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On properties of different notions of centers for convex cones

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Abstract

The points on the revolution axis of a circular cone are somewhat special: they are the "most interior" elements of the cone. This paper addresses the issue of formalizing the concept of center for a convex cone that is not circular. Four distinct proposals are studied in detail: the incenter, the circumcenter, the inner center, and the outer center. The discussion takes place in the context of a reflexive Banach space.

1 Introduction

The purpose of this work is studying four notions of center for a closed convex cone. For simplicity in the presentation, we ask the underlying space $(X, \|\cdot\|)$ to be Banach and reflexive. On some occasions we impose even further structural assumptions like rotundity or smoothness.

An axis, or ray, can be identified with a point on the unit sphere S_X. Defining a central axis in a convex cone is then a matter of identifying the unit vector that generates such axis. It is the unit vector, rather than the corresponding axis, what we have in mind when we refer to the center of a cone. Why should we care about the goal formulated in the title? By way of motivation, we mention three applications.

Example 1.1. The first motivation arises in numerical linear algebra. Let the space $\mathbb{S}_n$ of symmetric matrices of order n be equipped with the trace inner product $\langle A, B\rangle = \mathrm{tr}(AB)$. Suppose that $A_0 \in \mathbb{S}_n$ is positive definite, i.e., A_0 belongs to the interior of the Loewner cone

$$\mathcal{P}_n = \{A \in \mathbb{S}_n : x^T Ax \geq 0 \text{ for all } x \in \mathbb{R}^n\},$$

where the superscript "T" indicates transposition. Suppose also that A_0 has unit length. If A_0 is near the boundary of $\mathcal{P}_n$, then carrying out a Cholesky factorization could be problematic. Indeed, a small perturbation $E \in \mathbb{S}_n$ may produce a nearby matrix $A_0 + E$ that is no longer positive definite. On the contrary, if A_0 is somewhere in the center of $\mathcal{P}_n$, then more important pertubations can be tolerated because there is a long way to go before $A_0 + E$ looses its positive definiteness. Which is the safest location in $\mathcal{P}_n$ for placing the matrix A_0? An alternative formulation of the latter question reads as follows: which is the most positive definite matrix among all the positive definite matrices of unit length? In the same vein, one could ask also which is the most strictly copositive matrix, the most positive entrywise, and so on.

Example 1.2. The second motivation concerns the study of nonsmooth convex bodies. Let C be a convex body in a Hilbert space X and u be a nonsmooth boundary point of C. That u is nonsmooth means that the set

$$N_C(u) = \{y \in X : \langle y, x - u\rangle \leq 0 \text{ for all } x \in C\}$$

of normal vectors to C at u is not reduced to a ray (cf. Figure 1). Which one is the "most normal" among all normal vectors to C at u? This question was raised by Aubry and Löhner [3] for nonsmooth convex bodies in a three dimensional Euclidean space. The answer proposed in [3] is quite reasonable, but it is just one option among many. Everything depends, in fact, on what is to be understood by being the center of the closed convex cone $N_C(u)$.

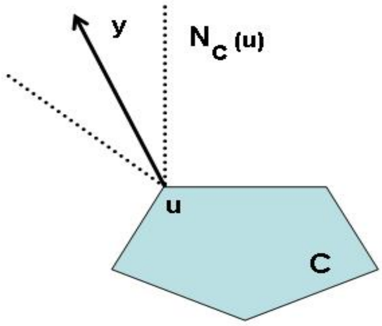

Figure 1: What means that y is the "most normal" vector to C at u?

Example 1.3. In the context of the previous example, consider the closed convex cone

$$T_C(u) = \mathrm{cl}\,[\mathbb{R}_+(C-u)]$$

of tangent directions to C at u. Here, the notation "cl" stands for the closure operation in X. If h is in the boundary of $T_C(u)$, then the half-line $u+\mathbb{R}_+ h$ may intersect C only at u. By contrast, if $h \in \mathrm{int}[T_C(u)]$, then there exists an $\varepsilon > 0$ such that $u + t\,h \in \mathrm{int}(C)$ for all $t \in]0,\varepsilon]$. In such a case, one says that h is an interior displacement direction to C at u. How to choose h if one wishes to get into the interior of C in the steepest possible way?

The concept of center of a closed convex cone, say K, can be formalized in many ways. In this work we explore four options: the *incenter*, which corresponds to the center of a certain largest ball inscribed in the cone; the *circumcenter*, which corrresponds to the center of a certain smallest ball that generates K; the *inner center*, which can be identified with the revolution axis of the largest revolution cone contained in K; the *outer center*, defined as previously, but now one looks for the smallest revolution cone containing K. These four types of center are different in general. Each concept has its own advantanges and inconveniences.

Before getting started we need to fix some terminology. The mathematical object K under analysis is an element of the hyperspace $\Xi(X)$ of nontrivial closed convex cones in X. That a convex cone is nontrivial means that it is different from the singleton $\{0\}$ and different from the whole space X. That K belongs to $\Xi(X)$ is the bare minimum. In practice, we ask K to satisfy further assumptions. Recall that $K \in \Xi(X)$ is *solid* if $\mathrm{int}(K)$ is nonempty, and it is *sharp* if there exists a nonzero vector f in the topological dual space X^* such that $\|x\| \leq \langle f, x \rangle$ for all $x \in K$. The symbol $\langle \cdot, \cdot \rangle$ stands for the duality product between X and X^*, that is to say, $\langle f, x \rangle = f(x)$ for all $(x, f) \in X \times X^*$. For convenience, we introduce the notation

$$\begin{aligned} \Xi_{\rm sol}(X) &= \{K \in \Xi(X) : K \text{ is solid}\}, \\ \Xi_{\rm sh}(X) &= \{K \in \Xi(X) : K \text{ is sharp}\}. \end{aligned}$$

In a reflexive Banach space setting, solidity and sharpness are dual properties (cf.[15, 20]). The use of duality arguments is ubiquitous throughout this work: solidity versus sharpness, smoothness versus rotundity, etc. On several occasions we move from X to X^*, and viceversa. This is done with the help of the duality map $I : X \rightrightarrows X^*$ and its inverse $I^{-1} : X^* \rightrightarrows X$. By definition, I is a multivalued map whose graph is given by

$$\mathrm{gr}(I) = \{(x, f) \in X \times X^* : \langle f, x \rangle = \|x\|^2 = \|f\|_*^2\}.$$

The norm on X^* is the usual one, i.e., $\|f\|_* = \sup_{\|x\|=1} \langle f, x \rangle$. Finally, recall that the set

$$K^+ = \{f \in X^* : \langle f, x \rangle \geq 0 \;\text{ for all } x \in K\}$$

is known as the *dual cone* of K. In the context of a reflexive Banach space, the dual cone of K^+ is nothing else than K itself. Further comments on duality will be given whenever the need arises.

2 The incenter of a convex cone

Likely the first idea of center of K introduced in the literature is that of a vector in the set

$$K \cap S_X = \{x \in K : \|x\| = 1\}$$

maximizing the distance to ∂K, i.e., to the boundary of K. If not the first chronologically, such an idea is at least quite natural and has a strong geometric appealing.

Definition 2.1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $K \in \Xi(X)$. An* incenter *of K is a solution to the variational problem*

$$\rho(K) = \sup_{x \in K \cap S_X} \mathrm{dist}[x, \partial K]. \tag{1}$$

The coefficient $\rho(K)$ is called the inradius *of K.*

The above definition makes sense in a general normed space, but we prefer to give it in a reflexive Banach space. In such a particular context, every element of $\Xi(X)$ admits at least one incenter.

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $K \in \Xi(X)$. Then*

$$\Pi_{\rm inc}(K) = \{x \in K \cap S_X : \mathrm{dist}[x, \partial K] = \rho(K)\}$$

is nonempty. If K happens to be solid, then $\Pi_{\rm inc}(K)$ is a convex set contained in $S_X \cap \mathrm{int}(K)$.

Proof. There is no loss of generality in assuming that K is solid, otherwise every point in $K \cap S_X$ is an incenter and the proposition is trivial. Let us shift the attention to the "convexified" problem

$$\begin{aligned} &\text{maximize } \operatorname{dist}[x, \partial K] \\ &x \in K \cap B_X \end{aligned} \tag{2}$$

with B_X standing for the closed unit ball of X. The boundary ∂K is nonempty because K is not the whole space X. The function $\operatorname{dist}[\,\cdot\,, \partial K]$ is continuous on X, and its restriction to the bounded convex closed set $K \cap B_X$ is concave. Since $(X, \|\cdot\|)$ is Banach and reflexive, (2) admits at least one solution. The solution set to the convexified problem is clearly convex. On the other hand, the solidity of K implies that any solution to (2) belongs to $\operatorname{int}(K)$. Since $\operatorname{dist}[\,\cdot\,, \partial K]$ is positively homogeneous, a solution to (2) must be a unit vector. This completes the proof of the proposition. $\square$

Solving the maximization problem (1) is often a challenging task. Such a maximization problem is worked out in the companion paper [14] for several convex cones arising in applications. A convenient way of representing the solution set to (1) is

$$\Pi_{\rm inc}(K) = \{x \in S_X : x + \rho(K) B_X \subset K\}.$$

This representation formula will be used on several occasions in the sequel. In order to proceed further with the presentation, we need to state a lemma on incenters of half-spaces. A *homogeneous half-space* of X is a set of the form

$$H_f = \{x \in X : \langle f, x\rangle \geq 0\}$$

with f standing for a unit vector of X^*.

Lemma 2.3. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $f \in S_{X^*}$. Then $\rho\,(H_f) = 1$ and $\Pi_{\rm inc}(H_f) = I^{-1}(f)$.*

Proof. As shown in [8, Theorem 1.1.2], the distance from $x \in X$ to the closed hyperplane

$$\partial H_f = \{x \in X : \langle f, x\rangle = 0\}$$

is given by $\operatorname{dist}[x, \partial H_f] = |\langle f, x\rangle|$. Hence, the maximization problem (1) takes the form

$$\rho(H_f) = \sup_{x \in H_f \cap S_X} \langle f, x\rangle.$$

But the constraint $x \in H_f$ is clearly redundant. Hence, $\rho(H_f) = \|f\|_* = 1$ and

$$\Pi_{\rm inc}(H_f) = \{x \in S_X : \langle f, x\rangle = 1\},$$

the set on the right-hand side being precisely $I^{-1}(f)$. $\square$

2.1 Uniqueness of the incenter

A normed space $(X, \|\cdot\|)$ is *rotund* (or strictly convex) if the unit sphere S_X contains no segment. By extension, the term rotundity applies also to the norm. The rotundity of $(X, \|\cdot\|)$ is necessary and sufficient for guaranteeing the uniqueness of solutions to the variational problem (1).

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then the following statements are equivalent:*

(a) Each $K \in \Xi_{\rm sol}(X)$ has a unique incenter.

(b) Each homogeneous half-space of X has a unique incenter.

(c) $(X, \|\cdot\|)$ is rotund.

Proof. The implication (a)⇒(b) is trivial because any homogeneous half-space of X is an element of $\Xi_{\rm sol}(X)$. In view of Lemma 2.3, what conditions (b) says is that $I^{-1}(f)$ is a singleton for all $f \in S_{X^*}$. By a general result on the geometry of Banach spaces (cf. [23, Chapter 5]), the latter condition implies the rotundity of $(X, \|\cdot\|)$. Finally, let (c) be true and let $K \in \Xi_{\rm sol}(X)$. The set $\Pi_{\rm inc}(K)$ being convex and contained in S_X, it must be a singleton. □

The book by Megginson [23] provides a good dozen of equivalent formulations of rotundity. The characterization (a) given in Theorem 2.4 is not surprising althogether, but, to the best of our knowledge, it is new.

In the sequel, whenever we refer to an incenter, we assume that the underlying space is rotund. The unique solution to (1) is then denoted by $\pi_{\rm inc}(K)$, and $\pi_{\rm inc} : \Xi_{\rm sol}(X) \to X$ is seen as an ordinary or single-valued function. It is helpful to think of $\pi_{\rm inc}(K)$ as the "most interior" unit vector of K.

There are various interpretations for $\rho(K)$ and, as a consequence, this coefficient does not have a universally accepted name. A few historical comments might help to put matters in perspective. Assuming that K is solid, Freund [11] writes (1) in the equivalent form

$$\rho(K) = \sup_{x \in {\rm int}(K)} \frac{{\rm dist}[x, \partial K]}{\|x\|} \tag{3}$$

and calls this number the min-width of K. In references [4, 10, 12], the min-width is called simply the width. Iusem and Seeger [18, 20] refer to (1) as a solidity coefficient of K, and write this number as the optimal-value of a variational problem

$$\begin{aligned} &\text{maximize } r \\ &\|x\| = 1 \\ &r \in [0,1] \\ &x + rB_X \subset K \end{aligned} \tag{4}$$

with feasible set in the product space $X \times \mathbb{R}$. View under this light, computing $\rho(K)$ amounts to finding the radius of the largest ball centered in a unit vector and contained in K. This observation explains why $\rho(K)$ measures to which extent the cone K is solid.

Corollary 2.5. *Let* $(X, \|\cdot\|)$ *be a rotund reflexive Banach space and let* $K \in \Xi_{\rm sol}(X)$*. Then the variational problem (4) has exactly one solution, namely* $(\bar{x}, \bar{r}) = (\pi_{\rm inc}(K), \rho(K))$*.*

The proof of the corollary is just a matter of exploiting Theorem 2.4 and the equality

$$\rho(K) = \sup_{x \in K \cap S_X} \sup_{\substack{r \in [0,1] \\ x + rB_X \subset K}} r\,.$$

Note that (4) is obtained by assembling the last two suprema. By mimicking the parlance of the theory of convex bodies, we refer to

$$\mathbb{B}_{\rm inc}(K) = \pi_{\rm inc}(K) + \rho(K)\, B_X$$

as the *inball* of K. There is no risk of confusion with the classical terminology because we are dealing here with cones and not with bounded sets.

2.2 Finding the incenter via least-norm minimization

There is yet another way of looking at the incenter of a convex cone. Recall that a set of the form

$$A \ominus B = \{z \in X : z + B \subset A\}$$

is referred to as the *erosion* of A by B. This name is commonly used in morphological analysis [2, 24], but control theorists refer to $A \ominus B$ as the Pontryagin difference of A and B.

Proposition 2.6. *Let* $(X, \|\cdot\|)$ *be a rotund reflexive Banach space and let* $K \in \Xi_{\rm sol}(X)$*. Then*

$$(\pi_{\rm inc}(K), \rho(K)) = \left(\frac{\xi(K)}{\|\xi(K)\|}, \frac{1}{\|\xi(K)\|} \right) \tag{5}$$

with $\xi(K) = \operatorname{argmin}_{z \in K \ominus B_X} \|z\|$ *denoting the least-norm element of* $K \ominus B_X$*.*

Proof. In view of Freund's representation formula (3), one can write

$$\rho(K) = \sup_{x \in {\rm int}(K)} \left\| \frac{x}{{\rm dist}[x, \partial K]} \right\|^{-1}.$$

The change of variables

$$z = \frac{x}{{\rm dist}[x, \partial K]} \tag{6}$$

leads to the least-norm problem

$$\frac{1}{\rho(K)} = \inf_{\substack{z \in K \\ {\rm dist}[z, \partial K] = 1}} \|z\|\,. \tag{7}$$

A standard homogeneity argument shows that the solution set to (7) remains unchanged if the minimization is carried over the larger set

$$K \ominus B_X = \{z \in K : {\rm dist}[z, \partial K] \geq 1\}. \tag{8}$$

Note that (8) is a closed convex subset of X. Hence, $\xi(K)$ is the unique solution to (7). The formula (5) is obtained by exploiting the relation (6) and the fact that $\pi_{\rm inc}(K)$ has unit length. □

Example 2.7. The incenter of the Pareto cone $\mathbb{R}^n_+$ can be easily found by solving a least-norm minimization problem. Let $B_{\mathbb{R}^n}$ be the closed unit ball of the standard Euclidean space $\mathbb{R}^n$. The least-norm element of the set

$$\mathbb{R}^n_+ \ominus B_{\mathbb{R}^n} = \{x \in \mathbb{R}^n : x_1 \geq 1, \ldots, x_n \geq 1\}$$

is clearly $\mathbf{1}_n = (1, \ldots, 1)^T$. Hence, $\rho(\mathbb{R}^n_+) = 1/\sqrt{n}$ and $\pi_{\mathrm{inc}}(\mathbb{R}^n_+) = \mathbf{1}_n/\sqrt{n}$.

Parenthetically, note that $\rho(\mathbb{R}^n_+) \to 0$ as $n \to \infty$. The asymptotic behavior of $\pi_{\mathrm{inc}}(\mathbb{R}^n_+)$ is more problematic. Let us give a quick look at the infinite dimensional version of the Pareto cone.

Example 2.8. Consider the space $\ell_2(\mathbb{R})$ of square summable real sequences equipped with the usual inner product $\langle y, x\rangle = \sum_{i\in\mathbb{N}} y_i x_i$. The closed convex cone

$$K = \{x \in \ell_2(\mathbb{R}) : x_i \geq 0 \text{ for all } i \in \mathbb{N}\}$$

has empty interior. Hence, $\rho(K) = 0$ and $\Pi_{\mathrm{inc}}(K) = K \cap S_{\ell_2(\mathbb{R})}$. On the other hand, since $K \ominus B_{\ell_2(\mathbb{R})}$ is empty, its least-norm element is not well defined. This example shows that the solidity assumption cannot be omitted in Proposition 2.6.

2.3 Stability of the incenter

This section concerns the stability of the incenter $\pi_{\mathrm{inc}}(K)$ with respect to perturbations in the argument K. In the sequel, the set $\Xi(X)$ is equipped with the truncated Pompeiu-Hausdorff metric

$$\varrho(K_1, K_2) = \mathrm{haus}(K_1 \cap B_X, K_2 \cap B_X).$$

The same metric is used on any subset of $\Xi(X)$. For alternative ways of measuring distances between closed convex cones, the reader may consult the survey paper [21].

In Theorem 2.9, the underlying normed space must enjoy a geometric property that is stronger than rotundity. A normed space $(X, \|\cdot\|)$ is *uniformly rotund* (or uniformly convex) if for all $\varepsilon > 0$ there exists $\eta > 0$ such that $\|u + v\| \leq 2(1 - \eta)$ whenever $u, v \in S_X$ and $\|u - v\| \geq \varepsilon$. It is known (cf. [5, Chapter III.7]) that a uniformly rotund Banach space $(X, \|\cdot\|)$ is rotund, reflexive, and satisfies the Kadec property

$$\{x_n\}_{n\in\mathbb{N}} \overset{weak}{\to} x \text{ and } \limsup_{n\to\infty} \|x_n\| \leq \|x\| \quad \Longrightarrow \quad \lim_{n\to\infty} \|x_n - x\| = 0 \tag{9}$$

for any $x \in X$ and any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X. The Kadec property is essential for passing from weak convergence to strong convergence.

Theorem 2.9. *Suppose that $(X, \|\cdot\|)$ is a uniformly rotund Banach space. Then the function $\pi_{\mathrm{inc}} : (\Xi_{\mathrm{sol}}(X), \varrho) \to (X, \|\cdot\|)$ is continuous.*

Proof. The proof simplifies if distances between closed convex cones are measured by means of the expression

$$\delta(K_1, K_2) = \max\left\{\sup_{a\in K_1\cap S_X} \mathrm{dist}[a, K_2], \sup_{b\in K_2\cap S_X} \mathrm{dist}[b, K_1]\right\}.$$

In a Hilbert space setting, ϱ and δ are exactly the same metric. In a general normed space, δ is not truely a metric because it does not satisfy the triangular inequality. This fact has no incidence in the proof of the theorem. The only thing one needs to know is that

$$\delta(K_1, K_2) \;\leq\; \varrho(K_1, K_2) \;\leq\; 2\,\delta(K_1, K_2). \tag{10}$$

This chain of inequalities can be found, for instance, in [20, Lemma 5]. We need to say also some words on the inradius function. As shown in [20, Corollary 9], whenever X is a reflexive Banach space, the function ρ satisfies

$$|\rho(K_1) - \rho(K_2)| \leq 2\,\delta(K_1, K_2) \tag{11}$$

for all $K_1, K_2 \in \Xi(X)$. The combination of (10) and the Lipschitz inequality (11) implies that $\rho : (\Xi(X), \varrho) \to \mathbb{R}$ is continuous. The continuity analysis of $\pi_{\rm inc}$ is more complicated. Consider a reference argument $K \in \Xi_{\rm sol}(X)$ and a sequence $\{K_n\}_{n\in\mathbb{N}}$ in $\Xi_{\rm sol}(X)$ such that

$$\lim_{n\to\infty} \varrho(K_n, K) = 0. \tag{12}$$

If one writes $c_n = \pi_{\rm inc}(K_n)$, then one has

$$c_n + \rho(K_n)\, B_X \subset K_n \tag{13}$$

for all $n \in \mathbb{N}$. Let us examine what happens with (13) as $n \to \infty$. We know already that

$$\lim_{n\to\infty} \rho(K_n) = \rho(K).$$

Since B_X is weakly sequentially compact, $\{c_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{c_{\varphi(n)}\}_{n\in\mathbb{N}}$ that converges weakly to some $\tilde{c} \in B_X$. We claim that

$$\tilde{c} + \rho(K) B_X \subset K. \tag{14}$$

Proving the inclusion (14) amounts to showing the inequality

$$\langle f, \tilde{c}\rangle + \rho(K)\, \langle f, u\rangle \geq 0 \tag{15}$$

for all $u \in B_X$ and $f \in K^+$. Pick then u and f as just indicated. By combining (10), (12), and the Walkup-Wets isometry theorem (cf. [26, Theorem 1]), one gets

$$\lim_{n\to\infty} \delta_*(K_n^+, K^+) = 0, \tag{16}$$

where δ_* is defined in an obvious way, i.e.,

$$\delta_*(Q_1, Q_2) = \max\left\{ \sup_{a \in Q_1 \cap S_{X^*}} \mathrm{dist}[a, Q_2], \sup_{b \in Q_2 \cap S_{X^*}} \mathrm{dist}[b, Q_1] \right\}.$$

In turn, (16) implies that

$$\lim_{n\to\infty} \mathrm{dist}[f, K_n^+] = \mathrm{dist}[f, K^+] = 0.$$

Hence, $\lim_{n\to\infty} \|f_n - f\|_* = 0$ for some sequence $\{f_n\}_{n\in\mathbb{N}}$ in X^* such that $f_n \in K_n^+$ for all $n \in \mathbb{N}$. We now take (13) into account. This inclusion yields in particular

$$\langle f_{\varphi(n)}, c_{\varphi(n)}\rangle + \rho(K_{\varphi(n)})\, \langle f_{\varphi(n)}, u\rangle \geq 0 \tag{17}$$

for all $n \in \mathbb{N}$. A passage to the limit in (17) leads to (15) and confirms the claim (14). Since $0 \notin \mathrm{int}(K)$, it is clear that $\tilde{c} \neq 0$. The case $0 < \|\tilde{c}\| < 1$ must also be ruled out, because the inclusion

$$\frac{\tilde{c}}{\|\tilde{c}\|} + \frac{\rho(K)}{\|\tilde{c}\|} B_X \subset K$$

would lead to the contradiction

$$\rho(K) \geq \|\tilde{c}\|^{-1} \rho(K) > \rho(K).$$

Hence, $\tilde{c}$ is a unit vector. Taking into account (14) and the uniqueness of the metric center, one deduces that $\tilde{c} = \pi_{\mathrm{inc}}(K)$. In conclusion, the whole sequence $\{c_n\}_{n\in\mathbb{N}}$ converges weakly to $\pi_{\mathrm{inc}}(K)$. Since everything takes place on S_X, the Kadec property (9) implies that $\{c_n\}_{n\in\mathbb{N}}$ converges strongly to $\pi_{\mathrm{inc}}(K)$. □

The proof of Theorem 2.9 shows that if $(X, \|\cdot\|)$ is a rotund reflexive Banach space, but not uniformly rotund, then π_{inc} is continuous from $(\Xi_{\mathrm{sol}}(X), \varrho)$ to X equipped with the weak topology. On the other hand, a super-reflexive Banach space can be given an equivalent norm that is uniformly rotund (cf. [9, Corollary 3]). Such a renorming principle enlarges the range of applicability of Theorem 2.9. However, one must be aware that a renorming of the space will modify the very definition of the metric center. Indeed, $\pi_{\mathrm{inc}}(K)$ depends not just on K but also on the choice of norm.

3 The circumcenter of a convex cone

We now consider the issue of defining a smallest ball associated with K or, more precisely, a smallest ball-generated cone containing K. By a *ball-generated cone* in a normed space $(X, \|\cdot\|)$ one understands a set of the form

$$M(w, s) = \mathrm{cl}\,[\mathbb{R}_+ (w + sB_X)] \tag{18}$$

with $w \in S_X$ and $s \in [0, 1]$. Note that the closure operation can be dropped when $s \in [0, 1[$. Ball-generated cones have a relatively simple structure and are used in the literature for various purposes (cf. [6, 13, 27]). We warn the reader that the concept of ball-generated cone is norm dependent: a set in X may be a ball-generated cone with respect $\|\cdot\|$, but not with respect to an equivalent norm.

Following a similar line of thought as in (4), we formulate the variational problem

$$\begin{aligned} &\text{minimize } s \\ &\|w\| = 1 \\ &s \in [0, 1] \\ &K \subset M(w, s) \end{aligned} \tag{19}$$

and denote by $\mu(K)$ its optimal value. Geometrically speaking, one must find a ball of smallest radius among all balls whose generated cone contains K. The formulation of the minimization problem (19) takes place in a space where the norm has been fixed once and for all. If one wishes to focus on the minimization variable w, then it is preferable to write (19) in the concatenated form

$$\mu(K) = \inf_{\|w\|=1} \inf_{\substack{s\in[0,1]\\ K\subset M(w,s)}} s\,.$$

Definition 3.1. *Let $(X,\|\cdot\|)$ be a reflexive Banach space and let $K \in \Xi(X)$ be contained in a ball-generated cone. The coefficient $\mu(K)$ is called the* circumradius *of K. A* circumcenter *of K is a minimizer of the extended-real-valued function*

$$w \in S_X \mapsto g_K(w) = \inf\{s : s \in [0,1]\,,\ K \subset M(w,s)\}.$$

The set of all circumcenters of K is denoted by $\Pi_{\rm circ}(K)$.

If $\Pi_{\rm circ}(K)$ happens to be a singleton, then its unique element is denoted by $\pi_{\rm circ}(K)$ and

$$\mathbb{B}_{circ}(K) = \pi_{\rm circ}(K) + \mu(K)\,B_X$$

is called the *circumball* of K. This name is inspired by a similar concept from the theory of convex bodies. In Definition 3.1 one asks $K \in \Xi(X)$ to be contained in a ball-generated cone, because otherwise the variational problem (19) is not feasible. As shown in the example below, a closed convex cone in a reflexive Banach space may not be contained in a ball-generated cone.

Example 3.2. Let $\mathbb{R}^2$ be equipped with the Manhattan norm $\|x\| = |x_1| + |x_2|$. If w is a unit vector and $s \in [0,1]$, then $M(w,s) \subset \{x \in \mathbb{R}^2 : \sigma_1 x_1 + \sigma_2 x_2 \geq 0\}$ with $\sigma_1, \sigma_2 \in \{-1,1\}$. One can check that $K = \{x \in \mathbb{R}^2 : c\,|x_1| \leq x_2\}$ is not contained in a ball-generated cone if the parameter c belongs to the interval $[0,1/2[$.

A normed space $(X,\|\cdot\|)$ is *smooth* if each boundary point of B_X admits a unique supporting hyperplane. By extension, the term smooth applies also to the norm. The theory of circumcenters simplifies considerably if the underlying space is smooth.

Lemma 3.3. *Let $(X,\|\cdot\|)$ be a reflexive Banach space. Then the following statements are equivalent:*

(a) Each $K \in \Xi(X)$ is contained in a ball-generated cone.

(b) Each homogeneous half-space of X is contained in a ball-generated cone.

(c) Each homogeneous half-space of X is a ball-generated cone.

(d) $(X,\|\cdot\|)$ is smooth.

Proof. For convenience, we divide the proof in several parts:
(a) $\Leftrightarrow$ (b) and (c) $\Rightarrow$ (b). This is immediate.
(d) $\Rightarrow$ (c). Pick $f \in S_{X^*}$ and $w \in I^{-1}(f)$. Since f is a unit vector, so is w. We claim that

$$H_f = M(w,1). \tag{20}$$

This equality can be obtained by relying on Lemma 3.1 by Zhuang [27], but we prefer to give here a short and self-contained proof. Note that

$$M(w,1) = \mathrm{cl}\,[\mathbb{R}_+\,(B_X - (-w))] = T_{B_X}(-w)\,,$$

i.e., $M(w,1)$ is equal to the tangent cone to B_X at $-w$. By passing to polars (or negative duals), one gets $[M(w,1)]^- = N_{B_X}(-w)$ with $N_{B_X}(-w)$ standing for the normal cone to B_X at $-w$. Since $-w$ is a smooth boundary point of B_X, the set $N_{B_X}(-w)$ is a ray. But,

$$\langle -f, x-(-w)\rangle = \langle -f, x\rangle - 1 \le 0 \quad \text{for all } x \in B_X,$$

i.e., $-f \in N_{B_X}(-w)$. We have shown in this way that $N_{B_X}(-w) = \mathbb{R}_+(-f)$. By passing to polars again, one arrives at

$$M(w,1) = [N_{B_X}(-w)]^- = [\mathbb{R}_+(-f)]^- = H_f.$$

The relation (20) confirms that (c) holds.
(b) $\Rightarrow$ (d). The proof of this implication has been suggested to us by Prof. J.P. Moreno (Madrid) to whom we express our appreciation. Suppose, on the contrary, that $(X, \|\cdot\|)$ is not smooth. Hence, $(X^*, \|\cdot\|_*)$ is not rotund. In such a case, there are distinct vectors $f_1, f_2 \in S_{X^*}$ such that

$$f := \frac{f_1+f_2}{2} \in S_{X^*}.$$

Consequently, every element of $I^{-1}(f)$ is a nonsmooth boundary point of B_X. This is a contradiction with the hypothesis (b). Indeed, (b) implies that $H_f \subset M(w,s)$ for some $(w,s) \in S_X \times [0,1]$. Clearly, the radius s must be equal to one. By taking polars in $H_f \subset M(w,1)$, one obtains

$$N_{B_X}(-w) \subset \mathbb{R}_+(-f). \tag{21}$$

Since the left-hand side of (21) contains a nonzero vector, the inclusion (21) is in fact an equality. Hence, w is smooth and $I(w) = f$. $\square$

The concept of circumcenter must be handled with care because there are plenty of situations leading to rather unexpected conclusions. For instance, even in a Hilbert space setting, there is no reason to believe that $\Pi_{\mathrm{circ}}(K)$ is a subset of K.

Example 3.4. Let y be a unit vector in a Hilbert space X. Then the homogeneous hyperplane

$$K = \{x \in X : \langle y, x\rangle = 0\}$$

admits exactly two circumcenters, namely, y and $-y$. Neither one of the circumcenters lies in K. By the way, this example also shows that $\Pi_{\mathrm{circ}}(K)$ may be topologically disconnected.

Example 3.5. In the same vein, consider a nontrivial closed linear subspace K in a Hilbert space X. A matter of computation yields $\Pi_{\mathrm{circ}}(K) = K^\perp \cap S_X$. If the orthogonal subspace $K^\perp$ is not a line, then $\Pi_{\mathrm{circ}}(K)$ is arc-connected. However, we still have the problem that $\Pi_{\mathrm{circ}}(K)$ does not intersect K.

That $\Pi_{\mathrm{circ}}(K)$ may be contained in the exterior of K is undoubtedly bad news. The reader may rightly argue that linear subspaces are uninteresting examples of convex cones. What happens if one considers a set that is "truly" conic?

Example 3.6. In the Euclidean space $\mathbb{R}^3$, consider the half-icecream cone

$$K = \left\{ x \in \mathbb{R}^3 : x_1 \geq 0,\ x_3 \geq \left[x_1^2 + x_2^2 \right]^{1/2} \right\}. \tag{22}$$

This cone is solid and sharp. It has a unique circumcenter, namely, $\pi_{\rm circ}(K) = (0, 0, 1)^T$. The smallest ball-generated cone containing (22) is the whole icecream cone

$$\Lambda_3 = \left\{ x \in \mathbb{R}^3 : x_3 \geq \left[x_1^2 + x_2^2 \right]^{1/2} \right\}.$$

Although $\pi_{\rm circ}(K)$ belongs to K, it does not belong to the interior of K.

Example 3.6 shows that incenters and circumcenters are different mathematical objects. The incenter of the half-icecream cone (22) lies in the interior of such cone. In fact, it is the vector

$$\bar{x} = \left(\alpha, 0, \sqrt{1 - \alpha^2} \right) \qquad \text{with} \qquad \alpha = \frac{\sqrt{2 - \sqrt{2}}}{2} \approx 0.3827.$$

Such $\bar{x}$ is found by solving explicitly the variational problem (1). The computations are facilitated by the fact that the boundary of (22) is the union of two very simple pieces. It is not worthwhile to enter into details.

3.1 Comparing inradii and circumradii

As shown in the next proposition, the inradius of a convex cone is always smaller than or equal to the circumradius. This fact is clear geometrically, but its formal proof is not immediate. We state first a preliminary lemma.

Lemma 3.7. *Let $(X, \|\cdot\|)$ be a reflexive Banach space of dimension greater than one. The implication*

$$M(x, r) \subset M(w, s) \quad \Longrightarrow \quad r \leq s \tag{23}$$

holds whenever $x, w \in S_X$ and $r, s \in [0, 1]$.

Proof. The case $s = 1$ is ruled out because it is trivial. The inclusion $M(x, r) \subset M(w, s)$ yields

$$x + rB_X \subset \mathbb{R}_+ (w + sB_X). \tag{24}$$

By taking the support function on each side of (24), one gets

$$\langle y, x \rangle + r\|y\|_* \leq \sup_{\alpha \in \mathbb{R}_+} \alpha \left(\langle y, w \rangle + s\|y\|_* \right)$$

for all $y \in X^*$. The above inequality is equivalent to saying that

$$\langle y, w \rangle + s\|y\|_* \leq 0 \quad \Longrightarrow \quad \langle y, x \rangle + r\|y\|_* \leq 0. \tag{25}$$

Next, we construct a unit vector $\tilde{y} \in X^*$ such that

$$\langle \tilde{y}, w \rangle + s = 0 \tag{26}$$

$$\langle \tilde{y}, x - w \rangle \geq 0. \tag{27}$$

To see that such $\tilde{y}$ exists, pick any $\ell \in S_{X^*}$ satisfying $\langle \ell, w\rangle = 0$. Changing ℓ by $-\ell$ if necessary, one may assume that $\langle \ell, x\rangle \geq 0$. We now take $f \in I(w)$ and define

$$\tilde{y} = -sf + \beta\ell.$$

Since $\|-sf\|_* = s < 1$, one may choose the scalar $\beta \geq 0$ so that $\|\tilde{y}\|_* = 1$. Note that

$$\begin{aligned} \langle \tilde{y}, w\rangle &= -s\langle f, w\rangle + \beta\langle \ell, w\rangle = -s \\ \langle \tilde{y}, x\rangle &= -s\langle f, x\rangle + \beta\langle \ell, x\rangle \ \geq \ -s \end{aligned}$$

takes care of (26)-(27). One gets in this way

$$-s + r = \langle \tilde{y}, w\rangle + r \leq \langle \tilde{y}, x\rangle + r \leq 0,$$

where the last inequality is due to (25). This proves that $r \leq s$. □

Proposition 3.8. *Let $(X, \|\cdot\|)$ be a reflexive Banach space of dimension greater than one, and let $K \in \Xi(X)$ be contained in a ball-generated cone. Then $\rho(K) \leq \mu(K)$.*

Proof. Let $\{(x_n, r_n)\}_{n\in\mathbb{N}}$ be a maximizing sequence for (4) and $\{(w_n, s_n)\}_{n\in\mathbb{N}}$ be a minimizing sequence for (19). By combining the double inclusion

$$M(x_n, r_n) \subset K \subset M(w_n, s_n)$$

and (23), one gets $r_n \leq s_n$. A passage to limit leads to $\rho(K) \leq \mu(K)$. □

Lemma 3.7 has several other consequences. The first corollary is consistent with intuition: for a ball-generated cone, the inradius and the circumradius coincide.

Corollary 3.9. *Let $(X, \|\cdot\|)$ be a reflexive Banach space of dimension greater than one. Then $\rho(M(w, s)) = \mu(M(w, s)) = s$ for all $w \in S_X$ and $s \in [0, 1]$.*

Proof. From the definition of a circumradius, it is clear that $\mu(M(w, s)) \leq s$. On the other hand, since the ball $w + sB_X$ is contained in $M(w, s)$, one has $s \leq \rho(M(w, s))$. For completing the proof we invoke Proposition 3.8. □

Remark 3.10. Corollary 3.9 admits a converse: if $(X, \|\cdot\|)$ is a reflexive Banach space and $K \in \Xi(X)$ is such that $\mathbb{B}_{\text{inc}}(K) = \mathbb{B}_{\text{circ}}(K)$, then K is a ball-generated cone.

The second corollary is not as intuitive as one might think. It works only if one assumes rotundity.

Corollary 3.11. *Let $(X, \|\cdot\|)$ be a rotund reflexive Banach space of dimension greater than one. Then $M : S_X \times [0, 1] \to \Xi(X)$ is injective.*

Proof. Let $x, w \in S_X$ and $r, s \in [0, 1]$ be such that $M(x, r) = M(w, s)$. By applying twice the implication (23), one gets $r = s$. On the other hand, one knows already that $\rho(M(w, s)) = s$. Since $w + sB_X \subset M(w, s)$ and the incenter of $M(w, s)$ is unique due to the rotundity assumption, it follows that $\pi_{\text{inc}}(M(w, s)) = w$. A similar formula is obtained for the cone $M(x, r)$. One gets in this way

$$x = \pi_{\text{inc}}(M(x, r)) = \pi_{\text{inc}}(M(w, s)) = w.$$

Hence, $(x, r) = (w, s)$, as needed for proving injectivity. □

3.2 An alternative characterization of the set of circumcenters

The next theorem characterizes the set $\Pi_{\mathrm{circ}}(K)$ when K is *ball-sharp*, i.e., when K is contained in a cone generated by a ball whose center is a unit vector and whose radius is smaller than 1. We comment in passing that ball-sharpness is stronger than pointedness, but in an Euclidean space setting both concepts coincide. Recall that a closed convex cone is *pointed* if it contains no line.

The basic idea behind Theorem 3.12 is that any $K \in \Xi(X)$ can be represented as closed conic hull

$$K = \mathrm{cl}\,[\mathrm{pos}(\Omega)] \tag{28}$$

of a subset Ω of X such that $0 \notin \Omega$. The definition of the conic hull is as usual, i.e.,

$$\mathrm{pos}(\Omega) = \left\{ \sum_{j=1}^{p} t_j g_j : \ p \in \mathbb{N},\ g_1, \ldots, g_p \in \Omega,\ t_1, \ldots, t_p \in \mathbb{R}_+ \right\}.$$

Theorem 3.12. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Let $K \in \Xi(X)$ be ball-sharp and represented as in (28) with $0 \notin \Omega$. Then*

$$\mu(K) = \inf_{\|w\|=1} \varphi_\Omega(w) \tag{29}$$

$$\Pi_{\mathrm{circ}}(K) = \{w \in S_X : \varphi_\Omega(w) = \mu(K)\}, \tag{30}$$

where $\varphi_\Omega : X \to \mathbb{R}$ stands for the sublinear function given by

$$\varphi_\Omega(w) = \sup_{a \in \Omega} \mathrm{dist}\,[w, \mathbb{R}_+ a]\,.$$

Proof. Sublinearity corresponds to the combination of convexity and positive homogeneity. It is clear that $\varphi_\Omega : X \to \mathbb{R}$ enjoys both properties. Given that K is ball-sharp, the variable s in (19) can be restricted to a closed interval $[0, \bar{s}]$ with $\bar{s} < 1$. The closure operation in the definition of $M\,(w, s)$ is superfluous if one takes (w, s) in $S_X \times [0, \bar{s}]$. Given the representation formula (28), an inclusion like $K \subset M\,(w, s)$ amounts to saying that

$$a \in M\,(w, s) \quad \text{for all } a \in \Omega. \tag{31}$$

Note that

$$\begin{aligned} a \in M\,(w, s) &\iff \left\| t^{-1} a - w \right\| \le s \quad \text{for some } t > 0 \\ &\iff \mathrm{dist}\,[w, \mathbb{R}_+ a] \le s. \end{aligned}$$

Hence, the condition (31) is equivalent to $\varphi_\Omega(w) \le s$. We have shown in this way that

$$\mu(K) = \inf_{\substack{(w,s) \in S_X \times [0,\bar{s}] \\ K \subset M(w,s)}} s \; = \inf_{\substack{(w,s) \in S_X \times [0,\bar{s}] \\ \varphi_\Omega(w) \le s}} s \; = \inf_{\|w\|=1} \inf_{\substack{s \in [0,\bar{s}] \\ \varphi_\Omega(w) \le s}} s\,.$$

By getting rid of the variable s, one ends up with (29)-(30). $\square$

Example 3.13. A polyhedral cone K is often represented as intersection of finitely many half-spaces, but sometimes it is given in terms of a finite collection of generators:

$$K = \left\{ \sum_{j=1}^{p} t_j g_j : t_1, \ldots, t_p \in \mathbb{R}_+ \right\}.$$

Here, $g_1, \ldots, g_p$ are unit vectors in the Euclidean space $\mathbb{R}^n$. There is no loss of generality in assuming that none of the g_i is a positive linear combination of the others. Suppose that K is pointed. A natural and convenient choice of Ω is the set of generators of K, i.e., $\Omega = \{g_1, \ldots, g_p\}$. According with Theorem 3.12, a circumcenter of K can be found by solving

$$\mu(K) = \inf_{\|w\|=1} \max_{1\leq j\leq p} \operatorname{dist}[w, \mathbb{R}_+ g_j]. \tag{32}$$

If w is a unit vector such that $g_j^T w \leq 0$ for some $j \in \{1, \ldots, p\}$, then the cost function of (32) takes the value 1. Hence, such a vector w cannot be a solution to (32). So, we are led to solve

$$\begin{aligned} &\text{minimize} \quad \max_{1\leq j\leq p} \|w - (g_j^T w) g_j\| \\ &\|w\| = 1 \\ &g_j^T w \geq 0 \;\; \forall j \in \{1, \ldots, p\}. \end{aligned} \tag{33}$$

One should be aware, however, that (33) is a nonconvex optimization problem.

Theorem 3.12 yields as a by-product the next existence result.

Corollary 3.14. *Let $(X, \|\cdot\|)$ be finite dimensional and let $K \in \Xi(X)$ be contained in a ball-generated cone. Then $\Pi_{\rm circ}(K)$ is nonempty.*

Proof. Suppose first that K is not ball-sharp, i.e., the only ball-generated cones that contain K are those of the form $M(w, 1)$ with $w \in S_X$. From the proof of Lemma 3.3 one sees that

$$K \subset M(w, 1) \quad \Longleftrightarrow \quad w \in I^{-1}(K^+ \cap S_{X^*}).$$

Hence, $\mu(K) = 1$ and $\Pi_{\rm circ}(K) = I^{-1}(K^+ \cap S_{X^*})$ is nonempty. Suppose now that K is ball-sharp. In such a case we are in the context of Theorem 3.12. Given that X is finite dimensional, the variational problem (29) is about minimizing a continuous function on a compact set. Again, $\Pi_{\rm circ}(K)$ is nonempty. □

4 The outer center and the inner center of a convex cone

4.1 Outer approximation by a revolution cone

Finite dimensional revolution cones are used in a conspicuous way in various fields of mathematics, including mathematical programming [7] and coding theory [25]. The usual definition of a revolution cone in the Euclidean space $\mathbb{R}^n$ is

$$\Gamma(y, \theta) = \{x \in \mathbb{R}^n : \; y^T x \geq \|x\| \cos\theta\}, \tag{34}$$

where $y \in \mathbb{R}^n$ is a unit vector that determines the revolution axis, and $\theta \in [0, \pi/2]$ is a parameter called the *half-aperture angle* (cf.[13]). Note that (34) is the set of vectors forming an angle not greater than θ with respect to y. The definition of a revolution cone extends to a Hilbert space setting without any substantial change. Beyond a Hilbert space setting, one adopts the definition

$$\Gamma(y, \theta) = \{x \in X : \langle y, x\rangle \geq \|x\| \cos\theta\} \tag{35}$$

with y standing for a unit vector in $(X^*, \|\cdot\|_*)$. We still call (35) a revolution cone, but this is obviously an abuse of language because the angular interpretation of the parameter θ is lost. Authors working in functional analysis and in vector optimization refer sometimes to (35) as a Bishop-Phelps cone (cf. [1, 17, 22]).

What about approximating a given $K \in \Xi(X)$ by a revolution cone? The first idea that comes to mind is searching for a revolution cone of smallest half-aperture angle that contains K. This leads to the minimization problem

$$\theta_{\rm out}(K) \;=\; \inf\{\theta : \|y\|_* = 1,\, \theta \in [0, \pi/2],\, K \subset \Gamma(y, \theta)\}. \tag{36}$$

The next three equivalent characterizations of $\theta_{\rm out}(K)$ are borrowed from [20, Section 5]. The notation "co" refers to the convex hull operation.

Proposition 4.1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $K \in \Xi(X)$. Then*

$$\begin{aligned} \cos\left[\theta_{\rm out}(K)\right] &= \sup_{\|y\|_*=1} \inf_{x\in K\cap S_X} \langle y, x\rangle && (37)\\ &= \operatorname{dist}\left[0, \operatorname{co}(K \cap S_X)\right] && (38)\\ &= \rho(K^+). && (39) \end{aligned}$$

Note that, in a reflexive Banach space setting, the function $\theta_{\rm out} : (\Xi(X), \varrho) \to \mathbb{R}$ is continuous. This is a direct consequence of (39). By the way, the duality relation

$$\rho(K^+) = \sup_{\|y\|_*=1} \inf_{x\in K\cap S_X} \langle y, x\rangle$$

can be found not just in [20], but also in an earlier paper by Freund and Vera [12, Proposition 2.1]. In the sequel, the set of maximizers of the function

$$y \in S_{X^*} \mapsto h_K(y) = \inf_{x\in K\cap S_X} \langle y, x\rangle \tag{40}$$

is denoted by $D_{\rm out}(K)$. The definition given below is inspired by the characterization (37). The presence of the inverse duality map $I^{-1} : X^* \rightrightarrows X$ may seem strange at first sight, so we shall comment on this point in a moment.

Definition 4.2. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $K \in \Xi(X)$. An* outer center *of K is an element of the set*

$$\Pi_{\rm out}(K) = I^{-1}\left[D_{\rm out}(K)\right]. \tag{41}$$

Definition 4.2 is proposed by [18] in an Euclidean space setting. In that reference a maximizer of (40) is called a *centroid* of K. Note that $D_{\rm out}(K)$ is a subset of X^* and not of the original space X. This explains why the map I^{-1} shows up in (41). Up to some extent, incenters and outer centers can be viewed as dual objects. In fact, one has the following duality result.

Theorem 4.3. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $K \in \Xi(X)$. Then*

$$D_{\rm out}(K) = \Pi_{\rm inc}(K^+). \tag{42}$$

In particular, K possesses at least one outer center.

Proof. The change of variable $r = \cos\theta$ brings the variational problem (36) to the equivalent form

$$\cos[\theta_{\rm out}(K)] \;=\; \sup_{\|y\|_*=1} \;\sup_{\substack{r\in[0,1] \\ K\subset\Gamma(y,\arccos r)}} r\,. \tag{43}$$

A key observation concerning the inclusion constraint in (43) is that

$$K \subset \Gamma(y, \arccos r) \quad\Longleftrightarrow\quad \langle y, x\rangle \geq r\|x\| \quad \text{for all } x \in K. \tag{44}$$

There are two ways of interpreting the right-hand side of (44). First of all, one can write such a condition as a ball inclusion, namely, $y + rB_{X^*} \subset K^+$. Hence, (43) is nothing else than the old problem of finding a largest ball in K^+. The second way of writing the right-hand side of (44) is

$$r \;\leq\; \inf_{x\in K\cap S_X} \langle y, x\rangle\,.$$

Hence, the second supremum in (43) is just $h_K(y)$. □

Theorem 4.4. *If $(X, \|\cdot\|)$ is a reflexive Banach space, then $K \in \Xi_{\rm sh}(X)$ possesses an outer center that lies in K itself.*

Proof. The proof of Proposition 4.7 in [18] provides us with the initial inspiration. Let $\bar y$ be a solution to the convexified problem

$$\text{maximize } \{h_K(y) : \|y\|_* \leq 1\}. \tag{45}$$

Since K is sharp, there exists a vector $y_0 \in S_{X^*}$ such that $h_K(y_0) > 0$. This fact and the positive homogeneity of h_K yield

$$\bar y \in D_{\rm out}(K). \tag{46}$$

Next we write down the standard optimality condition

$$0 \in \partial^{\rm fenchel}(-h_K)(\bar y) + N_{B_{X^*}}(\bar y)$$

for the convex problem (45). Here, the symbol $\partial^{\rm fenchel}$ indicates the usual subdifferential operator of convex analysis. Hence, there exists a vector $\bar x \in X$ such that

$$\bar x \;\in\; N_{B_{X^*}}(\bar y) \tag{47}$$
$$-\bar x \;\in\; \partial^{\rm fenchel}(-h_K)(\bar y). \tag{48}$$

The condition (48) decomposes into

$$\bar x \;\in\; {\rm cl}\,[{\rm co}(K\cap S_X)] \tag{49}$$
$$\langle \bar y, \bar x\rangle \;=\; h_K(\bar y). \tag{50}$$

Note that (50) forces $\bar{x}$ to be nonzero, whereas (49) forces $\bar{x}$ to be in K. On the other hand, the condition (47) yields

$$\hat{x} := \|\bar{x}\|^{-1}\bar{x} \in I^{-1}(\bar{y}). \tag{51}$$

In view of the relations (46) and (51), the vector $\hat{x} \in K$ belongs to $\Pi_{\rm out}(K)$. This completes the proof. $\square$

We now address the issue of uniqueness of outer centers. Given the duality formula (42), the following result is not surprising althogether.

Theorem 4.5. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then the following statements are equivalent:*

(a) $D_{\rm out}(K)$ is a singleton for each $K \in \Xi_{\rm sh}(X)$.

(b) $(X, \|\cdot\|)$ is smooth.

Proof. The proof is a matter of combining Theorems 2.4 and 4.3. One must bear in mind two facts: firstly, K is sharp if and only if K^+ is solid (cf. [15, 20]). And, secondly, the reflexive Banach space $(X, \|\cdot\|)$ is smooth if and only if $(X^*, \|\cdot\|_*)$ is rotund. $\square$

Consistent with our notational conventions, the symbol $d_{\rm out}(K)$ indicates the single element of $D_{\rm out}(K)$ in case the latter set is a singleton. The next two corollaries are immediate and so their proofs are omitted.

Corollary 4.6. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space and let $K \in \Xi_{\rm sh}(X)$. Then the variational problem (36) admits exactly one solution. Furthermore, the y-component of the solution is $d_{\rm out}(K) = \pi_{\rm inc}(K^+)$.*

Smoothness of the space and sharpness of the cone are essential assumptions in Corollary 4.6. By an obvious reason, one refers to the cone

$$K^{\rm out} = \Gamma\left(d_{\rm out}(K), \theta_{\rm out}(K)\right)$$

as the *outer revolution envelope* of K. A word of warning is here appropriate: the fact that $D_{\rm out}(K)$ is a singleton does not imply uniqueness of the outer center. This is simply because I^{-1} could be multivalued. In other words, the equation $I(x) = d_{\rm out}(K)$ could have more than one solution $x \in S_X$. However, this problem can be settled by asking not just smoothness but also rotundity.

Corollary 4.7. *Suppose that the reflexive Banach space $(X, \|\cdot\|)$ is smooth and rotund. Then each $K \in \Xi_{\rm sh}(X)$ possesses exactly one outer center, namely, $\pi_{\rm out}(K) = I^{-1}(d_{\rm out}(K))$. Furthermore, $\pi_{\rm out}(K)$ belongs to K.*

A normed space $(X, \|\cdot\|)$ is *uniformly smooth* if $\lim_{t\to 0^+} \tau(t)/t = 0$ with

$$\tau(t) = \sup_{x,h\in S_X} \left\{ \frac{\|x + th\| + \|x - th\|}{2} - 1 \right\}.$$

Uniform smoothness corresponds to the dual concept of uniform rotundity (cf.[23]). Theorem 2.9 can be combined with Theorem 4.3 in order to obtain a continuity result for the outer center.

Proposition 4.8. *Suppose that the reflexive Banach space $(X, \|\cdot\|)$ is uniformly smooth and uniformly rotund. Then the function $\pi_{\rm out} : (\Xi_{\rm sh}(X), \varrho) \to (X, \|\cdot\|)$ is continuous.*

Proof. Thanks to Corollaries 4.6 and Corollary 4.7, one has

$$\pi_{\rm out}(K) = I^{-1}(\pi_{\rm inc}(K^+))$$

for all $K \in \Xi_{\rm sh}(X)$. So, $\pi_{\rm out}$ is the composition of three continuous functions. The diagram

$$\begin{array}{ccc} (\Xi_{\rm sh}(X), \varrho) & \overset{(\cdot)^+}{\to} & (\Xi_{\rm sol}(X^*), \varrho_*) \\ \pi_{\rm out} \downarrow & & \downarrow \pi_{\rm inc} \\ (X, \|\cdot\|) & \overset{I^{-1}}{\leftarrow} & (X^*, \|\cdot\|_*) \end{array}$$

helps for visualizing the situation. Here, ϱ_* stands for the truncated Pompeiu-Hausdorff metric on $\Xi(X^*)$. Thanks to the Walkup-Wets isometry theorem, $K \mapsto K^+$ is continuous as function from $(\Xi_{\rm sh}(X), \varrho)$ to $(\Xi_{\rm sol}(X^*), \varrho_*)$. That $(X, \|\cdot\|)$ is uniformly smooth implies that $(X^*, \|\cdot\|_*)$ is uniformly rotund. Hence, in view of Theorem 2.9, the function $\pi_{\rm inc} : (\Xi_{\rm sol}(X^*), \varrho_*) \to (X^*, \|\cdot\|_*)$ is continuous. Finally, the uniform rotundity of $(X, \|\cdot\|)$ guarantees not just the single-valuedness, but also the continuity of the map $I^{-1} : (X^*, \|\cdot\|_*) \to (X, \|\cdot\|)$. □

4.2 Inner approximation by a revolution cone

As an alternative to the outer approximation technique, one may consider the problem of finding a revolution cone of largest half-aperture angle contained in K. This time one must solve a maximization problem of the form

$$\theta_{\rm inn}(K) \;=\; \sup\{\theta : \|y\|_* = 1,\, \theta \in [0, \pi/2],\, \Gamma(y, \theta) \subset K\}. \tag{52}$$

This resembles (36), but it is not quite the same problem. The "inner" counterpart of Definition 4.2 is formulated in Definition 4.9. Note that

$$\theta_{\rm inn}(K) \;=\; \sup_{\|y\|_*=1} \ell_K(y), \tag{53}$$

where ℓ_K is the extended-real-valued function defined on S_{X^*} by

$$\ell_K(y) = \sup\left\{\theta : \theta \in [0, \pi/2],\, \Gamma(y, \theta) \subset K\right\}.$$

Let $D_{\rm inn}(K)$ denote the solution set to (53).

Definition 4.9. *Suppose that $(X, \|\cdot\|)$ is a reflexive Banach space and that $K \in \Xi(X)$ contains a revolution cone. An* inner center *of K is an element of the set*

$$\Pi_{\rm inn}(K) = I^{-1}\left[D_{\rm inn}(K)\right].$$

If $D_{\text{inn}}(K)$ happens to be a singleton, then its single element is denoted by $d_{\text{inn}}(K)$. Needless to say, the set

$$K^{\text{inn}} = \Gamma\left(d_{\text{inn}}(K), \theta_{\text{inn}}(K)\right)$$

is referred to as the *inner revolution envelope* of K.

In Definition 4.9, one asks $K \in \Xi(X)$ to contain a revolution cone for making sure that the maximization problem (52) is feasible. Such a feasibility assumption is automatically satisfied if the reflexive Banach space $(X, \|\cdot\|)$ is rotund. This fact can be better understood with the help of the example below.

Example 4.10. Imagine that the cone K is very "small". Take, for instance, $K = \mathbb{R}_+ a$ with $a \in S_X$. Are we sure that $\Gamma(y,\theta) \subset \mathbb{R}_+ a$ for some $(y,\theta) \in S_{X^*} \times [0, \pi/2]$? The best chance of getting such an inclusion is to take $\theta = 0$. However, a set of the form

$$\Gamma(y, 0) = \mathbb{R}_+[I^{-1}(y)]$$

is not necessarily contained in a ray because I^{-1} is multivalued in general. If one choose $a \in S_X$ so that $I^{-1}(y)$ is multivalued for each $y \in I(a)$, then $\mathbb{R}_+ a$ will not contain a revolution cone.

Proposition 4.11. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. If $K \in \Xi(X)$ contains a revolution cone with positive half-aperture angle, then $\Pi_{\text{inn}}(K) \subset S_X \cap \text{int}(K)$.*

Proof. As a start, it is helpful to mention an abstract result on revolution cones according to which

$$\Gamma(y, \theta_1) \backslash \{0\} \subset \text{int}\left[\Gamma(y, \theta_2)\right]$$

whenever $y \in S_{X^*}$ and $0 \leq \theta_1 < \theta_2 \leq \pi/2$. Checking this inclusion offers no difficulty, so we omit the details. In particular, one can write

$$I^{-1}(y) \subset \text{int}\left[\Gamma(y, \theta)\right] \tag{54}$$

whenever $y \in S_{X^*}$ and $\theta \in]0, \pi/2]$. The proof of the proposition itself runs as follows. The assumption made on K ensures that $\theta_{\text{inn}}(K)$ is positive. If $\Pi_{\text{inn}}(K)$ is void, then there is nothing to prove. Let $x \in \Pi_{\text{inn}}(K)$, i.e., $x \in I^{-1}(y)$ for some $y \in D_{\text{inn}}(K)$. For any θ in the open interval $]0, \Pi_{\text{inn}}(K)[$, one clearly has $\Gamma(y,\theta) \subset K$. By applying (54), one gets

$$x \in \text{int}\left[\Gamma(y, \theta)\right] \subset \text{int}(K),$$

which completes the proof. □

The variational problem (52) can be dualized by exploiting the relationship existing between revolution cones and ball-generated cones. The class of ball-generated cones in X was introduced already in (18). Ball-generated cones in X^* are defined in a similar way.

Lemma 4.12. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then*

$$\begin{aligned} [\Gamma(y,\theta)]^+ &= M(y, \cos\theta) \\ [M(w,s)]^+ &= \Gamma(w, \arccos s) \end{aligned}$$

for all $w \in S_X$, $y \in S_{X^*}$, $s \in [0,1]$, and $\theta \in [0, \pi/2]$.

Proof. It is enough to prove the second formula. That $f \in [M(w,s)]^+$ is equivalent to

$$\langle f, w + sh \rangle \geq 0 \quad \text{for all } h \in B_X. \tag{55}$$

Since B_X is symmetric with respect to the origin, (55) is yet equivalent to $\langle f, w\rangle \geq s \|f\|_*$, that is to say, $f \in \Gamma(w, \arccos s)$. □

According to Lemma 4.12, the dual of a revolution cone is a ball-generated cone, and viceversa. This elementary duality result allows us to establish a link between inner centers and circumcenters.

Theorem 4.13. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. If $K \in \Xi(X)$ contains a revolution cone, then*

$$D_{\text{inn}}(K) = \Pi_{\text{circ}}(K^+) \tag{56}$$
$$\cos[\theta_{\text{inn}}(K)] = \mu(K^+). \tag{57}$$

Proof. That $K \in \Xi(X)$ contains a revolution cone amounts to saying that $K^+ \in \Xi(X^*)$ is contained in a ball-generated cone. In view of Lemma 4.12, the inclusion constraint $\Gamma(y,\theta) \subset K$ can be written in the equivalent form $K^+ \subset M(y, \cos\theta)$. Hence, the variational problem (52) becomes

$$\cos[\theta_{\text{inn}}(K)] = \inf\{\cos\theta : \|y\|_* = 1,\ \theta \in [0, \pi/2],\ K^+ \subset M(y, \cos\theta)\}.$$

The change of variables $s = \cos\theta$ leads finally to (56)-(57). □

By combining Theorem 4.13 and Corollary 3.14, one readily gets:

Corollary 4.14. *Let $(X, \|\cdot\|)$ be finite dimensional and let $K \in \Xi(X)$ contain a revolution cone, then $\Pi_{\text{inn}}(K)$ is nonempty.*

In the same vein, by combining Theorem 4.13 and Propositions 3.8 and 4.1, one obtains:

Corollary 4.15. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $K \in \Xi(X)$ contain a revolution cone. Then $\theta_{\text{inn}}(K) \leq \theta_{\text{out}}(K)$.*

5 Special results in Hilbert spaces

In a Hilbert space setting, the class of revolution cones coincides with the class of ball-generated cones. In particular, the dual of a revolution cone is a revolution cone and the dual of a ball-generated cone is a ball-generated cone:

Lemma 5.1. *Let $(X, \|\cdot\|)$ be a Hilbert space. Then*

$$[\Gamma(y,\theta)]^+ = \Gamma\left(y, \frac{\pi}{2} - \theta\right) \tag{58}$$
$$[M(w,s)]^+ = M\left(w, \sqrt{1-s^2}\right) \tag{59}$$

for all $w, y \in S_X$, $s \in [0,1]$, and $\theta \in [0, \pi/2]$.

Proof. The duality formula (58) is well known in an Euclidean space setting (cf. [13, 16]). Its proof in a Hilbert space runs as follows. Take $v \in X$ such that

$$v \in [\Gamma(y,\theta)]^+ . \tag{60}$$

We shall prove that

$$v \in \Gamma\left(y, \frac{\pi}{2} - \theta\right). \tag{61}$$

One may suppose that v is not a multiple of y, otherwise (61) holds trivially. Let $L = \mathrm{span}\{y, v\}$ be the two dimensional linear space spanned by the vectors y and v. Since $X = L \oplus L^\perp$, every $x \in X$ admits a unique decomposition as sum of two orthogonal vectors:

$$x = x_1 + x_2 \quad \text{with} \;\; x_1 \in L, x_2 \in L^\perp.$$

Denote by $\langle \cdot, \cdot \rangle_L$ and $\| \cdot \|_L$ the restriction to the closed linear subspace L of $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The symbol $\| \cdot \|_{L^\perp}$ is defined in a similar way. The hypothesis (60), i.e.,

$$\langle v, x \rangle \geq 0 \quad \text{whenever } \langle y, x \rangle \geq \|x\| \cos\theta,$$

takes then the form

$$\langle v, x_1 \rangle_L \geq 0 \quad \text{for all } (x_1, x_2) \in L \times L^\perp \text{ s.t. } \langle y, x_1 \rangle_L \geq \left[\|x_1\|_L^2 + \|x_2\|_{L^\perp}^2\right]^{1/2} \cos\theta. \tag{62}$$

The particular choice $x_2 = 0$ yields

$$\langle v, x_1 \rangle_L \geq 0 \quad \text{for all } x_1 \in L \text{ s.t. } \langle y, x_1 \rangle_L \geq \|x_1\|_L \cos\theta. \tag{63}$$

Since $(L, \| \cdot \|_L)$ is an Euclidean space, one gets

$$\langle y, v \rangle_L \geq \|v\|_L \cos\left(\frac{\pi}{2} - \theta\right). \tag{64}$$

The latter inequality is equivalent to (61) because one can view v as an element of L and, at the same time, as an element of X. Conversely, let (61) be true. One may assume that v is not a multiple of y, otherwise (60) holds trivially. We define L as before. The hypothesis (61) is equivalent (64), which in turn is equivalent to (63). Since

$$\|x_1\|_L \leq \left[\|x_1\|_L^2 + \|x_2\|_{L^\perp}^2\right]^{1/2} \quad \text{for all } (x_1, x_2) \in L \times L^\perp,$$

the condition (63) implies (62). One arrives in this way to (60). The duality formula (59) is obtained by combining (58) and Lemma 4.12. $\square$

The orthogonal decomposition technique used in Lemma 5.1 forces X to be Hilbert. As far as this work is concerned, the main impact of Lemma 5.1 is the next theorem.

Theorem 5.2. *Let X be a Hilbert space and $K \in \Xi(X)$. Then*

$$[\mu(K)]^2 + [\rho(K^+)]^2 = 1 \tag{65}$$
$$\theta_{\mathrm{inn}}(K) + \theta_{\mathrm{out}}(K^+) = \pi/2. \tag{66}$$

Furthermore,

$$\begin{aligned} \Pi_{\mathrm{circ}}(K) &= \Pi_{\mathrm{out}}(K) = \Pi_{\mathrm{inc}}(K^+) \\ \Pi_{\mathrm{inc}}(K) &= \Pi_{\mathrm{inn}}(K) = \Pi_{\mathrm{circ}}(K^+). \end{aligned}$$

Proof. In view of (58), the equality (52) becomes

$$\theta_{\rm inn}(K) \;=\; \sup\left\{\theta : \|y\|_* = 1,\, \theta \in [0,\pi/2],\, K^+ \subset \Gamma\left(y, \frac{\pi}{2} - \theta\right)\right\}.$$

The change of variable $\vartheta = (\pi/2) - \theta$ leads to

$$\theta_{\rm inn}(K) \quad = \quad (\pi/2) - \inf\left\{\vartheta : \|y\|_* = 1,\, \vartheta \in [0,\pi/2],\, K^+ \subset \Gamma(y,\vartheta)\right\},$$

which explains why we wrote the angular identity (66). By the same token, we got

$$\Pi_{\rm inn}(K) \;=\; \Pi_{\rm out}(K^+) \;=\; \Pi_{\rm inc}(K),$$

where the last equality is a consequence of Theorem 4.3. The remaining relations are obtained by invoking Proposition 4.1 and Theorem 4.13. □

The angular identity (66) is reminiscent of a similar looking formula of Iusem and Seeger [19, Theorem 3] relating the minimal angle of K and the maximal angle of K^+. Theorem 5.2 has a long list of consequences. For instance, in a Hilbert space setting every $K \in \Xi(X)$ admits an inner center and also a circumcenter; this is simply because an inner center is nothing but an incenter and a circumcenter is nothing but an outer center. In addition, one gets

$$\begin{array}{rclclr} \pi_{\rm circ}(K) & = & \pi_{\rm out}(K) & = & \pi_{\rm inc}(K^+) & \text{if } K \text{ is sharp,} \\ \pi_{\rm inc}(K) & = & \pi_{\rm inn}(K) & = & \pi_{\rm circ}(K^+) & \text{if } K \text{ is solid,} \end{array}$$

and many other by-products.

6 By way of conclusion

We have studied four kinds of centers for a closed convex cone K in a reflexive Banach space: the incenter, the circumcenter, the inner center, and the outer center. As mentioned already in the introduction, these concepts are different in general. The main lessons that can be drawn from this work are outlined below; see also Table 1.

type of center	conditions for existence	conditions for uniqueness	notation if uniqueness
incenter	extra hypotheses are not needed	K solid, X rotund	$\pi_{\rm inc}(K)$
circumcenter	X Hilbert, or X smooth and $\dim X < \infty$	K sharp X Hilbert	$\pi_{\rm circ}(K)$
inner center	X Hilbert, or X rotund and $\dim X < \infty$	K solid X Hilbert	$\pi_{\rm inn}(K)$
outer center	extra hypotheses are not needed	K sharp, X smooth and rotund	$\pi_{\rm out}(K)$

Table 1: Different types of center for a cone K in a reflexive Banach space.

- The existence of incenters is automatically guaranteed. For making sure that the incenter is unique, we ask the cone K to be solid and the underlying space $(X, \|\cdot\|)$ to be rotund. In such a case, the incenter $\pi_{\rm inc}(K)$ lies in the interior of K. This is consistent with the intuitive idea of being a center of a cone. The concept of incenter has further merits: the function $\pi_{\rm inc} : (\Xi_{\rm sol}(X), \varrho) \to (X, \|\cdot\|)$ is continuous if the rotundity assumption is uniform (cf. Theorem 2.9). It is reassuring to know that $\pi_{\rm inc}(K)$ behaves in a stable manner with respect to perturbations in the argument K.

- The existence of outer centers is also automatically guaranteed. This follows from a general formula that relates the set of outer centers of K and the set of incenters of the dual cone K^+. To make sure that the outer center is unique, we ask the cone K to be sharp and the underlying space to be smooth and rotund. If the latter structural properties are uniform, then $\pi_{\rm out} : (\Xi_{\rm sh}(X), \varrho) \to (X, \|\cdot\|)$ is continuous (cf. Proposition 4.8). An outer center does not need to be in the interior of the cone, even if the cone is solid and lives in an Euclidean space.

- The concept of circumcenter suffers from many drawbacks. First of all, a circumcenter could fail to be in the interior of the cone. In addition to this, computing a circumcenter is quite complicated in practice. And, finally, we do not see clearly how to guarantee the uniqueness of circumcenters without asking X to be Hilbert. Despite this long list of inconveniences, the concept of circumcenter is geometrically appealing and deserves some attention. We do not have yet a general existence result, but we have proven that K admits at least one circumcenter if K is contained in a ball-generated cone and $(X, \|\cdot\|)$ is finite dimensional. In a Hilbert space setting, every K admits a circumcenter.

- The concept of inner center has some pros and contras. On the positive side: provided K contains a revolution cone with positive half-aperture angle, every inner center of K belongs to $\mathrm{int}(K)$. One the negative side, we do not know how to guarantee the uniqueness of inner centers without asking X to be Hilbert. We do not have yet a general existence result, but we know that K admits at least one inner center if K contains a revolution cone and $(X, \|\cdot\|)$ is finite dimensional. In a Hilbert space setting, every K admits an inner center.

In this paper we have deliberately kept the discussion at an abstract level, so as to better understand the role of each assumption (smoothness, rotundity, etc). The whole theory of centers simplifies dramatically if the underlying space is Euclidean, i.e., Hilbert and finite dimensional. This special setting is treated exhaustively in the companion paper [14]. In addition to a few theoretical results, we compute there the incenter and the circumcenter for several convex cones arising in concrete applications.

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