Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Global existence and uniqueness for a singular/degenerate Cahn–Hilliard system with viscosity

Pierluigi Colli[1], Gianni Gilardi[1], Paolo Podio-Guidugli[2], Jürgen Sprekels[3]

submitted: June 8, 2012

[1] Dipartimento di Matematica "F. Casorati"
Università di Pavia
via Ferrata 1
27100 Pavia
Italy
E-Mail: pierluigi.colli@unipv.it
E-Mail: gianni.gilardi@unipv.it

[2] Dipartimento di Ingegneria Civile
Università di Roma "Tor Vergata"
via del Politecnico 1
00133 Roma
Italy
E-Mail: ppg@uniroma2.it

[3] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Juergen.Sprekels@wias-berlin.de

No. 1713
Berlin 2012

2010 *Mathematics Subject Classification.* 35K61, 35A05, 74A15.

Key words and phrases. Phase-field model, nonlinear laws, existence of solutions, new uniqueness proof.

PC and JS gratefully acknowledge the warm hospitality of the WIAS in Berlin and the IMATI of CNR in Pavia, respectively. Some financial support came from the MIUR-PRIN Grant 2008ZKHAHN *"Phase transitions, hysteresis and multiscaling"* for PC and GG, and from the FP7-IDEAS-ERC-StG Grant #200947 (BioSMA) for PC and JS. The work of JS was also supported by the DFG Research Center MATHEON in Berlin.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Existence and uniqueness are investigated for a nonlinear diffusion problem of phase-field type, consisting of a parabolic system of two partial differential equations, complemented by Neumann homogeneous boundary conditions and initial conditions. This system aims to model two-species phase segregation on an atomic lattice [19]; in the balance equations of microforces and microenergy, the two unknowns are the order parameter ρ and the chemical potential μ. A simpler version of the same system has recently been discussed in [8]. In this paper, a fairly more general phase-field equation for ρ is coupled with a genuinely nonlinear diffusion equation for μ. The existence of a global-in-time solution is proved with the help of suitable a priori estimates. In the case of costant atom mobility, a new and rather unusual uniqueness proof is given, based on a suitable combination of variables.

1 Introduction

In this paper, the last so far in a series [6, 7, 8, 9, 10, 11], we further our mathematical analysis of a mechanical model proposed by one of us [19] for phase segregation through atom rearrangement on a lattice. On postponing a detailed presentation of the model and its antecedents until next section, we begin by pointing out what features of the system we study are more general, and therefore more difficult to handle mathematically, than in our previous paper [8].

The initial and boundary value problem we here tackle consists in looking for two *fields*, the *chemical potential* $\mu > 0$ and the *order parameter* $\rho \in (0,1)$, solving

$$2h(\rho)\,\partial_t\mu + \mu\,h'(\rho)\,\partial_t\rho - \operatorname{div}\big(\kappa(\mu)\nabla\mu\big) = 0 \qquad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\delta\partial_t\rho - \Delta\rho + f'(\rho) = \mu h'(\rho), \qquad \text{in } \Omega\times(0,T), \tag{1.2}$$

$$(\kappa(\mu)\nabla\mu)\cdot\nu|_\Gamma = \partial_\nu\rho|_\Gamma = 0 \qquad \text{on } \Gamma\times(0,T), \tag{1.3}$$

$$\mu(\,\cdot\,,0) = \mu_0 \quad \text{and} \quad \rho(\,\cdot\,,0) = \rho_0 \qquad \text{in } \Omega, \tag{1.4}$$

where Ω denotes a bounded domain of $\mathbb{R}^3$ with conveniently smooth boundary Γ, $T > 0$, and ∂_ν denotes differentiation in the direction of the outward normal ν. In (1.1), the *atom mobility* is specified by a nonnegative, continuous and bounded, nonlinear function κ of μ (in particular, the degeneracy of κ around the critical value $\mu = 0$ is admitted). The problem is parameterized by two scalar-valued functions, h and f, and two positive numbers, ε and δ, both intended to be small. The parameter functions enter into the definition of the system's *free energy*

$$\psi = \widehat{\psi}(\rho,\nabla\rho,\mu) = -\mu\,h(\rho) + f(\rho) + \frac{1}{2}|\nabla\rho|^2, \tag{1.5}$$

where the last two terms favor phase segregation, the former because it introduces local energy minima and the latter because it penalizes spatial changes of the order parameter (we have set equal to 1 the relative material constant). For h, one can take any smooth function provided it is bounded from below by a positive constant:

$$h(\rho) \geq \frac{\varepsilon}{2}; \tag{1.6}$$

for f, the sum

$$f(\rho) = f_1(\rho) + f_2(\rho)$$

of a convex and lower semicontinuous function f_1, with proper domain $D(f_1) \subseteq \mathbb{R}$, and of a smooth function f_2 with no convexity properties, so as to allow for a double or multi-well potential f. Note that f_1 need not be differentiable in its domain, so that its possibly multivalued subdifferential $\beta := \partial f_1$ may appear in (1.2) in place of f_1'; in general, β is only a graph, not necessarily a function, and it may include vertical (and horizontal) lines, as for example when

$$f_1(\rho) = I_{[0,1]}(\rho) = \begin{cases} 0 & \text{if } 0 \leq \rho \leq 1 \\ +\infty & \text{elsewhere} \end{cases} \tag{1.7}$$

and $\beta = \partial I_{[0,1]}$ is specified by

$$\xi \in \beta(\rho) \quad \text{if and only if} \quad \xi \begin{cases} \leq 0 & \text{if } \rho = 0 \\ = 0 & \text{if } 0 < \rho < 1 \\ \geq 0 & \text{if } \rho = 1 \end{cases} . \tag{1.8}$$

The simpler situation dealt with in [8] obtains for κ constant-valued (and hence set equal to 1, without any loss of generality),

$$h(\rho) = \rho, \tag{1.9}$$

and f a double-well potential defined in $(0, 1)$, whose derivative f' is singular at the endpoints $\rho = 0$ and $\rho = 1$: e.g.,

$$f(\rho) = \alpha_1 \{\rho \ln(\rho) + (1 - \rho) \ln(1 - \rho)\} + \alpha_2 \, \rho \, (1 - \rho) \tag{1.10}$$

for some positive constants α_1 and α_2.[1] Under these less general circumstances, system (1.1)–(1.4) reduces to

$$\varepsilon \, \partial_t \mu + 2\rho \, \partial_t \mu + \mu \, \partial_t \rho - \Delta \mu = 0 \qquad \text{in } \Omega \times (0, T), \tag{1.11}$$

$$\delta \, \partial_t \rho - \Delta \rho + f'(\rho) = \mu \qquad \text{in } \Omega \times (0, T), \tag{1.12}$$

$$\partial_\nu \mu = \partial_\nu \rho = 0 \qquad \text{on } \Gamma \times (0, T), \tag{1.13}$$

$$\mu(\,\cdot\,, 0) = \mu_0 \quad \text{and} \quad \rho(\,\cdot\,, 0) = \rho_0 \qquad \text{in } \Omega. \tag{1.14}$$

Note that h might attain its lower bound for some significant values of ρ, that is, for some ρ's lying in the domain of f_1: actually, this was the case for h defined as in (1.9) over the interval

[1] Note that, according to whether or not $\alpha_1 \geq 2\alpha_2$, it turns out that f is convex in the whole of $[0, 1]$ or it exhibits two wells with a local maximum at $\rho = 1/2$.

$[0, 1]$, that is, over the effective domain of both potentials in (1.10) and (1.7). We were prompted to generalize (1.9) as in (1.6) by an interesting remark of Alexander Mielke, when one of us was lecturing on our results, namely, that the behavior of

$$h(\rho) = \rho + \text{ small parameter}$$

is different in a right neighbourhood of 0 ($h(\rho) \approx 0$) than in a left neighbourhood of 1 ($h(\rho) \approx 1$), whereas assuming only that h be bounded from below allows for many other instances like, e.g., a specular behavior of h around the extremal points of the domain of f.

Returning now to (1.1)–(1.4), we set

$$g(\rho) := h(\rho) - \frac{\varepsilon}{2} \geq 0 \quad \text{for all } \rho \in D(f_1),$$

and we reformulate our initial and boundary value problem as follows:

to find μ, ρ, and ξ, so as to solve

$$\big(\varepsilon + 2g(\rho)\big)\, \partial_t \mu + \mu\, g'(\rho)\, \partial_t \rho - \operatorname{div}\big(\kappa(\mu)\nabla\mu\big) = 0 \qquad \text{in } \Omega \times (0, T), \tag{1.15}$$

$$\delta\partial_t \rho - \Delta\rho + \xi + f_2'(\rho) = \mu\, g'(\rho)\,, \ \text{ with } \xi \in \beta(\rho), \qquad \text{in } \Omega \times (0, T), \tag{1.16}$$

$$(\kappa(\mu)\nabla\mu) \cdot \nu|_\Gamma = \partial_\nu \rho|_\Gamma = 0 \qquad \text{on } \Gamma \times (0, T), \tag{1.17}$$

$$\mu(\,\cdot\,, 0) = \mu_0 \quad \text{and} \quad \rho(\,\cdot\,, 0) = \rho_0 \qquad \text{in } \Omega. \tag{1.18}$$

The *global-in-time existence result* we derive is more general than in [8], for two reasons: because it holds when the potential f includes a multivalued graph β (possibly with vertical segments, e.g., for f_1 as in (1.7)) and only exploits the monotonicity property of β; and because the atom mobility $\kappa(\mu)$ is allowed to depend in a generic nonlinear way on the chemical potential. Note that problem (1.15)–(1.18) may become *singular* with respect to ρ due to the possible occurrence of singularities of β; on the other hand, it may be also *degenerate* with respect to μ since $\kappa(\mu)$ is allowed to vanish at $\mu = 0$.

Our *uniqueness result* is also more general than in [8], because nonsmooth potentials could not be handled with the technique there used, consisting in testing the difference of two equations (1.11) by the time derivative of the difference of the two ρ components; however, just as in [8], the proof is achieved under the assumption that $\kappa(\mu) =$ a constant.

Some directions for future research have already been explored by us under less general circumstances than those considered here: the long time behavior of system (1.11)–(1.14) and the structure of the relative omega-limit set have been analysed by us in [8] and in [9], where we also dealt with the asymptotics of (1.11)–(1.14) as $\varepsilon \to 0$ and found a weaker solution in the singular limit. Moreover, in [10] and [12] we studied two optimal control problems for systems similar to (1.11)–(1.14): a distributed control problem in [10] and a boundary control problem in [12]. Finally, in [11] we developed an existence theory for problem (1.1)–(1.4) when atom mobility is allowed to depend on both μ and ρ.

This paper is organized as follows. In the next section, as anticipated, we discuss the physical features of the phase segregation model we adopt. In Section 3, we state our assumptions and results with the necessary mathematical accuracy; since here we do not take up asymptotic procedures, without loss of generality we set $\varepsilon = 1$ in (1.15) and $\delta = 1$ in (1.16). The existence of solutions to problem (1.15)–(1.18) is proved in Section 4, their regularity properties in the successive section. Our last Section 6 is devoted to the uniqueness proof.

2 Short reasoned history of our mathematical model

The nonstandard phase-field model (1.11)–(1.14) can be regarded as a variant of the classic Cahn-Hilliard system for diffusion-driven phase segregation by atom rearrangement:

$$\partial_t\rho - \kappa\Delta\mu = 0\,, \qquad \mu = -\Delta\rho + f'(\rho). \tag{2.1}$$

Apart for the harmless choice $\kappa = 1$ for the mobility modulus in (1.11), one finds in (1.11)–(1.14) two awkward nonlinear terms involving time derivatives. Usually, equations (2.1) are combined in order to obtain the well-known *Cahn-Hilliard equation*:

$$\partial_t\rho = \kappa\Delta(-\Delta\rho + f'(\rho)). \tag{2.2}$$

Fried and Gurtin's generalization of Cahn-Hilliard equation. In [15, 17], a broad generalization of (2.2) was achieved by proposing the following:

(i) to interpret the second of (2.1) as a *balance of microforces:*

$$\operatorname{div}\boldsymbol{\xi} + \pi + \gamma = 0, \tag{2.3}$$

where the distance microforce per unit volume is split into an internal part π and an external part γ, and the contact microforce per unit area of a surface oriented by its normal $\boldsymbol{n}$ is measured by $\boldsymbol{\xi}\cdot\boldsymbol{n}$ in terms of the *microstress* vector $\boldsymbol{\xi}$;[2]

(ii) to regard the first of (2.1) as a *balance law for the order parameter*:

$$\partial_t\rho = -\operatorname{div}\boldsymbol{h} + \sigma, \tag{2.4}$$

where the pair $(\boldsymbol{h},\sigma)$ is the *inflow* of ρ;

(iii) to demand that the constitutive choices for $\pi, \boldsymbol{\xi}, \boldsymbol{h}$, and the *free energy density* ψ, be consistent in the sense of Coleman and Noll [5] with an *ad hoc* version of the Second Law of Continuum Thermodynamics:

$$\partial_t\psi + (\pi-\mu)\partial_t\rho - \boldsymbol{\xi}\cdot\nabla(\partial_t\rho) + \boldsymbol{h}\cdot\nabla\mu \le 0, \tag{2.5}$$

that is, a postulated "dissipation inequality that accommodates diffusion" (cf. equation (3.6) in [17]).

In [17], the following set of constitutive prescriptions was shown to be consistent with (iii):

$$\left\{\ \psi = \widehat{\psi}(\rho,\nabla\rho), \quad \widehat{\pi}(\rho,\nabla\rho,\mu) = \mu - \partial_\rho\widehat{\psi}(\rho,\nabla\rho), \quad \widehat{\boldsymbol{\xi}}(\rho,\nabla\rho) = \partial_{\nabla\rho}\widehat{\psi}(\rho,\nabla\rho)\ \right\}. \tag{2.6}$$

Moreover, it was presumed that

$$\boldsymbol{h} = -\boldsymbol{M}\nabla\mu, \quad \text{with } \boldsymbol{M} = \widehat{\boldsymbol{M}}(\rho,\nabla\rho,\mu,\nabla\mu), \tag{2.7}$$

[2] In [14], the microforce balance is stated under form of a principle of virtual powers for microscopic motions.

with the tensor-valued *mobility mapping* $\widehat{\boldsymbol{M}}$ satisfying the *residual dissipation inequality*

$$\nabla\mu \cdot \widehat{\boldsymbol{M}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu \geq 0.$$

With the help of (2.3), (2.4), (2.6), and $(2.7)_1$, a general equation for diffusive phase segregation processes is arrived at, namely,

$$\partial_t\rho = \operatorname{div}\Big(\boldsymbol{M}\nabla\Big(\partial_\rho\widehat{\psi}(\rho,\nabla\rho) - \operatorname{div}\big(\partial_{\nabla\rho}\widehat{\psi}(\rho,\nabla\rho)\big) - \gamma\Big)\Big) + \sigma. \tag{2.8}$$

The classic Cahn-Hilliard equation (2.2) is obtained from (2.8) by taking

$$\widehat{\psi}(\rho,\nabla\rho) = f(\rho) + \frac{1}{2}|\nabla\rho|^2, \qquad \boldsymbol{M} = \kappa\boldsymbol{1}, \tag{2.9}$$

and by letting the external distance microforce γ and the order-parameter source term σ be identically null.

An alternative generalization of Cahn-Hilliard equation. The Fried-Gurtin model was well accepted in the mathematical community. In 2006, a largely modified version of it was proposed [19]: while the crucial step (i) was retained, both the order-parameter balance (2.4) and the dissipation inequality (2.5) were dropped and replaced, respectively, by the *microenergy balance*

$$\partial_t\varepsilon = e + w, \quad e := -\operatorname{div}\overline{\boldsymbol{h}} + \overline{\sigma}, \quad w := -\pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho), \tag{2.10}$$

and the *microentropy imbalance*

$$\partial_t\eta \geq -\operatorname{div}\boldsymbol{h} + \sigma, \quad \boldsymbol{h} := \mu\overline{\boldsymbol{h}}, \quad \sigma := \mu\,\overline{\sigma}. \tag{2.11}$$

A further key feature of this new approach to modeling phase segregation by atomic rearrangement is that the *microentropy inflow* $(\boldsymbol{h},\sigma)$ is deemed proportional to the *microenergy inflow* $(\overline{\boldsymbol{h}},\overline{\sigma})$ through the *chemical potential* μ, a positive field; consistently, the free energy is defined to be

$$\psi := \varepsilon - \mu^{-1}\eta, \tag{2.12}$$

with chemical potential playing the same role as *coldness* in the deduction of the heat equation.[3]

Combining (2.10)-(2.12) yields

$$\partial_t\psi \leq -\eta\partial_t(\mu^{-1}) + \mu^{-1}\,\overline{\boldsymbol{h}}\cdot\nabla\mu - \pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho), \tag{2.13}$$

an inequality that replaces (2.5) in restricting *à la* Coleman-Noll the possible constitutive choices. On taking all of the constitutive mappings delivering $\pi, \boldsymbol{\xi}, \eta$, and $\overline{\boldsymbol{h}}$, dependent in principle on $\rho, \nabla\rho, \mu, \nabla\mu$, and on choosing

$$\psi = \widehat{\psi}(\rho,\nabla\rho,\mu) = -\mu\,\rho + f(\rho) + \frac{1}{2}|\nabla\rho|^2, \tag{2.14}$$

[3] As much as absolute temperature is a macroscopic measure of microscopic *agitation*, its inverse - the coldness - measures microscopic *quiet*; likewise, as argued in [19], the chemical potential can be seen as a macroscopic measure of microscopic *organization*.

compatibility with (2.13) implies that we must have:

$$\left\{\begin{array}{c} \widehat{\pi}(\rho,\nabla\rho,\mu) = -\partial_\rho\widehat{\psi}(\rho,\nabla\rho,\mu) = \mu - f'(\rho), \\ \widehat{\boldsymbol{\xi}}(\rho,\nabla\rho,\mu) = \partial_{\nabla\rho}\widehat{\psi}(\rho,\nabla\rho,\mu) = \nabla\rho, \\ \widehat{\eta}(\rho,\nabla\rho,\mu) = \mu^2\partial_\mu\widehat{\psi}(\rho,\nabla\rho,\mu) = -\mu^2\rho \end{array}\right\} \tag{2.15}$$

together with

$$\widehat{\widehat{\boldsymbol{h}}}(\rho,\nabla\rho,\mu,\nabla\mu) = \widehat{\boldsymbol{H}}(\rho,\nabla\rho,\mu,\nabla\mu)\nabla\mu, \quad \nabla\mu\cdot\widehat{\boldsymbol{H}}(\rho,\nabla\rho,\mu,\nabla\mu)\nabla\mu \geq 0.$$

If we now choose for $\widehat{\boldsymbol{H}}$ the simplest expression $\boldsymbol{H} = \kappa\boldsymbol{1}$, implying a constant and isotropic mobility, and if we once again assume that the external distance microforce γ and the source $\overline{\sigma}$ are null, then, with the use of (2.15) and (2.12), the microforce balance (2.3) and the energy balance (2.10) become, respectively,

$$\Delta\rho + \mu - f'(\rho) = 0 \tag{2.16}$$

and

$$2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \kappa\Delta\mu = 0, \tag{2.17}$$

a nonlinear system for the unknowns ρ and μ.

Insertion of the parameters ε and δ. Let us compare systems (2.16)–(2.17) and (2.1). Note that (2.16) and $(2.1)_2$ imply the same 'static' relation between μ and ρ; instead,(2.17) is rather different from $(2.1)_1$, for more than one reason: it is nonlinear; it features both time derivatives of ρ and μ; and, in front of both $\partial_t\mu$ and $\partial_t\rho$ there are nonconstant factors that should remain nonnegative during the evolution. Thus, the system (2.16)–(2.17) deserves a careful analysis.

We begun by attacking the problem as it was, prompted to optimism by the successful outcome of a previous joint research effort [6, 7] in which we tackled the system of Allen-Cahn type derived via the approach in [19] for no-diffusion phase-segregation processes. Unfortunately, the evolution problem ruled by (2.16)–(2.17) turned out to be too difficult for us. Therefore, we decided to study its regularized version (1.11)–(1.14), where equations (1.11) and (1.12) are obtained by introducing the extra terms $\varepsilon\,\partial_t\mu$ and $\delta\,\partial_t\rho$ in (2.17) and (2.16), respectively. Of course, the positive coefficients ε and δ were intended to be made smaller and smaller by way of an asymptotic procedure to be set up after the solvability of the regularized system were proved.

Mathematically, the introduction of the $\varepsilon-$term is motivated by the desire to have a strictly positive coefficient as a factor of $\partial_t\mu$ in (2.17), so as to guarantee the parabolic structure of equation (1.11); on the other hand, the introduction of the $\delta-$term transforms (2.16) into an Allen-Cahn equation with source μ, and is strongly reminiscent of a sort of regularization already employed in various approaches to the so-called *viscous Cahn-Hilliard equations* (examples can be found in [2, 3, 16, 18, 20] and in the references therein).

It is also possible to make clear what additional physics the regularizing perturbations we introduced incorporate into the model. As to the term $\varepsilon\,\partial_t\mu$, it can be made to appear in the microenergy balance (1.11) by modifying as follows the choice for the free energy in (2.14):

$$\psi = -\mu\Big(\rho + \frac{\varepsilon}{2}\Big) + f(\rho) + \frac{1}{2}|\nabla\rho|^2. \tag{2.18}$$

As to the term $\delta\,\partial_t\rho$, it is enough to note that all is needed to make that term appear in the microforce balance (1.12) is to add $\partial_t\rho$ to the list of state variables we considered to analyze the constitutive consequences of (2.13). This measure brings in the dissipation mechanism typical of Allen-Cahn nondiffusional segregation processes, where dissipation depends essentially on $(\partial_t\rho)^2$, in addition to Cahn-Hilliard's $|\nabla\mu|^2-$ dissipation (cf. [19]); and it opens the way to *splitting the distance microforce additively* into an equilibrium and a nonequilibrium part, with $\pi^{eq} = -\partial_\rho\widehat{\psi}(\rho,\nabla\rho,\mu) = \mu - f'(\rho)$ the equilibrium part, just as in $(2.15)_1$, and with $\pi^{neq} = -\delta\,\partial_t\rho$ the nonequilibrium part.

3 Main results

In this section, we state precisely the mathematical problem under investigation, fix our assumptions, and present our results. Let Ω to be a bounded connected open set in $\mathbb{R}^3$ with smooth boundary Γ (the lower-dimensional cases can be treated with minor changes). We also introduce a final time $T \in (0,+\infty)$ and set $Q := \Omega \times (0,T)$. Moreover, we set for convenience:

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad \text{and} \quad W := \{v \in H^2(\Omega) :\ \partial_\nu v = 0 \text{ on } \Gamma\}, \tag{3.1}$$

and we endow these spaces with their standard norms, for which we use a self-explanatory notation like $\|\cdot\|_V$. For $p \in [1,+\infty]$, we write $\|\cdot\|_p$ for the usual norm in $L^p(\Omega)$. As no confusion can arise, the symbol $\|\cdot\|_p$ is used for the norm in $L^p(Q)$ as well. Moreover, any of the above symbols for the norms is used even for any power of these spaces. We remark that the embeddings $W \subset V \subset H$ are compact, since Ω is bounded and smooth. As V is dense in H, we can identify H with a subspace of V^* in the usual way (i.e., so as to have ${}_{V^*}\langle u,v\rangle_V = (u,v)_H$ for every $u \in H$ and $v \in V$); the embedding $H \subset V^*$ is also compact.

We are now concerned with the structural assumptions to set on our system. As the chemical potential is expected to be at least nonnegative, we assume that the function κ is defined just for nonnegative arguments. However, one could study the more general mathematical problem of finding solutions whose component μ might change its sign. In such a case, κ has to be defined on the whole of $\mathbb{R}$ and must satisfy similar assumptions. We require that:

$$\kappa : [0,+\infty) \to \mathbb{R} \text{ is continuous}, \tag{3.2}$$

$$\kappa_*, \kappa^* \in (0,+\infty) \quad \text{and} \quad r_* \in [0,+\infty), \tag{3.3}$$

$$\kappa(r) \le \kappa^* \quad \text{for every } r \ge 0 \quad \text{and} \quad \kappa(r) \ge \kappa_* \quad \text{for every } r \ge r_*, \tag{3.4}$$

$$K(r) := \int_0^r \kappa(s)\,ds \quad \text{for } r \ge 0; \quad K \text{ is strictly increasing}, \tag{3.5}$$

$$f = f_1 + f_2\,, \quad f_1 : \mathbb{R} \to [0,+\infty], \quad f_2 : \mathbb{R} \to \mathbb{R}, \quad g : \mathbb{R} \to [0,+\infty), \tag{3.6}$$

$$f_1 \text{ is convex, proper, l.s.c., and } f_2 \text{ and } g \text{ are } C^2 \text{ functions}, \tag{3.7}$$

$$f_2',\ g, \text{ and } g' \text{ are Lipschitz continuous}, \tag{3.8}$$

$$\beta := \partial f_1 \quad \text{and} \quad \pi := f_2'\,. \tag{3.9}$$

In the following, $D(f_1)$ and $D(\beta)$ $(\subseteq D(f_1))$ denote the effective domains of f_1 and β, respectively.

Remark 3.1. We observe that our assumptions on f and κ allow for strong singularities (at the boundary of $D(\beta)$) and a possible degeneracy (in a right neighbourhood of 0) in the equations for ρ and μ, respectively. The former fact is clear. As far as the latter is concerned, we note that (3.5) is satisfied if and only if κ is nonnegative and the set where κ vanishes has empty interior. So, $r_* = 0$ means uniform parabolicity for equation (1.1). On the contrary, if $r_* > 0$, the equation can degenerate, e.g., at the origin (or even in rather big set of small values). An example is given by $\kappa(r) = \tanh r^{m-1}$ with $m > 1$. In this case, (1.1) roughly behaves like the porous medium equation (slow diffusion) in the region where μ is small.

Remark 3.2. It is known that any proper, convex and lower semicontinuous function is bounded from below by an affine function (see, e.g., [1, Prop. 2.1, p. 51]). Hence, our assumption $f_1 \geq 0$ looks reasonable, because one can suitably modify the smooth perturbation f_2 by adding a straight line. Moreover, the other positivity condition, $g \geq 0$, is just needed on the set $D(\beta)$, while g can take negative values outside of $D(\beta)$. Finally, (3.8) implies that, within the range of relevant values of r, the functions $f_2'(r)$, $g(r)$, and $g'(r)$ grow at most linearly with respect to $|r|$, while $f_2(r)$ grows at most quadratically in $|r|$.

For the initial data, we postulate:

$$\mu_0 \in V, \quad \rho_0 \in W, \quad \mu_0 \geq 0 \quad \text{and} \quad \rho_0 \in D(\beta) \quad \text{a.e. in } \Omega, \tag{3.10}$$

$$\text{and there exists some } \xi_0 \in H \text{ such that } \xi_0 \in \beta(\rho_0) \text{ a.e. in } \Omega. \tag{3.11}$$

Since f_1 is convex and f_2 is smooth, the above assumptions entail $f(\rho_0) \in L^1(\Omega)$.

Now, we introduce the a priori regularity that we require from any solution (μ, ρ, ξ) to our problem. Note that equation (1.2) reduces for any given μ to a rather standard phase-field equation. Therefore, it is natural to look for pairs (ρ, ξ) that satisfy

$$\rho \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{3.12}$$

$$\xi \in L^\infty(0,T;H), \tag{3.13}$$

and to solve the subproblem in a strong form, namely

$$\partial_t \rho - \Delta\rho + \xi + \pi(\rho) = \mu\, g'(\rho) \quad \text{and} \quad \xi \in \beta(\rho) \qquad \text{a.e. in } Q, \tag{3.14}$$

$$\rho(0) = \rho_0 \qquad \text{a.e. in } \Omega. \tag{3.15}$$

We note that (3.12) also contains the Neumann boundary condition for ρ (see (3.1) for the definition of W). On the contrary, the situation is different for the component μ. In the case of uniform parabolicity, i.e., if $r_* = 0$, the coefficient $\kappa(\mu)$ is bounded away from zero, and we can require that

$$\mu \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad \mu \geq 0 \quad \text{a.e. in } Q, \tag{3.16}$$

$$\operatorname{div}\bigl(\kappa(\mu)\nabla\mu\bigr) \in L^2(0,T;H), \tag{3.17}$$

and that μ satisfy

$$\int_\Omega \big(1+2g(\rho(t))\big)\,\partial_t\mu(t)\,v + \int_\Omega \mu(t)\,g'(\rho(t))\,\partial_t\rho(t)\,v + \int_\Omega \kappa(\mu(t))\nabla\mu(t)\cdot\nabla v = 0$$

for every $v \in V$ and for a.a. $t \in (0,T)$, (3.18)

$$\mu(0) = \mu_0 \quad \text{a.e. in } \Omega. \tag{3.19}$$

Thus, the equation holds in a strong sense, i.e.,

$$\big(1+2g(\rho)\big)\,\partial_t\mu + \mu\,g'(\rho)\,\partial_t\rho - \operatorname{div}\big(\kappa(\mu)\nabla\mu\big) = 0 \quad \text{a.e. in } Q, \tag{3.20}$$

while the Neumann boundary condition is understood in the usual weak sense. Furthermore, we observe that (3.16)–(3.18) imply further regularity for μ whenever κ is smoother, thanks to the regularity theory of quasilinear elliptic equations.

If instead we allow r_* to be positive, such a formulation is too strong, since no sufficient information on the gradient $\nabla\mu$ can be obtained. As a consequence, the same happens for the time derivative $\partial_t\mu$. For that reason, we rewrite equation (3.20) in the different form

$$\partial_t\big(1+2g(\rho)\mu\big) - \mu g'(\mu)\partial_t\rho - \Delta K(\mu) = 0. \tag{3.21}$$

More precisely, we also account for the initial and Neumann boundary conditions and accordingly rewrite (3.18)–(3.19). In conclusion, we require the lower regularity

$$\mu \in L^\infty(0,T;H), \quad \mu \ge 0 \quad \text{a.e. in } Q, \quad K(\mu) \in H^1(0,T;H)\cap L^\infty(0,T;V), \tag{3.22}$$

$$(1+2g(\rho))\mu \in H^1(0,T;V^*), \tag{3.23}$$

and replace (3.18)–(3.19) by

$$\langle \partial_t\big((1+2g(\rho))\mu\big)(t), v\rangle - \int_\Omega \big(\mu g'(\rho)\partial_t\rho\big)(t)\,v + \int_\Omega \nabla K(\mu(t))\cdot\nabla v = 0$$

for every $v \in V$ and for a.a. $t \in (0,T)$, (3.24)

$$\big((1+2g(\rho))\mu\big)(0) = \big(1+2g(\rho_0)\big)\,\mu_0. \tag{3.25}$$

In this situation, (3.21) is satisfied in the sense of distributions, only.

Remark 3.3. We observe that even the middle term of (3.24) is meaningful, as we immediately see. First, we note that

$$\rho \in C^0([0,T];C^0(\overline{\Omega})) = C^0(\overline{Q}), \tag{3.26}$$

directly from (3.12) and the compact embedding $W \subset C^0(\overline{\Omega})$ (see, e.g., [21, Sect. 8, Cor. 4]), whence $g'(\rho) \in C^0(\overline{Q})$. Next, (3.22) and the embedding $V \subset L^4(\Omega)$ imply that $K(\mu) \in L^\infty(0,T;L^4(\Omega))$, whence also $\mu \in L^\infty(0,T;L^4(\Omega))$, since $K(r)$ behaves like r for big $|r|$ by (3.4). Finally, $\partial_t\rho \in L^\infty(0,T;H)$. Therefore, $\mu g'(\rho)\partial_t\rho \in L^\infty(0,T;L^{4/3}(\Omega))$. On the other hand, $v \in L^4(\Omega)$ whenever $v \in V$.

Remark 3.4. Note that (3.25) makes sense because $(1 + 2g(\rho))\mu \in C^0([0,T]; V^*)$ (due to (3.23)). However, by accounting for (3.12) and the regularity of g, we see that (3.25) can be read in the simpler form (3.19) also in this case. Indeed, the function $(1 + 2g(\rho))\mu$ also belongs to $L^\infty(0,T;H)$. As is well known (and easy to prove), this implies that it actually is an H-valued function which is weakly continuous, in addition. It easily follows that μ enjoys the same property.

Here are our results. The first two state that the strong and weak formulations are equivalent in the case $r_* = 0$ and that there exists a weak solution in the general case. Due to the former, the latter also proves the existence of a strong solution if $r_* = 0$. Both results will be proved in Section 4.

Proposition 3.5. *Assume* (3.2)–(3.9)*,* (3.10)–(3.11)*, and* $r_* = 0$*. Then, any triplet* (μ, ρ, ξ) *satisfing* (3.12)–(3.13)*,* (3.22)–(3.23) *and solving problem* (3.14)–(3.15)*,* (3.24)–(3.25) *also satisfies* (3.16)–(3.19)*.*

Theorem 3.6. *Assume* (3.2)–(3.9) *and* (3.10)–(3.11)*. Then, there exists at least one triplet* (μ, ρ, ξ) *satisfing* (3.12)–(3.13)*,* (3.22)–(3.23) *and solving problem* (3.14)–(3.15)*,* (3.24)–(3.25)*.*

We notice that no further assumptions are needed to ensure boundedness for ρ, due to (3.26). As far as the first component is concerned, we have the following boundedness result.

Theorem 3.7. *Assume* (3.2)–(3.9)*,* (3.10)–(3.11)*, and let* $\mu_0 \in L^\infty(\Omega)$*. Then, the component* μ *of any triplet* (μ, ρ, ξ) *satisfing* (3.12)–(3.13)*,* (3.22)–(3.23) *and solving problem* (3.14)–(3.15) *and* (3.24)–(3.25) *is essentially bounded.*

The next results hold if we assume that κ is constant. We notice that we could weaken this assumption in our regularity result, while we are not able to do the same as far as uniqueness is concerned, unfortunately. In order to simplify the regularity proof, we take $\kappa = 1$ at once. In the forthcoming Remark 5.4, we will sketch how to deduce even further regularity.

Theorem 3.8. *Assume* (3.2)–(3.9)*,* (3.10)–(3.11)*,* $\mu_0 \in W$*, and* $\kappa = 1$*. Then, any triplet* (μ, ρ, ξ) *satisfing* (3.12)–(3.13)*,* (3.16) *and solving problem* (3.14)–(3.15) *and* (3.18)–(3.19) *enjoys the regularity property*

$$\mu \in W^{1,p}(0,T;H) \cap L^p(0,T;W) \quad \text{for every } p \in [1,+\infty). \tag{3.27}$$

Theorem 3.9. *Assume* (3.2)–(3.9)*,* (3.10)–(3.11)*,* $\mu_0 \in W$*, and* $\kappa = 1$*. Then, the triplet* (μ, ρ, ξ) *satisfing* (3.12)–(3.13)*,* (3.16) *and solving problem* (3.14)–(3.15) *and* (3.18)–(3.19) *is unique.*

Throughout the paper, we account for the well-known embedding $V \subset L^p(\Omega)$ for $1 \le p \le 6$ and the related Sobolev inequality:

$$\|v\|_p \le C\|v\|_V \quad \text{for every } v \in V \text{ and } 1 \le p \le 6, \tag{3.28}$$

where C depends on Ω only. Moreover, we recall that the embeddings $V \subset L^4(\Omega)$ (more generally $V \subset L^p(\Omega)$ with $p < 6$) and $W \subset C^0(\overline{\Omega})$ are compact and use the corresponding inequality

$$\|v\|_4 \leq \varepsilon \|\nabla v\|_H + C_\varepsilon \|v\|_H \quad \text{for every } v \in V \text{ and } \varepsilon > 0, \tag{3.29}$$

where C_ε depends on Ω and ε, only. Furthermore, we make repeated use of the notation

$$Q_t := \Omega \times (0,t) \quad \text{for } t \in [0,T], \tag{3.30}$$

and of the well-known Hölder inequality and the elementary Young inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}\, b^2 \quad \text{for every } a, b \geq 0 \text{ and } \varepsilon > 0. \tag{3.31}$$

Finally, throughout the paper we use a small-case italic c for different constants that may only depend on Ω, the final time T, the shape of the nonlinearities f and g, and the properties of the data involved in the statements at hand; a notation like c_ε signals a constant that depends also on the parameter ε. The reader should keep in mind that the meaning of c and c_ε might change from line to line and even in the same chain of inequalities, whereas those constants that we need to refer to are always denoted by capital letters, just like C in (3.28).

4 Existence

In this section, we first show the equivalence result stated in Proposition 3.5. Then, we prove Theorem 3.6, which ensures the existence of a weak solution.

Proof of Proposition 3.5. As $r_* = 0$, we have $\kappa(r) \geq \kappa_*$ for every $r \geq 0$. This implies that the inverse function $K^{-1} : [0,+\infty) \to [0,+\infty)$ is Lipschitz continuous. Hence, (3.22) implies that

$$\mu = K^{-1}(K(\mu)) \in H^1(0,T;H) \cap L^\infty(0,T;V),$$

i.e., that (3.16) holds. In particular, we can write

$$\nabla K(\mu) = \kappa(\mu)\nabla\mu \quad \text{and} \quad \partial_t\big((1+2g(\rho))\mu\big) = \mu\partial_t(1+2g(\rho)) + (1+2g(\rho))\partial_t\mu$$

and thus replace the weak formulation by the strong one. Next, we note that (3.18) implies that (3.20) holds in the sense of distribution, whence (3.17) follows by comparison. Finally, (3.19) holds even in the general case, as we have observed in Remark 3.4. □

Now we prove Theorem 3.6. Even though our proof closely follows the argumentation of [8], we present the whole procedure and sometimes give some detail, since the changes with respect to the quoted paper are spread in the whole calculation. The starting point is an approximating problem which is still based on a time delay in the right-hand side of (3.14). Namely, we define the translation operator $\mathcal{T}_\tau : L^1(0,T;H) \to L^1(0,T;H)$ depending on a time step $\tau > 0$ by setting, for $v \in L^1(0,T;H)$ and for a.a. $t \in (0,T)$,

$$(\mathcal{T}_\tau v)(t) := v(t-\tau) \quad \text{if } t > \tau \quad \text{and} \quad (\mathcal{T}_\tau v)(t) := \mu_0 \quad \text{if } t < \tau \tag{4.1}$$

(but the same notation $\mathcal{T}_\tau v$ will be used even for functions v that are defined in some subinterval $[0,T']$ of $[0,T]$), and replace μ by $\mathcal{T}_\tau\mu$ in (3.14), essentially. However, we modify the equation for μ at the same time. Precisely, we force uniform parabolicity and allow the solution to take negative values, if possible. To do that, we define $\kappa_\tau : \mathbb{R} \to \mathbb{R}$ and the related function K_τ used later on by

$$\kappa_\tau(r) := \kappa(|r|) + \tau \quad \text{and} \quad K_\tau(r) := \int_0^r \kappa_\tau(s)\,ds \quad \text{for } r \in \mathbb{R}. \tag{4.2}$$

So, the approximating problem consists of the following equations

$$\bigl(1 + 2g(\rho_\tau)\bigr)\,\partial_t\mu_\tau + \mu_\tau\, g'(\rho_\tau)\,\partial_t\rho_\tau - \operatorname{div}\bigl(\kappa_\tau(\mu_\tau)\nabla\mu_\tau\bigr) = 0 \qquad \text{a.e. in } Q, \tag{4.3}$$

$$\partial_t\rho_\tau - \Delta\rho_\tau + \xi_\tau + \pi(\rho_\tau) = (\mathcal{T}_\tau\mu_\tau)\,g'(\rho_\tau) \quad \text{and} \quad \xi_\tau \in \beta(\rho_\tau) \qquad \text{a.e. in } Q, \tag{4.4}$$

complemented with the homogeneous Neumann boundary conditions for both μ_τ and ρ_τ and the initial conditions $\mu_\tau(0) = \mu_0$ and $\rho_\tau(0) = \rho_0$. For convenience, we allow τ to take just discrete values, namely, $\tau = T/N$, where N is any positive integer. ☐

Lemma 4.1. *The approximating problem has a solution* $(\mu_\tau, \rho_\tau, \xi_\tau)$ *satisfying the analogues of* (3.12)–(3.13) *and* (3.16)–(3.17).

Proof. We just give a sketch. As in [8], we inductively solve N problems on the time intervals $I_n = [0, t_n]$, $n = 1, \dots, N$, by constructing the solution directly on the whole of I_n at each step. Namely, given μ_{n-1}, which is defined in $\Omega \times I_{n-1}$, we note that $\mathcal{T}_\tau\mu_{n-1}$ is well defined and known in $\Omega \times I_n$ (even in the starting case $n = 1$) and solves the boundary value problem for ρ_n given by the phase-field equations

$$\partial_t\rho_n - \Delta\rho_n + \xi_n + \pi(\rho_n) = (\mathcal{T}_\tau\mu_n)\,g'(\rho_n) \quad \text{and} \quad \xi_n \in \beta(\rho_n) \quad \text{in } \Omega \times I_n, \tag{4.5}$$

complemented with the boundary and initial conditions just mentioned for ρ_τ. Such a problem is quite standard and has a unique solution in a proper (rather weak) functional analytic framework. Once ρ_n is constructed, we solve the parabolic equation

$$\bigl(1 + 2g(\rho_n)\bigr)\,\partial_t\mu_n + \mu_n\, g'(\rho_n)\,\partial_t\rho_n - \operatorname{div}\bigl(\kappa_\tau(\mu_n)\nabla\mu_n\bigr) = 0 \quad \text{in } \Omega \times I_n, \tag{4.6}$$

together with the boundary and initial conditions prescribed for μ_τ. We note that $g \ge 0$ and $\kappa_\tau(r) \ge \tau$ for every $r \in \mathbb{R}$, so that the equation is uniformly parabolic. Therefore, the problem to be solved has a unique solution in a proper space provided that the coefficient $g'(\rho_n)\partial_t\rho_n$ is not too irregular. So, we should prove that, step by step, we get the right regularity for ρ_n and μ_n. This could be done by induction, as in [8], with some modifications due to our more general framework. We omit this detail and just observe that the needed a priori estimates are close (and even simpler, since τ is fixed here) to the ones performed later on in order to let τ tend to zero. The final point is $\mu_n \ge 0$. We give the proof in detail. We multiply equation (4.6) by $-\mu_n^- := -(-\mu_n)^+$, the negative part of μ_n, and integrate over Q_t with any $t \in I_n$. We observe that

$$\bigl[\bigl(1 + 2g(\rho_n(t))\bigr)\,\partial_t\mu_n + \mu_n\, g(\rho_n)\,\partial_t\rho_n\bigr]\,(-\mu_n^-) = \frac{1}{2}\,\partial_t\bigl((1 + 2g(\rho_n))\,|\mu_n^-|^2\bigr).$$

Hence, by using $\mu_0 \geq 0$, and owing to the boundary condition, we have

$$\frac{1}{2}\int_\Omega (1+2g(\rho_n(t)))\,|\mu_n^-(t)|^2 + \int_{Q_t} \kappa_\tau(\mu_n)|\nabla \mu_n^-|^2 = 0 \quad \text{for every } t \in I_n.$$

Since g and κ_τ are nonnegative, this implies $\mu_n^- = 0$, that is, $\mu_n \geq 0$ a.e. in $\Omega \times I_n$. Once all this is checked, the finite sequence (μ_n, ρ_n, ξ_n), $n = 1, \dots, N$, is actually constructed, and it is clear that a solution to the approximating problem we are looking for is obained by simply taking $n = N$. □

Although the solution to the approximating problem is unique, we do not need uniqueness in the following and just fix a solution $(\mu_\tau, \rho_\tau, \xi_\tau)$ for each τ. Our aim is to let τ tend to zero in order to obtain a solution as stated in Theorem 3.6. Our proof uses compactness arguments and thus relies on a number of uniform (with respect to τ) a priori estimates. Clearly, in performing them, we can take τ as small as we desire, and it will be suitable to assume that $\tau \leq \kappa^*$. In order to make the formulas more readable, we shall omit the index τ in the calculations, waiting for writing μ_τ and ρ_τ only when each estimate is established.

First a priori estimate. Let us test (4.3) by μ_τ and point out that

$$\big[\big(1+2g(\rho)\big)\,\partial_t\mu + \mu\, g'(\rho)\,\partial_t\rho\big]\mu = \frac{1}{2}\,\partial_t\big[(1+2g(\rho))\mu^2\big].$$

Therefore, by integrating over $(0,t)$, where $t \in [0,T]$ is arbitrary, we obtain

$$\int_\Omega \big(1+2g(\rho(t))\big)|\mu(t)|^2 + \int_{Q_t} \kappa_\tau(\mu)|\nabla\mu|^2 = \int_\Omega (1+2g(\rho_0))\mu_0^2\,.$$

Hence, we recall that $g \geq 0$ and observe that $\kappa_\tau^2(r) \leq 2\kappa^*\kappa_\tau(r)$ for every $r \in \mathbb{R}$ by (3.4) and $\tau \leq \kappa^*$. We conclude that

$$\|\mu_\tau\|_{L^\infty(0,T;H)} + \|K_\tau(\mu_\tau)\|_{L^2(0,T;V)} \leq c. \tag{4.7}$$

Actually, we have proved more, namely

$$\|K_\tau^*(\mu_\tau)\|_{L^2(0,T;V)} \leq c \quad \text{where} \quad K_\tau^*(r) := \int_0^r (\kappa_\tau(s))^{1/2}\,ds \quad \text{for } r \in \mathbb{R}.$$

Moreover, we observe that K has a linear growth, so that (4.7) also yields

$$\|K_\tau(\mu_\tau)\|_{L^\infty(0,T;H)} \leq c. \tag{4.8}$$

An implication of (4.7)–(4.8), along with (4.1) and (3.10), is

$$\|\mathfrak{T}_\tau\mu_\tau\|_{L^\infty(0,T;H)} + \|\mathfrak{T}_\tau K_\tau(\mu_\tau)\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c. \tag{4.9}$$

Consequence. The Sobolev inequality (3.28) and estimate (4.7) imply that

$$\|K_\tau(\mu_\tau)\|_{L^2(0,T;L^6(\Omega))} \leq c.$$

On the other hand, (3.4) implies that $K_\tau(r) \geq \kappa_* r - c$ for every $r \geq 0$. We deduce that

$$\|\mu_\tau\|_{L^2(0,T;L^6(\Omega))} \leq c \tag{4.10}$$

Second a priori estimate. Let us add ρ_τ on both sides of (4.4) and test by $\partial_t \rho_\tau$. We have that

$$\begin{aligned}
&\int_{Q_t} |\partial_t \rho|^2 + \frac{1}{2} \|\rho(t)\|_V^2 + \int_\Omega f_1(\rho(t)) \\
&= \frac{1}{2} \|\rho_0\|_V^2 + \int_\Omega f(\rho_0) + \frac{1}{2} \int_\Omega \big(\rho^2(t) - 2 f_2(\rho(t))\big) + \int_{Q_t} g'(\rho)(\mathfrak{T}_\tau \mu) \partial_t \rho \\
&\leq c + c \int_\Omega |\rho(t)|^2 + \frac{1}{4} \int_{Q_t} |\partial_t \rho|^2 + c \|\mathfrak{T}_\tau \mu\|_{L^\infty(0,T;H)}^2,
\end{aligned}$$

for every $t \in [0, T]$. In view of the chain rule and Young's inequality (3.31), we observe that

$$c \int_\Omega |\rho(t)|^2 \leq c \int_\Omega |\rho_0|^2 + \frac{1}{4} \int_{Q_t} |\partial_t \rho|^2 + c \int_0^t \|\rho(s)\|_H^2 \, ds.$$

Hence, as f_1 is nonnegative, on account of (4.9), and with the help of the Gronwall lemma, we deduce that

$$\int_{Q_t} |\partial_t \rho|^2 + \|\rho(t)\|_V^2 + \int_\Omega f_1(\rho(t)) \leq c.$$

Therefore, we obtain:

$$\|\rho_\tau\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq c \quad \text{and} \quad \|f(\rho_\tau)\|_{L^\infty(0,T;L^1(\Omega))} \leq c. \tag{4.11}$$

Third a priori estimate. We rewrite (4.4) as

$$-\Delta \rho + \beta(\rho) \ni -\partial_t \rho - \pi(\rho) + (\mathfrak{T}_\tau \mu) g'(\rho)$$

and note that the right-hand side is bounded in $L^2(0, T; H)$, thanks to the Lipschitz continuity of π and g' and to the previous estimates. By a standard argument (formally test by either $-\Delta \rho$ or $\beta(\rho)$ and use the regularity theory for elliptic equations), we first recover that

$$\|\Delta \rho(s)\|_H^2 + \|\xi(s)\|_H^2 \leq 2 \|-\partial_t \rho(s) - \pi(\rho(s)) + ((\mathfrak{T}_\tau \mu) g'(\rho))(s)\|_H^2 \tag{4.12}$$

for a.a. $s \in (0, T)$, and finally conclude that

$$\|\rho_\tau\|_{L^2(0,T;W)} \leq c \quad \text{and} \quad \|\xi_\tau\|_{L^2(0,T;H)} \leq c. \tag{4.13}$$

Fourth a priori estimate. Our aim is to improve the estimates (4.11) and (4.13). To this end, we proceed formally, at least at the beginning, for the sake of simplicity. However, this procedure could be made rigorous by suitably regularizing equation (4.4) (with respect to ρ, only, i.e., keeping μ fixed), the main tool being the Yosida regularization of maximal monotone operators (see, e.g., [4, p. 28]; see also the proof of Lemma 3.1 of [8] for a further regularization). Such a theory yields, in particular, the estimate

$$\|\partial_t u(0)\|_H \le \|\psi(0) + \Delta\rho_0\|_H + \min_{\eta\in\beta(\rho_0)} \|\eta\|_H \tag{4.14}$$

for the unique solution (u, ω) to the equations

$$\partial_t u - \Delta u + \omega = \psi := g'(\rho)\mathcal{T}_\tau\mu - \pi(\rho) \quad \text{and} \quad \omega \in \beta(u),$$

complemented with the same initial and boundary conditions as those prescribed for ρ. Note that in (4.14) β is understood as the induced maximal monotone operator from H to H. Observe that $(u, \omega) = (\rho, \xi)$; then the application of (4.14), in combination with our assumptions on ρ_0 (see (3.11), in particular), leads to

$$\|\partial_t \rho_\tau(0)\|_H \le c\big(\|\mu_0\|_H + \|\rho_0\|_W + 1 + \|\xi_0\|_H\big) = c. \tag{4.15}$$

We use (4.15) in the subsequent calculation, where we proceed formally, as announced (however, our procedure becomes completely rigorous after a while). In particular, we write $\beta(\rho)$ in place of ξ and treat β as if it were a smooth function. We differentiate (4.4) with respect to time and test the resulting equation by $\partial_t\rho$. We obtain:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\partial_t\rho(t)|^2 + \int_{Q_t} |\nabla\partial_t\rho|^2 + \int_{Q_t} \beta'(\rho)|\partial_t\rho|^2 \\
&= \frac{1}{2}\int_\Omega |(\partial_t\rho)(0)|^2 - \int_{Q_t} \pi'(\rho)|\partial_t\rho|^2 + \int_{Q_t} g''(\rho)(\mathcal{T}_\tau\mu)|\partial_t\rho|^2 \\
&\quad + \int_{Q_t} g'(\rho)\partial_t(\mathcal{T}_\tau\mu)\,\partial_t\rho \\
&\le \frac{1}{2}\int_\Omega |(\partial_t\rho)(0)|^2 + c\int_{Q_t} (1 + \mathcal{T}_\tau\mu)|\partial_t\rho|^2 + \int_{Q_t} g'(\rho)\partial_t(\mathcal{T}_\tau\mu)\,\partial_t\rho.
\end{aligned} \tag{4.16}$$

We treat each term on the right-hand side separately. The first one is estimated by (4.15). In order to deal with the second one, we first account for the Hölder inequality. Then, we also invoke (4.7), the compact embedding $V \subset L^4(\Omega)$ (see (3.29)), and (4.11). For every $\varepsilon > 0$ we infer that

$$\begin{aligned}
&\int_{Q_t} (1 + \mathcal{T}_\tau\mu)|\partial_t\rho|^2 \le \int_0^t \|1 + (\mathcal{T}_\tau\mu)(s)\|_H \|\partial_t\rho(s)\|_4^2\,ds \\
&\le c\int_0^t \|\partial_t\rho(s)\|_4^2\,ds \le \varepsilon\int_{Q_t} |\nabla\partial_t\rho|^2 + c_\varepsilon\int_{Q_t} |\partial_t\rho|^2 \\
&\le \varepsilon\int_{Q_t} |\nabla\partial_t\rho|^2 + c_\varepsilon\,.
\end{aligned} \tag{4.17}$$

The estimate of the last term of (4.16) needs much more work. We recall that $\mathcal{T}_\tau\mu$ is constant with respect to time on the interval $(0, \tau)$ and first compute $\partial_t\mu$ from (4.3). Then we integrate by parts and repeatedly exploit the Hölder, Sobolev, and Young inequalities. We obtain:

$$\int_{Q_t} g'(\rho)\partial_t(\mathfrak{T}_\tau\mu)\,\partial_t\rho = \int_0^{t-\tau}\!\!\int_\Omega \partial_t\mu(s)\,g'(\rho(s+\tau))\partial_t\rho(s+\tau)\,ds$$

$$= \int_0^{t-\tau}\!\!\int_\Omega \frac{1}{1+2g(\rho(s))}\big[\operatorname{div}\big(\kappa_\tau(\mu)(s)\nabla\mu(s)\big) - \mu(s)g'(\rho(s))\partial_t\rho(s)\big]\partial_t g(\rho(s+\tau))\,ds$$

$$= \int_0^{t-\tau}\!\!\int_\Omega \kappa_\tau(\mu)(s)\nabla\mu(s)\cdot\nabla\frac{\partial_t g(\rho(s+\tau))}{1+2g(\rho(s))}\,ds$$

$$\quad - \int_0^{t-\tau}\!\!\int_\Omega \frac{g'(\rho(s))\,g'(\rho(s+\tau))}{1+2g(\rho(s))}\,\mu(s)\partial_t\rho(s)\partial_t\rho(s+\tau)\,ds, \tag{4.18}$$

and now treat the last two integrals separately, by accounting for our structural assumptions. We have

$$\int_0^{t-\tau}\!\!\int_\Omega \kappa_\tau(\mu)(s)\nabla\mu(s)\cdot\nabla\frac{\partial_t g(\rho(s+\tau))}{1+2g(\rho(s))}\,ds$$

$$= \int_0^{t-\tau}\!\!\int_\Omega \nabla K_\tau(\mu)(s)\cdot\nabla\,\frac{g'(\rho(s+\tau))\partial_t\rho(s+\tau)}{1+2g(\rho(s))}\,ds$$

$$\le c\int_0^{t-\tau}\!\!\int_\Omega |\nabla K_\tau(\mu)(s)|\,|\nabla\partial_t\rho(s+\tau)|\,ds$$

$$\quad + c\int_0^{t-\tau}\!\!\int_\Omega |\nabla K_\tau(\mu)(s)|\,|\nabla\rho(s)|\,|\partial_t\rho(s+\tau)|\,ds$$

$$\quad + c\int_0^{t-\tau}\!\!\int_\Omega |\nabla K_\tau(\mu)(s)|\,|\nabla\rho(s+\tau)|\,|\partial_t\rho(s+\tau)|\,ds. \tag{4.19}$$

For every $\varepsilon\in(0,1)$ there holds

$$\int_0^{t-\tau}\!\!\int_\Omega |\nabla K_\tau(\mu)(s)|\,|\nabla\partial_t\rho(s+\tau)|\,ds$$

$$\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_{Q_t}|\nabla K_\tau(\mu)|^2 \;\le\; \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon, \tag{4.20}$$

thanks to (4.7). On the other hand, we also have

$$\int_0^{t-\tau}\!\!\int_\Omega |\nabla K_\tau(\mu)(s)|\,|\nabla\rho(s)|\,|\partial_t\rho(s+\tau)|\,ds$$

$$\le \int_0^{t-\tau}\|\nabla K_\tau(\mu)(s)\|_2\|\nabla\rho(s)\|_4\|\partial_t\rho(s+\tau)\|_4\,ds$$

$$\le \varepsilon\int_0^t\|\partial_t\rho(s)\|_V^2\,ds + c_\varepsilon\int_0^{t-\tau}\|\nabla K_\tau(\mu)(s)\|_H^2\|\nabla\rho(s)\|_V^2\,ds$$

$$\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c\int_{Q_t}|\partial_t\rho|^2 + c_\varepsilon\int_0^{t-\tau}\|\nabla K_\tau(\mu)(s)\|_H^2\|\nabla\rho(s)\|_V^2\,ds$$

$$\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c + c_\varepsilon\int_0^{t-\tau}\|\nabla K_\tau(\mu)(s)\|_H^2\|\nabla\rho(s)\|_V^2\,ds. \tag{4.21}$$

In the last inequality we have used (4.11). However, the above estimate has to be improved. To this end, we use the regularity theory for linear elliptic equations and estimates (4.9) and (4.11) again. Indeed, with the help of (4.12) we have:

$$\|\nabla\rho(s)\|_V^2 \le c\big(\|\rho(s)\|_V^2 + \|\Delta\rho(s)\|_H^2\big) \le c\big(\|\partial_t\rho(s)\|_H^2 + 1\big).$$

Therefore, the above estimate becomes

$$\int_0^{t-\tau}\!\!\int_\Omega |\nabla K_\tau(\mu)(s)|\,|\nabla\rho(s)|\,|\partial_t\rho(s+\tau)|\,ds$$
$$\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_0^t \|\nabla K_\tau(\mu)(s)\|_H^2\|\partial_t\rho(s)\|_H^2\,ds + c_\varepsilon\,. \tag{4.22}$$

Analogously, one shows that

$$\int_0^{t-\tau}\!\!\int_\Omega |\nabla K_\tau(\mu)(s)|\,|\nabla\rho(s+\tau)|\,|\partial_t\rho(s+\tau)|\,ds$$
$$\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_0^t \|\nabla(\mathcal{T}_\tau K_\tau(\mu))(s)\|_H^2\,\|\partial_t\rho(s)\|_H^2\,ds + c_\varepsilon\,. \tag{4.23}$$

Thus, by collecting (4.20) and (4.22)–(4.23), we deduce that (4.19) yields

$$\int_0^{t-\tau}\!\!\int_\Omega \kappa_\tau(\mu)(s)\nabla\mu(s)\cdot\nabla\frac{\partial_t\rho(s+\tau)}{1+2g(\rho(s))}\,ds \le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2$$
$$+ c_\varepsilon\int_0^t \Big(\|\nabla K_\tau(\mu)(s)\|_H^2 + \|\nabla(\mathcal{T}_\tau K_\tau(\mu))(s)\|_H^2\Big)\|\partial_t\rho(s)\|_H^2\,ds + c_\varepsilon \tag{4.24}$$

for every $\varepsilon > 0$. Let us come to the last term of (4.18). By using the compacness inequality (3.29), and (4.7) as well, we have

$$-\int_0^{t-\tau}\!\!\int_\Omega \frac{g'(\rho(s))g'(\rho(s+\tau))}{1+2g(\rho(s))}\,\mu(s)\partial_t\rho(s)\partial_t\rho(s+\tau)\,ds$$
$$\le c\int_0^{t-\tau}\|\mu(s)\|_4\|\partial_t\rho(s+\tau)\|_4\|\partial_t\rho(s)\|_2\,ds$$
$$\le \varepsilon\int_0^{t-\tau}\|\partial_t\rho(s+\tau)\|_V^2\,ds + c_\varepsilon\int_0^t\|\mu(s)\|_4^2\|\partial_t\rho(s)\|_H^2\,ds$$
$$\le \varepsilon\int_0^t\|\nabla\partial_t\rho(s)\|_H^2\,ds + c + c_\varepsilon\int_0^t\|\mu(s)\|_4^2\|\partial_t\rho(s)\|_H^2\,ds. \tag{4.25}$$

Therefore, due to (4.24) and (4.25), (4.18) becomes

$$\int_{Q_t} g'(\rho)\partial_t(\mathcal{T}_\tau\mu)\,\partial_t\rho \le 2\varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_0^t\|\mu(s)\|_4^2\|\partial_t\rho(s)\|_H^2\,ds$$
$$+ c_\varepsilon\int_0^t\Big(\|\nabla K_\tau(\mu)(s)\|_H^2 + \|\nabla(\mathcal{T}_\tau K_\tau(\mu))(s)\|_H^2\Big)\|\partial_t\rho(s)\|_H^2\,ds + c_\varepsilon\,. \tag{4.26}$$

At this point, we combine (4.15), (4.17), (4.26) with (4.16) and choose ε small enough. As the last integral on the left-hand side is nonnegative since f_1 is convex, we obtain

$$\int_\Omega |\partial_t\rho(t)|^2 + \int_{Q_t} |\nabla\partial_t\rho|^2 \le c\int_0^t \phi(s)\|\partial_t\rho(s)\|_H^2\,ds + c$$
$$\text{where}\quad \phi(s) := \|\mu(s)\|_4^2 + \|\nabla K_\tau(\mu)(s)\|_H^2 + \|\nabla(\mathfrak{T}_\tau K_\tau(\mu))(s)\|_H^2\,.$$

As $\phi \in L^1(0,T)$ by (4.7) and (4.10), we can apply the Gronwall lemma and conclude that

$$\|\partial_t\rho_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c. \tag{4.27}$$

Consequence. We have $-\Delta\rho_\tau + \xi_\tau = \psi := -\partial_t\rho_\tau + g'(\rho_\tau)\mathfrak{T}_\tau\mu_\tau \in L^\infty(0,T;H)$ due to (4.7) and (4.27). Therefore, by a standard argument (formally multiply by $-\Delta\rho_\tau$ at any fixed time), we deduce that both $-\Delta\rho_\tau$ and ξ_τ belong to $L^\infty(0,T;H)$. Owing to elliptic regularity, we conclude that

$$\|\rho_\tau\|_{L^\infty(0,T;W)} \le c \quad\text{and}\quad \|\xi_\tau\|_{L^\infty(0,T;H)} \le c, \tag{4.28}$$

whence also

$$\|\rho_\tau\|_{L^\infty(Q)} + \|g(\rho_\tau))\|_{L^\infty(Q)} + \|g'(\rho_\tau))\|_{L^\infty(Q)} + \|\pi(\rho_\tau))\|_{L^\infty(Q)} \le c, \tag{4.29}$$

due to the continuous embedding $W \subset L^\infty(\Omega)$ and the continuity of g, g', and π (note however that g' is a bounded function since g is Lipschitz continuous).

Fifth a priori estimate. We write (4.3) as $\partial_t\big((1+2g(\rho))\mu\big) = \Delta K_\tau(\mu) + g'(\rho)\mu\partial_t\rho$. Thus, we have for every $v \in L^2(0,T;V)$

$$\begin{aligned}
&\left|\int_Q \partial_t\big((1+2g(\rho))\mu\big)\,v\right| = \left|-\int_Q \nabla K_\tau(\mu)\cdot\nabla v + \int_Q g'(\rho)\mu\partial_t\rho\,v\right|\\
&\le \|K_\tau(\mu)\|_{L^2(0,T;V)}\|v\|_{L^2(0,T;V)} + \|\partial_t\rho\|_{L^\infty(0,T;H)}\|\mu\|_{L^2(0,T;L^4(\Omega))}\|v\|_{L^2(0,T;L^4(\Omega))}\\
&\le \big(\|K_\tau(\mu)\|_{L^2(0,T;V)} + c\|\partial_t\rho\|_{L^\infty(0,T;H)}\|\mu\|_{L^2(0,T;L^4(\Omega))}\big)\|v\|_{L^2(0,T;V)}.
\end{aligned}$$

By accounting for (4.7), (4.27), and (4.10), we deduce that

$$\|\partial_t\big((1+2g(\rho_\tau))\mu_\tau\big)\|_{L^2(0,T;V^*)} \le c. \tag{4.30}$$

Sixth a priori estimate. We test (4.3) by $\partial_t K_\tau(\mu) = \kappa_\tau(\mu)\partial_t\mu$ and obtain

$$\begin{aligned}
&\int_{Q_t} (1+2g(\rho))\kappa_\tau(\mu)|\partial_t\mu|^2 + \frac{1}{2}\int_\Omega |\nabla K_\tau(\mu(t))|^2\\
&= \frac{1}{2}\int_\Omega |\nabla K_\tau(\mu_0)|^2 - \int_{Q_t} g'(\rho)\partial_t\rho\,\mu\partial_t K_\tau(\mu)
\end{aligned} \tag{4.31}$$

for every $t \in (0,T)$. Note that the first term on the left-hand side can be estimated from below as follows

$$\int_{Q_t} (1+2g(\rho))\kappa_\tau(\mu)|\partial_t\mu|^2 \ge \int_{Q_t} \frac{\kappa_\tau^2(\mu)}{2\kappa^*}\,|\partial_t\mu|^2 = \frac{1}{2\kappa^*}\int_{Q_t} |\partial_t K_\tau(\mu)|^2. \tag{4.32}$$

Now, we deal with the right-hand side of (4.31). The first term being trivial thanks to $(3.10)_1$, we come to the second one. We invoke the Young, Hölder, and Sobolev inequalities and have

$$
\begin{aligned}
&-\int_{Q_t} g'(\rho)\partial_t\rho\,\mu\partial_t K_\tau(\mu) \le \frac{1}{4\kappa^*}\int_{Q_t} |\partial_t K_\tau(\mu)|^2 + c\int_0^t \|\mu(s)\|_4^2 \|\partial_t\rho(s)\|_4^2\,ds \\
&\le \frac{1}{4\kappa^*}\int_{Q_t} |\partial_t K_\tau(\mu)|^2 + c\int_0^t \|\mu(s)\|_4^2 \|\partial_t\rho(s)\|_V^2\,ds. \qquad (4.33)
\end{aligned}
$$

Now, we observe that (3.4) yields $K_\tau(r) \ge \kappa_* r - c_*$ for every $r \ge 0$, where c_* depends on the structural assumptions, only. Hence, by owing to (4.8) as well, we deduce

$$
\begin{aligned}
&\|\mu(s)\|_4^2 \le c\big(\|K_\tau(\mu)(s)\|_4^2 + 1\big) \le c\big(\|K_\tau(\mu)(s)\|_V^2 + 1\big) \\
&\le c\|\nabla K_\tau(\mu)(s)\|_H^2 + c\|K_\tau(\mu)(s)\|_H^2 + c \le c\|\nabla K_\tau(\mu)(s)\|_H^2 + c
\end{aligned}
$$

for a.a. $s \in (0,T)$. By combining (4.32) and (4.33) with (4.31), we obtain

$$
\frac{1}{4\kappa^*}\int_{Q_t} |\partial_t K_\tau(\mu)|^2 + \frac{1}{2}\int_\Omega |\nabla K_\tau(\mu(t))|^2 \le c + c\int_0^t \phi(s)\big(\|\nabla K_\tau(\mu)(s)\|_H^2 + 1\big)\,ds
$$
$$
\text{where} \quad \phi(s) := \|\partial_t\rho(s)\|_V^2.
$$

As $\phi \in L^1(0,T)$ by (4.27), we can apply the Gronwall lemma and conclude that

$$
\|K_\tau(\mu_\tau)\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c. \qquad (4.34)
$$

Consequence. By arguing as we did for (4.10), we derive that

$$
\|\mu_\tau\|_{L^\infty(0,T;L^6(\Omega))} \le c. \qquad (4.35)
$$

Limit and conclusion. By the above estimates, there exist a triplet (μ,ρ,ξ), with $\mu \ge 0$ a.e. in Q, and functions k and ζ such that

$\mu_\tau \to \mu$	weakly star in $L^\infty(0,T;L^6(\Omega))$,	(4.36)
$\rho_\tau \to \rho$	weakly star in $L^\infty(0,T;W)$,	(4.37)
$\partial_t\rho_\tau \to \partial_t\rho$	weakly star in $L^\infty(0,T;H) \cap L^2(0,T;V)$,	(4.38)
$\xi_\tau \to \xi$	weakly star in $L^\infty(0,T;H)$,	(4.39)
$K_\tau(\mu_\tau) \to k$	weakly star in $H^1(0,T;H) \cap L^\infty(0,T;V)$,	(4.40)
$\zeta_\tau := (1+2g(\rho_\tau))\mu_\tau \to \zeta$	weakly star in $H^1(0,T;V^*) \cap L^\infty(0,T;L^6(\Omega))$,	(4.41)

at least for a susequence $\tau = \tau_i \searrow 0$. By (4.37)–(4.38), (4.40), and the compact embeddings $W \subset C^0(\overline{\Omega})$ and $V \subset H$, we can apply well-known strong compactness results (see, e.g., [21, Sect. 8, Cor. 4]) and have that

$\rho_\tau \to \rho$	strongly in $C^0(\overline{Q})$	(4.42)
$K_\tau(\mu_\tau) \to k$	strongly in $C^0([0,T];H)$ and a.e. in Q.	(4.43)

The weak convergence (4.39) and (4.42) imply that $\xi \in \beta(\rho)$ a.e. in Q, as is well known (see, e.g., [4, Prop. 2.5, p. 27]). The strong convergence (4.42) also implies the Cauchy condition (3.15) and that $\phi(\rho_\tau) \to \phi(\rho)$ strongly in $C^0(\overline{Q})$ for every continuous function $\phi : \mathbb{R} \to \mathbb{R}$. We can apply this fact to the functions g, g', and π (see (3.8)). In particular, we infer that $\mu_\tau g'(\rho_\tau)$ has some weak limit in $L^\infty(0,T;L^6(\Omega))$: thus, we can identify it with the help of (4.36) and conclude that (3.14) holds. Now, we prove that μ_τ converges to μ a.e. in Q. To this aim, we note that K_τ^{-1} converges to K^{-1} uniformly on $[0,R]$ for every $R > 0$. Hence, we see that (4.43) implies $\mu_\tau \to K^{-1}(k)$ a.e. in Q. By comparison with (4.36), we deduce that $\mu_\tau \to \mu$ a.e. in Q. Next, let us deal with the subproblem for μ. The identity $K^{-1}(k) = \mu$ just proved means that $k = K(\mu)$. From the convergence almost everywhere of μ_τ to μ we also infer that ζ_τ converges to $(1+2g(\rho))\mu$ a.e. in Q, whence $\zeta = (1+2g(\rho))\mu$ by comparing with (4.41). The last term to be identified is the limit of $\eta_\tau := \mu_\tau g'(\rho_\tau)\partial_t\rho_\tau$. Precisely, we prove that η_τ converges to $\eta := \mu g'(\rho)\partial_t\rho$ weakly in some L^p-type space. We observe that (4.36) and the convergence almost everywhere of μ_τ imply that

$$\mu_\tau \to \mu \quad \text{strongly in } L^p(0,T;L^q(\Omega)) \text{ for every } p < +\infty \text{ and } q < 6 \tag{4.44}$$

as is well-known (via the Severini-Egorov theorem). By choosing, e.g., $p = q = 4$ and combining with the weak star convergence of $\partial_t\rho_\tau$ in $L^\infty(0,T;H)$ (see (4.38)) and the uniform convergence of $g'(\rho_\tau)$, we deduce that η_τ converges to η weakly in $L^4(0,T;L^{4/3}(\Omega))$, thus weakly in $L^2(0,T;L^{4/3}(\Omega))$. At this point, it is straightforward to derive (3.24) in an integrated form, namely

$$\int_0^T \langle \partial_t\big((1+2g(\rho))\mu\big)(t), v(t)\rangle\, dt - \int_Q \mu g'(\rho)\partial_t\rho\, v + \int_Q \nabla K(\mu)\cdot\nabla v = 0 \tag{4.45}$$

for any $v \in L^2(0,T;V) \subset L^2(0,T;L^4(\Omega))$, whence also the time-pointwise version (3.24) itself. Finally, (4.41) implies that $\zeta_\tau \to \zeta$ strongly in $C^0([0,T];V^*)$, thus, $\zeta_\tau(0) \to \zeta(0)$ strongly in V^*, so that the Cauchy condition (3.25) is verified as well. This concludes the proof.

5 Further properties

In this section, we prove Theorems 3.7 and 3.8 and make some remarks on the regularity of solutions. As far as the first result is concerned, we adapt the argument used in [8]. However, as the first estimate of the proof has to be derived in a different way, we prepare a technical lemma.

Lemma 5.1. *Assume*

$$u \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \quad \textit{and} \quad u^+ \in H^1(0,T;H) \cap L^\infty(0,T;V),$$

and let $\gamma \in H^1(0,T;V) \cap L^\infty(0,T;W)$. *Then, for every* $t \in [0,T]$, *we have that*

$$\int_0^t \langle \partial_t u(s), \gamma(s)u^+(s)\rangle\, ds = \int_{Q_t} u\gamma\partial_t u^+. \tag{5.1}$$

Proof. As in Remark 3.4, u is a weakly continuous H-valued function and the pointwise values of u and u^+ make sense. We start from the formula

$$\int_0^t \langle \partial_t u(s), v(s)\rangle \, ds = \langle u(t), v(t)\rangle - \langle u(0), v(0)\rangle - \int_0^t \langle u(s), \partial_t v(s)\rangle \, ds$$
$$= \int_\Omega u(t)v(t) - \int_\Omega u(0)v(0) - \int_{Q_t} u \, \partial_t v,$$

which holds if $v \in H^1(0,T;V)$. By an easy regularization, one sees that it still holds if $v \in H^1(0,T;H) \cap L^2(0,T;V)$. Now, by applying our assumptions on u^+ and γ and also owing to the Sobolev inequality (3.28), we have

$$u^+ \in L^\infty(0,T;L^4(\Omega)), \quad \nabla u^+ \in L^2(Q), \quad \partial_t u^+ \in L^2(Q)$$
$$\gamma \in L^\infty(Q), \quad \nabla\gamma \in L^\infty(0,T;L^4(\Omega)), \quad \partial_t\gamma \in L^2(0,T;L^4(\Omega)). \tag{5.2}$$

It follows that $\gamma u^+ \in H^1(0,T;H) \cap L^2(0,T;V)$. Therefore, we obtain

$$\int_0^t \langle \partial_t u(s), \gamma(s) u^+(s)\rangle \, ds$$
$$= \int_\Omega u(t)\gamma(t)u^+(t) - \int_\Omega u(0)\gamma(0)u^+(0) - \int_{Q_t} u \, \partial_t(\gamma u^+)$$
$$= \int_\Omega u(t)\gamma(t)u^+(t) - \int_\Omega u(0)\gamma(0)u^+(0) - \int_{Q_t} u^+(u^+\partial_t\gamma + \gamma\partial_t u^+)$$
$$= \int_\Omega \gamma(t)|u^+(t)|^2 - \int_\Omega \gamma(0)|u^+(0)|^2 - \int_{Q_t} |u^+|^2\partial_t\gamma - \int_{Q_t} \gamma\partial_t|u^+|^2 + \int_{Q_t} u^+\gamma\partial_t u^+$$
$$= \int_{Q_t} u\gamma\partial_t u^+$$

for all $t \in [0,T]$, and the lemma is proved. □

Proof of Theorem 3.7. Set $\mu_0^* := \max\{1, \|u_0\|_\infty\}$. In the proof performed in [8] the quantity $\int_\Omega |(\mu(t)-k)^+|^2 + \int_{Q_t} |\nabla(\mu-k)^+|^2$ is estimated for any $k \geq \mu_0^*$ by testing (3.18) by $(\mu-k)^+$. In the present case, the equation to be tested is (3.24) instead of (3.18), and a more elaborate procedure is needed. First of all, we check that $(\mu-k)^+$ is an admissible test function (this is not obvious since $\nabla\mu$ might not exist in the usual sense). We recall that K is a strictly increasing mapping from $[0,+\infty)$ onto itself and that K^{-1} is Lipschitz continuous on the interval $[s_*,+\infty)$, where $s_* := K(r_*)$, due to (3.4). Therefore, we can choose a strictly increasing map $K_* : [0,+\infty) \to [0,+\infty)$ that is globally Lipschitz continuous and coincides with K^{-1} on $[s_*,+\infty)$. Hence, we have $K_*(K(r)) = r$ for every $r \geq r_*$ and $K_*(K(r)) < r_*$ for $r < r_*$. It follows that $(r-k)^+ = (K_*(K(r))-k)^+$ for every $r \geq 0$ if $k \geq r_*$. On the other hand, $K_*(K(\mu)) \in H^1(0,T;H) \cap L^2(0,T;V)$ by (3.22). Hence, $(\mu-k)^+$ enjoys the same regularity and is an admissible test function in (3.24) provided that $k \geq r_*$. Thus, we assume $k \geq \max\{\mu_0^*, r_*\}$ from now on. We have from (3.24)

$$\int_0^t \langle \partial_t\big[(1+2g(\rho))\mu\big](s), (\mu(s)-k)^+\rangle \, ds + \int_{Q_t} \nabla K(\mu)\cdot\nabla(\mu-k)^+$$
$$= \int_{Q_t} \mu\partial_t g(\rho)\,(\mu-k)^+$$

for every $t \in [0,T]$, and a simple rearrangement yields

$$\int_0^t \langle \partial_t \big[(1+2g(\rho))(\mu-k)\big](s), (\mu(s)-k)^+\rangle \, ds + \int_{Q_t} \nabla K(\mu)\cdot\nabla(\mu-k)^+ \\ = \int_{Q_t} \partial_t g(\rho)\,|(\mu-k)^+|^2 - k\int_{Q_t} \partial_t g(\rho)\,(\mu-k)^+. \tag{5.3}$$

Now, noting that $1/(1+2g(\rho)) \in H^1(0,T;V)\cap L^\infty(0,T;W)$ by (3.12) and our assumptions on g (recall (3.6)–(3.8)), we apply Lemma 5.1 with $u := (1+2g(\rho))(\mu-k)$ and $\gamma := 1/(1+2g(\rho))$ and transform the first term on the left-hand side as follows:

$$\int_0^t \langle \partial_t \big[(1+2g(\rho))(\mu-k)\big](s), (\mu(s)-k)^+\rangle \, ds = \int_{Q_t} (\mu-k)\partial_t\big[(1+2g(\rho))(\mu-k)^+\big] \\ = \int_{Q_t} 2\partial_t g(\rho)\,|(\mu-k)^+|^2 + \int_{Q_t} (\mu-k)(1+2g(\rho))\,\partial_t(\mu-k)^+ \\ = \frac{1}{2}\int_{Q_t} \partial_t\big[(1+2g(\rho))|(\mu-k)^+|^2\big] + \int_{Q_t} \partial_t g(\rho)\,|(\mu-k)^+|^2.$$

On the other hand, we have a.e. in the set where $\mu \ge k$

$$\nabla(\mu-k)^+ = \nabla\mu = \nabla K^{-1}(K(\mu)) = (K^{-1})'(K(\mu))\nabla K(\mu) = \frac{1}{\kappa(\mu)}\,\nabla K(\mu).$$

Finally, $(\mu(0)-k)^+ = 0$ since $k \ge \mu_0^*$. Therefore, (5.3) becomes

$$\frac{1}{2}\int_\Omega (1+2g(\rho(t)))|(\mu(t)-k)^+|^2 + \int_{Q_t} \kappa(\mu)|\nabla(\mu-k)^+|^2 = -k\int_{Q_t} \partial_t g(\rho)\,(\mu-k)^+.$$

Since g is nonnegative and $\kappa(r) \ge \kappa_*$ for $r \ge k$ (because $k \ge \kappa_*$), we obtain

$$\frac{1}{2}\int_\Omega |(\mu(t)-k)^+|^2 + \kappa_*\int_{Q_t} |\nabla(\mu-k)^+|^2 \le k\int_{Q_t} |\partial_t g(\rho)|\,(\mu-k)^+.$$

At this point, the argument used in [8] can be repeated without changes, essentially. Indeed, the analog of (3.14) is never used there, and the whole proof is based just on the regularity $\partial_t\rho \in L^\infty(0,T;H)\cap L^2(0,T;V)$. In the present case, we have to exploit the same regularity for $\partial_t g(\rho)$ which follows from (3.12). □

Remark 5.2. As observed in Remark 3.4, the component μ of any weak solution is a weakly continuous H-valued function. We show that

$$\mu \in C^0([0,T];L^p(\Omega)) \quad \text{for } p\in[1,2), \quad \text{and for } p\in[1,+\infty) \text{ if } \mu_0 \in L^\infty(\Omega). \tag{5.4}$$

Assume $t_n \to t \in [0,T]$. We prove that $\mu(t_n) \to \mu(t)$ strongly in $L^p(\Omega)$ for p like in (5.4). In fact, $\mu(t_n)$ is bounded in $L^2(\Omega)$ in the general case, and in $L^\infty(\Omega)$ if μ_0 is bounded (thanks to Theorem 3.7). So, the desired convergence is proved once we show that $\mu(t_n) \to \mu(t)$ a.e. in Ω, at least for a subsequence. We observe that (3.22) implies $K(\mu) \in C^0([0,T];H)$. Hence, $K(\mu(t_n)) \to K(\mu)$ a.e. in Ω, at least for a subsequence. As K^{-1} is continuous, the claim follows.

Proof of Theorem 3.8. As (3.26) holds, $g(\rho)$ is continuous and $g'(\rho)$ is bounded. On the other hand, μ is bounded too (by Theorem 3.7 since μ_0 is bounded). Hence, (3.18) can be seen as a linear uniformly parabolic equation for μ with continuous coefficients and a right-hand side belonging to $L^\infty(0,T;H)$. We have indeed

$$\partial_t\mu - \bigl(1+2g(\rho)\bigr)^{-1}\Delta\mu = -\bigl(1+2g(\rho)\bigr)^{-1}\,\mu\, g'(\rho)\,\partial_t\rho.$$

By owing to $\mu_0 \in W$ and optimal L^p-L^q-regularity results (see, e.g., [13, Thm. 2.3]), we infer that (3.27) holds. $\square$

Remark 5.3. We notice that the same result (3.27) holds under an assumption on μ_0 that is weaker than $\mu_0 \in W$. The optimal condition involves a proper Besov space and can give a similar result for a fixed p. We are going to use (3.27) just with $p = 4$ in our proof of the uniqueness of the solution. It follows that uniqueness still holds for a less regular μ_0.

Remark 5.4. We observe that the case corresponding to an empty interior of $D(\beta)$ is completely trivial. Indeed, if $D(\beta) = \{r_0\}$, then ρ takes the constant value r_0, μ solves the corresponding heat equation, and ξ is computed from (3.14). In the opposite case, further regularity can be proved under suitable assumptions on the initial data. For instance, by supposing μ_0 to be bounded and nonnegative, we note that (3.14) yields

$$\partial_t\rho - \Delta\rho + \xi = \mu g'(\rho) - \pi(\rho) \in L^\infty(Q).$$

So, by assuming that $\inf\rho_0$ and $\sup\rho_0$ belong to the interior of $D(\beta)$, one can easily derive that $\xi \in L^\infty(Q)$. Indeed, one can formally multiply by $|\xi|^{p-1}\operatorname{sign}\xi$ and estimate $\|\xi\|_p$ uniformly with respect to p if this assumption on ρ_0 is satisfied. This implies that $\rho \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ for every $p < +\infty$ whenever ρ_0 is smooth enough. However, no further regularity can be proved, in general, since (3.14) cannot be differentiated, unless β is particular, e.g., like in [8]. By the way, in that case, the condition $\xi \in L^\infty(Q)$ is equivalent to $\inf\rho > 0$ and $\sup\rho < 1$. More generally, if $D(\beta)$ is an open interval (a,b) and β is a smooth function, then $\xi = \beta(\rho) \in L^\infty(Q)$ implies that $\inf\rho > a$ and $\sup\rho < b$ and a bootstrap technique using both equations can lead to higher regularity.

6 Uniqueness

In this section, we prove Theorem 3.9. We observe that the uniqueness of the third component ξ follows by comparison in (3.14) once we prove that the pair (μ,ρ) is unique. So, we deal with the first two components, only, and remind the reader that we can use the further regularity given by Theorem 3.8. In particular, by accounting also for (3.12) and (3.28), we have

$$|\nabla\mu| \in L^4(0,T;L^6(\Omega)) \quad \text{and} \quad |\nabla\rho| \in L^4(0,T;L^6(\Omega)) \tag{6.1}$$

for every solution. First of all, we rewrite equation (3.18) in the form

$$\partial_t\bigl(\mu/\alpha(\rho)\bigr) - \alpha(\rho)\Delta\mu = 0, \tag{6.2}$$

where the function $\alpha : [0, +\infty) \to (0, +\infty)$ is defined by

$$\alpha(r) := \bigl(1 + 2g(r)\bigr)^{-1/2} \quad \text{for } r \geq 0. \tag{6.3}$$

More precisely, we consider the variational formulation of (6.2) that accounts for the homogeneous Neumann boundary condition and involves a related unknown function, namely

$$z := \frac{\mu}{\alpha(\rho)} \quad \text{and} \quad \int_\Omega \partial_t z(t)\, v + \int_\Omega \nabla\bigl(\alpha(\rho(t))\, z(t)\bigr) \cdot \nabla\bigl(\alpha(\rho(t))v\bigr) = 0$$
$$\text{for a.a. } t \in (0, T) \text{ and for every } v \in V. \tag{6.4}$$

Notice that z is bounded since both μ and ρ are. Indeed, (3.26) holds and Theorem 3.7 can be applied since $W \subset L^\infty(\Omega)$. Moreover, z satisfies the analogue of (6.1). At this point, we pick two solutions (μ_i, ρ_i, ξ_i), $i = 1, 2$, and set

$$a_i := \alpha(\rho_i) \quad \text{and} \quad z_i := \mu_i / a_i \quad \text{for } i = 1, 2$$

so that (z_i, ρ_i) satisfy (6.4). In the subsequent estimates the (varying) value of the constant c may even depend on the considered solutions, e.g., through $\|z_i\|_\infty$. Our method proceeds as follows. We write (6.4) for both solutions and choose $v = z_1 - z_2$ in the difference. Then we integrate over $(0, t)$, where $t \in (0, T)$ is arbitrary. At the same time, we write (3.14) for both solutions and multiply the difference by $\rho_1 - \rho_2$. Then we integrate over Q_t. Finally, we take a suitable linear combination of the resulting equalities and perform a number of estimates that lead us to apply the Gronwall lemma. However, in order to simplify notation and make the proof more readable, we set

$$\mu := \mu_1 - \mu_2, \quad \rho := \rho_1 - \rho_2, \quad \xi := \xi_1 - \xi_2, \quad z := z_1 - z_2, \quad \text{and} \quad a := a_1 - a_2,$$

and prepare some auxiliary material before starting. The next inequalities account for the boundedness and the Lipschitz continuity of α, α', and $1/\alpha$ on the range of ρ (recall that ρ is bounded). We have

$$\begin{aligned}
&|a| = |\alpha(\rho_1) - \alpha(\rho_2)| \leq c|\rho|, \\
&|\nabla a| = |\alpha'(\rho_1)\nabla\rho + \bigl(\alpha'(\rho_1) - \alpha'(\rho_2)\bigr)\nabla\rho_2| \leq c|\nabla\rho| + c|\nabla\rho_2|\,|\rho|, \\
&|\nabla a_i^{-1}| \leq c|\nabla\rho_i|, \\
&|\mu| \leq |a|\,|z_2| + a_2|z| \leq c|a| + c|z| \leq c|\rho| + c|z|, \\
&|\nabla z| = |\nabla\bigl(a_1^{-1}(a_1 z)\bigr)| \leq c|\nabla(a_1 z)| + c|\nabla\rho_1|\,|z|.
\end{aligned}$$

In what follows, we will repeatedly use these inequalities without reminding the reader.

Lemma 6.1. *We have, for every $t \in [0, T]$,*

$$\int_{Q_t} |\nabla z|^2 \leq c \int_{Q_t} |\nabla(a_1 z)|^2 + c \int_0^t \bigl(1 + \|\nabla\rho_1(s)\|_6^4\bigr)\, \|z(s)\|_2^2\, ds. \tag{6.5}$$

Proof. By the preliminary inequalities just stated, we have

$$\int_{Q_t} |\nabla z|^2 \le c \int_{Q_t} |\nabla(a_1 z)|^2 + c \int_{Q_t} |\nabla \rho_1|^2 \, |z|^2. \tag{6.6}$$

Now, by using the Hölder and Sobolev inequalities (see (3.28)), we obtain

$$\begin{aligned}
&\int_{Q_t} |\nabla \rho_1|^2 \, |z|^2 \le \int_0^t \|\nabla \rho_1(s)\|_6^2 \, \|z(s)\|_6 \, \|z(s)\|_2 \, ds \\
&\le c \int_0^t \|\nabla \rho_1(s)\|_6^2 \left(\|\nabla z(s)\|_2 + \|z(s)\|_2 \right) \|z(s)\|_2 \, ds \\
&\le c \int_0^t \|\nabla \rho_1(s)\|_6^2 \, \|z(s)\|_2^2 \, ds \\
&\quad + \varepsilon \int_{Q_t} |\nabla z|^2 + c_\varepsilon \int_0^t \|\nabla \rho_1(s)\|_6^4 \, \|z(s)\|_2^2 \, ds \\
&\le \varepsilon \int_{Q_t} |\nabla z|^2 + c_\varepsilon \int_0^t \left(1 + \|\nabla \rho_1(s)\|_6^4\right) \|z(s)\|_2^2 \, ds,
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Hence, (6.5) follows by combining this with (6.6) and then choosing ε small enough. □

Lemma 6.2. *Let $k \in L^4(0,T; L^6(\Omega))$. Then we have*

$$\begin{aligned}
\int_{Q_t} k^2 (|z|^2 + |\rho|^2) &\le \varepsilon \int_{Q_t} \left(|\nabla(a_1 z)|^2 + |\nabla \rho|^2 \right) \\
&\quad + c_\varepsilon \int_0^t \left(1 + \|\nabla \rho_1(s)\|_6^4 + \|k(s)\|_6^4\right) \left(\|z(s)\|_2^2 + \|\rho(s)\|_2^2 \right) ds
\end{aligned} \tag{6.7}$$

for every $\varepsilon > 0$ and every $t \in [0,T]$.

Proof. By the Hölder and Sobolev inequalities (see (3.28)), we have

$$\begin{aligned}
&\int_{Q_t} k^2 (|z|^2 + |\rho|^2) \le \int_0^t \|k(s)\|_6^2 \left(\|z(s)\|_6 \, \|z(s)\|_2 + \|\rho(s)\|_6 \, \|\rho(s)\|_2 \right) ds \\
&\le c \int_0^t \|k(s)\|_6^2 \left(\|\nabla z(s)\|_2 \, \|z(s)\|_2 + \|z(s)\|_2^2 + \|\nabla \rho(s)\|_2 \, \|\rho(s)\|_2 + \|\rho(s)\|_2^2 \right) ds \\
&\le \varepsilon \int_0^t \left(\|\nabla z(s)\|_2^2 + \|\nabla \rho(s)\|_2^2 \right) ds \\
&\quad + c_\varepsilon \int_0^t \left(\|k(s)\|_6^4 + \|k(s)\|_6^2 \right) \left(\|z(s)\|_2^2 + \|\rho(s)\|_2^2 \right) ds.
\end{aligned}$$

By applying Lemma 6.1 and denoting the constant that appears in (6.5) by C, we can continue and obtain

$$
\begin{aligned}
&\int_{Q_t} k^2(|z|^2+|\rho|^2) \\
&\le \varepsilon\left(C\int_{Q_t}|\nabla(a_1z)|^2 + C\int_0^t\bigl(1+\|\nabla\rho_1(s)\|_6^4\bigr)\,\|z(s)\|_2^2\,ds + \int_{Q_t}|\nabla\rho|^2\right) \\
&\quad + c_\varepsilon\int_0^t\bigl(1+\|k(s)\|_6^4\bigr)\bigl(\|z(s)\|_2^2+\|\rho(s)\|_2^2\bigr)\,ds.
\end{aligned}
$$

Hence, (6.7) immediately follows. □

At this point, we can start with our program. However, in order to make the argument more transparent, we deal with the first equation only, for a while. We have

$$
\frac{1}{2}\int_\Omega |z(t)|^2 + \int_{Q_t}\bigl(\nabla(a_1z_1)\cdot\nabla(a_1z) - \nabla(a_2z_2)\cdot\nabla(a_2z)\bigr) = 0.
$$

It is convenient to transform the last integrand as follows:

$$
\begin{aligned}
&\nabla(a_1z_1)\cdot\nabla(a_1z) - \nabla(a_2z_2)\cdot\nabla(a_2z) \\
&= |\nabla(a_1z)|^2 + \nabla(a_1z_2)\cdot\nabla(a_1z) - \nabla(a_2z_2)\cdot\nabla(a_1z) + \nabla(a_2z_2)\cdot\nabla(az) \\
&= |\nabla(a_1z)|^2 + \nabla(az_2)\cdot\nabla(a_1z) + \nabla\mu_2\cdot\nabla(az).
\end{aligned}
$$

Then, the above equality becomes

$$
\frac{1}{2}\int_\Omega |z(t)|^2 + \int_{Q_t}|\nabla(a_1z)|^2 = -\int_{Q_t}\nabla(az_2)\cdot\nabla(a_1z) - \int_{Q_t}\nabla\mu_2\cdot\nabla(az), \tag{6.8}
$$

and we estimate each term of the right-hand side separately. We immediately have

$$
\begin{aligned}
&-\int_{Q_t}\nabla(az_2)\cdot\nabla(a_1z) \le \frac{1}{4}\int_{Q_t}|\nabla(a_1z)|^2 + 2\int_{Q_t}\bigl(z_2^2|\nabla a|^2 + a^2|\nabla z_2|^2\bigr) \\
&\le \frac{1}{4}\int_{Q_t}|\nabla(a_1z)|^2 + c\int_{Q_t}\bigl(|\nabla\rho|^2 + |\nabla\rho_2|^2\,|\rho|^2\bigr) + c\int_{Q_t}|\nabla z_2|^2\,|\rho|^2 \\
&\le \frac{1}{4}\int_{Q_t}|\nabla(a_1z)|^2 + C_1\int_{Q_t}|\nabla\rho|^2 + c\int_{Q_t}\bigl(|\nabla\rho_2|^2 + |\nabla z_2|^2\bigr)\,|\rho|^2,
\end{aligned} \tag{6.9}
$$

where we have marked the constant that we want to refer to by terming it C_1. We treat the last term of (6.8) as follows:

$$
\begin{aligned}
&-\int_{Q_t}\nabla\mu_2\cdot\nabla(az) \le \int_{Q_t}|\nabla\mu_2|\bigl(|a|\,|\nabla z| + |z|\,|\nabla a|\bigr) \\
&\le c\int_{Q_t}|\nabla\mu_2|\bigl(|\nabla(a_1z)|\,|\rho| + |z|\,|\nabla\rho_1|\,|\rho| + |z|\,|\nabla\rho| + |z|\,|\nabla\rho_2|\,|\rho|\bigr) \\
&\le \frac{1}{4}\int_{Q_t}|\nabla(a_1z)|^2 + c\int_{Q_t}|\nabla\mu_2|^2\,|\rho|^2 + \int_{Q_t}|\nabla\rho|^2 \\
&\quad + c\int_{Q_t}|\nabla\mu_2|^2\,|z|^2 + c\int_{Q_t}\bigl(|\nabla\rho_1|^2 + |\nabla\rho_2|^2\bigr)\,|\rho|^2.
\end{aligned} \tag{6.10}
$$

Now, we deal with the second equation. Testing the difference of (3.14) by ρ as mentioned at the beginning, easily yields

$$\frac{1}{2}\int_\Omega |\rho(t)|^2 + \int_{Q_t} |\nabla\rho|^2 + \int_{Q_t} \xi\rho = \int_{Q_t} \bigl(\mu_1 g'(\rho_1) - \mu_2 g'(\rho_2) - \pi(\rho_1) + \pi(\rho_2)\bigr)\rho\,. \quad (6.11)$$

We note that the last integral on the left-hand side of (6.11) is nonnegative by monotonicity, while the integrand on the right-hand side can be estimated as follows:

$$\begin{aligned}
&\bigl(\mu_1 g'(\rho_1) - \mu_2 g'(\rho_2) - \pi(\rho_1) + \pi(\rho_2)\bigr)\rho \\
&\leq \bigl(|\mu|\,|g'(\rho_1)| + |\mu_2|\,|g'(\rho_1) - g'(\rho_2)| + |\pi(\rho_1) - \pi(\rho_2)|\bigr)|\rho| \\
&\leq |g'(\rho_1)|\,|\mu|\,|\rho| + c|\mu_2|\,|\rho|^2 + c|\rho|^2 \leq c\bigl(|\mu|^2 + |\rho|^2\bigr) \leq c\bigl(|z|^2 + |\rho|^2\bigr).
\end{aligned}$$

Thus, by inspecting the coefficients of the integral $\int_{Q_t} |\nabla\rho|^2$ that appear on the right-hand sides of (6.9) and (6.10), it is clear that it is convenient to multiply (6.11) by $C_1 + 2$ before adding it to (6.8). Once such a care is taken, it is straightforward to deduce that

$$\begin{aligned}
&\int_\Omega |z(t)|^2 + \int_{Q_t} |\nabla(a_1 z)|^2 + \int_\Omega |\rho(t)|^2 + \int_{Q_t} |\nabla\rho|^2 \\
&\leq c\int_{Q_t} \bigl(|\nabla\mu_2|^2 + 1\bigr)\,|z|^2 + c\int_{Q_t} \bigl(|\nabla\mu_2|^2 + |\nabla\rho_1|^2 + |\nabla\rho_2|^2 + |\nabla z_2|^2 + 1\bigr)\,|\rho|^2 \\
&\leq c\int_{Q_t} \bigl(|\nabla\mu_2| + |\nabla\rho_1| + |\nabla\rho_2| + |\nabla z_2| + 1\bigr)^2\,\bigl(|z|^2 + |\rho|^2\bigr).
\end{aligned}$$

At this point, we recall that (6.1) hold for μ_i, ρ_i, and z_i, so that we can apply Lemma 6.2 with $k = |\nabla\mu_2| + |\nabla\rho_1| + |\nabla\rho_2| + |\nabla z_2| + 1$. Then, we choose $\varepsilon > 0$ small enough and use the Gronwall lemma. We conclude that $z = 0$ and $\rho = 0$, whence $(\mu_1, \rho_1) = (\mu_2, \rho_2)$ follows.

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