markov chain is an object of independent interest, as it may serve also for other long-time investigations of the model, as well as for computer simulations.

This chain is defined as follows. We consider the times $0 \le S_1 < S_2 < \dots$ at which any of the two walkers arrives at her/his waypoint. Formally, $S_0 = 0$ and

$$S_j = \inf\Big\{t > S_{j-1} \colon W^{(1)}_{N^{(1)}(t)} \ne W^{(1)}_{N^{(1)}(S_{j-1})} \text{ or } W^{(2)}_{N^{(2)}(t)} \ne W^{(2)}_{N^{(2)}(S_{j-1})}\Big\}, \qquad j \in \mathbb{N}, \tag{1.12}$$

where the superscripts (1) and (2) mark the two walkers. Put

$$\begin{aligned} Z_j &= \big(Y^{(1)}(S_j), Y^{(2)}(S_j)\big) \\ &= \Big(\big(X^{(1)}_{S_j}, W^{(1)}_{N^{(1)}(S_j)}, V^{(1)}_{N^{(1)}(S_j)}\big), \big(X^{(2)}_{S_j}, W^{(2)}_{N^{(2)}(S_j)}, V^{(2)}_{N^{(2)}(S_j)}\big)\Big) \in \mathcal{D}^2, \quad j \in \mathbb{N}_0. \end{aligned} \tag{1.13}$$

We will see in Section 3.4 that $(Z_j)_{j\in\mathbb{N}_0}$ is a time-homogeneous, ψ-mixing and Harris ergodic Markov chain on $\mathcal{D}^2$. We will give a proof of the ergodic limit in Lemma 1.4(i) in Section 3.5, giving another formula for the limit $p_*^{(>)}$. The main object in the proof of Theorem 1.5 in Section 3.6 is the empirical pair measure of the chain $(Z_j)_{j\in\mathbb{N}_0}$, for which a large-deviation principle is known to hold.

2. Proof of Theorem 1.2

In this section, we prove our first main result, Theorem 1.2. As a preparation, we first summarise in Section 2.1 all relevant available information about continuum percolation. In Section 2.2 we find the limit of the expectation of the connection time, and in Section 2.3 we finish the proof.

2.1. **Static continuum percolation.** Let us collect some facts from (static) continuum percolation, see [MR96] or [P03]. Throughout the paper we assume that $d \ge 2$. Let $(Z_i)_{i\in\mathbb{N}}$ be a Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$. Fix a radius $R > 0$ and consider the union U_R of the balls $B(Z_i, R)$ over $i \in \mathbb{N}$. We say that two sites $x, y \in \mathbb{R}^d$ are *connected* if they belong to the same connected component of U_R. Connected components of U_R are called *clusters*. By $\mathcal{C}(x)$ we denote the cluster that contains $x \in \mathbb{R}^d$. The *percolation probability* $\Theta(\lambda, R)$ is defined as the probability that $\mathcal{C}(0)$ has infinite Lebesgue measure, which we phrase that 0 is connected with ∞. By scaling, $\Theta(\lambda, R) = \Theta(\lambda R^d, 1)$. There is a *critical threshold* $\lambda^{(\mathrm{cr})}(R) = R^{-d}\lambda^{(\mathrm{cr})}(1)$, defined by $\Theta(\lambda, R)$ being 0 for $\lambda < \lambda^{(\mathrm{cr})}(R)$ and positive for $\lambda > \lambda^{(\mathrm{cr})}(R)$. Another characterization of the critical threshold is that $|\mathcal{C}(0)| = \infty$ with positive probability for $\lambda > \lambda^{(\mathrm{cr})}(R)$ and $|\mathcal{C}(0)| < \infty$ with probability one for $\lambda < \lambda^{(\mathrm{cr})}(R)$. In the supercritical case, there exists, with probability one, a unique cluster with infinite Lebesgue measure, which we call $\mathcal{C}_\infty$. In the subcritical case, there is no cluster with infinite Lebesgue measure, almost surely, and the random variable $|\mathcal{C}(0)|$ has finite exponential moments. The map $\lambda \mapsto \Theta(\lambda, R)$ is continuous in any point, with a possible exception at the critical point, $\lambda^{(\mathrm{cr})}(R)$ [S97, Theorem 1.1]. The continuity at the critical point is an open question, but is widely conjectured to be true.

Actually, it is not Θ that we will work with in our model, for the following reason. Certainly, the points Z_i play the rôle of the locations of the participants in our telecommunication system. It will turn out that a given participant located at Z_i is well connected with the main part of the system if $B(Z_i, R)$ has a non-trivial intersection with $\mathcal{C}_\infty$; it is not necessary that Z_i itself belongs to $\mathcal{C}_\infty$. Hence, we will be working with a slightly different notion of percolation: Define $\overline{\Theta}(\lambda, R)$ as the probability that the ball $B(0, R)$ is connected with ∞, i.e., has a non-trivial intersection with $\mathcal{C}_\infty$. Obviously, $\Theta \leq \overline{\Theta}$, and $\overline{\Theta}$ shares the above mentioned properties with Θ, however with possibly different numerical values. In particular, $\overline{\Theta}$ possesses the same scaling properties, and is an increasing function of λ, and is positive above some threshold $\overline{\lambda}^{(\mathrm{cr})}(R)$ and zero below.

The two percolation notions are not much different from each other, as the following lemma shows. We believe that these facts appear in the literature, but could not find it, so we give proofs.

Lemma 2.1 (Comparison of percolation notions). (i) $\overline{\lambda}^{(\mathrm{cr})}(R) = \lambda^{(\mathrm{cr})}(R)$.
(ii) *The map* $\lambda \mapsto \overline{\Theta}(\lambda, R)$ *is continuous in* $\mathbb{R} \setminus \{\lambda^{(\mathrm{cr})}(R)\}$.

Proof. By scaling we can restrict to $R = 1$.

Proof of (i)*:* Recalling the trivial assertion $\Theta \leq \overline{\Theta}$, we deduce $\overline{\lambda}^{(\mathrm{cr})}(1) \leq \lambda^{(\mathrm{cr})}(1)$. But for $\lambda < \lambda^{(\mathrm{cr})}(1)$, there is no infinite connected component. Therefore $B(0, 1)$ cannot be connected to infinity, thus $\overline{\Theta}(\lambda, 1) = 0$, which implies $\overline{\lambda}^{(\mathrm{cr})}(1) = \lambda^{(\mathrm{cr})}(1)$.

Proof of (ii)*:* Pick $\lambda^{(\mathrm{cr})}(1) < \lambda < \lambda'$. We use a standard coupling: let Z_λ and $Z_{\lambda,\lambda'}$ be two independent standard Poisson processes on $\mathbb{R}^d$ with intensity λ and $\lambda' - \lambda$, respectively, then their union is a Poisson process with parameter λ'. By $\mathcal{C}_\infty(\lambda)$ and $\mathcal{C}_\infty(\lambda')$ we denote the unique infinite clusters (with radii of the ball equal to one) for the processes with parameters λ and λ', respectively. Certainly, $\mathcal{C}_\infty(\lambda) \subset \mathcal{C}_\infty(\lambda')$; furthermore, $\overline{\Theta}(\lambda, 1)$ is the probability that $\mathcal{C}_\infty(\lambda)$ intersects $B(0, 1)$, analogously for λ'.

Hence, $\overline{\Theta}(\lambda', 1) - \overline{\Theta}(\lambda, 1)$ is equal to the probability that $B(0, 1)$ intersects $\mathcal{C}_\infty(\lambda')$, but not $\mathcal{C}_\infty(\lambda)$. This event is contained in the event A that Z_λ does not intersect $B(0, 2)$, but there is at least one point of $Z_{\lambda,\lambda'}$ in $B(0, 2) \cap \mathcal{C}_\infty(\lambda')$. Introducing a small auxiliary parameter $\varepsilon > 0$, we distinguish now the event $E(\varepsilon)$ that the annulus $B(0, 2 + \varepsilon) \setminus B(0, 2 - \varepsilon)$ contains a point from $Z_{\lambda'}$ and its complement. On the event $A \cap E(\varepsilon)^{\mathrm{c}}$, there is one point of $Z_{\lambda,\lambda'}$ in $B(0, 2 - \varepsilon) \cap \mathcal{C}_\infty(\lambda')$. As the 1-ball around that point also belongs to $\mathcal{C}_\infty(\lambda')$, at least a certain part of $B(0, 1)$ belongs to $\mathcal{C}_\infty(\lambda')$ with positive Lebesgue measure $M_\varepsilon(d)$. However, since we are still on the event A, this part does not belong to $\mathcal{C}_\infty(\lambda)$. Hence, the volume of $[\mathcal{C}_\infty(\lambda') \cap B(0, 3R)] \setminus [\mathcal{C}_\infty(\lambda) \cap B(0, 3R)]$ is at least $M_\varepsilon(d)$.

Summarizing and writing P for the coupling probability measure, we get the upper bound

$$\begin{aligned}\overline{\Theta}(\lambda',1)-\overline{\Theta}(\lambda,1) &\le P(E(\varepsilon))+P(E(\varepsilon)^{\mathrm{c}}\cap A)\\ &\le P(E(\varepsilon))+P\Big(\big|[\mathcal{C}_\infty(\lambda')\cap B(0,3R)]\setminus[\mathcal{C}_\infty(\lambda)\cap B(0,3R)]\big|\ge M_\varepsilon(d)\Big)\\ &\le P(E(\varepsilon))+\frac{1}{M_\varepsilon(d)}E\Big[\big|[\mathcal{C}_\infty(\lambda')\cap B(0,3R)]\setminus[\mathcal{C}_\infty(\lambda)\cap B(0,3R)]\big|\Big]\\ &= P(E(\varepsilon))+\frac{1}{M_\varepsilon(d)}\int_{B(0,3)}\mathrm{d}x\,\Big(P(x\in\mathcal{C}_\infty(\lambda'))-P(x\in\mathcal{C}_\infty(\lambda))\Big)\\ &= P(E(\varepsilon))+\frac{1}{M_\varepsilon(d)}|B(0,3)|\big(\Theta(\lambda',1)-\Theta(\lambda,1)\big),\end{aligned}$$

where in the third line we used Markov's inequality, and in the fourth line we used Fubini's theorem. This yields the result, picking ε small (obviously, $P(E(\varepsilon))$ vanishes) and making then $\lambda'\downarrow\lambda$. □

Rigorous bounds in $d=2$ are $0.174<\lambda^{(\mathrm{cr})}(1)<0.843$, the numerical value is $\lambda^{(\mathrm{cr})}(1)\approx 0.6763475$, derived by computer simulations [QZ07].

Furthermore, we have, that $\overline{\Theta}(\lambda,R)=\overline{\Theta}(\lambda R^d,1)$ converges super-exponentially quickly towards one for $R\to\infty$. More precisely, [P91], Corollary of Theorem 3 states that

Theorem 2.2 (Exponential bounds for $\overline{\Theta}$)**.**

$$\lim_{\lambda\to\infty}\frac{\log\big(1-\overline{\Theta}(\lambda,1)\big)}{\lambda|B(0,2)|}=-1. \tag{2.1}$$

2.2. **Limiting expectation of the connection time.** We fix $T>0$ for the remainder of the section. In the following, we abbreviate

$$\mathbb{P}_{1,2}(\cdot)=\mathbb{P}\big(\cdot\,|\,X^{(1)},X^{(2)}\big)\qquad\text{and}\qquad\mathbb{E}_{1,2}[\,\cdot\,]=\mathbb{E}\big[\cdot\,|\,X^{(1)},X^{(2)}\big].$$

Use (1.2) and Fubini's theorem to see that

$$\mathbb{E}_{1,2}[\tau_T^{(N)}]=\int_0^T\mathrm{d}s\,\mathbb{P}_{1,2}\Big(X_s^{(1)}\xleftrightarrow[s]{N}X_s^{(2)}\Big).$$

We are going to approximate the event $\{X_s^{(1)}\xleftrightarrow[s]{N}X_s^{(2)}\}$ by the event that $X_s^{(1)}$ and $X_s^{(2)}$ are separated from each other, but connected through either $\{f_s>\lambda^{(\mathrm{cr})}(R)\}$ or through $\{f_s\ge\lambda^{(\mathrm{cr})}(R)\}$ and belong locally to the macroscopic part of the communication zone. More precisely, for $s\in[0,T]$, $\delta>0$ and $N\in\mathbb{N}$, we introduce the events

$$G_{N,s,\delta}^{(i)}=\Big\{X_s^{(i)}\xleftrightarrow[s]{N}\partial\big[X_s^{(i)}+(-\delta/2,\delta/2)^d\big]\Big\},\qquad i\in\{1,2\},$$

that $X_s^{(i)}$ and at least some point of the boundary of the $\delta/2$-box around $X_s^{(i)}$ lie in the same connected component of the union of the $RN^{-1/d}$-balls around $X_s^{(1)},\dots,X_s^{(N)}$. Note that $G_{N,s,\delta}^{(i)}$ only depends on the walkers within the δ-box around $X_s^{(i)}$.

We will give bounds for the connection time $\tau_T^{(N)}$ in terms of

$$\tau_T^{(N,\delta,\diamond)}(X^{(1)},X^{(2)})=\int_0^T\mathrm{d}s\prod_{i=1}^2\mathbb{1}_{G_{N,s,\delta}^{(i)}}\mathbb{1}\{|X_s^{(1)}-X_s^{(2)}|>3\delta\}\mathbb{1}\{X_s^{(1)}\xleftrightarrow[s]{\diamond}X_s^{(2)}\}, \tag{2.2}$$

in the limit $N\to\infty$, followed by $\delta\downarrow 0$. We will use $\tau_T^{(N,\delta,>)}$ as a lower bound and $\tau_T^{(N,\delta,\ge)}$ as an upper bound for $\tau_T^{(N)}$. Recall the quantities $\tau_T^{(\diamond)}$ defined in (1.3), which will serve as limiting objects of $\tau_T^{(N,\delta,\diamond)}$.

Proposition 2.3 (Limiting expectation of $\tau_T^{(N)}$)**.** *Let the distributions of the N i.i.d. walkers satisfy Assumption 1.1(i). Then, for $\mathbb{P}$-almost all $X^{(1)}$ and $X^{(2)}$, provided that R is choosen such that $\int_0^t \mathrm{d}s\, \mathbb{1}\{f_s(X_s^{(i)}) = \lambda^{(\mathrm{cr})}(R)\} = 0$, for $i = 1, 2$,*

(i)

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{E}_{1,2}\big(\tau_T^{(N)} - \tau_T^{(N,\delta,\geq)}\big) \;\leq\; 0, \tag{2.3}$$

$$\liminf_{\delta\downarrow 0}\liminf_{N\to\infty}\mathbb{E}_{1,2}\big(\tau_T^{(N)} - \tau_T^{(N,\delta,>)}\big) \;\geq\; 0. \tag{2.4}$$

(ii) *For any $\diamond \in \{>, \geq\}$,*

$$\lim_{\delta\downarrow 0}\lim_{N\to\infty}\mathbb{E}_{1,2}[\tau_T^{(N,\delta,\diamond)}] = \tau_T^{(\diamond)}(X^{(1)}, X^{(2)}). \tag{2.5}$$

The main step in the proof is the following.

Lemma 2.4. *Let the distributions of the N i.i.d. walkers satisfy Assumption 1.1(i). Then, for $\mathbb{P}$-almost all $X^{(1)}$ and $X^{(2)}$, for almost any $s \in [0, T]$ and on the event $\{f_s(X_s^{(1)}) \neq \lambda^{(\mathrm{cr})}(R)\} \cap \{f_s(X_s^{(2)}) \neq \lambda^{(\mathrm{cr})}(R)\} \cap \{X_s^{(1)} \neq X_s^{(2)}\}$,*

(i)

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{P}_{1,2}\Big[\Big(X_s^{(1)} \overset{N}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\Big)\Big\backslash\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)} \cap \{X_s^{(1)} \overset{\geq}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}\Big)\Big] \;\leq\; 0, \tag{2.6}$$

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{P}_{1,2}\Big[\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)} \cap \{X_s^{(1)} \overset{>}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}\Big)\Big\backslash\Big(X_s^{(1)} \overset{N}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\Big)\Big] \;\leq\; 0. \tag{2.7}$$

(ii)

$$\begin{aligned}\overline{\Theta}(f_s(X_s^{(1)})-, R)\overline{\Theta}(f_s(X_s^{(2)})-, R) &\leq \liminf_{\delta\downarrow 0}\liminf_{N\to\infty}\mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big)\\ &\leq \limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big)\\ &\leq \overline{\Theta}(f_s(X_s^{(1)})+, R)\overline{\Theta}(f_s(X_s^{(2)})+, R).\end{aligned} \tag{2.8}$$

Proof. Fix s and let us abbreviate $x = X_s^{(1)}$ and $y = X_s^{(2)}$. Under $\mathbb{P}_{1,2}$, only the sites $X_s^{(3)}, \dots, X_s^{(N)}$ are random (in fact, they are i.i.d. with density f_s), but the notion of connectedness and components induced by the point process refer to *all* the balls $B(X_s^{(i)}, RN^{-1/d})$ with $i = 1, 2, \dots, N$.

We are going to decompose into separate cases, and for all cases show (i) and (ii).

First we consider the case that $f_s(x) < \lambda^{(\mathrm{cr})}(R)$ or $f_s(y) < \lambda^{(\mathrm{cr})}(R)$, in which case the events $\{X_s^{(1)} \overset{>}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}$ and $\{X_s^{(1)} \overset{\geq}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}$ are not fulfilled. Without loss of generality, let us assume that $f_s(x) < \lambda^{(\mathrm{cr})}(R)$. Choose $\delta > 0$ so small that the δ-box around x does not contain y and that $f_s < \lambda^{(\mathrm{cr})}(R)$ within that box. We apply [P95, Proposition 2] for $\varepsilon = \delta/4$ and obtain that, with $\mathbb{P}_{1,2}$-probability tending to one as $N \to \infty$, any connected component of $\bigcup_{i=3}^N B(X_s^{(i)}, RN^{-1/d})$ in this cube has a diameter $\leq \varepsilon$. In particular, with $\mathbb{P}_{1,2}$-probability tending to one, x is not connected with the boundary of the cube $x + (-\delta, \delta)^d$ and therefore neither with y. This proves (2.6) and (2.7) in this case. (2.8) is trivial, as all terms are zero, by the previous argument.

In the second part of the proof, we assume that x and y belong to the same component of $\{f_s > \lambda^{(\mathrm{cr})}(R)\}$, in which case both events $\{X_s^{(1)} \overset{>}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}$ and $\{X_s^{(1)} \overset{\geq}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}$ are fulfilled. Pick some auxiliary parameter $\eta > 0$ that is smaller than $f_s(x) - \lambda^{(\mathrm{cr})}(R)$ and smaller than $f_s(y) - \lambda^{(\mathrm{cr})}(R)$. Now, using the continuity of f_s in accordance with Assumption 1.1(i), pick $\delta > 0$ so small that $x + (-\delta, \delta)^d$ and $y + (-\delta, \delta)^d$ have positive distance and that f_s takes values in $[f_s(x) - \eta, f_s(x) + \eta]$ in $x + (-\delta, \delta)^d$ and values in $[f_s(y) - \eta, f_s(y) + \eta]$ in $y + (-\delta, \delta)^d$ and such that there exists a set of the form $U = \bigcup_{i=0}^m 2\delta z_i + [-\delta, \delta]^d$ in $\{f_s > \lambda^{(\mathrm{cr})}(R)\}$ with $m \in \mathbb{N}$, $z_1, \dots, z_m \in \mathbb{Z}^d$ such that z_i and z_{i-1} are nearest neighbors for any $i = 1, \dots, m$ and $x + (-\delta, \delta)^d \subset U$ and $y + (-\delta, \delta)^d \subset U$ and $f_s > \lambda^{(\mathrm{cr})}(R)$ inside U. That this is possible is easy to see by elementary continuity and compactness arguments. Since U is a compact subset of $\{f_s > \lambda^{(\mathrm{cr})}(R)\}$, the density f_s is even bounded away from $\lambda^{(\mathrm{cr})}(R)$ on U.

Let $\mathcal{C}_{x,\delta}^{(s,N)}$ and $\mathcal{C}_{y,\delta}^{(s,N)}$, respectively, denote the largest component of the union of the $RN^{-1/d}$-balls around the points $X_s^{(1)}, \dots, X_s^{(N)}$ which lie in $x + (-\delta, \delta)^d$, respectively in $y + (-\delta, \delta)^d$. According to [P95, Proposition 3], with $\mathbb{P}_{1,2}$-probability tending to one as $N \to \infty$, these are the only ones in the respective boxes whose size (measured in terms of the number of i such that $X_s^{(i)}$ belongs to it) is of order N, and they are also uniquely determined by requiring their diameter of positive order. In particular, as $N \to \infty$, the probability of the symmetric difference between the events $\{x \in \mathcal{C}_{x,\delta}^{(s,N)}\}$ and $G_{N,s,\delta}^{(1)}$ (respectively $\{y \in \mathcal{C}_{y,\delta}^{(s,N)}\}$ and $G_{N,s,\delta}^{(2)}$) goes to zero. By [P95, Proposition 4], such a unique cluster, $\mathcal{C}_U^{(s,N)}$ also exists for the set U. Hence, with $\mathbb{P}_{1,2}$-probability tending to one, both $\mathcal{C}_{x,\delta}^{(s,N)}$ and $\mathcal{C}_{y,\delta}^{(s,N)}$ belong to $\mathcal{C}_U^{(s,N)}$. This implies that with probability tending to one as $N \to \infty$, the symmetric difference between the event $\{x \overset{N}{\underset{s}{\longleftrightarrow}} y\}$ and the event $G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}$ goes to zero, which implies (2.7) and (2.6).

To prove (2.8), we show now that the two events $G_{N,s,\delta}^{(1)}$ and $G_{N,s,\delta}^{(2)}$ are asymptotically independent with $\mathbb{P}_{1,2}$-probabilities tending to $\overline{\Theta}(f_s(x), R)$ and $\overline{\Theta}(f_s(y), R)$, respectively. Indeed, first note that, for every sufficiently large N such that the ball diameter $2RN^{-1/d}$ is less than the distance between $x + (-\delta, \delta)^d$ and $y + (-\delta, \delta)^d$. Hence, the positions of the points falling in $x + (-\delta, \delta)^d$ and $y + (-\delta, \delta)^d$ are independent, conditionally on their numbers. These two numbers are binomially distributed with parameters N and $\mu_s(x+(-\delta, \delta)^d))$ and $\mu_s(y+(-\delta, \delta)^d)$, respectively. Therefore, by classical arguments, they stochastically dominate, with $\mathbb{P}_{1,2}$-probability tending to one, the Poisson law with parameters $N(\mu_s(x + (-\delta, \delta)^d) - \eta)$ and $N(\mu_s(y + (-\delta, \delta)^d) - \eta)$, respectively. Note that the events $G_{N,s,\delta}^{(1)}$ and $G_{N,s,\delta}^{(2)}$ are monotonic in the intensity, i.e., their $\mathbb{P}_{1,2}$-probability is not larger than the $\mathbb{P}_{1,2}$-probability of the same event under continuum percolation in $x + (-\delta, \delta)^d$ and $y + (-\delta, \delta)^d$ with intensity parameters $f_s(x) - 2\eta$ and $f_s(y) - 2\eta$, respectively, and ball diameter $RN^{-1/d}$. Since we are now considering Poisson point processes, the events are independent. Their respective probabilities converge towards $\overline{\Theta}(f_s(x) - 2\eta, R)$ and $\overline{\Theta}(f_s(y) - 2\eta, R)$. Since this is true for any η, we can use the continuity of $\overline{\Theta}(\cdot, R)$, to obtain the lower bound in (2.8). The upper bound is proved in a similar manner, using that $\overline{\Theta}(\lambda)$ is the limiting probability that the origin is connected with the boundary of a centered cube for diverging radius.

In the third case, we have $f_s(x) > \lambda^{(\mathrm{cr})}(R)$ and $f_s(y) > \lambda^{(\mathrm{cr})}(R)$, and $x \overset{\geq}{\underset{s}{\longleftrightarrow}} y$, but not $x \overset{>}{\underset{s}{\longleftrightarrow}} y$, in which case (2.7) is trivial, as the event inside the probability is empty. To prove (2.6), it is enough to see that, deterministically, the existence of a path between x and y implies $G_{N,s,\delta}^{(1)}$ and $G_{N,s,\delta}^{(2)}$. (2.8) follows here from the same arguments as before.

In the fourth case, we have $f_s(x) > \lambda^{(\mathrm{cr})}(R)$ and $f_s(y) > \lambda^{(\mathrm{cr})}(R)$, but not $x \overset{\geq}{\underset{s}{\longleftrightarrow}} y$. Indeed, to prove (2.7) and (2.6) it is enough to check that, with probability tending to one, x and y are not connected in the union of the $RN^{-1/d}$-balls around the points $X_s^{(1)}, \dots, X_s^{(N)}$. Here it is intuitively clear that any path between x and y has to cross a non-trivial zone where $f_s < \lambda^{(\mathrm{cr})}(R)$ and that this disconnects x and y in the limit. Let us give a proof.

First we argue that there is a (deterministic) compact set $\Gamma \subset D$ and $\varepsilon, \gamma > 0$ such that $\Gamma \subset \{f_s \leq \lambda^{(\mathrm{cr})}(R) - \varepsilon\}$ and every path connecting x and y passes through Γ for at least γ space units. Indeed, since $x \overset{\geq}{\underset{s}{\longleftrightarrow}} y$ does not hold, x and y lie in disjoint components of $\{f_s \geq \lambda^{\rm (cr)}(R)\}$. Hence, both these components have a positive distance η to the remainder of $\{f_s \geq \lambda^{\rm (cr)}(R)\}$, since these three sets are compact and mutually disjoint. Abbreviate

$$\Gamma_\alpha = \{z \in D\colon \mathrm{dist}(z, \{f_s \geq \lambda^{\rm (cr)}(R)\}) \geq \alpha\}, \qquad \alpha > 0,$$

and pick $\Gamma = \Gamma_{\eta/16}$. Then every path from x to y passes at least a distance $\geq \gamma = \eta - 2\eta/16 = 7\eta/8$ through Γ. By continuity of f_s, this set Γ is compact and is contained in $\{f_s \leq \lambda^{\rm (cr)}(R) - \varepsilon\}$ for some $\varepsilon > 0$.

Second, we argue that, with $\mathbb{P}_{1,2}$-probability tending to one as $N \to \infty$, any connected component of $\bigcup_{i=3}^N B(X_s^{(i)}, RN^{-1/d})$ in Γ has a diameter $\leq \gamma/2$. Indeed, consider the neighborhood $\widetilde{\Gamma} = \Gamma_{\eta/32}$ of Γ, then, for N sufficiently large, the connected components inside Γ do not depend on the configuration outside $\widetilde{\Gamma}$. By continuity of f_s, on $\widetilde{\Gamma}$, the function f_s is still bounded away from $\lambda^{\rm (cr)}(R)$, say it is $\leq \lambda^{\rm (cr)}(R) - \widetilde{\varepsilon}$ for some $\widetilde{\varepsilon} > 0$. We upper bound the probability of having any connected component inside $\widetilde{\Gamma}$ of diameter $> \gamma/2$ against the same probability under the homogeneous Poisson point process with intensity parameter $\lambda^{\rm (cr)}(R) - \widetilde{\varepsilon}/2$ on some cube that contains $\widetilde{\Gamma}$ (see the above argument). Now, as this intensity parameter is subcritical, this probability tends to zero as $N \to \infty$.

Now we finish the proof of (2.6) and (2.7) in the fourth case. Indeed, the existence of a connection from x to y through $\bigcup_{i=1}^N B(X_s^{(i)}, RN^{-1/d})$ implies the existence of at least one connected component of this set in Γ of diameter $\geq \gamma$, since any path from x to y passes at least a distance $\geq \gamma$ through Γ. But, as we saw in the second step, the probability of this existence tends to zero as $N \to \infty$.

As before, (2.8) follows here from the arguments of the second case. □

Proof of Proposition 2.3. Observe that

$$\begin{aligned}
&\mathbb{E}_{1,2}\big(\tau_T^{(N)} - \tau_T^{(N,\delta,\geq)}\big)\\
&\leq \int_0^T \mathrm{d}s\, \Big(\mathbb{P}_{1,2}\Big[\Big(X_s^{(1)} \overset{N}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\Big) \Big\backslash \Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)} \cap \{X_s^{(1)} \overset{\geq}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}\Big)\Big] \mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| \geq 3\delta\}\\
&\qquad \cdot \mathbb{1}\{f_s(X_s^{(1)}) \neq \lambda^{\rm (cr)}(R)\} \mathbb{1}\{f_s(X_s^{(2)}) \neq \lambda^{\rm (cr)}(R)\}\\
&\qquad + \mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| < 3\delta\} + \mathbb{1}\{f_s(X_s^{(1)}) = \lambda^{\rm (cr)}(R)\} + \mathbb{1}\{f_s(X_s^{(2)}) = \lambda^{\rm (cr)}(R)\}\Big).
\end{aligned} \tag{2.9}$$

Hence, by (2.6),

$$\limsup_{\delta\downarrow 0} \limsup_{N\to\infty} \mathbb{E}_{1,2}\big(\tau_T^{(N)} - \tau_T^{(N,\delta,\geq)}\big) \leq \int_0^T \mathrm{d}s\, \mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| = 0\},$$

according to our assumption on R. Note that, almost surely, $\int_0^T \mathrm{d}s\, \mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| = 0\} = 0$, since $X_s^{(1)}$ and $X_s^{(2)}$ are independent with density f_s for any $s \in [0, T]$. Hence, the proof of (2.3) is finished. The proof of (2.4) is done in the same way using (2.7). Hence, part (i) is proved.

Now we turn to the proof of (ii).

Note that our assumptions exclude that $f_s(X_s^{(i)}) = \lambda^{\rm (cr)}(R)$ outside a set of measure zero. Therefore this does not appear in the integral. Furthermore by Lemma 2.1, $\overline{\Theta}$ is continuous excepted maybe for $\lambda^{\rm (cr)}(R)$. Therefore for almost every s, (2.8) reformulates to

$$\begin{aligned}
\lim_{\delta\downarrow 0} \liminf_{N\to\infty} \mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big) &= \lim_{\delta\downarrow 0} \limsup_{N\to\infty} \mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big)\\
&= \overline{\Theta}(f_s(X_s^{(1)}), R)\overline{\Theta}(f_s(X_s^{(2)}), R).
\end{aligned} \tag{2.10}$$

Thus (ii) follows by Lebesgue's theorem. □

2.3. **Finish of the proof.** The second main step in proving Theorem 1.2 is the following lemma. Recall that $\mathbb{P}_{1,2}$ denotes the conditional distribution given $X^{(1)}$ and $X^{(2)}$.

Lemma 2.5 ($\tau_T^{(N,\delta,\diamond)}$ is asymptotically deterministic)**.** *Let the distributions of the N i.i.d. walkers satisfy Assumption 1.1(i) and (ii). Then, for any $\diamond \in \{>, \geq\}$, for almost every paths $X^{(1)}, X^{(2)}$, the difference $\tau_T^{(N,\delta,\diamond)} - \mathbb{E}_{1,2}[\tau_T^{(N,\delta,\diamond)}]$ vanishes as $N \to \infty$, followed by $\delta \downarrow 0$, in $\mathbb{P}_{1,2}$-probability, provided that R is choosen such that $\int_0^T \mathrm{d}s\, \mathbb{1}\{f_s(X_s^{(i)}) = \lambda^{(\mathrm{cr})}(R)\} = 0$ for $i = 1, 2$.*

Proof. The claimed convergence follows, by Chebyshev's inequality, from the fact that the $\mathbb{P}_{1,2}$-variance of $\tau_T^{(N,\delta,\diamond)}$ vanishes. Writing $\mathbb{V}_{1,2}$ for the $\mathbb{P}_{1,2}$-variance, this is equal to

$$\begin{aligned}\mathbb{V}_{1,2}(\tau_T^{(N,\delta,\diamond)}) = \int_0^T \mathrm{d}s \int_0^T \mathrm{d}\widetilde{s}\, \mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| > 3\delta\} \mathbb{1}\{X_s^{(1)} \overset{\diamond}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\} \\ \cdot \mathbb{1}\{|X_{\widetilde{s}}^{(1)} - X_{\widetilde{s}}^{(2)}| > 3\delta\} \mathbb{1}\{X_{\widetilde{s}}^{(1)} \overset{\diamond}{\underset{\widetilde{s}}{\longleftrightarrow}} X_{\widetilde{s}}^{(2)}\}\end{aligned} \tag{2.11}$$

$$\cdot \Big[\mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)} \cap G_{N,\widetilde{s},\delta}^{(1)} \cap G_{N,\widetilde{s},\delta}^{(2)}\Big) - \mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big)\mathbb{P}_{1,2}\Big(G_{N,\widetilde{s},\delta}^{(1)} \cap G_{N,\widetilde{s},\delta}^{(2)}\Big)\Big].$$

We now show, for any $s \neq \widetilde{s}$, that the limit superior of the term in $[\cdots]$ is not positive. This finishes the proof by Lebesgue's theorem.

We abbreviate $x = X_s^{(1)}$ and $\widetilde{x} = X_{\widetilde{s}}^{(1)}$ and $y = X_s^{(2)}$ and $\widetilde{y} = X_{\widetilde{s}}^{(2)}$. Without loss of generality, we assume that $s < \widetilde{s}$, $x \neq y$ and $\widetilde{x} \neq \widetilde{y}$. Furthermore we also may and will assume that $x \overset{\geq}{\underset{s}{\longleftrightarrow}} y$ and $\widetilde{x} \overset{\geq}{\underset{\widetilde{s}}{\longleftrightarrow}} \widetilde{y}$. Without loss of generality, all the four terms $f_s(x)$, $f_s(y)$, $f_{\widetilde{s}}(\widetilde{x})$ and $f_{\widetilde{s}}(\widetilde{y})$ are larger than $\lambda^{(\mathrm{cr})}(R)$. Let, as in the proof of Lemma 2.4, $\mathcal{C}_{x,\delta}^{(s,N)}$ denote the biggest component of the union of the $R/N^{1/d}$-balls around $X_s^{(3)}, X_s^{(4)}, \dots, X_s^{(N)}$ within $x + (-\delta, \delta)^d$, analogously for $y, \widetilde{s}, \widetilde{x}$ and $\widetilde{y}$.

We recall from the proof of Lemma 2.4 that the probability of the symmetric difference between $G_{N,t,\delta}^{(i)}$ and the event $\big\{X_t^{(i)} \in \mathcal{C}_{X_t^{(i)},\delta}^{(t,N)}\big\}$, $i = 1, 2$ and $t = s, \widetilde{s}$, tends to zero as N goes to infinity, followed by $\delta \to 0$. This reduces the problem to showing that

$$\begin{aligned}\limsup_{\delta\to 0} \limsup_{N\to\infty} \Big[\mathbb{P}_{1,2}\big(x \in \mathcal{C}_{x,\delta}^{(s,N)}, y \in \mathcal{C}_{y,\delta}^{(s,N)}, \widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)}, \widetilde{y} \in \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},N)}\big) \\ - \mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big)\mathbb{P}_{1,2}\Big(G_{N,\widetilde{s},\delta}^{(1)} \cap G_{N,\widetilde{s},\delta}^{(2)}\Big)\Big] \leq 0.\end{aligned} \tag{2.12}$$

We pick $\delta > 0$ smaller than $\frac{1}{3}\min\{|x-y|, |\widetilde{x}-\widetilde{y}|\}$. Let us give some heuristic explanation of the following argument. To get (2.12), we only have to prove that, with probability tending to one as $N \to \infty$, the partial clusters $\mathcal{C}_{x,\delta}^{(s,N)} \cup \mathcal{C}_{y,\delta}^{(s,N)}$, and $\mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)} \cup \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},N)}$, depend only on two disjoint subcollections of $X^{(3)}, \dots, X^{(N)}$ or at least on subcollections with a small overlap. This will follow from Assumption 1.1(ii). In more technical terms, it says the following. By $\mathcal{B}(D)$ we denote the Borel-σ-field on D. Let a version of the conditional distribution of X_s given $X_{\widetilde{s}} = y$ be given, i.e., a Markov kernel $K_{s,\widetilde{s}}\colon D \times \mathcal{B}(D) \to \mathcal{B}(D)$ such that, almost surely, $\mathbb{P}(X_s \in A \mid X_{\widetilde{s}} = y) = K_{s,\widetilde{s}}(y, A)$ for any $A \in \mathcal{B}(D)$. Then we require that $K_{s,\widetilde{s}}(y, \{x\}) = 0$ for any $x \in D$. Indeed, this assumption implies that, for any $y \in D$,

$$\lim_{\delta\downarrow 0} \mathbb{P}(X_s \in B(x, \delta) \mid X_{\widetilde{s}} = y) = \lim_{\delta\downarrow 0} K_{s,\widetilde{s}}(y, B(x, \delta)) = K_{s,\widetilde{s}}(y, \{x\}) = 0. \tag{2.13}$$

Since the probability on the left-hand side is continuous in y and monotonous in δ, the convergence is even uniform in $y \in D$, according to Dini's theorem. Hence, we can multiply this term with $f_{\widetilde{s}}(y)$, integrate over $y \in D$ and interchange this integration with the limit $\delta \downarrow 0$. Now we can see heuristically the statement as follows. According to a large-N ergodic theorem, there are only of order $N\delta^{2d}$ walkers that are at time s in $B(x,\delta)$ and at time $\widetilde{s}$ in $B(\widetilde{x},\delta)$, analogously with y and $\widetilde{y}$. Hence, among all the $\asymp N\delta^d$ walkers present in $B(\widetilde{x},\delta)$ at time $\widetilde{s}$, those ones who were in $B(x,\delta)$ at time s are negligible for small δ. This implies the claimed asymptotic independence.

Let us turn to the proof. We need to introduce a bit of notation. For $A \subset \{1,\dots,N\}$, we write $\mathcal{C}_{x,\delta}^{(s,A)}$ for the largest cluster in the δ-box around x that is built out of all the $X_s^{(i)}$ with $i \in A$ only. We put

$$A_s^{(N)} = \{i \in \{3,\dots,N\} \colon X_s^{(i)} \notin B(x,\delta) \cup B(y,\delta)\}.$$

Now we use the triangle inequality to bound

$$\begin{aligned} \mathbb{P}_{1,2}\Big(x \in \mathcal{C}_{x,\delta}^{(s,N)}, y \in \mathcal{C}_{y,\delta}^{(s,N)}, \widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)}, \widetilde{y} \in \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},N)}\Big) \\ \le \mathbb{P}_{1,2}\Big(x \in \mathcal{C}_{x,\delta}^{(s,N)}, y \in \mathcal{C}_{y,\delta}^{(s,N)}, \widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(s,A_s^{(N)})}, \widetilde{y} \in \mathcal{C}_{\widetilde{y},\delta}^{(s,A_s^{(N)})}\Big) \\ + \mathbb{P}_{1,2}\Big(\widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(s,N)} \setminus \mathcal{C}_{\widetilde{x},\delta}^{(s,A_s^{(N)})}\Big) + \mathbb{P}_{1,2}\Big(\widetilde{y} \in \mathcal{C}_{\widetilde{y},\delta}^{(s,N)} \setminus \mathcal{C}_{\widetilde{y},\delta}^{(s,A_s^{(N)})}\Big). \end{aligned} \tag{2.14}$$

Since $\mathcal{C}_{x,\delta}^{(s,N)}$ and $\mathcal{C}_{y,\delta}^{(s,N)}$ depend only on the $X_s^{(i)}$ with i in the complement of $A_s^{(N)}$, the first two events in the first term on the right-hand side are independent from the last two events. Lemma 2.4(ii) and the continuity of $\overline{\Theta}(\cdot,R)$ imply that the probability of the intersection of the first two events converges towards $\overline{\Theta}(f_s(x),R)\overline{\Theta}(f_s(y),R)$. Note that the particles that the point processes $\mathcal{C}_{\widetilde{x},\delta}^{(s,A_s^{(N)})}$ and $\mathcal{C}_{\widetilde{y},\delta}^{(s,A_s^{(N)})}$ puts are given by trajectories that do not visit any of the two balls $B(x,\delta)$ and $B(y,\delta)$ at time s; more precisely, they are picked according to the density

$$f_{\widetilde{s}}^{(s,\delta)}(z) = \mathbb{P}(X_s \notin B(x,\delta) \cup B(y,\delta), X_{\widetilde{s}} \in \mathrm{d}z)/\mathrm{d}z = K_{s,\widetilde{s}}\big(z, (B(x,\delta) \cup B(y,\delta))^{\mathrm{c}}\big)\, f_{\widetilde{s}}(z). \tag{2.15}$$

Hence, the probability of the intersection of the last two events converges towards $\overline{\Theta}(f_{\widetilde{s}}^{(s,\delta)}(\widetilde{x}),R)\overline{\Theta}(f_{\widetilde{s}}^{(s,\delta)}(\widetilde{y}),R)$.

A glance at (2.15) shows that $f_{\widetilde{s}}^{(s,\delta)}(z)$ converges, as $\delta \to 0$, for any $z \in D$, towards $\mathbb{P}(X_s \neq x, X_s \neq y, X_{\widetilde{s}} \in \mathrm{d}z)/\mathrm{d}z$, which is, by Assumption 1.1(ii), equal to $f_{\widetilde{s}}(z)$. Since $f_{\widetilde{s}}(\widetilde{x})$ and $f_{\widetilde{s}}(\widetilde{y})$ are larger than the critical value, we may use continuity of $\overline{\Theta}$.

All together, we have that the first term of the right-hand side of (2.14) converges, as $N \to \infty$ followed by $\delta \to 0$, towards

$$\overline{\Theta}(f_s(x),R)\overline{\Theta}(f_s(y),R)\overline{\Theta}(f_{\widetilde{s}}(\widetilde{x}),R)\overline{\Theta}(f_{\widetilde{s}}(\widetilde{y}),R). \tag{2.16}$$

Furthermore, Assumption 1.1(ii) also implies that

$$\limsup_{N\to\infty} \mathbb{P}_{1,2}\Big(\widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(s,N)} \setminus \mathcal{C}_{\widetilde{x},\delta}^{(s,A_s^{(N)})}\Big) \tag{2.17}$$

vanishes as $\delta \to 0$. Indeed, we know that $\mathcal{C}_{\widetilde{x},\delta}^{(s,A_s^{(N)})} \subset \mathcal{C}_{\widetilde{x},\delta}^{(s,N)}$, therefore the above limit superior is equal to $\overline{\Theta}(f_{\widetilde{s}}(\widetilde{x})) - \overline{\Theta}(f_{\widetilde{s}}^{(s,\delta)}(\widetilde{x}))$. Hence, the convergence of $f_{\widetilde{s}}^{(s,\delta)}$ and the continuity of $\overline{\Theta}$ give the result. We proceed analogously for the last term in (2.14) and get that the limit superior as $N \to \infty$ and $\delta \to 0$ of the left-hand side of (2.14) is not larger than the expression in (2.16). Now use Lemma 2.4(ii) for the second term in (2.12) to see that from this the desired assertion follows. □

Proof of Theorem 1.2. First note that both assertions of (1.4) easily follow from Proposition 2.3, in conjunction with Lemma 2.5, provided that R is chosen such that

$$\int_0^T \mathrm{d}s\, \mathbb{1}\{f_s(X_s^{(i)}) = \lambda^{(\mathrm{cr})}(R)\} = 0 \qquad \text{for } i = 1, 2. \tag{2.18}$$

Furthermore, note that, almost surely, (2.18) holds for almost all R. Indeed, this follows from

$$\mathbb{E}\Big(\int_0^\infty \mathrm{d}R \int_0^T \mathrm{d}s\, \mathbb{1}\{f_s(X_s^{(i)}) = \lambda^{(\mathrm{cr})}(R)\}\Big) = \int_0^T \mathrm{d}s \int_D \mathrm{d}x \int_0^\infty \mathrm{d}R\, \mathbb{1}\{f_s(x) = R^{-d}\lambda^{(\mathrm{cr})}(1)\} = 0.$$

Hence, for a given (random) exceptional R, we pick sequences $(R_k)_{k\in\mathbb{N}}$ and $(R'_k)_{k\in\mathbb{N}}$ such that $R_k \downarrow R$ and $R'_k \uparrow R$ and R_k and R'_k satisfy (2.18) for any k in place of R. Since $\tau_T^{(N)}$ is an increasing function of R, we may estimate it from above and below by replacing R with R_k and R'_k, respectively, and applying Proposition 2.3 and Lemma 2.5 with these. This yields (1.4) with $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ replaced by their versions for R replaced with R_k and with R'_k, respectively.

The only thing that we need to do is to show the right-uppersemicontinuity of the map $R \mapsto \tau_T^{(\geq)}$ and the left-lowersemicontinuity of the map $R \mapsto \tau_T^{(>)}$. To show these, note that $\overline{\Theta}^{(\geq)}(\cdot, R) = \overline{\Theta}(R^d \cdot +, 1)$ is right-continuous and $\overline{\Theta}^{(>)}(\cdot, R) = \overline{\Theta}(R^d \cdot -, 1)$ is left-continuous. Furthermore, for any $x, y \in D$ and any $s \in [0, T]$, the map $R \mapsto \mathbb{1}\{x \overset{\geq}{\underset{s}{\longleftrightarrow}} y\}$ is right-uppersemicontinuous, and the map $R \mapsto \mathbb{1}\{x \overset{>}{\underset{s}{\longleftrightarrow}} y\}$ is left-lowersemicontinuous. The latter assertion is quite easy to see; let us show the former. Assume that, for all $\varepsilon > 0$, x and y are connected through the set $\{f_s \geq \lambda^{(\mathrm{cr})}(R+\varepsilon)\}$. Recall that $\lambda^{(\mathrm{cr})}(R) = R^{-d}\lambda^{(\mathrm{cr})}(1)$ is decreasing in R. If x and y were not connected through the set $\{f_s \geq \lambda^{(\mathrm{cr})}(R)\}$, then they would lie in different components of this set. By compactness, these components have a positive distance to each other. Hence, there is a hyperplane in D through the complement of $\{f_s \geq \lambda^{(\mathrm{cr})}(R)\}$ that separates these two components. Since this hyperplane is compact, f_s assumes a maximum on it, which is strictly smaller than $\lambda^{(\mathrm{cr})}(R)$. Hence, every curve from x to y must cross this hyperplane, i.e.. must pass a point with an f_s-value bounded away from $\lambda^{(\mathrm{cr})}(R)$. This means that, for some sufficiently small $\varepsilon > 0$, x and y are not connected through $\{f_s \geq \lambda^{(\mathrm{cr})}(R+\varepsilon)\}$. Hence, $\limsup_{\varepsilon\downarrow 0} \mathbb{1}\{x \overset{\geq, R+\varepsilon}{\underset{s}{\longleftrightarrow}} y\} \leq \mathbb{1}\{x \overset{\geq, R}{\underset{s}{\longleftrightarrow}} y\}$, where we wrote $\overset{\geq, R}{\underset{s}{\longleftrightarrow}}$ for connectedness through the set $\{f_s \geq \lambda^{(\mathrm{cr})}(R)\}$. Using Lebesgue's theorem shows the claimed continuity properties of $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ in R and finishes the proof of Theorem 1.2. □

3. Long-time investigations for the random waypoint model

In this section, we prove Lemmas 1.3 and 1.4 and Theorem 1.5, that is, we restrict ourselves to the random waypoint model (RWP) introduced in Section 1.4, show that it satisfies Assumption 1.1 and study the long time behaviour of the limiting connection time both in terms of an ergodic theorem and a large-deviations result. First we prove Lemma 1.3 in Section 3.1. Then we show in Section 3.2 the convergence of the RWP to its invariant distribution. Actually, we give two proofs of Lemma 1.4(i), one of which is based on the ergodicity of the RWP and is found in Section 3.3. The other one is based on a certain discrete-time Markov chain, whose ergodic and mixing properties are derived in Section 3.4. The second proof of Lemma 1.4(i) then follows in Section 3.5. Finally, we prove Theorem 1.5 in Section 3.6.

3.1. The RWP satisfies Assumption 1.1.

In this section we prove Lemma 1.3, namely that the RWP is an example for the assumed movement scheme. We first show that Assumption 1.1(i) is satisfied. Indeed, fix

$s \in (0,\infty)$ and note that, on the event $\{s \le R_1\}$,

$$X_s = X_0 + sV_1 \frac{W_1 - W_0}{|W_1 - W_0|},$$

which has obviously a continuous density, since W_0, V_1 and W_1 have and are independent. On the event $\{R_j < s \le R_{j+1}\}$ with $j \in \mathbb{N}$, we represent

$$X_s = W_j + (s - R_j)V_{j+1} \frac{W_{j+1} - W_j}{|W_{j+1} - W_j|},$$

which also has a continuous density, since W_j, V_{j+1} and W_{j+1} have and are independent (and R_j is a continuous function of them). Hence, $X_s \mathbb{1}\{R_j < s \le R_{j+1}\}$ has a continuous density. Summing on $j \in \mathbb{N}_0$, we also see by use of Dini's theorem that also X_s has a continuous density.

Let us now verify Assumption 1.1(ii). For any $x \in D$, $\mathbb{P}(X_s = x | X_{\widetilde{s}} = y) = 0$ is clear on the event $\bigcup_{j\in\mathbb{N}}\{s \le R_j < \widetilde{s}\}$, since there was a change of direction between time s and $\widetilde{s}$. On the counterevent, $\bigcup_{j\in\mathbb{N}_0}\{R_j < s < \widetilde{s} \le R_{j+1}\}$, we have

$$\begin{aligned}\mathbb{P}(X_s = x | X_{\widetilde{s}} = y) &= \mathbb{P}\Big(V_{j+1} = \frac{|X_{\widetilde{s}} - x|}{\widetilde{s} - s},\ \frac{W_{j+1} - W_j}{|W_{j+1} - W_j|} = \frac{X_{\widetilde{s}} - x}{|X_{\widetilde{s}} - x|}\Big| X_{\widetilde{s}} = y\Big)\\ &\le \mathbb{P}\Big(V_{j+1} = \frac{|y - x|}{\widetilde{s} - s}\Big| X_{\widetilde{s}} = y\Big) = 0\end{aligned}$$

because the speed is independent from the location and has a continuous density.

3.2. **Recurrence and ergodicity of the RWP.** Since we want to study long-time properties of the connection time, we will need recurrence and ergodic properties of the RWP, which we provide in this section. For the special case of μ being the uniform distribution on D, most of our results in this section are already contained in [LV06], but our Proposition 3.2 below also contains a statement on convergence in total variation, which will be important in Lemma 3.5 below. For the reader's convenience, we provide all necessary proofs; they are independent of [LV06], but use different variants of the Markov renewal theorem available in the literature.

The trajectory is divided into *trips*, by which we mean the parts from leaving a waypoint to arriving at the next one. $\mathbb{P}^{(0)}$ and $\mathbb{E}^{(0)}$ denote probability and expectation if the process starts at time 0 at the beginning of a trip at the zeroth of the waypoints, i.e., if the initial waypoint W_0 has distribution μ.

In [LV06, Theorem 6], another variant of Y is considered, and it is argued that that process possesses a unique invariant distribution. Projecting on our first coordinate, the location of the walker, the distribution of X in equilibrium is given by the formula

$$\mu_*(\mathrm{d}x) = \frac{1}{Z}\int_0^1 \mathrm{d}s\, \mathbb{E}^{(0)}\Big(\frac{V_1}{|W_1 - W_0|}; W_0 + s(W_1 - W_0) \in \mathrm{d}x\Big), \tag{3.1}$$

where Z is a normalization. It turns out below that this formula persists also for a general waypoint measure. In particular μ_* has a continuous density. We refer in particular to [L04] for a general methodology to describe this measure. See [BW02, Section 5] and [HLV06, Section III and IV] for explicit formulas, approximations and simulations for special cases of domains D and waypoint measures μ, like uniform distributions on rectangles and balls.

For the sake of illustration, we give an explicit value in $d = 2$ in the simplest case where the domain is the unit disk, the waypoint measure μ is the uniform measure on it and the velocity is chosen to be constant. In this case, the density of the waypoint location in the invariant distribution is given by

$$f_*(x) = \frac{45}{64\,\pi}(1-|x|^2)\int_0^\pi \sqrt{1-|x|^2\cos^2(\varphi)}\,\mathrm{d}\varphi, \qquad x \in B(0,1).$$

An approximation with a mean square error ≤ 0.0065 and an absolute error ≤ 0.067 is given by $f_*(x) = \frac{2}{\pi}(1-|x|^2)$, see [QZ07] and [BW02, eq. (18)].

In the following, we give detailed proofs for ergodic properties of the RWP, based on the Markov renewal theorem in the form provided by [K74]. Alternative proofs could be based on the form given in [LV06, Theorem 6].

We first show that the sequence of the trips is positive Harris recurrent. More precisely, we consider the sequence $\mathcal{T} = (\mathcal{T}_n)_{n\in\mathbb{N}} = (W_{n-1}, W_n, V_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$. Since $(W_n)_{n\in\mathbb{N}_0}$ and $(V_n)_{n\in\mathbb{N}}$ are independent i.i.d. sequences, $\mathcal{T}$ is obviously a Markov chain. Furthermore, it is also easy to see that $\mathcal{T}$ is positive Harris recurrent, since it satisfies

$$\mathbb{P}_y(\mathcal{T}_n \in A) = \mu\otimes\mu\otimes\mathcal{V}(A), \qquad n\geq 2, y\in\mathcal{D}, A\subset\mathcal{D}\text{ mb.}, \tag{3.2}$$

where we wrote $\mathbb{P}_y$ for the probability measure under which the walker starts from $Y(0) = y$. We use this to prove the convergence of Y_t introduced in (1.6). We give two different proofs. The first one (see Lemma 3.1) applies the Markov renewal theorem using the fact that Y_t is a time change of $\mathcal{T}$ and gives a good understanding of the limit law. However, as we will see, this approach only gives weak convergence. A more conceptual approach using Harris recurrence (see Proposition 3.2) gives actually convergence in total variation. By $\mathbb{P}_\alpha$ we denote the probability measure under which the process $(Y_t)_{t\in[0,\infty)}$ starts from the distribution α.

Lemma 3.1. *For any bounded continuous function $g\colon \mathcal{D}\times\mathbb{R}^+ \to \mathbb{R}^+$, and for any $y\in\mathcal{D}$,*

$$\lim_{t\to\infty}\mathbb{E}_y[g(\mathcal{T}_{N(t)}, R_{N(t)}-t)] = \frac{1}{\mathbb{E}[U_1]}\int_{\mathcal{D}}\mathbb{P}_{\mu\otimes\mu\otimes\mathcal{V}}[\mathcal{T}_1\in\mathrm{d}z, U_1\in\mathrm{d}\lambda]\int_0^\lambda g(z,s)\,\mathrm{d}s. \tag{3.3}$$

Proof. We apply [K74, Theorem 1], which immediately implies the assertion, noting that the measure ψ in [K74] is indeed equal to $\mu\otimes\mu\otimes\mathcal{V}$ by [K74, Lemma 2]. That is, we only have to check the validity of Conditions I.1-4 of [K74].

Conditions I.1 and I.2 are trivial here, while Condition I.3 is the usual non-lattice assumption. It states that there is a non-lattice sequence $(\zeta_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{R}$ such that, for each $\nu\in\mathbb{N}$ and $\delta>0$, there exists some $y\in\mathcal{D}$, such that, for every $\epsilon>0$, there exists a measurable set A with positive $\mu\otimes\mu\otimes\mathcal{V}$-measure, integers m_1, m_2 and $\tau\in\mathbb{R}$ such that, for $x\in A$,

$$\mathbb{P}_x[d(\mathcal{T}_{m_1}, y)<\epsilon, |R_{m_1}-\tau|\leq\delta]>0 \quad\text{and}\quad \mathbb{P}_x[d(\mathcal{T}_{m_2}, y)<\epsilon, |R_{m_2}-\tau-\zeta_\nu|\leq\delta]>0, \tag{3.4}$$

d being the usual Euclidean distance on $\mathcal{D}$.

We will prove this assumption with an arbitrary $y = (w_0, w_1, v_1)$ inside the support of $\mu\otimes\mu\otimes\mathcal{V}$, not depending on ν nor on δ, and with $A = \{x\in\mathcal{D}\colon d(x,y)<\epsilon\}$, where we assumed without loss of generality that $2\epsilon v_-^{-1} + \mathrm{diam}(D)\epsilon v_-^{-2} < \delta/3$. Furthermore, we put $\tau := |w_1-w_0|/v_1$ and pick any non-lattice sequence $(\zeta_\nu)_{\nu\in\mathbb{N}}$ inside the support of $\tau+|w_0-w_1|/V_1$. Furthermore, put $m_1 = 1$ and $m_2 = 3$. By continuity of the densities of μ and $\mathcal{V}$, the $\mu\otimes\mu\otimes\mathcal{V}$-measure of A is positive. Putting $x = (w_0', w_1', v_1')\in A$ and denoting by $R_1(x) = |w_1'-w_0'|/v_1'$ the (deterministic) value of R_1 starting from x, we see that

$$|R_1(x)-\tau| \leq \frac{|w_1'-w_0'-(w_1-w_0)|}{v_1'} + |w_1-w_0|\Big|\frac{1}{v_1}-\frac{1}{v_1'}\Big| \leq \frac{2\epsilon}{v_-} + \frac{\mathrm{diam}(D)\epsilon}{v_-^2} < \frac{\delta}{3}. \tag{3.5}$$

Noting that $\mathcal{T}_1 = x$ with $\mathbb{P}_x$-probability one, we see that the first part of (3.4) is satisfied; the probability is even equal to one.

Now we turn to the proof of the second. Keep $x \in A$ fixed. Recall that $R_n = U_0 + U_1 + \cdots + U_{n-1}$ and that $U_n = |W_{n+1} - W_n|/V_{n-1}$ for any n. Note that, under $\mathbb{P}_x$, $\mathcal{T}_3$ has distribution $\mu \otimes \mu \otimes \mathcal{V}$, and therefore $\mathbb{P}_x(d(\mathcal{T}_3, y) < \epsilon) = \mu \otimes \mu \otimes \mathcal{V}(A) > 0$. On the event $\{d(\mathcal{T}_3, y) < \epsilon\}$, with $\mathbb{P}_x$-probability one, (3.5) shows that $|U_0 - \tau| < \delta/3$, and a the same calculation with x replaced by $\mathcal{T}_3$ shows that $|U_2 - \tau| < \delta/3$. By our choice of ζ_ν and by continuity of the densities of μ and $\mathcal{V}$, we easily see that the event $\{|U_1 + \tau - \zeta_\nu| \leq \delta/3\}$ has positive $\mathbb{P}_x$-probability on $\{d(\mathcal{T}_3, y) < \epsilon\}$, since

$$|U_1 + \tau - \zeta_\nu| \leq \frac{|W_2 - W_1 - (w_0 - w_1)|}{v_-} + \Big|\frac{|w_0 - w_1|}{V_2} - (\zeta_\nu - \tau)\Big| \leq \frac{2\epsilon}{v_-} + \Big|\frac{|w_0 - w_1|}{V_2} - (\zeta_\nu - \tau)\Big|,$$

and the probability (with respect to V_2) to have the last term smaller than $\mathrm{diam}(D)\epsilon v_-^{-2}$ is positive. Since

$$|R_3 - \tau - \zeta_\nu| = |U_0 + U_1 + U_2 - \tau - \zeta_\nu| \leq |U_0 - \tau| + |U_1 + \tau - \zeta_\nu| + |U_2 - \tau|,$$

we now see that also the last condition in (3.4) is satisfied.

Condition I.4 states that, for any $x \in \mathcal{D}$, $\delta > 0$, there exists $r_0(x, \delta) > 0$ such that for any measurable function $f \colon \mathcal{D}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}$, and for all y with $d(y, x) < r_0(x, \delta)$,

$$\begin{aligned} &\mathbb{E}_x\big[f\big((\mathcal{T}_i)_{i\in\mathbb{N}}, (U_i)_{i\in\mathbb{N}_0}\big)\big] \\ &\qquad \leq \mathbb{E}_y\big[\lim_{n\to\infty} \sup\big\{f\big((t_i)_{i\in\mathbb{N}}, (u_i)_{i\in\mathbb{N}_0}\big) \colon d(t_i, \mathcal{T}_i) + |u_i - U_i| < \delta \text{ for } i \leq n\big\}\big] + \delta \sup|f|. \end{aligned} \tag{3.6}$$

This assumption is in general difficult to prove, but here things are simple, as $\mathcal{T}_i$ and U_i are independent of the starting point for $i \geq 3$. We can do the following coupling: write $x = (w_0^{(x)}, w_1^{(x)}, v_1^{(x)})$ and $y = (w_0^{(y)}, w_1^{(y)}, v_1^{(y)})$. We draw a sequence of i.i.d. waypoints and speeds $(W_i, V_i)_{i\geq 2}$ according to $\mu \otimes \mathcal{V}$. Define, for $z \in \{x, y\}$,

$$W_0^{(z)} = w_0^{(z)}, \qquad W_1^{(z)} = w_1^{(z)}, \qquad V_1^{(z)} = v_1^{(z)}, \qquad (W_i^{(z)}, V_i^{(z)})_{i\geq 2} = (W_i, V_i)_{i\geq 2}, \tag{3.7}$$

and put $\mathcal{T}_i^{(z)} = (W_{i-1}^{(z)}, W_i^{(z)}, V_i^{(z)})$. It is then clear that $(\mathcal{T}_i^{(z)})_{i\in\mathbb{N}}$ is a realisation of $(\mathcal{T}_i)_{i\in\mathbb{N}}$ under $\mathbb{P}_z$ and that for any $i \geq 3$, $\mathcal{T}_i^{(x)} = \mathcal{T}_i^{(y)}$. We saw in the verification of Condition I.3 that, if $d(x, y) < r$, then with obvious notation,

$$d(\mathcal{T}_i^{(x)}, \mathcal{T}_i^{(y)}) < r, \qquad d(U_i^{(x)}, U_i^{(y)}) < r\Big(\frac{2}{v_-} + \frac{\mathrm{diam}(D)}{v_-^2}\Big).$$

Taking $r_0(\delta)$ such that both right-hand sides are $< \delta$, immediately gives Condition I.4. $\square$

Using (1.5), we easily derive the above mentioned weak convergence of X_t towards μ_* identified in (3.1), as X_t may be written as an explicit continuous function of $\mathcal{T}_{N(t)}$ and $R_{N(t)} - t$. We now give a refined result, using the notion of Harris recurrence for continuous-time Markov chains. First note that the process

$$\mathcal{Y} = \big(\mathcal{Y}_t\big)_{t\in[0,\infty)} = \Big(\mathcal{T}_{N(t)}, \frac{R_{N(t)} - t}{U_{N(t)-1}}\Big)_{t\in[0,\infty)}$$

is a continuous-time Markov chain on $\mathcal{D} \times [0, 1]$ with right-continuous paths. The second component of $\mathcal{Y}$ runs from 0 to 1 with linear speed between the arrival times at the waypoints. It is also easy to express Y_t as a continuous functional of $\mathcal{Y}_t$.

Proposition 3.2. $(\mathcal{Y}_t)_{t\in[0,\infty)}$ *is a strongly aperiodic Harris recurrent chain, and its distribution converges in total variation towards the unique invariant distribution. As a consequence, the convergence in Lemma 3.1 is true for any measurable bounded function* g*. Furthermore, an ergodic theorem holds for* $(\mathcal{Y}_t)_{t\in[0,\infty)}$*.*

Proof. We use the characterization of Harris recurrence given in [KM94, Theorem 1], with the measure ν given by $\mu \otimes \mu \otimes \mathcal{V} \otimes \lambda$, where λ is the Lebesgue measure on $[0,1]$. It is easy to see that any set A with positive ν-measure will be hit by the process $(\mathcal{Y}_t)_{t\in[0,\infty)}$. Indeed, without loss of generality, we can assume that A is a product set. By independence it will certainly happen that one of the $\mathcal{T}_n$ will fall into the $\mathcal{D}$-component of A. Then as $\frac{R_{N(t)}-t}{U_{N(t)-1}}$ visits all of $[0,1]$ between two waypoints, it follows that also A will be hit by $\mathcal{Y}$, implying Harris recurrence.

This implies in particular the existence of a unique (up to multiplicative constants) invariant measure. It is not difficult to check that this measure has to be the one appearing in Lemma 3.1, up to the normalization. In particular, it has finite total mass. As a consequence, $\mathcal{Y}$ is strongly Harris recurrent. We also have that this process has spread-out cycles, in the sense of [A03, p. 202]. In fact, the hitting times of any set under any starting point are spread out. Indeed, the first hitting times might be deterministic (if the initial condition implies that the set is hit during the first travel of the walker), but then one can easily check that, due to the existence of a density for the speed, the hitting times also have a continuous density. Therefore, using [A03, Proposition VII.3.8], this implies convergence in total variation of $\mathcal{Y}_t$ towards its invariant distribution. The ergodic theorem can be found in [A03, Proposition VII.3.7]. $\square$

3.3. **Longtime average of the connection time I.** In this section we give our first proof of Lemma 1.4, using the results from the preceding section. Note that

$$\tau_T^{(\diamond,*)} = \int_0^T G_\diamond(X_s^{(1)}, X_s^{(2)})\,\mathrm{d}s,$$

where

$$G_\diamond(x,y) = \mathbb{1}\{x \overset{\diamond}{\underset{*}{\longleftrightarrow}} y\}\overline{\Theta}^{(\diamond)}(f_*(x),R)\overline{\Theta}^{(\diamond)}(f_*(y),R), \qquad x,y\in D.$$

Since $G_\diamond$ is bounded and measurable, the application of the ergodic theorem from Proposition 3.2 immediately implies that Lemma 1.4(i) holds. A second proof is given in Section 3.5 in terms of the discrete-time Markov chain $(Z_j)_{j\in\mathbb{N}_0}$.

It is clear that Lemma 1.4(ii) follows from (i), together with the following lemma (using an approximation of exceptional R's like in the proof of Theorem 1.2).

Lemma 3.3 ($\tau_T^{(>)} \approx \tau_T^{(>,*)}$). *Assume that R is chosen such that $\{f_* = \lambda^{(\mathrm{cr})}(R)\}$ has Lebesgue measure zero. Furthermore, assume that $f_s \to f_*$ as $s\to\infty$ uniformly on D. Assume that the assumptions of Proposition 3.2 are true. Then, under any starting measure, in probability,*

$$\liminf_{T\to\infty} \frac{1}{T}\big(\tau_T^{(>)} - \tau_T^{(>,*)}\big) \geq 0 \qquad \textit{and} \qquad \limsup_{T\to\infty} \frac{1}{T}\big(\tau_T^{(\geq)} - \tau_T^{(\geq,*)}\big) \leq 0.$$

Proof. We estimate, for any $0<S<T$,

$$\begin{aligned}\mathbb{E}\Big[\frac{1}{T}\big(\tau_T^{(>)} - \tau_T^{(>,*)}\big)\Big] &\geq -\frac{S}{T} - \sum_{i=1}^2 \frac{1}{T}\int_S^T \mathrm{d}s\,\mathbb{E}\Big[\big|\overline{\Theta}^{(>)}\big(f_*(X_s^{(i)}),R\big) - \overline{\Theta}^{(>)}\big(f_s(X_s^{(i)}),R\big)\big|\Big] \\ &\quad - \frac{1}{T}\int_S^T \mathrm{d}s\,\mathbb{E}\Big[\Big(\mathbb{1}\{X_s^{(1)} \overset{>}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\} - \mathbb{1}\{X_s^{(1)} \overset{>}{\underset{*}{\longleftrightarrow}} X_s^{(2)}\}\Big)_-\Big] \\ &= -\frac{S}{T} - I_1 - I_2 - II,\end{aligned}$$

where $(a)_-$ is the negative part of a. The term II is small as $T\to\infty$, followed by $S\to\infty$. Indeed, write the distribution of $X_s^{(1)}$ and $X_s^{(2)}$ with the help of two integrals $\int_D \mathrm{d}x\, f_s(x) \int_D \mathrm{d}y\, f_s(y)$, and use that

$\liminf_{s\to\infty} 1\!\!1\{x \overset{>}{\underset{s}{\longleftrightarrow}} y\} \geq 1\!\!1\{x \overset{>}{\underset{*}{\longleftrightarrow}} y\}$, which is easily derived from the uniform convergence of f_s towards f_*.

Let us show now that also I_1 gets small for $T \to \infty$, followed by $S \to \infty$ (the term I_2 is identical). We further estimate, for any $\delta, \omega > 0$,

$$\begin{aligned} I_1 \leq \frac{1}{T}\int_S^T \mathrm{d}s \int_D \mathrm{d}x\, f_*(x)\big|\overline{\Theta}^{(>)}(f_s(x),R) - \overline{\Theta}^{(>)}(f_*(x),R)\big| \\ 1\!\!1\{|f_*(x) - \lambda^{(\mathrm{cr})}(R)| \geq \delta\} 1\!\!1\{|f_s(x) - f_*(x)| \leq \omega\} \\ + \frac{1}{T}\int_S^T \mathrm{d}s \int_D \mathrm{d}x\, f_*(x) 2 1\!\!1\{|f_*(x) - \lambda^{(\mathrm{cr})}(R)| < \delta\} \\ + \frac{1}{T}\int_S^T \mathrm{d}s\, 2 f_*(x) 1\!\!1\{|f_s(x) - f_*(x)| > \omega\}. \end{aligned}$$

In the second summand, estimate $f_*(x)$ against the maximum of f_* and use that, as $\delta \downarrow 0$, the $\mathrm{d}x$-integral goes over the Lebesgue-null set $\{f_* = \lambda^{(\mathrm{cr})}(R)\}$. Hence, for given $\varepsilon > 0$, we may pick $\delta > 0$ so small that the second term is below $\varepsilon/3$. On the set $[0, \lambda^{(\mathrm{cr})}(R) - \delta/2] \cup [\lambda^{(\mathrm{cr})}(R) + \delta/2, C]$, with C chosen such that $f_s(x) \leq C$ for all s and $f_*(x) \leq C$, we know that $\overline{\Theta}^{(>)}(\cdot, R)$ is continuous and hence uniformly continuous. Hence, we may pick ω (depending on δ) so small that, for any a, b in that set satisfying $|a-b| \leq \omega$, we have $|\overline{\Theta}^{(>)}(a,R) - \overline{\Theta}^{(>)}(b,R)| \leq \varepsilon/3C$. Hence, the first term is not larger than $\varepsilon/3$. In the last term, estimate $f_*(x)$ against C and the indicator against $\frac{1}{\omega}|f_s(x) - f_*(x)|$, then it is not larger than $\frac{2C}{\omega}\sup_{s\geq S}\|f_s - f_*\|_1$. For S sufficiently large, this is below $\varepsilon/3$. This shows that I_1 gets arbitrarily small for $T \to \infty$, followed by $S \to \infty$, and finishes the proof of the first assertion. The second assertion is proved in the same way, noting that $\limsup_{s\to\infty} 1\!\!1\{x \overset{\geq}{\underset{s}{\longleftrightarrow}} y\} \leq 1\!\!1\{x \overset{\geq}{\underset{*}{\longleftrightarrow}} y\}$ for $x, y \in D$. □

3.4. **Recurrence and mixing properties of Z.** In this section, we study the recurrence and the mixing properties of the discrete-time Markov chain $Z = (Z_j)_j$ defined in (1.13). The reason that we introduced this chain is that, for the proof of the large-deviations result of Theorem 1.5, we need to apply a large-deviation principle for the empirical pair measures of a *discrete-time* Markov chain, as such principles in continuous time are not known, to the best of our knowledge. Since this deviations result needs the identification of the ergodic limit in terms of the same objects, we also provide a proof of the ergodic limit in terms of Z here.

For the convenience of the reader, we repeat the definition here. By the superscripts, we mark the two walkers; $Y^{(1)}$ and $Y^{(2)}$ are two independent copies of the RWP. Put $S_0 = 0$ and, for $j \in \mathbb{N}$,

$$S_j = \inf\{t > S_{j-1} \colon W^{(1)}_{N^{(1)}(t)} \neq W^{(1)}_{N^{(1)}(S_{j-1})} \text{ or } W^{(2)}_{N^{(2)}(t)} \neq W^{(2)}_{N^{(2)}(S_{j-1})}\}$$

and $Z_j = (Y^{(1)}_{S_j}, Y^{(2)}_{S_j})$. That is, $Z = (Z_j)_{j\in\mathbb{N}_0}$ is the trace-Markov chain of two independent copies of the RWP, observed at the times at which any of the two arrives at a waypoint; it is a time-change of $(Y^{(1)}, Y^{(2)})$. It is easy to see that $(Z_j)_j$ is a time-homogeneous Markov chain on $\mathcal{D}^2$. This chain does not explicitly record the location of the random walker at any fixed time, but the time that passes between the waypoint arrivals can be deduced from the information contained in Z. Hence, it is well-suitable for deducing asymptotic assertions for long time. First we derive a mixing property, which will later be used for the large-deviations principle.

Lemma 3.4. *The sequence $(Z_j)_j$ is ψ-mixing under any starting distribution, i.e.,*

$$\lim_{k\to\infty} \sup_{A\in\mathcal{F}_0^0, B\in\mathcal{F}_k^\infty} \Big|\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)\mathbb{P}(B)} - 1\Big| = 0,$$

where $\mathcal{F}_m^k := \sigma(Z_m, \dots, Z_k)$.

Proof. Introduce the event U_k that both walkers choose at least two new waypoints by time S_k. Then, conditional on U_k, any $A \in \mathcal{F}_0^0$ and $B \in \mathcal{F}_k^\infty$ are independent. Using this, a small calculation yields that

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)\mathbb{P}(B)} = \frac{\mathbb{P}(U_k|A)\mathbb{P}(U_k|B)}{\mathbb{P}(U_k)} + \mathbb{P}(U_k^{\mathrm{c}}|A \cap B)\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)\mathbb{P}(B)}.$$

Hence, the assertion follows from

$$\lim_{k\to\infty} \sup_{A\in\mathcal{F}_0^0, B\in\mathcal{F}_k^\infty} \mathbb{P}(U_k^{\mathrm{c}}|A \cap B) = 0. \tag{3.8}$$

We show now that (3.8) holds. The event U_k^{c} splits into the event that the first walker has chosen not more than one new waypoint by time S_k, but the second has chosen at least $k-1$ new waypoints, and the same event with first and second walker reversed. Let us only look at the first of these two events. On this event, the time S_k is not larger than $2\mathrm{diam}(D)/v_-$, since a choice of a new waypoint is done after $\mathrm{diam}(D)/v_-$ time units at the latest, since all ways are no longer than $\mathrm{diam}(D)$ and all velocities are no less than v_-. Since the time that passes between the second walker picks his $(l-1)$-st and the l-th waypoint is $|W_l^{(2)} - W_{l-1}^{(2)}|/V_l^{(2)}$, we have that its sum over $l \in \{1, \dots, k-1\}$ is not larger than $2\mathrm{diam}(D)/v_-$. Hence, on this event we have

$$\sum_{l=1}^{k-1} \big|W_l^{(2)} - W_{l-1}^{(2)}\big| \le 2\frac{v_+}{v_-}\mathrm{diam}(D).$$

Leaving out the summands for $l = 1$ and $l = k-1$, this remaining sum is still upper bounded by the right-hand side, and it does not depend on Z_0 nor on $Z_k, Z_{k+1}, \dots$. Hence, the probability for this sum being smaller than the right-hand side is an upper bound for the half of $\mathbb{P}(U_k^{\mathrm{c}}|A \cap B)$ that we are considering, and it does not depend on A nor on B. Since the right-hand side is constant and since the waypoints are not deterministic, the probability for this event tends to zero as $k \to \infty$. This shows that (3.8) holds and ends the proof. □

The following lemma says that Z is Harris recurrent, has a unique invariant distribution and is non-lattice, which is summarized by saying that it is Harris ergodic. In particular, it satisfies an ergodic theorem, i.e., for any bounded measurable function f, the averages $\frac{1}{N}\sum_{i=1}^N f(Z_i)$ converge almost surely to the integral of f with respect to the invariant distribution.

Lemma 3.5. *The chain* Z *is Harris ergodic.*

Proof. Harris recurrence of Z is equivalent to the existence of a nontrivial σ-finite measure φ such that Z is φ-recurrent, see [A03, Cor. VII.3.12]. Therefore we have to show that there exists some σ-finite measure φ such that every measurable set $F \subset \mathcal{D}^2$ with $\varphi(F) > 0$ is recurrent.

We denote the invariant measure of the process $(Y_t^{(1)})_{t\in[0,\infty)}$ by γ. Define $\varphi = \gamma \otimes \mu \otimes \mu \otimes \mathcal{V}$, which is obviously σ-finite. Let $F \subset \mathcal{D}^2$ be measurable. Without loss of generality, F contains a product set $F^{(1)} \times F^{(2)}$ satisfying $\gamma(F^{(1)}) > 0$ and $\mu \otimes \mu \otimes \mathcal{V}(F^{(2)}) > 0$. We are going to show that the hitting time of F is almost surely finite. First consider the sequence $(n_k)_{k\in\mathbb{N}_0}$ of times at which the second walker arrives at a waypoint, that is, $(S_{n_k})_{k\in\mathbb{N}_0} = (R_k^{(2)})_{k\in\mathbb{N}_0}$.

$(Z_{n_k})_{k\in\mathbb{N}_0}$ is now a process whose first component is a RWP sampled at times which are given by an independent renewal process, and the second component has the same law as $(\mathcal{T}_{k+1})_{k\in\mathbb{N}_0}$. According to (3.2) and [A03, Cor. VII.3.12] the second component is $(\mu \otimes \mu \otimes \mathcal{V})$-positive recurrent. In particular there exists a subsequence $(\widetilde{n}_k)_k$ of $(n_k)_k$ such that the second component of $Z_{\widetilde{n}_k}$ belongs to $F^{(2)}$ for any

$k \in \mathbb{N}_0$. Also $(S_{\widetilde{n}_k})_{k\in\mathbb{N}_0}$ is a transient Markov renewal process, independent of $Y^{(1)}$. Since we know that $Y^{(1)}$ is Harris ergodic by Proposition 3.2, a slight modification of [A03, Prop. VII.3.8 (i)] implies that also $(Y^{(1)}_{S_{\widetilde{n}_k}})_{k\in\mathbb{N}_0}$ is a Harris chain. Moreover, it is easy to see that it has the same invariant measure, γ, as $Y^{(1)}$, as obviously a measure invariant for $(Y^{(1)}_t)_{t\in[0,\infty)}$ is invariant for $(Y^{(1)}_{S_{\widetilde{n}_k}})_{k\in\mathbb{N}_0}$, and such a measure is unique. Therefore, as $\gamma(F^{(1)}) > 0$, $F^{(1)}$ is positive recurrent for $(Y^{(1)}_{S_{\widetilde{n}_k}})_{k\in\mathbb{N}_0}$.

Since $Y^{(2)}_{S_{\widetilde{n}_k}}$ automatically belongs to $F^{(2)}$, this implies in particular that F is recurrent. According to [A03, Cor. VII.3.12], this proves Harris recurrence of $(Z_n)_{n\in\mathbb{N}}$, and in particular the existence of a unique invariant measure, [A03, Thm. VII.3.5]. Now as we want positive Harris recurrence, we are going to show that this measure is finite.

Note that the previous arguments, together with the ergodic theorem [A03, Prop. VII.3.7] give that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\mathbb{1}_{\{Z_{\widetilde{n}_k}\in F\}} = \gamma(F^{(1)}) > 0.$$

Note that $n_k/k \to 2$, since the arrival times of $Y^{(1)}$ and $Y^{(2)}$ are disjoint and have asymptotically the same distribution. Hence, since $\mu\otimes\mu\otimes\mathcal{V}(F^{(2)})$ is equal to the probability that $Y^{(2)}$ hits $F^{(2)}$, we have $\widetilde{n}_k/k \to 2/\mu\otimes\mu\otimes\mathcal{V}(F^{(2)})$ by the classical renewal theorem. Noting the symmetry in the two components, we see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\mathbb{1}_{\{Z_k\in F\}} = \frac{1}{2}\big(\gamma\otimes\mu\otimes\mu\otimes\mathcal{V}(F) + \mu\otimes\mu\otimes\mathcal{V}\otimes\gamma(F)\big).$$

Since the right-hand side is a probability measure in F, $(Z_n)_{n\in\mathbb{N}}$ is positive Harris recurrent. Note that we proved the ergodic theorem in the course of the proof, as well as gave an explicit form for the invariant measure.

We also see from this proof that the sequence of hitting times of F is non-lattice, since the sequence $(\widetilde{n}_k)_{k\in\mathbb{N}}$ is non-lattice, because $(n_k)_{k\in\mathbb{N}}$ is non-lattice. □

3.5. **Longtime average of the connection time II.** Here we give a second proof of the ergodic limit in Lemma 1.4(i). This proof is based on a description of the connection time in terms of the discrete-time Markov chain $(Z_k)_{k\in\mathbb{N}_0}$ defined in (1.13) and the convergence of Z_k to equilibrium, as stated in Lemma 3.5.

We are going to express $\tau_T^{(\diamond,*)}$ in terms of Z. To this end, we define, for any $z_k = \big((x_k^{(1)}, w_k^{(1)}, v_k^{(1)}); (x_k^{(2)}, w_k^{(2)}, v_k^{(2)})\big) \in \mathcal{D}^2$,

$$M^{(1)}(z_1, z_2) = \frac{|x_2^{(1)} - x_1^{(1)}|}{v_1^{(1)}}, \qquad (3.9)$$

$$F_\diamond(z_1, z_2) = \int_0^1 \mathrm{d}s\, \overline{\Theta}^{(\diamond)}\big(f_*(p_1(s)), R\big)\overline{\Theta}^{(\diamond)}\big(f_*(p_2(s)), R\big)\mathbb{1}\{p_1(s) \overset{\diamond}{\underset{*}{\longleftrightarrow}} p_2(s)\}, \qquad (3.10)$$

where $p_i(s) = sx_2^{(i)} + (1-s)x_1^{(i)}$, $s\in[0,1]$, denotes the path of the i-th walker from $x_1^{(i)}$ to $x_2^{(i)}$. Then $M^{(1)}$ is the time that elapses while the two walkers move from one waypoint arrival to the next one, and $F_\diamond$ describes the proportion of time that the two are connected with each other on that way.

We write $Z_j = ((Z_j^{(1,1)}, \dots, Z_j^{(1,3)}), (Z_j^{(2,1)}, \dots, Z_j^{(2,3)}))$ and have, for any $n\in\mathbb{N}$,

$$S_n = \sum_{j=1}^{n}(S_j - S_{j-1}) = \sum_{j=1}^{n}\frac{|Z_j^{(1,1)} - Z_{j-1}^{(1,1)}|}{Z_j^{(1,2)}} = \sum_{j=1}^{n} M^{(1)}(Z_{j-1}, Z_j). \qquad (3.11)$$

Now we express $\tau_T^{(\diamond,*)}$ for T replaced by the waypoint arrival time. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}\tau_{S_n}^{(\diamond,*)} &= \sum_{j=1}^n \int_{S_{j-1}}^{S_j} \mathrm{d}s\, \overline{\Theta}^{(\diamond)}(f_*(X_s^{(1)}), R)\overline{\Theta}^{(\diamond)}(f_*(X_s^{(2)}), R)\mathbb{1}\{X_s^{(1)} \overset{\diamond}{\underset{*}{\longleftrightarrow}} X_s^{(2)}\} \\ &= \sum_{j=1}^n (S_j - S_{j-1}) \int_0^1 \mathrm{d}s\, \overline{\Theta}^{(\diamond)}(f_*(p_1(s)), R)\overline{\Theta}^{(\diamond)}(f_*(p_2(s)), R)\mathbb{1}\{p_1(s) \overset{\diamond}{\underset{*}{\longleftrightarrow}} p_2(s)\} \\ &= \sum_{j=1}^n M^{(1)}(Z_{j-1}, Z_j)F_\diamond(Z_{j-1}, Z_j),\end{aligned} \tag{3.12}$$

where $p_i(s) = Z_{j-1}^{(i,1)} + s(Z_j^{(i,1)} - Z_{j-1}^{(i,1)})$.

Now the proof of Lemma 1.4(i) is quite obvious. According to [A03, Th. VII.3.6], based on Lemma 3.5, implies that the distribution of Z_k converges towards its invariant distribution, which we want to call π. Hence, (Z_{j-1}, Z_j) converges to its invariant distribution $\pi \otimes P$, where we wrote $P\colon \mathcal{D} \times \mathcal{F} \to [0,1]$ for its transition kernel, writing $\mathcal{F}$ for the σ algebra on $\mathcal{D}$. This convergence is in total variation sense. Since $M^{(1)}$ and $F_\diamond$ are bounded and measurable, we have that

$$\lim_{n\to\infty} \frac{1}{n} S_n = \int M^{(1)}\, \mathrm{d}(\pi \otimes P) \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{n} \tau_{S_n}^{(\diamond,*)} = \int M^{(1)} F_\diamond\, \mathrm{d}(\pi \otimes P).$$

Pick $n_T = \sup\{n \in \mathbb{N}\colon S_n \leq T\}$, then it is easy to see that $\frac{1}{T} n_T \to 1$ as $T \to \infty$ in probability. Even though n_T is random, it is only an exercise to prove that the above limits are also true if n is replaced by n_T. Furthermore, it is also easy to see that $\frac{1}{T}(\tau_T^{(\diamond,*)} - \tau_{S_{n_T}}^{(\diamond,*)})$ vanishes in probability as $T \to \infty$. Hence, we have

$$p_*^{(\diamond)} = \lim_{T\to\infty} \frac{1}{T} \tau_T^{(\diamond,*)} = \frac{\langle M^{(1)} F_\diamond, \pi \otimes P\rangle}{\langle M^{(1)}, \pi \otimes P\rangle}, \tag{3.13}$$

where we now preferred the notation $\langle f, P\rangle$ for the integral of a function f with respect to a measure P. This ends the proof of Lemma 1.4(i) with the identification of the limit $p_*^{(\diamond)}$ as the right-hand side of (3.13).

3.6. **Proof of Theorem 1.5.** Now we turn to the proof of Theorem 1.5, i.e., we prove the upper bound for the downwards deviations of the normalized connection time, $\frac{1}{T}\tau_T^{(>,*)}$, for the RWP in the limit $T \to \infty$. Let us abbreviate $\tau_T^{(>,*)}$ by τ_T. We are going to give an explicit upper bound for the probability of the event $\{\tau_T \leq Tp\}$ for any $p \in (0, p_*^{(>)})$. In order to formulate it, we need to introduce some more notation, which mostly stems from the theory of large deviations. See [DZ10] for more about this theory.

As a consequence of Lemma 3.4, also $(Z_{j-1}, Z_j)_{j\in\mathbb{N}}$ is a ψ-mixing and bounded Markov chain. As a nice consequence, we now have a large-deviation principle (LDP) for the empirical pair measure of the Z_n, defined as

$$Q_n := \frac{1}{n} \sum_{j=1}^n \delta_{(Z_{j-1}, Z_j)} \in \mathcal{M}_1(\mathcal{D} \times \mathcal{D}). \tag{3.14}$$

The LDP is on $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D} \times \mathcal{D})$, the set of probability measures on $\mathcal{D} \times \mathcal{D}$ whose two marginals coincide. Indeed, combine the ψ-mixing property with [DZ10, Theorem 6.4.4] and the following remark, and use [DZ10, Corollary 6.5.10] to identify the rate function as the function

$$I(Q) = H(Q \mid \overline{Q} \otimes P) = \begin{cases} \int_{\mathcal{D}} \int_{\mathcal{D}} Q(\mathrm{d}x, \mathrm{d}y) \log \frac{Q(\mathrm{d}x, \mathrm{d}y)}{\overline{Q}(\mathrm{d}x) P(x, \mathrm{d}y)} & \text{if } Q \ll \overline{Q} \otimes P, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.15}$$

where we denote any of the two marginals of Q by $\overline{Q}$, i.e., $\overline{Q}(A) = Q(A\times\mathcal{D}) = Q(\mathcal{D}\times A)$ for $A \in \mathcal{B}(\mathcal{D})$. Then the LDP states that the level sets $\{Q \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}\times\mathcal{D})\colon I(Q) \le c\}$ are compact for any $c \in \mathbb{R}$, and that we have the estimates

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_*(Q_n \in F) \le -\inf_F I \qquad \text{and} \qquad \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_*(Q_n \in G) \ge -\inf_G I,$$

for any closed, respectively open, subset F and G of $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}\times\mathcal{D})$.

Theorem 1.5 follows from the following theorem. We recall from (3.13) that $p_*^{(>)} = \langle M^{(1)}F_>, \pi\otimes P\rangle/\langle M^{(1)}, \pi\otimes P\rangle$, where π is the invariant distribution of Z.

Theorem 3.6. *For any $p \in (0, p_*^{(>)})$,*

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{P}_*(\tau_T \le Tp) \le -\chi_p, \tag{3.16}$$

where

$$\chi_p = \inf\Big\{ \frac{I(Q)}{\langle M^{(1)}, Q\rangle} \colon Q \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}\times\mathcal{D}), \frac{\langle M^{(1)}F_>, Q\rangle}{\langle M^{(1)}, Q\rangle} \le p \Big\}. \tag{3.17}$$

Moreover, the infimum is attained, and χ_p is positive.

The term $\langle M^{(1)}, Q\rangle$ is the average time that elapses between two subsequent arrivals at waypoints, if the two walkers move in such a way that the distribution of the location, velocity and next waypoint at two subsequent such arrivals is given by Q, and $\langle M^{(1)}F_>, Q\rangle$ is the average portion of connection time on such a way, and $I(Q)$ is the negative rate of the probability that the two follow that strategy Q per number of waypoints. Hence, the upper bound in (3.16) is intuitive and can be interpreted. Note that $F_>$ is lower semicontinuous, as the indicator of connectedness of two points through $\{f_* > \lambda^{(\mathrm{cr})}(R)\}$ is a countable sum of indicators of open sets. However, in general $F_>$ may not be upper semicontinuous. This makes it questionable whether or not also the lower bound in (3.16) holds, since the map $Q \mapsto \langle Q, M^{(1)}F_>\rangle$ is in general not continuous.

Proof of Theorem 3.6. That the infimum in (3.17) is attained is easily seen as follows. By lower semicontinuity of $F_>$ and [DZ10, Theorem D.12], the map $Q \mapsto \langle Q, M^{(1)}F_>\rangle$ is also lower semicontinuous. Since also I is lower semicontinuous and has compact level sets and the map $Q \mapsto \langle Q, M^{(1)}\rangle$ is continuous, it easily follows that the infimum in (3.17) is even a minimum.

Now we argue that χ_p is positive. Indeed, the only minimiser of I on $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}\times\mathcal{D})$ is the measure $\pi\otimes P$, where we recall that π is the invariant distribution of Z and P its transition kernel. To see this, note that, for any Q satisfying $I(Q) = 0$, we have $Q(\mathrm{d}x, \mathrm{d}y) = \overline{Q}(\mathrm{d}x)P(x, \mathrm{d}y)$ by the equality discussion in Jensen's inequality, and from the marginal property it follows that $\overline{Q}$ is invariant for P, i.e., equal to π by uniqueness of the invariant distribution for the chain Z. Hence, also the only minimiser of $Q \mapsto I(Q)/\langle M^{(1)}, Q\rangle$ is $\pi\otimes P$, and it satisfies $p_*^{(>)} = \langle M^{(1)}F_>, \pi\otimes P\rangle/\langle M^{(1)}, \pi\otimes P\rangle$, see below (3.13). Therefore, it is not contained in the admissibility set on the right of (3.17) and is therefore not equal to its minimiser. Hence, χ_p is positive.

Now we prove (3.16). We are going to express the time T and the variable τ_T in terms of integrals over Q_n. First the time. We write $Z_j = ((Z_j^{(1,1)}, \dots, Z_j^{(1,3)}), (Z_j^{(2,1)}, \dots, Z_j^{(2,3)}))$. From (3.11) and (3.12) we have, for any $n \in \mathbb{N}$,

$$S_n = n\langle M^{(1)}, Q_n\rangle \qquad \text{and} \qquad \tau_{S_n} = n\langle M^{(1)}F_>, Q_n\rangle,$$

recalling the definition of $M^{(1)}$ and of $F_>$ in (3.9), where $p_i(s) = Z_{j-1}^{(i,1)} + s(Z_j^{(i,1)} - Z_{j-1}^{(i,1)})$. Hence, we can already give a heuristic proof of Theorem 3.6 as follows. The large-deviation principle of $(Q_n)_{n\in\mathbb{N}}$ roughly says that $\mathbb{P}_*(Q_n \approx Q) \approx \mathrm{e}^{-nI(Q)}$ for any strategy $Q \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}^2)$. Taking n such that $T \approx S_n$, we have that $n \approx T/\langle M^{(1)}, Q_n\rangle$ and $\tau_T/T \approx \langle M^{(1)}F_>, Q_n\rangle/\langle M^{(1)}, Q_n\rangle$. Hence, we should have

$$\begin{aligned}\mathbb{P}_*(\tau_T \le pT) &\approx \mathbb{P}_*\Big(\langle M^{(1)}F_>, Q_n\rangle/\langle M^{(1)}, Q_n\rangle \le p\Big)\\ &\approx \exp\Big(-n\inf\Big\{I(Q)\colon Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}^2), \frac{\langle M^{(1)}F_>, Q\rangle}{\langle M^{(1)}, Q\rangle}\le p\Big\}\Big)\\ &\approx \mathrm{e}^{-T\chi_p},\end{aligned}$$

with χ_p as defined in Theorem 3.6. The main difficulty in making this line of argument rigorous lies in the randomness of n.

Let us now give a rigorous proof of the upper bound in (3.16). Fix $p \in (0, p_*^{(>)})$. Oberserve that $M^{(1)}$ is bounded from above by $L = \mathrm{diam}(D)/v_-$, with probability one with respect to Q for any $Q \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}^2)$, since D is bounded and all velocities are $\ge v_-$. In other terms, we have $S_j - S_{j-1} \le L$ for any $j \in \mathbb{N}$ and therefore also $0 < S_n/n \le L$ for any $n \in \mathbb{N}$.

Now we estimate

$$1 \le \sum_{n=\lfloor T/L\rfloor}^{\lfloor KT\rfloor} 1\!\!1\{S_n \le T < S_{n+1}\} + 1\!\!1\{T \ge S_{\lfloor KT\rfloor+1}\},$$

for any $K \in (0,\infty)$, which we will later pick very large. On the first event, $\{S_n \le T < S_{n+1}\}$, we have, for any $\delta \in (0,1)$,

$$\tau_T \ge \tau_{S_n} = n\langle Q_n, M^{(1)}F_>\rangle \ge (T-L)\frac{\langle M^{(1)}F_>, Q_n\rangle}{\langle M^{(1)}, Q_n\rangle} \ge T(1-\delta)\frac{\langle M^{(1)}F_>, Q_n\rangle}{\langle M^{(1)}, Q_n\rangle},$$

where the last inequality is true for all sufficiently large T, which we want to assume from now. The event itself can be upper bounded as

$$1\!\!1\{S_n \le T < S_{n+1}\} \le 1\!\!1\{T-L \le S_n \le T\} \le 1\!\!1\Big\{(1-\delta)\frac{T}{n} \le \langle M^{(1)}, Q_n\rangle \le \frac{T}{n}\Big\}.$$

This implies the upper bound

$$\mathbb{P}_*(\tau_T \le pT) \le \sum_{n=\lfloor T/L\rfloor}^{\lfloor KT\rfloor} \mathbb{P}_*\Big(\frac{\langle M^{(1)}F_>, Q_n\rangle}{\langle M^{(1)}, Q_n\rangle} \le \frac{p}{1-\delta}, (1-\delta)\frac{T}{n} \le \langle M^{(1)}, Q_n\rangle \le \frac{T}{n}\Big) + \mathbb{P}_*(T \ge S_{\lfloor KT\rfloor+1}).$$

The last term is an error term, as we will show later that

$$\lim_{K\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}_*(T \ge S_{\lfloor KT\rfloor+1}) = -\infty. \tag{3.18}$$

Now we cut the sum over n into pieces of length $T\varepsilon$, where $\varepsilon > 0$ is a small auxiliary parameter:

$$\sum_{n=\lfloor T/L\rfloor}^{\lfloor KT\rfloor} = \sum_{i=1+\lfloor 1/L\varepsilon\rfloor}^{\lfloor K/\varepsilon\rfloor}\ \sum_{(i-1)T\varepsilon < n \le iT\varepsilon}.$$

Recall that $F_>$ is lower semicontinuous. By [DZ10, Theorem D.12], the map $Q \mapsto \langle Q, M^{(1)}F_>\rangle$ is also lower semicontinuous. Hence, the set

$$A = \Big\{Q \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}^2)\colon \frac{\langle M^{(1)}F_>, Q\rangle}{\langle M^{(1)}, Q\rangle} \le \frac{p}{1-\delta}, \frac{1-\delta}{i\varepsilon} \le \langle M^{(1)}, Q\rangle \le \frac{1}{(i-1)\varepsilon}\Big\}$$

is closed in the weak topology.

For fixed i and $(i-1)T\varepsilon < n \leq iT\varepsilon$, we can estimate, with the help of the upper bound in the above mentioned LDP, as $T\to\infty$,

$$\mathbb{P}_*\Big(\frac{\langle M^{(1)}F_>,Q_n\rangle}{\langle M^{(1)},Q_n\rangle}\leq\frac{p}{1-\delta},(1-\delta)\frac{T}{n}\leq\langle M^{(1)},Q_n\rangle\leq\frac{T}{n}\Big)\leq\mathbb{P}_*(Q_n\in A)\leq \mathrm{e}^{-T\widetilde{\chi}_p(\delta,\varepsilon)}\,\mathrm{e}^{o(T)}, \tag{3.19}$$

where

$$\begin{aligned}
\widetilde{\chi}_p(\delta,\varepsilon)&=(i-1)\varepsilon\inf\Big\{I(Q)\colon Q\in A\Big\}\\
&=(i-1)\varepsilon\inf\Big\{I(Q)\colon Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}^2),\frac{\langle M^{(1)}F_>,Q\rangle}{\langle M^{(1)},Q\rangle}\leq\frac{p}{1-\delta},\frac{1-\delta}{i\varepsilon}\leq\langle M^{(1)},Q\rangle\leq\frac{1}{(i-1)\varepsilon}\Big\}\\
&\geq\inf\Big\{I(Q)\Big(\frac{1-\delta}{\langle M^{(1)},Q\rangle}-\varepsilon\Big)\colon Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}^2),\frac{\langle M^{(1)}F_>,Q\rangle}{\langle M^{(1)},Q\rangle}\leq\frac{p}{1-\delta},\\
&\qquad\qquad\frac{1-\delta}{i\varepsilon}\leq\langle M^{(1)},Q\rangle\leq\frac{1}{(i-1)\varepsilon}\Big\}\\
&\geq\inf\Big\{I(Q)\Big(\frac{1-\delta}{\langle M^{(1)},Q\rangle}-\varepsilon\Big)\colon Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}^2),\frac{\langle M^{(1)}F_>,Q\rangle}{\langle M^{(1)},Q\rangle}\leq\frac{p}{1-\delta}\Big\}\\
&=:\chi_p(\delta,\varepsilon).
\end{aligned}$$

It is easy to see that $\lim_{\varepsilon\downarrow 0,\delta\downarrow 0}\chi_p(\delta,\varepsilon)=\chi_p$ as defined in (3.17). Hence, the upper bound in (3.16) is proved, subject to (3.18), which we prove now.

Note that $\{S_n\colon n\in\mathbb{N}_0\}=\{R_n^{(1)}\colon n\in\mathbb{N}_0\}\cup\{R_n^{(2)}\colon n\in\mathbb{N}_0\}$, where $R_n^{(i)}$ denotes the arrival time of the i-th walker at the n-th waypoint. Hence,

$$\mathbb{P}_*(T\geq S_{\lfloor KT\rfloor+1})\leq 2\mathbb{P}_*(T\geq R^{(1)}_{\lfloor KT/2\rfloor+1}).$$

Hence, we are looking at downwards deviations of the random walk $(R_n^{(1)})_{n\in\mathbb{N}}$, whose j-th step, B_j, is the duration of the first walker's travel from the $(j-1)$-st to the j-th waypoint. These steps are i.i.d. with support in $[0,L]$. Therefore, Cramér's theorem yields

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}_*(R^{(1)}_{\lfloor KT/2\rfloor}\leq T)\leq\frac{K}{2}\limsup_{T\to\infty}\frac{1}{KT/2}\log\mathbb{P}_*\Big(R^{(1)}_{\lfloor KT/2\rfloor}\leq\frac{2}{K}\lfloor KT/2\rfloor\Big)\\
&\leq-\frac{K}{2}\sup_{\lambda<0}\Big(\lambda\frac{2}{K}-\log\mathbb{E}_*[\mathrm{e}^{\lambda B_1}]\Big)=-\sup_{\lambda<0}\Big(\lambda-\frac{K}{2}\log\mathbb{E}_*[\mathrm{e}^{\lambda B_1}]\Big).
\end{aligned}$$

Note that the essential infimum of B_1 is equal to zero, as we assumed that the waypoint measure has a continuous density. Indeed, if the waypoint walker stands in his waypoint, with probability one there is a nontrivial ball around the location in which the waypoint measure has a positive density, and therefore arbitrarily small travels to the next waypoint have a positive probability.

Hence, $\log\mathbb{E}_*[\mathrm{e}^{\lambda B_1}]=o(|\lambda|)$ as $\lambda\to-\infty$, and therefore it is possible to pick a sequence $\lambda_K\to\infty$ as $K\to\infty$ such that $\lambda_K-\frac{K}{2}\log\mathbb{E}_*[\mathrm{e}^{\lambda_K B_1}]\to\infty$ as $K\to\infty$. This implies that (3.18) holds and finishes the proof of Theorem 3.6.

□

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