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# Finite element error analysis for state-constrained optimal control of the Stokes equations

Juan Carlos de los Reyes[1], Christian Meyer[2], Boris Vexler[3]

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[1] Departamento de Matemática
EPN Quito
Ladrón de Guevara E11-253
Quito
Ecuador
E-Mail: jcdelosreyes@andinanet.net

[2] Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: meyer@wias-berlin.de

[3] Johann Radon Institute
for Computational and Applied Mathematics
Austrian Academy of Sciences
Altenberger Straße 69
A-4040 Linz
Austria
E-Mail: boris.vexler@oeaw.ac.at

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

**Abstract.** An optimal control problem for 2d and 3d Stokes equations is investigated with pointwise inequality constraints on the state and the control. The paper is concerened with the full discretization of the control problem allowing for different types of discretization of both the control and the state. For instance, piecewise linear and continuous approximations of the control are included in the present theory. Under certain assumptions on the $L^\infty$-error of the finite element discretization of the state, error estimates for the control are derived which can be seen to be optimal since their order of convergence coincides with the one of the interpolation error. The assumptions of the $L^\infty$-finite-element-error can be verified for different numerical settings. The theoretical results are confirmed by numerical examples.

**1. Introduction.** This paper is concerned with the finite element discretization for the following linear quadratic optimal control problem subject to the Stokes equations and additional constraints on the control and the state:

$$
\text{(P)} \quad \begin{cases} \text{minimize} & J(v,u) := \dfrac{1}{2}\displaystyle\int_\Omega |v-z|^2_{\mathbb{R}^d}\,dx + \dfrac{\alpha}{2}\int_\Omega |u|^2_{\mathbb{R}^d}\,dx \\ \text{subject to} & -\Delta\, v + \nabla p = u \quad \text{in } \Omega \\ & \nabla\cdot v = 0 \quad \text{in } \Omega \\ & v = 0 \quad \text{on } \Gamma := \partial\Omega \\ \text{and} & v \in K \subset L^\infty(\Omega')^d \\ & a \le u(x) \le b \quad \text{a.e. in } \Omega, \end{cases}
$$

where $u$ denotes the control, $v$ and $p$ are velocity and pressure, respectively, and $z$ is the given desired state. Furthermore, $\Omega \subset \mathbb{R}^d$, $d = 2,3$ is a bounded domain with boundary $\Gamma$ and $\alpha > 0$ is a given number. Moreover, $a, b \in \mathbb{R}^d$ are given vectors, whereas $K$ denotes a closed and convex subset of $L^\infty(\Omega')^d$, where $\Omega'$ is a fixed (not necessarily proper) subset of $\Omega$. Possible examples for $K$ are box constraints for $v$ or restrictions on the Euclidian norm of $v$, i.e.,

$$
K^{(1)} := \left\{ v \in L^\infty(\Omega')^d | v_a \le v(x) \le v_b \text{ a.e. in } \Omega' \right\}
$$
$$
K^{(2)} := \left\{ v \in L^\infty(\Omega')^d | |v(x)|^2_{\mathbb{R}^d} \le \varrho \text{ a.e. in } \Omega' \right\}
$$

with given bounds $v_a, v_b \in \mathbb{R}^d$, and $\varrho > 0$. In view of the no-slip conditions on the boundary, it might be reasonable to require the state constraints only in the interior of $\Omega$. The presented theory is applicable for both cases, i.e. $\Omega' \neq \Omega$ and $\Omega' = \Omega$.

It is well known that, if certain constraint qualifications are satisfied, then the generalized Karush-Kuhn-Tucker theory allows to derive first-order necessary conditions that include the existence of Lagrange multipliers associated to the state constraints in $(L^\infty(\Omega')^d)^*$, i.e., the dual of $L^\infty(\Omega')^d$ with respect to the inner product of $L^2(\Omega')^d$ (cf. [35] or [7]). This lack of regularity of the multipliers complicates the numerical analysis of state-constrained optimal control problems. Nevertheless, in the recent past, some progress has been achieved concerning the finite element error analysis of state-constrained elliptic problems. We exemplarily mention Casas [8], where a semilinear elliptic control problem with finitely many state constraints is considered, and Casas and Mateos [9], where convergence of a finite element discretization for state-constrained semilinear elliptic problems is proved in a general setting. Moreover, we refer to Deckelnick and Hinze [14, 15], where a variational discretization of state-constrained elliptic problems is considered, and to [17] for problems with pointwise constraints on the gradient of the state variable. Furthermore, in [16], Deckelnick and Hinze also investigated piecewise constant approximations of the control in the presence of pointwise state constraints and obtained an order of convergence of $h|\log h|$ in the two dimensional case and $h^{1/2}$ in case of three dimensions. Afore, slightly worse results for the same setting are proven in [30] by employing a completely different analysis.

In this paper, we show that the analysis of [30] can be transferred to the Stokes equations and *piecewise linear and continuous* ansatz functions for the control. In particular, the use of piecewise linear ansatz functions requires to significantly modify the theory presented in [30], which is performed by using a particular quasi-interpolant introduced by Carstensen in [6]. Moreover, to deal with different discretization techniques for the Stokes equations, we have to allow for discrete states which may not be feasible for the continuous problem. This constitutes another significant difference to the existing theory. The presented analysis covers results for different settings such as for instance the following: Let $\Omega \in \mathbb{R}^2$ be a convex polygon and $\Omega'$ be strictly contained in $\Omega$ and suppose that the Stokes equations are discretized with the Taylor-Hood element, while we use piecewise linear ansatz functions for the control. Then there holds for every $\varepsilon > 0$

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^2} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq C\, h^{1-\varepsilon},$$

where $(\bar{u}, \bar{v}, \bar{p})$ is the solution of (P), while $(\bar{u}_h, \bar{v}_h, \bar{p}_h)$ denotes the solution of its discrete counterpart.

To the authors' best knowledge, this is the first note that deals with the discretization error for the optimal control of the Stokes equations in the presence of pointwise state constraints. There are several papers considering finite element discretizations of the unconstrained optimal control of the Stokes and Navier-Stokes equations (see for instance [3, 13, 24, 25]) as well as contributions for the purely control-constrained case [32]. However, the analysis in case of pointwise state constraints differs significantly from these settings since, among other things, optimal $L^\infty$-error estimates for the finite element discretization of the Stokes equations are required.

The paper is organized as follows: after stating the main assumptions and known results for the continuous problem (P) in the following section, we introduce a general framework for a discretization of (P) in Section 3, which covers different concrete discrete schemes. Thereafter, in Section 4 we discuss some special interpolation results to be used in Section 5, where a priori error analysis for the problem under consideration is presented. Finally, Section 6 is devoted to concrete discretization schemes and their practical realization, whereas the numerical examples are presented in Section 7.

**2. Notation and Assumptions.** In all what follows, $|z|_{\mathbb{R}^d} = \big(\sum_{i=1}^d z_i^2\big)^{1/2}$ denotes the Euclidian norm and inequalities of the form $z \leq w$ with $w, z \in \mathbb{R}^d$, are understood componentwise. Moreover the natural inner product of $L^2(\Omega)^d$ and the associated norm are abbreviated by $(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega)^d}$ and $\|\cdot\| := \|\cdot\|_{L^2(\Omega)^d}$. Furthermore, we introduce the Hilbert spaces

$$V := H_0^1(\Omega)^d, \quad L := \big\{p \in L^2(\Omega) \,|\, \int_\Omega p(x)\,dx = 0\big\}.$$

Throughout this article, let $\sigma$ be a real number satisfying $1 < \sigma < d/(d-1)$. Then we define the conjugate exponent by $\sigma' = \sigma/(\sigma-1)$. In addition, $W_\sigma$ denotes the Sobolev space $W^{1,\sigma}(\Omega)^d$, whereas we set $V_\infty := L^\infty(\Omega')^d$. The dual spaces associated to $W_\sigma$ and $V_\infty$ with respect to the inner product of $L^2(\Omega)^d$ and $L^2(\Omega')^d$, respectively, are denoted by $W_\sigma^*$ and $V_\infty^*$.

Assumption 2.1. *On the quantities in* (P), *we impose the following conditions:*

- $\Omega$ *is an open, simply connected domain* $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, *while* $\Omega'$ *denotes an open subset of* $\Omega$.
- $K$ *is a closed and convex subset of* $L^\infty(\Omega')^d$
- $a, b \in \mathbb{R}^d$ *with* $a \leq b$
- $z \in L^2(\Omega)^d$.

Let us introduce the variational formulation of the Stokes equations by

$$(\nabla v, \nabla\varphi) - (p, \nabla\cdot\varphi) + (\nabla\cdot v, \psi) = (u, \varphi) \quad \forall\,(\varphi,\psi) \in V \times L. \tag{2.1}$$

It is well known that, for a given right-hand side $u \in L^2(\Omega)^d$, there exists a unique solution to (2.1) and the associated solution operator, denoted by $G : u \mapsto (v, p)$, is continuous from $L^2(\Omega)^d$ to $V \times L$. Moreover, we introduce the control-to-state operator $S\colon L^2(\Omega)^d \to V$ which maps the control variable $u$ to the velocity component of the solution $G u$, i.e., $S\colon u \mapsto v$. Sometimes $S$ and $G$ are considered in different spaces (e.g. $L^\infty(\Omega')^d$), for simplicity also denoted by $S$ and $G$, respectively. Based on the control-to-state operator, we define the reduced control problem by:

$$(\mathrm{P}) \quad \begin{cases} \displaystyle\min_{u \in L^2(\Omega)^d} & f(u) := J(S\,u, u) \\ \quad\text{s.t.} & S\,u \in K \\ & a \leq u(x) \leq b \quad \text{a.e. in } \Omega. \end{cases}$$

We assume the following mapping properties of $S$:

Assumption 2.2. *There is a positive number $\bar\sigma < d/(d-1)$ such that $S$ continuously maps $W_\sigma^* = (W^{1,\sigma}(\Omega)^d)^*$ to $W^{1,\sigma'}(\Omega)^d$ for all $\sigma \in [\bar\sigma, d/(d-1)[$. Hence, due to $\bar\sigma' > d$, Sobolev embedding theorems give*

$$S : L^2(\Omega)^d \hookrightarrow W_\sigma^* \to W^{1,\sigma'}(\Omega)^d \hookrightarrow V_\infty \quad \forall\, \sigma \in [\bar\sigma, d/(d-1)[. \tag{2.2}$$

For the rest of the paper, let $\sigma$ be a fixed, but arbitrary number in $[\bar\sigma, d/(d-1)[$. We point out that, if $\Omega$ has a smooth boundary, then (2.2) is satisfied, see Temam [34, Ch. I, Prop. 2.3]. Furthermore, in case of Lipschitz domains, (2.2) is proven in three dimensions by Brown and Shen [5, Theorem 2.9] and, under certain conditions on the Lipschitz constant of $\Gamma$ (the angles in the corners should not be too acute), Galdi et al. proved (2.2) for two and three dimensions [22, Theorem 2.1].

As already mentioned in the introduction, a certain constraint qualification is needed to derive the existence of Lagrange multipliers by means of the generalized Karush-Kuhn-Tucker theory. Here, we rely on

Assumption 2.3 (Slater condition). *There is a $\hat u \in L^\infty(\Omega)^d \cap W_\sigma$, satisfying*

$$\begin{gathered} S\,\hat u \in \operatorname{int} K \\ a \le \hat u(x) \le b \text{ a.e. in } \Omega. \end{gathered}$$

In order to state necessary optimality conditions for the solution of (P) we introduce the set of admissible controls which incorporates both the control and the state constraints:

$$U_{\mathrm{ad}} := \big\{ u \in L^\infty(\Omega)^d | a \le u(x) \le b \text{ a.e. in } \Omega,\ S\,u \in K \big\}.$$

Theorem 2.4. *Under Assumption 2.3 there exists a unique solution of* (P)*, denoted by $\bar u$. This solution provides some additional regularity, namely $\bar u \in W_\sigma$, and satisfies the following variational inequality*

$$(S\,\bar u - z\,,\, S\,u - S\,\bar u) + \alpha\,(\bar u\,,\, u - \bar u) \ge 0 \quad \forall\, u \in U_{\mathrm{ad}} \tag{2.3}$$

*where $U_{\mathrm{ad}}$ is defined as above.*

*Proof.* The existence and uniqueness result is standard. To show the additional regularity of $\bar u$, we make use of the generalized Karush-Kuhn-Tucker theory (cf. Zowe and Kurcyusz [35]). To this end, set $\bar v = S(\bar u)$. Under the Slater condition in Assumption 2.3, the generalized KKT theory guarantees the existence of a Lagrange multiplier $\bar\mu \in V_\infty^*$ such that $\bar u$ satisfies

$$\bar u = \Pi_{[a,b]}\big\{ -\frac{1}{\alpha}\, S^*\big(E_2(\bar v - z) + E_\infty \bar\mu\big)\big\} \tag{2.4}$$

with the adjoint operator $S^* \colon (W^{1,\sigma'}(\Omega)^d)^* \to W_\sigma$ (see Assumption 2.2). Moreover, $E_2 \colon L^2(\Omega)^2 \to (W^{1,\sigma'}(\Omega)^d)^*$ and $E_\infty \colon V_\infty^* \to (W^{1,\sigma'}(\Omega)^d)^*$ are the associated embedding operators. Furthermore, $\Pi_{[a,b]}$ denotes the component- and pointwise projection operator on the interval $[a,b]$. Since this projection operator maps $W_\sigma$ to itself, we have $\bar u \in W_\sigma$. Finally, the derivation of the variational inequality follows standard arguments. ∎

Remark 2.5. *We point out that the convergence analysis, presented below, does not involve dual variables, i.e., the adjoint state or Lagrange multipliers. In this context, the existence of Lagrange multipliers is just required to guarantee the additional regularity of $\bar u$ which is needed for the derivation of interpolation error estimates (see Lemma 4.4 and 4.5 below).*

**3. Discretization.** Now we turn to the discretization of (P). First, let us introduce a family of meshes $\{\mathcal{T}_h\}$ with mesh size $h > 0$. The mesh $\mathcal{T}_h$ consists of open cells $T$ (triangles, tetrahedra, quadrilaterals, hexahedra) such that

$$\bar\Omega = \bigcup_{T \in \mathcal{T}_h} \bar T$$

fulfilling usual assumptions on the finite element mesh, see, e.g., [4]. Notice that this implies that the cells lying on the boundary of $\Omega$ may be curved if $\Gamma$ is smooth (see Section 6.1 for details). The mesh size is defined by

$$h := \max_{T \in \mathcal{T}_h} h_T \quad \text{with} \quad h_T := \operatorname{diam}(T).$$

With each $T \in \mathcal{T}_h$, we associate the diameter of the largest ball contained in $T$, denoted by $R_T$. We suppose the following regularity assumptions for $\{\mathcal{T}_h\}$:

Assumption 3.1. *There exist two positive constants $\rho$ and $R$ such that*

$$\frac{h_T}{R_T} \le R\,, \quad \frac{h}{h_T} \le \rho$$

*hold for all cells $T \in \cup_{h>0} \mathcal{T}_h$.*

To each mesh, we associate finite dimensional subspaces of $V$ and $L$, denoted by $V_h$ and $L_h$. The discrete counterpart of (2.1) is then given by

$$(\nabla v_h, \nabla \varphi_h) - (p_h, \nabla \cdot \varphi_h) + (\nabla \cdot v_h, \psi_h) = (u, \varphi_h) \quad \forall\, (\varphi_h, \psi_h) \in V_h \times L_h \tag{3.1}$$

with associated solution operator $G_h u = (v_h, p_h) \in V_h \times L_h$. Concrete choices for the pairs $(V_h, L_h)$, allowing for existence of the solution operator $G_h$, will be discussed in Section 6. Analogously to above, we define the discrete control-to-state operator $S_h$ mapping given control $u$ to the velocity component $v_h$ of $G_h u$. In all what follows, we rely on the following conditions on $S_h$, that will be verified in Section 6 for different settings.

Assumption 3.2. *The following error estimates hold true*

$$\|S\, u - S_h\, u\|_{V_\infty} \le c\, \delta(h)\, \|u\|_{L^\infty(\Omega)^d} \tag{3.2}$$

*with some function $\delta\colon \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\delta(h) \to 0$ if $h \downarrow 0$, and a constant $c$ independent of $h$ and $u$.*

Next, we turn to the discretization of the control. To this end, we define the associated ansatz functions.

Assumption 3.3. *Let $n \in \mathbb{N}$ be given and suppose that $n$ ansatz functions $\phi_i \in L^\infty(\Omega)$, $1 \le i \le n$, are given such that for every $i \in \{1, ..., n\}$*

$$\max_{x \in \bar\Omega} \phi_i(x) = 1, \quad \phi_i(x) \ge 0 \ \text{a.e. in } \Omega, \quad \sum_{i=1}^{n} \phi_i(x) = 1 \ \text{a.e. in } \Omega. \tag{3.3}$$

*Moreover, we assume that the patch $\omega_i := supp\, \phi_i$ is a set of positive measure and contained in the union of $M_i$ adjacent cells that share at least one common vertex.*

Notice that Assumption 3.1 implies the existence of a constant $M \in \mathbb{N}$, independent of $h$, such that $M_i \le M$ for all $i \in \{1, ..., n\}$.

Remark 3.4. *If $\Omega$ is a polygon ($d = 2$) or polyhedron ($d = 3$), the assumptions on the ansatz functions $\phi_i$, $i = 1, ..., n$ are clearly fulfilled for different common finite elements such as:*

- piecewise constant *elements,*
- linear *finite elements in case of triangles and tetrahedrons, respectively,*
- bi-/trilinear *elements for quadrilaterals and hexahedrons, respectively.*

*The assumption $\phi_i(x) \ge 0$ a.e. in $\Omega$ is not needed for the derivation of interpolation error estimates, but for the feasibility of interpolated controls (see Lemma 5.5).*

The discrete control space is given by $U_h := \operatorname{span}\{\phi_i \,|\, 1 \le i \le n\}^d$. Now we are in the position to define the discrete counterpart to (P):

$$(\mathrm{P}_h) \qquad \begin{cases} \displaystyle\min_{u_h \in U_h} \ f(u) := J(S_h\, u_h, u_h) \\ \text{s.t.} \ \ S_h\, u_h \in K, \\ \qquad a \le u_h(x) \le b \quad \text{a.e. in } \Omega. \end{cases}$$

Notice that $(\mathrm{P}_h)$ is not a fully discrete problem, since $K$ and $z$ are not discretized. The discretization of $K$ and $z$ is postponed to Section 6.3. One shows by standard arguments:

THEOREM 3.5. *Assume that the feasible set for* $(\mathrm{P}_h)$ *is not empty, i.e., there exists a discrete control* $u_h \in U_h$ *with* $a \le u_h(x) \le b$ *a.e. in* $\Omega$ *and* $S_h\, u_h \in K$. *Then there exists unique solution of* $(\mathrm{P}_h)$, *denoted by* $\bar u_h \in U_h$, *which satisfies the following discrete variational inequality*

$$(S_h\, \bar u_h - z\,,\, S_h\, u_h - S_h\, \bar u_h) + \alpha\,(\bar u_h\,,\, u_h - \bar u_h) \ge 0 \quad \forall\, u_h \in U^h_{\mathrm{ad}} \tag{3.4}$$

*with*

$$U^h_{\mathrm{ad}} := \big\{u_h \in U_h | a \le u_h(x) \le b \text{ a.e. in } \Omega,\ S_h\, u_h \in K\big\}.$$

**4. Interpolation estimates.** In this section we discuss some interpolation estimates for functions in $W_\sigma$. For the error analysis in the next section we need an interpolation operator which provides interpolation estimates of optimal order among other things in negative Sobolev norms (cf. Lemma 4.5) and additionally has the following property:

$$a \le u(x) \le b \text{ a.e. in } \Omega \quad \Rightarrow \quad a \le (\Pi_h u)(x) \le b \text{ a.e. in } \Omega. \tag{4.1}$$

To this end we consider the quasi-interpolation operator introduced in [6]. For an arbitrary $u \in L^1(\Omega)$, the construction is as follows:

$$\Pi_h u = \sum_i \pi_i(u)\phi_i, \tag{4.2}$$

where

$$\pi_i(u) = \frac{\int_{\omega_i} u\phi_i\, dx}{\int_{\omega_i} \phi_i\, dx}\,. \tag{4.3}$$

Analogously, the quasi-interpolation operator for vectorial quantities is defined componentwise for simplicity also denoted by $\Pi_h$. The property (4.1) is obviously fulfilled due to the above construction and Assumption 3.3.

In the following we discuss error estimates for $u - \Pi_h u$ in different norms on the computational domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. To keep the discussion concise, we argue for a single component for the rest of this section. The results for vectorial quantities immediately follow from norm equivalence in $\mathbb{R}^d$.

LEMMA 4.1. *For each* $i \in \{1, ..., n\}$, *there is a constant* $c_i$ *which may depend on diam* $\omega_i$ *such that*

$$\|u - \pi_i(u)\|_{L^2(\omega_i)} \le c_i\, \|\nabla u\|_{L^s(\omega_i)} \quad \forall\, u \in W^{1,s}(\omega_i)$$

*for all* $\frac{2d}{d+2} \le s \le 2$.

REMARK 4.2. *The condition* $s \ge \frac{2d}{d+2}$ *is required for the embedding* $W^{1,s}(\omega_i) \hookrightarrow L^2(\omega_i)$. *It obviously holds:*

$$\frac{2d}{d+2} < \frac{d}{d-1} \quad \textit{for } d = 2, 3.$$

*Proof.* Let $i \in \{1, ..., n\}$ be arbitrary. For the proof we use an indirect argument. If the proposed assertion is false, there exists a sequence $\{u_k\} \subset W^{1,s}(\omega_i)$ with

$$\|u_k - \pi_i(u_k)\|_{L^2(\omega_i)} = 1 \quad \text{and} \quad \|\nabla u_k\|_{L^s(\omega_i)} \le \frac{1}{k} \quad \forall\, k \in \mathbb{N}.$$

We consider $v_k = u_k - \pi_i(u_k)$ and obtain in view of $\nabla \pi_i(u_k) = 0$

$$\|v_k\|_{L^2(\omega_i)} = 1 \quad \text{and} \quad \|\nabla v_k\|_{L^s(\omega_i)} \le \frac{1}{k} \quad \forall\, k \in \mathbb{N}. \tag{4.4}$$

Therefore, thanks to $s \le 2$, $\{v_k\}$ is bounded in $W^{1,s}(\omega_i)$ and there exists a subsequence denoted again by $\{v_k\}$ with

$$v_k \rightharpoonup v \quad \text{in } W^{1,s}(\omega_i)$$

and therefore

$$v_k \to v \quad \text{in } L^s(\omega_i).$$

Due to (4.4), $\nabla v_k$ is a Cauchy sequence in $L^s(\omega_i)$ and therefore

$$v_k \to v \quad \text{in } W^{1,s}(\omega_i).$$

Hence, $\nabla v = 0$ and $v = \mathit{const}$. Moreover there holds by the definition of $\pi_i$:

$$\int_{\omega_i} v_k \phi_i \, dx = 0,$$

and therefore

$$\int_{\omega_i} v \phi_i \, dx = 0,$$

which implies $v = 0$. Due to the embedding $W^{1,s}(\omega_i) \hookrightarrow L^2(\omega_i)$, we have $v_k \to v$ in $L^2(\omega_i)$ and therefore $\|v\|_{L^2(\omega_i)} = 1$. This is a contradiction. ∎

LEMMA 4.3. *There is a constant $c$ which is independent of $h$ such that*

$$\|u - \pi_i(u)\|_{L^2(\omega_i)} \le c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1} \|\nabla u\|_{L^s(\omega_i)} \quad \forall\, u \in W^{1,s}(\omega_i)$$

*for all $i \in \{1, ..., n\}$ and all $\frac{2d}{d+2} \le s \le 2$.*

*Proof.* The proof uses the assertion from Lemma 4.1 on a reference patch $\hat{\omega}_i$ and a standard transformation argument. For convenience of the reader, we shortly sketch the arguments for a domain with polygonal ($d = 2$) or polyhedral ($d = 3$) boundary and the case of triangles and tetrahedra, respectively. Let $\omega_i$ be an arbitrary patch consisting of the cells $T_j^{(i)}$, $j = 1, ..., M_i$. As mentioned above, $M = \max_i\{M_i\}$ is bounded independently of $h$. To each patch $\omega_i$, we associate a reference patch $\hat{\omega}_i$ whose vertices lie on the surface of the unit ball in $\mathbb{R}^d$. Moreover, it consists of $M_i$ congruent cells $\hat{T}_j^{(i)}$. Due to $M_i \le M$, the number of possible references patches is finite and they can be constructed such that $|\hat{T}_j^{(i)}|$ is bounded from below and above by constants independent of $h$. Now denote by $F_i$, $F_i\hat{x} = x$, the bi-Lipschitz transformation from $\hat{\omega}_i$ to $\omega_i$, and set $F_j^{(i)} := F_i|_{\hat{T}_j^{(i)}}$, i.e. the affine-linear transformation from $\hat{T}_j^{(i)}$ to $T_j^{(i)}$. Analogously to (4.3), let $\hat{\pi}_i$ be defined by

$$\hat{\pi}_i(v) := \frac{\int_{\hat{\omega}_i} \hat{\phi}_i \, v \, d\hat{x}}{\int_{\hat{\omega}_i} \hat{\phi}_i \, d\hat{x}} = \frac{\int_{\hat{\omega}_i} (\phi_i \circ F_i)\, v \, d\hat{x}}{\int_{\hat{\omega}_i} \phi_i \circ F_i \, d\hat{x}},$$

where $\hat{\phi}_i$ denote the ansatz function on $\hat{\omega}_i$. Then, due to $u \circ F_i \in W^{1,s}(\hat{\omega}_i)$, we obtain

$$\begin{aligned}
\|u - \pi_i(u)\|_{L^2(\omega_i)}^2 &= \sum_{j=1}^{M_i} \frac{|T_j^{(i)}|}{|\hat{T}_j^{(i)}|} \int_{\hat{T}_j^{(i)}} \Big(u(F_j^{(i)}\hat{x}) - \pi_i(u)\Big)^2 d\hat{x} \\
&\le c\, h^d \int_{\hat{\omega}_i} \big(u \circ F_i - \hat{\pi}_i(u \circ F_i)\big)^2 d\hat{x} \le c\, h^d \bigg(\int_{\hat{\omega}_i} |\nabla_{\hat{x}}(u \circ F_i)|^s d\hat{x}\bigg)^{\frac{2}{s}} \\
&\le c\, h^d \bigg(\sum_{j=1}^{M_i} \frac{|\hat{T}_j^{(i)}|}{|T_j^{(i)}|} \int_{T_j} |\nabla_x u|^s \Big|\frac{\partial x}{\partial \hat{x}}\Big|^s dx\bigg)^{\frac{2}{s}} \le c\, h^{d\left(1-\frac{2}{s}\right)+2} \|\nabla u\|_{L^s(\omega_i)}^2
\end{aligned}$$

with a constant $c > 0$ independent of $h$. If quadrilaterals or hexahedra are used, one argues analogously using suitably defined reference patches. In case of smooth boundaries, where $F_j^{(i)}$ is not longer affine-linear, the result follows from similar transformation arguments known from the theory of interpolation on curved domains (see [2, Lemma 2.3]). ∎

Lemma 4.4. *There is a constant $c$ which is independent of $h$ such that*

$$\|u - \Pi_h u\|_{L^2(\Omega)} \le c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1} \|\nabla u\|_{L^s(\Omega)} \quad \forall\, u \in W^{1,s}(\Omega),$$

*with* $\frac{2d}{d+2} \le s \le 2$.

*Proof.* For all $v \in L^2(\Omega)$, we find

$$\begin{aligned}
(u - \Pi_h u, v) = \left( u \sum_{i=1}^n \phi_i - \sum_{i=1}^n \pi_i(u)\phi_i, v \right) &= \sum_{i=1}^n \int_{\omega_i} (u - \pi_i(u))\, \phi_i\, v\, dx, \\
&\le c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1} \sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)} \|v\|_{L^2(\omega_i)} \\
&\le c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1} \left( \sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)}^s \right)^{1/s} \left( \sum_{i=1}^n \|v\|_{L^2(\omega_i)}^{s'} \right)^{1/s'}.
\end{aligned}$$

Using the fact that $\frac{s'}{2} \ge 1$ since $s \le 2$, we have

$$\sum_{i=1}^n \|v\|_{L^2(\omega_i)}^{s'} = \sum_{i=1}^n \left( \|v\|_{L^2(\omega_i)}^2 \right)^{\frac{s'}{2}} \le \left( \sum_{i=1}^n \|v\|_{L^2(\omega_i)}^2 \right)^{\frac{s'}{2}}.$$

Hence,

$$|(u - \Pi_h u, v)| \le c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1} \|\nabla u\|_{L^s(\Omega)} \|v\|_{L^2(\Omega)}.$$

Notice that Assumption 3.3 implies $\sum_{i=1}^n \|\nabla w\|_{L^q(\omega_i)}^q \le c\, \|\nabla w\|_{L^q(\Omega)}^q$ for every $w \in W^{1,q}(\Omega)$ and every $1 \le q < \infty$. Setting $v = u - \Pi_h u$, we complete the proof. ∎

Lemma 4.5. *There exists a constant $c$, independent of $h$, such that*

$$\|u - \Pi_h u\|_{W^{1,s}(\Omega)^*} \le c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \|u\|_{W^{1,s}(\Omega)} \quad \forall\, u \in W^{1,s}(\Omega)$$

*with* $\frac{2d}{d+2} \le s \le 2$.

*Proof.* We consider for all $v \in W^{1,s}(\Omega)$:

$$(u - \Pi_h u, v) = \left( u \sum_{i=1}^n \phi_i - \sum_{i=1}^n \pi_i(u)\phi_i, v \right) = \sum_{i=1}^n \int_{\omega_i} (u - \pi_i(u))\, \phi_i\, v\, dx,$$

where we have used $\sum_{i=1}^n \phi_i \equiv 1$ and the definition of $\Pi_h$. Due to definition of $\pi_i$, we have

$$\int_{\omega_i} (u - \pi_i(u))\, \phi_i\, dx = 0,$$

and therefore we continue with

$$\begin{aligned}
(u - \Pi_h u, v) &= \sum_{i=1}^n \int_{\omega_i} (u - \pi_i(u))\, \phi_i\, (v - \pi_i(v))\, dx \\
&\le c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)} \|\nabla v\|_{L^s(\omega_i)} \\
&\le c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \left( \sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)}^s \right)^{\frac{1}{s}} \left( \sum_{i=1}^n \|\nabla v\|_{L^s(\omega_i)}^{s'} \right)^{\frac{1}{s'}},
\end{aligned}$$

Using the fact $\frac{s'}{s} \geq 1$ since $s \leq 2$, we obtain

$$\sum_{i=1}^{n} \|\nabla v\|_{L^s(\omega_i)}^{s'} = \sum_{i=1}^{n} \Big(\|\nabla v\|_{L^s(\omega_i)}^{s}\Big)^{\frac{s'}{s}} \leq \Big(\sum_{i=1}^{n} \|\nabla v\|_{L^s(\omega_i)}^{s}\Big)^{\frac{s'}{s}}$$

such that

$$\begin{aligned}|(u - \Pi_h u, v)| \leq c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \Big(\sum_{i=1}^{n} \|\nabla u\|_{L^s(\omega_i)}^{s}\Big)^{\frac{1}{s}} \Big(\sum_{i=1}^{n} \|\nabla v\|_{L^s(\omega_i)}^{s}\Big)^{\frac{1}{s}} \\ \leq c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \|u\|_{W^{1,s}(\Omega)} \|v\|_{W^{1,s}(\Omega)}.\end{aligned}$$

This completes the proof. ∎

LEMMA 4.6. *For every $u \in L^\infty(\Omega)$, there holds*

$$\|\Pi_h u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}.$$

*Proof.* In view of (4.3), we obtain

$$|\pi_i(u)| \leq \|u\|_{L^\infty(\Omega)} \quad \forall\, i \in \{1, ..., n\}.$$

Together with (3.3), this implies

$$\left|\sum_{i=1}^{n} \pi_i(u)\, \phi_i(x)\right| \leq \max_i\{|\pi_i(u)|\} \sum_{i=1}^{n} \phi_i(x) \leq \|u\|_{L^\infty(\Omega)} \quad \forall\, x \in \Omega,$$

which gives the assertion. ∎

**5. Convergence analysis.** With the above results at hand, in particular Lemma 4.4 and 4.5, one can extend the theory from [30] to problem (P). The analysis of [30] is mainly based on the existence of functions $u_d \in U_h$ and $u_c \in U$ which are feasible for one of the problems (P) or ($P_h$), but in some sense close to the solution of the other problem. In [30], the proofs are presented for the case of box constraints on the state. With the help of the support functional, the arguments can easily be adapted to the more general state constraint in (P). For convenience of the reader, this is demonstrated in the following section. We characterize the convex set $K$ by means of the support functional: since the interior of $K$ is not empty by Assumption 2.3, the supporting hyperplane theorem implies

$$\operatorname{int} K = \bigcap_{\mu \in V_\infty^*,\, \mu \neq 0} \{v \in V_\infty \,|\, \langle \mu\,,\, v\rangle_{V_\infty^*, V_\infty} < s(\mu)\}, \tag{5.1}$$

where $s : V_\infty^* \to \mathbb{R}$ denotes the support functional, i.e. $s(\mu) = \sup_{v \in K} \langle \mu\,,\, v\rangle_{V_\infty^*, V_\infty}$ (see, e.g., Luenberger [29]). Hence, in view of Assumption 2.3, there is a $\tau > 0$ such that

$$\langle \mu\,,\, S\,\hat{u}\rangle_{V_\infty^*, V_\infty} \leq s(\mu) - \tau \quad \forall\, \mu \in V_\infty^*, \mu \neq 0. \tag{5.2}$$

Recall that $\sigma$ is a fixed, but arbitrary number in $[\bar{\sigma}, d/(d-1)[$ and $W_\sigma = W^{1,\sigma}(\Omega)^d$.

DEFINITION 5.1. *Given $\sigma \in [\bar{\sigma}, d/(d-1)[$ and $h > 0$, we set*

$$\begin{aligned}\eta(\sigma, h) &:= h^{2d\left(\frac{1}{2}-\frac{1}{\sigma}\right)+2} \\ \beta(\sigma, h) &:= \max\{\eta(\sigma, h), \delta(h)\},\end{aligned}$$

*where $\delta(h)$ is defined as in Assumption 3.2. Moreover, we define*

$$\begin{aligned}u_c &:= \bar{u}_h + \gamma_c \delta(h)(\hat{u} - \bar{u}_h) \\ u_d &:= \Pi_h\, \bar{u} + \gamma_d\, \beta(\sigma, h)(\Pi_h\, \hat{u} - \Pi_h\, \bar{u}),\end{aligned}$$

*with constants $\gamma_c, \gamma_d > 0$ defined in the subsequent.*

LEMMA 5.2. *There exist a constant $\gamma_c$ independent of $h$ and an $h_1 > 0$ such that the function $u_c$ is feasible for* (P) *for all $h < h_1$.*

*Proof.* First we show $S\,u_c \in K$. To this end, let $\mu \in V_\infty^*$, $\mu \neq 0$, be arbitrary and define

$$\tilde\mu := \frac{1}{\|\mu\|_{V_\infty^*}}\,\mu.$$

such that $\|\tilde\mu\|_{V_\infty^*} = 1$. Then, one obtains

$$\begin{aligned}\langle\tilde\mu\,,\,S\,u_c\rangle_{V_\infty^*,V_\infty} &= (1-\gamma_c\delta(h))\langle\tilde\mu\,,\,S\,\bar u_h\rangle_{V_\infty^*,V_\infty} + \gamma_c\delta(h)\langle\tilde\mu\,,\,S\hat u\rangle_{V_\infty^*,V_\infty}\\ &\le (1-\gamma\delta(h))\big[\langle\tilde\mu\,,\,S_h\,\bar u_h\rangle_{V_\infty^*,V_\infty} + \langle\tilde\mu\,,\,(S-S_h)\bar u_h\rangle_{V_\infty^*,V_\infty}\big] + \gamma_c\delta(h)\big(s(\tilde\mu)-\tau\big)\\ &\le s(\tilde\mu) - \gamma_c\delta(h)\tau + (1-\gamma_c\delta(h))\|\tilde\mu\|_{V_\infty^*}\,\|(S-S_h)\bar u_h\|_{V_\infty}\\ &\le s(\tilde\mu) - \delta(h)\big(\gamma_c\tau - c(1-\gamma_c\delta(h))\|\bar u_h\|_{L^\infty(\Omega)^d}\big)\end{aligned} \tag{5.3}$$

where we used Assumption 3.2, (5.2), and the feasibility of $\bar u_h$ for (P$_h$) which implies $\langle\tilde\mu\,,\,S_h\,\bar u_h\rangle \le s(\tilde\mu)$. In view of the control constraints in (P), we obtain for the second addend in the last inequality

$$\gamma_c\tau - c(1-\gamma_c\delta(h))\|\bar u_h\|_{L^\infty(\Omega)^d} \ge \gamma_c\tau - c\,\max\{|a|,|b|\}$$

such that $\langle\tilde\mu\,,\,S\,u_c\rangle < s(\tilde\mu)$ is fulfilled if we choose $\gamma_c > c\,\max\{|a|,|b|\}/\tau$. Hence, $u_c$ satisfies

$$\langle\mu\,,\,S\,u_c\rangle_{V_\infty^*,V_\infty} = \|\mu\|_{V_\infty^*}\,\langle\tilde\mu\,,\,S\,u_c\rangle_{V_\infty^*,V_\infty} < \|\mu\|_{V_\infty^*}\,s(\tilde\mu) = s(\mu),$$

since the support functional is clearly sublinear. As $\mu$ was chosen arbitrary, (5.1) implies $S\,u_c \in K$ if $\gamma_c > c\,\max\{|a|,|b|\}/\tau$. Furthermore, if we choose $h$ small enough, then $u_c$ is a convex linear combination of two functions in $\{u \in L^\infty(\Omega)^d \,|\, a \le u(x) \le b \text{ a.e. in } \Omega\}$ and therefore also satisfies the control constraints in (P). Consequently the assertion holds true. ∎

To prove a similar result for the other direction, i.e., the feasibility of $u_d$ for (P$_h$), we need some auxiliary results which are presented in the subsequent.

LEMMA 5.3. *Suppose $u \in W_\sigma$ is given. Then*

$$\|S(u-\Pi_h\,u)\|_{V_\infty} \le c\,\eta(\sigma,h)\,\|u\|_{W_\sigma}$$

*holds true with a constant $c$ only depending on $\Omega$.*

*Proof.* The mapping properties of $S$ in Assumption 2.2 imply

$$\|S(u-\Pi_h\,u)\|_{V_\infty} \le c\,\|S\|_{\mathcal{L}(W_\sigma^*,W^{1,\sigma'}(\Omega)^d)}\,\|u-\Pi_h\,u\|_{W_\sigma^*} \le c\,\eta(\sigma,h)\,\|u\|_{W_\sigma},$$

where we used Lemma 4.5 and the definition of $\eta$. ∎

LEMMA 5.4. *Let $\tilde\mu \in V_\infty^*$ with $\|\tilde\mu\|_{V_\infty^*} = 1$ be arbitrary. Then, for every $u \in W_\sigma \cap L^\infty(\Omega)^d$,*

$$\langle\tilde\mu\,,\,S_h\,\Pi_h\,u\rangle_{V_\infty^*,V_\infty} \le \langle\tilde\mu\,,\,S\,u\rangle_{V_\infty^*,V_\infty} + c\,\beta(\sigma,h)\,\big(\|u\|_{W_\sigma} + \|u\|_{L^\infty(\Omega)^d}\big)$$

*is satisfied with a constant $c > 0$ independent of $h$ and $u$.*

*Proof.* In view of $\|\tilde\mu\|_{V_\infty^*} = 1$, we find

$$\begin{aligned}&\langle\tilde\mu\,,\,S_h\,\Pi_h\,u\rangle_{V_\infty^*,V_\infty}\\ &\quad= \langle\tilde\mu\,,\,S\,u\rangle_{V_\infty^*,V_\infty} + \langle\tilde\mu\,,\,S(\Pi_h\,u-u)\rangle_{V_\infty^*,V_\infty} + \langle\tilde\mu\,,\,(S_h-S)\Pi_h\,u\rangle_{V_\infty^*,V_\infty}\\ &\quad\le \langle\tilde\mu\,,\,S\,u\rangle_{V_\infty^*,V_\infty} + \|\tilde\mu\|_{V_\infty^*}\,\|S(\Pi_h\,u-u)\|_{V_\infty} + \|\tilde\mu\|_{V_\infty^*}\,\|(S_h-S)\Pi_h\,u\|_{V_\infty}\\ &\quad\le \langle\tilde\mu\,,\,S\,u\rangle_{V_\infty^*,V_\infty} + c\,\big(\eta(\sigma,h)\,\|u\|_{W_\sigma} + \delta(h)\,\|u\|_{L^\infty(\Omega)^d}\big)\end{aligned}$$

where we used Lemma 5.3, Assumption 3.2, and Lemma 4.6. With the definition of $\beta$ (cf. Definition 5.1), the assertion is verified. ∎

LEMMA 5.5. *There exist a constant $\gamma_d$ depending on $\bar u$ and $\sigma$, but not on $h$, and a mesh size $h_2$ so that $u_d$ is feasible for* ($\mathrm{P}_h$) *if $h < h_2$.*

*Proof.* Let $\mu \in V_\infty^*$ again be arbitrary and define $\tilde\mu = \mu/\|\mu\|_{V_\infty^*}$ as in the proof of Lemma 5.2. Similarly to (5.3), we estimate

$$\begin{aligned}
&\langle \tilde\mu\,,\, S_h\, u_d\rangle_{V_\infty^*,V_\infty}\\
&= (1-\gamma_d\beta(\sigma,h))\langle \tilde\mu\,,\, S_h\,\Pi_h\,\bar u\rangle_{V_\infty^*,V_\infty} + \gamma_d\beta(\sigma,h)\,\langle \tilde\mu\,,\, S_h\,\Pi_h\,\hat u\rangle_{V_\infty^*,V_\infty}\\
&\le\ (1-\gamma_d\beta(\sigma,h))\Big[\langle \tilde\mu\,,\, S\,\bar u\rangle_{V_\infty^*,V_\infty} + c\,\beta(\sigma,h)\left(\|\bar u\|_{W_\sigma} + \|\bar u\|_{L^\infty(\Omega)^d}\right)\Big]\\
&\qquad + \gamma_d\beta(\sigma,h)\Big[\langle \tilde\mu\,,\, S\,\hat u\rangle_{V_\infty^*,V_\infty} + c\,\beta(\sigma,h)\left(\|\hat u\|_{W_\sigma} + \|\hat u\|_{L^\infty(\Omega)^d}\right)\Big]\\
&\le\ s(\tilde\mu) - \beta(\sigma,h)\Big[\gamma_d\tau - c\,\big(\underbrace{\|\bar u\|_{W_\sigma} + \|\bar u\|_{L^\infty(\Omega)^d} + \|\hat u\|_{W_\sigma} + \|\hat u\|_{L^\infty(\Omega)^d}}_{:=\,c_u}\big)\Big].
\end{aligned} \tag{5.4}$$

Hence, if we choose $\gamma_d > c\,c_u/\tau$, then one obtains $\langle \tilde\mu\,,\, S_h\, u_d\rangle < s(\tilde\mu)$ which gives in turn $S_h\, u_d \in K$ by the same arguments as in the proof of Lemma 5.2. Notice that $\gamma_d$ is independent of $h$, but depends on $\|\bar u\|_{W_\sigma}$ and therefore on $\bar u$ and $\sigma$. Moreover, we have that

$$a \le (\Pi_h u)(x) \le b \quad \text{a.e. in } \Omega,$$

see (4.1). Hence, the same arguments as in the proof of Lemma 5.2 give

$$a \le u_d(x) \le b \quad \text{a.e. in } \Omega,$$

if $h$ is sufficiently small. Since $u_d \in U_h$ by construction, we therefore end up with $u_d \in U_{\mathrm{ad}}^h$. ∎

Now we are in the position to prove our main result which reads as follows:

THEOREM 5.6. *Let $\bar u$ and $\bar u_h$ denote the optimal solutions of* (P) *and* ($\mathrm{P}_h$)*, respectively. Then, under Assumptions 2.1–2.3 and 3.1–3.3, the following estimate holds true*

$$\|\bar u - \bar u_h\| + \|S\,\bar u - S_h\,\bar u_h\| \le C\,\sqrt{\max\{\eta(\sigma,h), \delta(h)\}}$$

*with a constant $C > 0$ which depends on $\bar u$ and $\sigma$, but not on $h$.*

*Proof.* Based on a technique introduced in Falk [20], it is shown in [30] that the variational inequalities (2.3) and (3.4) imply

$$\begin{aligned}
&\frac{\alpha}{2}\,\|\bar u - \bar u_h\|^2 + \frac12\,\|S\,\bar u - S_h\,\bar u_h\|^2\\
&\quad\le c\,\Big[\|u_h - \bar u\|^2 + \big(\|\bar u\|_{W_\sigma} + \|S\,\bar u - z\|\big)\big(\|u - \bar u_h\|_{W_\sigma^*} + \|u_h - \bar u\|_{W_\sigma^*}\big)\\
&\qquad + \|u_h - \bar u\|^2_{W_\sigma^*} + \|(S - S_h)u_h\|^2\\
&\qquad + \|S\,\bar u - z\|\big(\|(S - S_h)\bar u_h\| + \|(S - S_h)u_h\|\big)\Big] \quad \forall\, u \in U_{\mathrm{ad}},\ u_h \in U_{\mathrm{ad}}^h.
\end{aligned} \tag{5.5}$$

Here, the constant $c$ depends on $\alpha$, but not on $\bar u$, $\bar u_h$, $u$, and $u_h$. Thanks to Lemma 5.2 and 5.5, we are allowed to insert $u = u_c$ and $u_h = u_d$ in (5.5). Then, by means of Lemma 4.4 and 4.5 and the definition of $\beta$, we obtain

$$\begin{aligned}
\|u_d - \bar u\| &\le \|\Pi_h\bar u - \bar u\| + \gamma_d\,\beta(\sigma,h)\,\|\Pi_h\hat u - \Pi_h\bar u\|\\
&\le c\left(\|\bar u\|_{W_\sigma} + \|\hat u\|_{W_\sigma}\right)\,\max\{\sqrt{\eta(\sigma,h)}, \beta(\sigma,h)\}
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\|u_d - \bar u\|_{W_\sigma^*} &\le \|\Pi_h\bar u - \bar u\|_{W_\sigma^*} + \gamma_d\,\beta(\sigma,h)\,\|\Pi_h\hat u - \Pi_h\bar u\|_{W_\sigma^*}\\
&\le c\left(\|\bar u\|_{W_\sigma} + \|\hat u\|_{W_\sigma}\right)\beta(\sigma,h).
\end{aligned} \tag{5.7}$$

In case of $u = u_c$, we have

$$\|u_c - \bar u_h\|_{W_\sigma^*} \le c\,\gamma_c\,\delta(h)\,\|\hat u - \bar u_h\|_{W_\sigma^*}. \tag{5.8}$$

For the remaining expressions in (5.5), (3.2) implies

$$\begin{aligned}\|(S_h - S)u_d\| &\le c\,\delta(h)\,\|\Pi_h\bar u - \gamma_d\,\delta(h)(\Pi_h\hat u - \Pi_h\bar u)\|_{L^\infty(\Omega)^d}\\ &\le c\left(\|\bar u\|_{L^\infty(\Omega)^d} + \|\hat u\|_{L^\infty(\Omega)^d}\right)\delta(h)\end{aligned} \tag{5.9}$$

$$\|(S_h - S)\bar u_h\| \le c\,\delta(h)\,\|\bar u_h\|_{L^\infty(\Omega)^d}, \tag{5.10}$$

where we used Lemma 4.6 for the estimation of the right hand side in (5.9). Notice that $\bar u$ and $\hat u$ are bounded in $W_\sigma$ and $L^\infty(\Omega)^d$ due to Assumption 2.3 and Theorem 2.4, whereas $\bar u_h$ is uniformly bounded in $L^\infty(\Omega)^d$ due to the control constraints. Inserting (5.6)–(5.10) in (5.5) finally implies

$$\begin{aligned}&\frac{\alpha}{2}\,\|\bar u{-}\bar u_h\|^2 + \frac{1}{2}\,\|S\,\bar u - S_h\,\bar u_h\|^2\\ &\quad\le c\Big[\max\{\eta(\sigma,h),\beta(\sigma,h)^2\} + \big(\|\bar u\|_{W_\sigma} + \|S\,\bar u - z\|\big)\big(\beta(\sigma,h) + \delta(h)\big)\\ &\qquad\quad + \beta(\sigma,h)^2 + \delta(h)^2 + \|S\,\bar u - z\|\,\delta(h)^2\Big]\\ &\quad\le C^2\,\max\{\eta(\sigma,h),\delta(h)\}\end{aligned}$$

thanks to the definition of $\beta$. An inspection of the proof yields that $C$ depends on $\bar u$ and $\sigma$, but not on $h$. ◻

Corollary 5.7. *Suppose that, in addition to the assumptions of Theorem 5.6,*

$$\|(G_h - G)u\|_{H^1(\Omega)^d\times L^2(\Omega)} \le c\,\vartheta(h)\,\|u\|_{L^\infty(\Omega)^d}$$

*is fulfilled with* $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$, $\vartheta(h) \to 0$ *as* $h \downarrow 0$. *Then,* $(\bar v, \bar p) = G\,\bar u$ *and* $(\bar v_h, \bar p_h) = G_h\,\bar u_h$ *satisfy*

$$\|\bar v - \bar v_h\|_{H^1(\Omega)^d} + \|\bar p - \bar p_h\|_{L^2(\Omega)} \le C\,\max\{\vartheta(h), \sqrt{\delta(h)}, \sqrt{\eta(\sigma,h)}\}$$

*with a constant* $C$ *independent of* $h$.

*Proof.* The proof is almost standard. The mapping properties of $S$ imply

$$\begin{aligned}&\|G\bar v - G_h\bar v_h\|_{H^1(\Omega)^d\times L^2(\Omega)}\\ &\quad\le \|G(\bar u - \bar u_h)\|_{H^1(\Omega)^d\times L^2(\Omega)} + \|(G - G_h)\bar u_h\|_{H^1(\Omega)^d\times L^2(\Omega)}\\ &\quad\le \|G\|_{\mathcal{L}(L^2(\Omega)^d, H^1(\Omega)^d\times L^2(\Omega))}\|\bar u - \bar u_h\| + c\,\vartheta(h)\,\|\bar u_h\|_{L^\infty(\Omega)^d},\end{aligned}$$

such that Theorem 5.6 yields the assertion. ◻

**6. Concrete numerical settings.** In the subsequent, several control problems and discretization techniques are discussed that are covered by the above theory. The critical point is to verify (3.2) for a concrete discretization such that $\delta(h)$, i.e., the $L^\infty$-error of the finite element approximation, is not worse than $\eta(\sigma,h)$, i.e., the interpolation error. To keep the discussion concise, we restrict ourselves to discretization schemes that fulfill the discrete inf-sup condition so that there is no need for stabilization. We rely on the following assumptions:

Assumption 6.1. *The spaces* $V_h \subset V$ *and* $L_h \subset L$ *satisfy the following conditions*

- *There is a number* $k \in \mathbb{N}$, $k \ge 1$, *such that*

$$V_h \subset C(\bar\Omega)^d,\ \mathcal{P}_k(T)^d \subseteq V_h|_T, \quad \mathcal{P}_{k-1}(T) \subseteq L_h|_T \quad \forall\, T \in \mathcal{T}_h. \tag{6.1}$$

*Consequently, there exist interpolation operators* $i_h^v$ *and* $i_h^p$ *that fulfill standard* approximation properties. *In particular, if* $t \in \{0,1\}$ *and* $q, r, s \in [1,\infty]$ *are given such that* $W^{2,r}(\Omega) \hookrightarrow W^{t,q}(\Omega)$ *and* $W^{1,s}(\Omega) \hookrightarrow L^q(\Omega)$, *then there holds:*

$$\|\nabla^t(v - i_h^v\,v)\|_{L^q(T)} \le c\,h^{2-t+d(1/q-1/r)}\,\|\nabla^2 v\|_{L^r(T)} \quad \forall\, v \in W^{2,r}(T) \tag{6.2}$$

$$\|p - i_h^p\,p\|_{L^q(T)} \le c\,h^{1+d(1/q-1/s)}\,\|\nabla p\|_{L^s(\omega_T)} \quad \forall\, p \in W^{1,s}(\omega_T) \tag{6.3}$$

*for all* $T \in \mathcal{T}_h$. *Here,* $\omega_T$ *denotes the union of patches associated to the ansatz funtions that are non-zero on* $T$.

- Inverse property*: For all $v_h \in V_h$,

$$\|v_h\|_{L^\infty(T)^d} \le c\, h^{-\frac{d}{2}} \|v_h\|_{L^2(T)^d} \quad \forall\, T \in \mathcal{T}_h \tag{6.4}$$

*is valid.*

- Discrete inf-sup condition*: There is a real number $\gamma > 0$ such that*

$$\sup_{\phi_h \in V_h} \frac{(p_h, \nabla \cdot \phi_h)}{\|\nabla \phi_h\|} \ge \gamma \|p_h\| \quad \forall\, p_h \in L_h.$$

The conditions in Assumption 6.1 are fulfilled by many standard finite elements, in particular by all examples mentioned in the following. *Beside Assumption 6.1, we suppose Assumptions 2.1–2.3, 3.1, and 3.3 to be satisfied in all what follows. The aim of the subsequent sections is to verify Assumption 3.2.*

**6.1. Smooth domains with $\Omega' = \Omega$.** Before we start the discussion, let us point out that we assume a triangulation that exactly fits the boundary which is fairly artifical in case of a smooth boundary. Moreover, we tacitly supposed that the integrals in (3.1) are exactly evaluated which is clearly hard to implement if $\Omega$ is not polygonally bounded. Therefore, a realistic discretization would cause other types of errors which are neglected here since this would go beyond the scope of this paper. Notice that these problems do clearly not arise if $\Omega$ has a polygonal boundary as in case of the subsequent sections. We apply a result of Chen [10], which requires some additional assumptions on the discretization, in particular a local $L^2$-error estimate of the Ritz-projection, see [10, Section 2] for details. The additional conditions are verified by Arnold and Liu [1] for different types of finite elements such as

- all stable discretizations formed with *Lagrange elements* such as for instance the *Taylor-Hood element* (i.e. $\mathcal{P}_2/\mathcal{P}_1$-element)
- the *Mini element*, i.e., the unstable $\mathcal{P}_1/\mathcal{P}_1$-element enriched with bubble functions.

Using a technique developed in [33], Chen proved the following result:

Theorem 6.2. *Assume that the solution of* (2.1) *satisfies* $(v,p) \in W^{1,\infty}(\Omega)^d \times L^\infty(\Omega)$. *There is a constant $c > 0$, independent of $h$, $v$, and $p$, such that the solution of* (3.1)*, denoted by* $(v_h, p_h) \in V_h \times L_h$*, satisfies*

$$\|v - v_h\|_{L^\infty(\Omega)^d} \le c\, h\, |\log(h)|^m \Big( \inf_{w \in V_h} \|v - w\|_{W^{1,\infty}(\Omega)^d} + \inf_{q \in V_h} \|q - p\|_{L^\infty(\Omega)} \Big),$$

*where $m = 0$ if $k > 1$ and $m = 1$ if $k = 1$.*

If $\Omega$ is of class $C^2$, then $G\colon L^p(\Omega)^d \to W^{2,p}(\Omega)^d \times W^{1,p}(\Omega)$ for all $1 < p < \infty$ (see Temam [34, Proposition 2.3]). Therefore, together with (6.2) and (6.3), Chen's result yields

Corollary 6.3. *For every $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$, independent of $h$ and $u$, so that*

$$\|v - v_h\|_{L^\infty(\Omega)^d} \le c_\varepsilon\, h^{2-\varepsilon} \|u\|_{L^\infty(\Omega)^d}.$$

Theorem 6.4. *For every $\varepsilon > 0$, there holds*

$$\|\bar u - \bar u_h\|_{L^2(\Omega)^d} + \|\bar v - \bar v_h\|_{H^1(\Omega)^d} + \|\bar p - \bar p_h\|_{L^2(\Omega)} \le C\, h^{2-\frac{d}{2}-\varepsilon} \tag{6.5}$$

*with a constant $C > 0$ which depends on $\varepsilon$, but not on $h$.*

*Proof.* Let $\varepsilon > 0$ be given. In view of Corollary 6.3, Assumption 3.2 is fulfilled with a constant $c$ depending on $\varepsilon$ and $\delta(h) = h^{2-2\varepsilon}$. Moreover, by choosing $\sigma = \max\left\{\bar\sigma, \frac{d}{d-1+\varepsilon}\right\}$, we obtain $\eta(\sigma, h) \le h^{4-d-2\varepsilon}$ (cf. Definition 5.1). Thus, Theorem 5.6 and Corollary 5.7 together with standard finite element results give the assertion. ∎

Remark 6.5. *Notice that $C$ depends on $\varepsilon$ firstly because of the constant $c_\varepsilon$ from Corollary 6.3 and secondly due to the coupling of $\sigma$ and $\varepsilon$.*

Remark 6.6. *As above, let $\sigma = \sigma(\varepsilon) = \max\left\{\bar\sigma, \frac{d}{d-1+\varepsilon}\right\}$ with a fixed, but arbitrary $\varepsilon > 0$. Then Lemma 4.4 implies*

$$\|u - \Pi_h u\|_{L^2(\Omega)^d} \le c\, h^{2-\frac{d}{2}-\varepsilon} \|u\|_{W_{\sigma(\varepsilon)}} \quad \forall\, u \in W_{\sigma(\varepsilon)} \tag{6.6}$$

*and therefore, the order in* (6.5) *coincides with the one of the interpolation error.*

**6.2. Convex domains with polygonal or polyhedral boundary.** First, we consider the case $\Omega' = \Omega$. In case of polygons and polyhedrons, respectively, the following regularity result is known. For the proof, we refer to [12] and [27].

THEOREM 6.7. *Let $\Omega$ be a convex domain with polygonal ($d = 2$) or polyhedral ($d = 3$) boundary. Then, for all $u \in L^2(\Omega)^d$, the unique solution $(v, p) \in V \times L$ of* (2.1) *belongs to $H^2(\Omega)^d \times H^1(\Omega)$.*

Based on this result and standard finite element error estimates, one proves for an arbitrary $u \in L^2(\Omega)$

$$\|v - v_h\|_{L^\infty(\Omega)^d} \le c\, h^{2-\frac{d}{2}}\, \|u\|_{L^2(\Omega)^d},$$

where $v = S\,u$ and $v_h = S_h\,u$ and $c > 0$ only depends on $\Omega$ (see for instance [32, Lemma 3.2]). Therefore, by setting $\delta(h) = h^{2-d/2}$ and $\sigma = \max\{\bar\sigma, 4/3\}$ (notice that $4/3 < d/(d-1)$ for $d = 2, 3$) such that $\eta(\delta, h) \le h^{2-d/2}$, Theorem 5.6 and Corollary 5.7 imply

THEOREM 6.8. *Suppose that $\Omega$ is a convex domain with polygonal ($d = 2$) or polyhedral ($d = 3$) boundary. Then, we have*

$$\|\bar u - \bar u_h\|_{L^2(\Omega)} + \|\bar v - \bar v_h\|_{H^1(\Omega)} + \|\bar p - \bar p_h\|_{L^2(\Omega)} \le C\, h^{1-\frac{d}{4}}$$

*with a constant $C > 0$ independent of $h$.*

Notice that the order of convergence now differs from the one of the interpolation error. The situation changes if we restrict to two dimensional domains with polygonal boundary and a maximum angle of less or equal $\pi/2$. To see this, let us define the weighted $L^2$-norm as follows:

$$\|q\|^2_{\varsigma^\nu} := \int_\Omega |q(x)|^2\, \varsigma(x)^\nu\, dx, \quad q \in L^2(\Omega)^d, \tag{6.7}$$

where $\varsigma : \bar\Omega \to \mathbb{R}_+$ is defined by

$$\varsigma(x) := \sqrt{|x - x_0|^2 + \theta^2}, \tag{6.8}$$

with given $x_0 \in \Omega$ and $\theta > h > 0$.

THEOREM 6.9. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon whose maximum aperture angle is less or equal $\pi/2$. Moreover, suppose that $(V_h, L_h)$ satisfies the* discrete weighted inf-sup condition, *i.e. there is a constant $c > 0$ independent of $h$ such that, for every $\theta > 0$,*

$$\sup_{\phi_h \in V_h} \frac{(p_h, \nabla\cdot\phi_h)}{\|\nabla\phi_h\|_{\varsigma^2}} \ge c\,|\log\theta|^{-1/2}\|p_h\|_{\varsigma^{-2}} \quad \forall\, p_h \in L_h. \tag{6.9}$$

*Then, for every $\varepsilon > 0$, the discrete solution satisfies*

$$\|\bar u - \bar u_h\|_{L^2(\Omega)^2} + \|\bar v - \bar v_h\|_{H^1(\Omega)^2} + \|\bar p - \bar p_h\|_{L^2(\Omega)} \le C\, h^{1-\varepsilon}$$

*with a constant $C > 0$ which depends on $\varepsilon$, but not on $h$.*

*Proof.* According to a result of Mazya et al. [28, Section 5.8.1], for all $q \in [1, \infty[$, the solution $v \in (V \times L)$ of (2.1) belongs to $W^{2,q}(\Omega)^2 \times W^{1,q}(\Omega)$, provided that $u \in L^q(\Omega)^2$, and there holds

$$\|v\|_{W^{2,q}(\Omega)^2} + \|p\|_{W^{1,q}(\Omega)} \le c\,\|u\|_{L^q(\Omega)^2}. \tag{6.10}$$

Moreover, Duran and Nochetto proved in [19] that, for all discretizations fulfilling Assumption 6.1 and (6.9), there exists a constant $c > 0$ independent of $h$ such that

$$\|v - v_h\|_{L^\infty(\Omega)^2} \le c\,h\,|\log(h)|^3 \big(\inf_{w \in V_h} \|v - w\|_{W^{1,\infty}(\Omega)^2} + \inf_{q \in V_h} \|q - p\|_{L^\infty(\Omega)}\big).$$

Hence, together with (6.10), (6.2) and (6.3) give the existence of a constant $c_\varepsilon > 0$, depending on $\varepsilon$, but not on $h$, such that for every $\varepsilon > 0$

$$\|v - v_h\|_{L^\infty(\Omega)^2} \le c_\varepsilon\, h^{2-\varepsilon}\, \|u\|_{L^\infty(\Omega)^2}.$$

Then an argument, analogous to the proof of Theorem 6.4, finally implies the assertion. ∎

REMARK 6.10. *The discrete weighted inf-sup condition* (6.9) *is satisfied by various common stable finite elements, as proven in [19]. We only mention*

- *the Taylor-Hood element on triangles or quadrilaterals (i.e., $\mathcal{P}_2/\mathcal{P}_1$- and $\mathcal{Q}_2/\mathcal{Q}_1$-elements, respectively)*
- *the Mini element*
- *the Crouzeix-Raviart element of different order $k \geq 2$, i.e., the $\mathcal{P}_k/\mathcal{P}_{k-1}$-element enriched with bubble functions.*

If the state constraints are only imposed in the interior of $\Omega$, the results of [19] allow to get same the order of convergence as in the interpolation error (6.6), even if the maximum angle is larger than $\pi/2$. Notice that, in the presence of no-slip boundary conditions, it appears natural to consider the state constraints only in the interior of $\Omega$, as illustrated in the introduction.

THEOREM 6.11. *Assume that $\Omega$ is a convex polygon and let $\Omega' \subset \Omega$ be given. Furthermore, we assume that, for every $h$, a union of cells of $\mathcal{T}_h$, denoted by $\Omega''$, exists that contains $\Omega'$ and fulfills* $\operatorname{dist}(\overline{\Omega'}, \overline{\Omega \setminus \Omega''}) =: d > 0$ *and* $\operatorname{dist}(\overline{\Omega''}, \Gamma) =: \delta > 0$ *with $d$ and $\delta$ independent of $h$. Furthermore, suppose that $(V_h, L_h)$ satisfies the* discrete weighted inf-sup condition (6.9). *Then, for every $\varepsilon > 0$, there is a constant $C > 0$ depending on $\varepsilon$, but not on $h$, such that*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^2} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq C\, h^{1-\varepsilon}.$$

*Proof.* The proof is similar to the proof of [19, Theorem 4.1]. In view of Theorem 6.7 and embedding theorems for $d = 2$, we have $\nabla v \in L^q(\Omega)$ for all $q < \infty$. Thus, Theorem 4.1 in [21] yields for every $q \in [1, \infty[$ that $(v, p) \in W^{2,q}_{loc}(\Omega)^2 \times W^{1,q}_{loc}(\Omega)$ if $u \in L^q_{loc}(\Omega)^2$ which is clearly fulfilled due to the control constraints. Thus we obtain $(v, p) \in W^{2,q}(\Omega'')^2 \times W^{1,q}(\Omega'')$ for all $q < \infty$. Based on (6.9), it is shown in [19] that

$$\|v - v_h\|^2_{\varsigma^{-4}} \leq c\, \frac{h^2}{\theta^2}\, |\log\theta|^3 \Big(\|\nabla(v - i^v_h v)\|^2_{\varsigma^{-2}} + \|v - i^v_h v\|^2_{\varsigma^{-4}} + \|p - i^p_h p\|^2_{\varsigma^{-2}}\Big) \tag{6.11}$$

holds for all $\theta > h > 0$ provided that $\Omega$ is a convex polygon. Here, $\varsigma$ and the associated norms are defined as in (6.8) and (6.7). Recall that $V_\infty = L^\infty(\Omega')^2$. We start by estimating

$$\|v - v_h\|_{V_\infty} \leq \|v - i^v_h v\|_{V_\infty} + \|v_h - i^v_h v\|_{V_\infty}.$$

Since $|v_h - i^v_h v| \in C(\bar{\Omega}')$, there is an $x_0 \in \bar{T}_0 \subseteq \overline{\Omega'}$ such that $\|v_h - i^v_h v\|_{V^\infty} = |v_h(x_0) - i^v_h v(x_0)|$. In all what follows, we use this $x_0$ in the definition of $\varsigma$ in (6.8). The inverse estimate (6.4) implies

$$\begin{aligned}|v_h(x_0) - i^v_h v(x_0)| &\leq \|v_h - i^v_h v\|_{L^\infty(T_0)^2}\\ &\leq c\, h^{-1}\, \|v_h - i^v_h v\|_{L^2(T_0)^2} \leq c\, \frac{\theta^2}{h}\, \|v_h - i^v_h v\|_{\varsigma^{-4}},\end{aligned}$$

where the last estimate follows from the definition of $\|\cdot\|_{\varsigma^{-4}}$ because of $\theta > h$. Now, one can apply (6.11) and continue with

$$\begin{aligned}\|v - v_h\|_{V_\infty} \leq\ &\|v - i^v_h v\|_{V_\infty} + c\,\theta|\log\theta|^{\frac{3}{2}} \Big(\|\nabla(v - i^v_h v)\|_{\varsigma^{-2}} + \|p - i^p_h p\|_{\varsigma^{-2}}\Big)\\ &+ c\Big(\frac{\theta^2}{h} + \theta|\log\theta|^{3/2}\Big)\|v - i^v_h v\|_{\varsigma^{-4}}\end{aligned}$$

For an arbitrary $w \in L^\infty(\Omega)$ and $\nu \geq 0$, we obtain

$$\begin{aligned}\|w\|_{\varsigma^{-(2+\nu)}} &\leq \|w\,\varsigma^{-(1+\nu/2)}\|_{L^2(\Omega'')} + \|w\,\varsigma^{-(1+\nu/2)}\|_{L^2(\Omega\setminus\Omega'')}\\ &\leq \|w\|_{L^\infty(\Omega')} \int_{\Omega''} \varsigma^{-(2+\nu)}\, dx^{\frac{1}{2}} + c\,\|w\|_{L^2(\Omega)},\end{aligned}$$

where we used the norm equivalence of $\|\cdot\|_{\varsigma^{-(2+\nu)}}$ and $\|\cdot\|_{L^2}$ on $\Omega\backslash\Omega''$ which holds due to $\operatorname{dist}(x_0, \overline{\Omega \setminus \Omega''}) \geq d > 0$. Together with

$$\int_{\Omega''} \varsigma^{-(2+\nu)}\, dx \leq \begin{cases} c\,\theta^{-\nu}, & \nu > 0 \\ c\,|\log\theta|, & \nu = 0 \end{cases}$$

(see [19]), it follows with $\nu = 0$ and $\nu = 2$, respectively, that

$$\begin{aligned} \|v - v_h\|_{V_\infty} \leq\; & \|v - i_h^v v\|_{L^\infty(\Omega'')^2} \\ & + c\,\theta|\log\theta|^2 \Big( \|\nabla(v - i_h^v v)\|_{L^\infty(\Omega'')^2} + \|\nabla(v - i_h^v v)\|_{L^2(\Omega)^2} \\ & \qquad + \|p - i_h^p p\|_{L^\infty(\Omega'')} + \|p - i_h^p p\|_{L^2(\Omega)} \Big) \\ & + c\Big(\frac{\theta}{h} + |\log\theta|^{3/2}\Big)\Big( \|v - i_h^v v\|_{L^\infty(\Omega'')^2} + \|v - i_h^v v\|_{L^2(\Omega)^2} \Big). \end{aligned}$$

Because of the regularity of $(v, p)$ stated at the beginning of the proof, choosing $\theta = h|\log h| > h$ and applying (6.2) and (6.3) yields the existence of a constant $c_\varepsilon > 0$, depending on $\varepsilon$, such that

$$\|v - v_h\|_{V_\infty} \leq c_\varepsilon\, h^{2-\varepsilon}\, \|u\|_{L^\infty(\Omega)^2} \quad \forall\, \varepsilon > 0.$$

Notice that the assumption $\operatorname{dist}(\overline{\Omega''}, \Gamma) =: \delta > 0$ implies $\operatorname{dist}(\omega_T, \Gamma) > 0$ for all $T \in \mathcal{T}_h \subset \Omega''$ if $h$ is sufficiently small. Hence, the above regularity result implies

$$p \in W^{1,q}\Bigg( \bigcup_{T\subset\Omega''} \omega_T \Bigg) \qquad \forall\, q < \infty$$

such that (6.3) applies to $\|p - i_h^p p\|_{L^\infty(\Omega'')}$. For the rest of the proof, we argue as in the proof of Theorem 6.4, which gives the assertion. ∎

**6.3. Discretization of the data.** Up to now, problem $(\mathrm{P}_h)$ is no finite dimensional optimization problem since we have not discretized the problem data, i.e., the desired state $z$ and the set $K$. To this end, let us introduce the space of linear (bilinear) finite elements $V_h^{(1)} \subset V_h$ and a nodewise interpolant $i_h^{(1)} : C(\bar\Omega)^d \to V_h^{(1)}$. In addition, we introduce a discretization of $K$, denoted by $K_h \subset V_\infty$. The corresponding completely discrete problem for

$$u_h = \sum_{i=1}^{n} u_i \phi_i,$$

for simplicity also denoted by $(\mathrm{P}_h)$, is then given with

$$(\mathrm{P}_h) \qquad \begin{cases} \min & J_h(v_h, u_h) := \dfrac{1}{2}\, \|v_h - i_h^{(1)} z\|^2_{L^2(\Omega)^d} + \dfrac{\alpha}{2}\, \|u_h\|^2_{L^2(\Omega)^d} \\ \text{s.t.} & v_h = S_h\, u_h \\ \text{and} & i_h^{(1)} v_h \in K_h \\ & u_h \in U_h,\ a \leq u_i \leq b \quad \forall\, i \in \{1, ..., n\}. \end{cases}$$

Remark 6.12. *Notice that it depends on the concrete structure of $K$ and its discretization whether $(\mathrm{P}_h)$ represents a finite dimensional optimization problem or not. In the cases, discussed in this paper, the linear (bilinear) interpolation operator $i_h^{(1)}$ allows to formulate $(\mathrm{P}_h)$ as a finite dimensional problem, which can be solved numerically (see below).*

To shorten the description, we assume in all what follows that Assumption 3.2 is fulfilled with $\delta(h) = c\,h^{2-\varepsilon}$ with a fixed but arbitrary $\varepsilon > 0$ (see Sections 6.1 and 6.2). If this is not fulfilled, the subsequent analysis can easily be modified.

Assumption 6.13. *Beside Assumptions 2.1–2.3 and 3.1–3.3, assume that* $z \in H^2(\Omega)^d$. *Furthermore, let Assumption 3.2 hold with*

$$\delta(h) = c\, h^{2-\varepsilon} \tag{6.12}$$

*with some fixed but arbitrary* $\varepsilon > 0$ *and assume that* $S\colon L^\infty(\Omega)^d \to W^{2,q}(\Omega'')^d$ *for all* $q < \infty$, *where* $\Omega''$ *is a union of cells containing* $\Omega'$. *Moreover, suppose that* $K_h$ *is convex with associated support functional* $s_h : V_\infty^* \to \mathbb{R}$ *that fulfills*

$$|s(\mu) - s_h(\mu)| \leq c_s\, h^{2-\varepsilon}\, \|\mu\|_{V_\infty^*} \quad \forall\, \mu \in V_\infty^* \tag{6.13}$$

*with a constant* $c_s > 0$. *To guarantee the existence of a solution to* $(\mathrm{P}_h)$, *we require the existence of a feasible point, i.e., there is a* $\hat{u} \in U_h$ *with* $a \leq \hat{u}_i \leq b\, \forall\, i \in \{1, ..., n\}$ *and* $i_h^{(1)}\, S_h\, \hat{u}_h \in K_h$.

Remark 6.14. *Notice that the hypothesis on* $S$ *and* $\delta(h)$ *agree with the theory presented in Sections 6.1 and 6.2 (cf. in particular Corollary 6.3 and the proofs of Theorem 6.9 and 6.11).*

Lemma 6.15. *Suppose that Assumption 6.13 holds. Let* $u \in L^\infty(\Omega)^d$ *be arbitrary and set as before* $v_h = S_h\, u$. *Then, for every* $\varepsilon > 0$, *there is a constant* $c > 0$, *independent of* $u$ *and* $h$, *such that*

$$\|v_h - i_h^{(1)} v_h\|_{V^\infty} \leq c\, h^{2-\varepsilon}\, \|u\|_{L^\infty(\Omega)^d}.$$

*Proof.* The arguments are standard. For convenience of the reader, we sketch the proof for a single component of $v_h$, for simplicity also denoted by $v_h$. Let $\varepsilon > 0$ be arbitrary. We start by estimating

$$\|v_h - i_h^{(1)} v_h\|_{V^\infty} \leq \|i_h^{(1)}(v - v_h)\|_{V^\infty} + \|v - i_h^{(1)} v\|_{V^\infty} + \|v - v_h\|_{V^\infty}$$

with $v = S\, u$. Similarly to Lemma 4.6, one proves

$$\|i_h^{(1)}(v - v_h)\|_{V^\infty} \leq \|v - v_h\|_{L^\infty(\Omega'')}.$$

Moreover, the standard linear (bilinear) interpolation operator satisfies

$$\|v - i_h^{(1)} v\|_{L^\infty(\Omega'')} \leq c\, h^{2-d/q} \|\nabla^2 v\|_{L^q(\Omega'')} \quad \forall\, q < \infty$$

(cf. [4] or [2]). Thus, by choosing $q = d/\varepsilon < \infty$, the mapping properties of $S$ together with Assumption 3.2 and (6.12), i.e.

$$\|v - v_h\|_{V^\infty} \leq c\, h^{2-\varepsilon} \|u\|_{L^\infty(\Omega)},$$

gives the assertion. ∎

Theorem 6.16. *Assume that Assumption 6.13 is fulfilled. Then, for every* $\varepsilon > 0$, *the unique solution of* $(\mathrm{P}_h)$ *satisfies*

$$\|\bar{u} - \bar{u}_h\| + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^d} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)^d} \leq C\, h^{2-\frac{d}{2}-\varepsilon}$$

*where the constant* $C > 0$ *depends on* $\varepsilon$ *but not on* $h$.

*Proof.* Since $z$ is sufficiently smooth by assumption, we have $\|z - i_h^{(1)} z\|_{L^2(\Omega)^d} \leq c\, h^2\, \|z\|_{H^2(\Omega)^d}$ due to standard interpolation estimates. In view of this, the discretization of $z$ can easily incorporated in the presented analysis. The underlying arguments are presented in detail in [30, Section 7]. In addition, due to Assumption 3.3, it is sufficient to require the control constraints only in the nodes as done in $(\mathrm{P}_h)$. If $K$ is discretized, then the proofs of Lemma 5.2 and 5.5 have to be modified, more precisely (5.3) and (5.4), respectively. We exemplarily consider (5.4), the arguments in case of (5.3) are similar. Using (6.13) and Lemma 6.15, we obtain for all $\tilde{\mu}$ with $\|\tilde{\mu}\|_{V_\infty^*} = 1$

$$\begin{aligned}
\langle \tilde{\mu}\,,\, i_h^{(1)}\, S_h\, u_d \rangle_{V_\infty^*, V_\infty} &\leq \langle \tilde{\mu}\,,\, S_h\, u_d \rangle_{V_\infty^*, V_\infty} + \|S_h\, u_d - i_h^{(1)}\, S_h\, u_d\|_{V_\infty} \\
&\leq s(\tilde{\mu}) - c\, h^{2-\frac{d}{2}-\varepsilon}(\gamma_d \tau - c_u) + c\, h^{2-\varepsilon}\, \|u_d\|_{L^\infty(\Omega)^d} \\
&\leq s_h(\tilde{\mu}) - c\, h^{2-\frac{d}{2}-\varepsilon}(\gamma_d \tau - c_u - c_s),
\end{aligned}$$

where $c_u$ is defined as in (5.4). Hence, if we choose $\gamma_d > (c_u + c_s)/\tau$, then the same arguments as in the proof of Lemma 5.5 imply that $u_d$ is feasible for $(\mathrm{P}_h)$. Again $\gamma_d$ depends on $\bar{u}$ and $\sigma$, but not on $h$. Based on the feasibility of $u_c$ and $u_d$, one can argue as in the proof of Theorem 5.6 to verify the assertion. ☐

Let us investigate two exemplary state constraints that are also used for the numerical tests in Section 7:

$$K^{(1)} := \{v \in V_\infty | v_a(x) \le v(x) \le v_b(x) \text{ a.e. in } \Omega'\}$$
$$K^{(2)} := \{v \in V_\infty || v(x)|^2_{\mathbb{R}^d} \le \varrho \text{ a.e. in } \Omega'\}.$$

First, we consider $K^{(1)}$, i.e., the cases of box constraints. Let us assume that $\Omega'$ coincides with a union of cells of $\mathcal{T}_h$ and denote the set of all nodes of $\mathcal{T}_h$ by $\mathcal{N}(\mathcal{T}_h)$. We consider the following finite dimensional optimization problem

$$(\mathrm{P}_h^{(1)}) \quad \begin{cases} \min\limits_{u_h \in U_h} & J_h(v_h, u_h) \\ \quad \text{s.t.} & v_h = S_h\, u_h \\ \quad \text{and} & v_{a,h}(x_i) \le v_h(x_i) \le v_{b,h}(x_i) \quad \forall\, x_i \in \mathcal{N}(\mathcal{T}_h) \cap \overline{\Omega'} \\ & a \le u_i \le b \quad \forall\, i \in \{1, ..., n\}, \end{cases}$$

with $v_{b,h} = i_h^{(1)}\, v_b$ and $v_{a,h}$ defined analogously.

Corollary 6.17. *Suppose that $\Omega$ is a convex polygon and let $\Omega' \subset \Omega$ be a union of cells of $\mathcal{T}_h$ for all $h > 0$. Assume in addition that $\Omega'$ fulfills the assumptions of Theorem 6.11. Furthermore, suppose that $z \in H^2(\Omega)^d$ and $v_a, v_b \in W^{2,\infty}(\Omega')^d$. Then the solution of $(\mathrm{P}_h^{(1)})$ satisfies for every $\varepsilon > 0$*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^2} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le C\, h^{1-\varepsilon},$$

*where the constant $C > 0$ depends on $\varepsilon$, but not on $h$.*

*Proof.* We apply Theorem 6.16. Thus, we have to verify (6.13). To shorten the demonstration, we just consider the upper bound $v_b$. The case with lower constraint can be discussed analogously. Let $\varphi_i$, $i = 1, ..., m$, denote the ansatz functions associated to the linear (bilinear) interpolant $i_h^{(1)}$. Since they are non-negative and satisfy $\varphi_i(x_j) = \delta_{ij}$, the state constraints in $(\mathrm{P}_h^{(1)})$ are equivalent to $(i_h^{(1)}\, v_h)(x) \le v_{b,h}(x)$ a.e. in $\Omega'$. Thus, $K_h^{(1)}$ is given by

$$K_h^{(1)} := \{v \in V_\infty \,|\, v(x) \le v_{b,h}(x) \text{ a.e. in } \Omega'\}.$$

Given an arbitrary $v \in K^{(1)}$, we define $\Pi_{K_h^{(1)}}(v)(x) := \min\{v(x), v_{b,h}(x)\}$, hence $\|v - \Pi_{K_h^{(1)}}(v)\|_{V_\infty} \le \|v_b - v_{b,h}\|_{V_\infty}$. Therefore we have for every $\mu \in V_\infty^*$

$$\langle \mu\,,\, v\rangle_{V_\infty^*, V_\infty} \le \langle \mu\,,\, \Pi_{K_h^{(1)}}(v)\rangle_{V_\infty^*, V_\infty} + \|\mu\|_{V_\infty^*}\, \|v_b - v_{b,h}\|_{V_\infty} \quad \forall\, v \in K^{(1)}.$$

Since $\Pi_{K_h^{(1)}}(v) \in K_h^{(1)}$, this gives

$$s(\mu) \le s_h(\mu) + \|\mu\|_{V_\infty^*}\, \|v_b - v_{b,h}\|_{V_\infty} \tag{6.14}$$

An analogous argument with $\Pi_{K^{(1)}}(v)(x) := \min\{v(x), v_b(x)\}$, $v \in K_h^{(1)}$, implies

$$s_h(\mu) \le s(\mu) + \|\mu\|_{V_\infty^*}\, \|v_b - v_{b,h}\|_{V_\infty}. \tag{6.15}$$

Together with (6.14), this verifies (6.13) provided that $v_b$ is sufficiently smooth, for instance $v_b \in W^{2,\infty}(\Omega')^d$. The remaining conditions in Assumption 6.13, in particular (6.12), are verified by the proof of Theorem 6.11 which gives the assertion. ☐

Now, let us turn to $K^{(2)}$, i.e., constraints on the Euclidian norm of $v$. For this case we set $K_h^{(2)} = K^{(2)}$. The completely discrete problem is now given by

$$(\mathrm{P}_h^{(2)}) \quad \begin{cases} \min\limits_{u_h \in U_h} & J_h(v_h, u_h) \\ \quad \text{s.t.} & v_h = S_h\, u_h \\ \quad \text{and} & |v_h(x_i)|^2_{\mathbb{R}^2} \le \varrho \quad \forall\, x_i \in \mathcal{N}(\mathcal{T}_h) \cap \overline{\Omega'} \\ & a \le u_i \le b \quad \forall\, i \in \{1, ..., n\}. \end{cases}$$

COROLLARY 6.18. *Suppose that $\Omega$ is a convex polygon and $\Omega' \subset \Omega$ fulfills the assumptions of Corollary 6.17. Furthermore, assume that $z \in H^2(\Omega)^d$. Then, the solution of $(\mathrm{P}_h^{(2)})$ satisfies for every $\varepsilon > 0$*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^2} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq C\, h^{1-\varepsilon},$$

*where the constant $C > 0$ depends on $\varepsilon$, but not on $h$.*

*Proof.* Similar arguments as in the proof of Corollary 6.17 together with the convexity of $|\cdot|_{\mathbb{R}^2}^2$ imply that the state constraints in $(\mathrm{P}_h^{(2)})$ are equivalent to $|(i_h^{(1)} v_h)(x)|_{\mathbb{R}^2}^2 \leq \varrho$ a.e. in $\Omega'$. Thus, Theorem 6.16 and the same arguments as in the proof of Theorem 6.11 give the assertion. ∎

**7. Numerical experiments.** In this section we perform numerical tests in order to verify the finite element error estimates obtained in the previous sections. The convex polygonal domain $\Omega = (0,1)\times(0,1)$ was discretized using a uniform triangular mesh. Boundary conditions of Dirichlet type were imposed on the boundary. On the upper boundary the horizontal velocity takes the value one, while the vertical component is zero. On the remaining boundary the condition is of no slip type. This problem is known in the literature as the "driven cavity flow". It is easy to see that the non-homogeneous Dirichlet boundary conditions do not influence the above theory since the solution of the Stokes equation can be seen as a superposition of a fixed contribution caused by the inhomogeneity on the boundary and a variable part associated to the control to which the presented analysis applies. For the finite element discretization, we use Taylor-Hood elements with quadratic ansatz functions for the velocity and linear functions for the pressure. The controls were also discretized using piecewise linear polynomials consistent with the conditions in Assumption 3.3. The discretized inequality constrained optimization problems are solved by applying a semi-smooth Newton method as stated in [26]. The inequality state constraints are added to the cost functional through a penalized Moreau-Yosida regularization term, see, e.g., [18]. For the solution of the discretized systems appearing in each semi-smooth Newton step a penalty method is applied (cf. [23, p. 125]). This method considers, for $0 < \epsilon << 1$, the modified Stokes system

$$\begin{pmatrix} A & B^T \\ B & \epsilon I \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} M\vec{u} \\ 0 \end{pmatrix},$$

where $A$, $B$, and $M$ are the matrices resulting from the finite element discretization of (2.1), $I$ is the identity matrix, and $\vec{v}, \vec{p}$, and $\vec{u}$ are the vectors for the velocity, pressure, and control, respectively. A similar penalty scheme was used for the adjoint equations. For convergence results on this approach we refer to [23].

The semi-smooth Newton algorithm stops if the $L^2$-residuum of the discretized control is lower than a given tolerance, typically set as $10^{-4}$. The method is initialized setting the controls equal to 0 and solving successively the Stokes and the adjoint equations. With this values at hand, the active and inactive sets are determined for the first iteration.

The resulting linear systems in each semismooth Newton iteration were solved using MATLAB exact solver. All algorithms were implemented in MATLAB 7.4 and run on a 300 GHz machine with 24 GByte RAM and a precision of eps=2.2204e-16.

**7.1. Example 1: box constraints.** First, we consider simple box constraints on the state, i.e., constraints of the form $K^{(1)}$. To be more precise, the state constraint is given by $y_1 \geq -0.15$ in $\Omega_s = [0.1, 0.9] \times [0.1, 0.9]$. The target is to diminish the backward flow velocity and, as a consequence, the intensity of the vortex. The desired state is given by $z_d \equiv 0$. Thus, the example fits to the setting of Corollary 6.17. The Tikhonov regularization parameter is set to $\alpha = 0.1$, while we choose $10^5$ as penalization parameter for the state constraints.

With a mesh size $h = \sqrt{2}/32$ the algorithm stops after 20 iterations. The horizontal and vertical components of the optimal control are depicted in Figure 7.1, for $h = \sqrt{2}/64$. In Figure 7.2 the active set for the horizontal velocity component is depicted. From the graphics, the concentration of the irregular part of the horizontal control on the active set can be observed.

In Table 7.1 the convergence history is registered. The experimental error norms for different values of $h$ are tabulated. We consider as optimal solution the one obtained numerically with a mesh step size $h = \sqrt{2}/160$. The quantity #it refers to the number of semi-smooth Newton iterations. We observe that

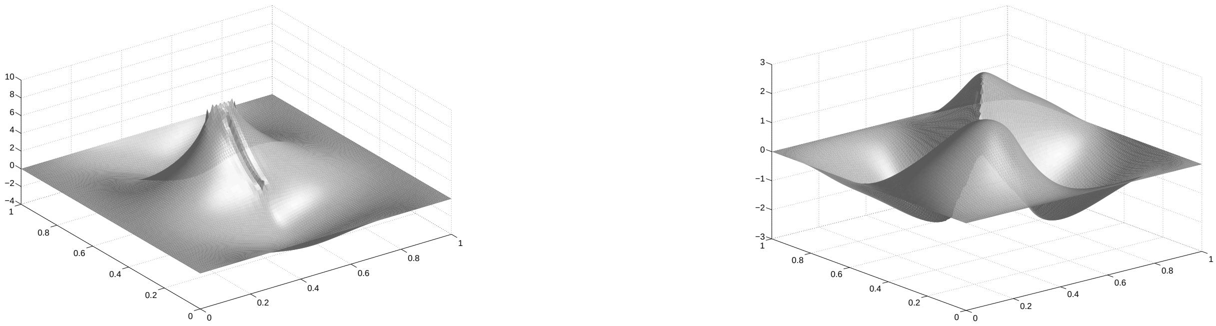

Fig. 7.1. *Example 1: horizontal and vertical components of the optimal control;* $h = \sqrt{2}/64$.

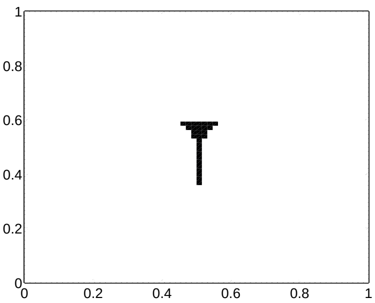


Fig. 7.2. *Example 1: active set for the horizontal component of the velocity;* $h = \sqrt{2}/32$.

Table 7.1
*Example 1, convergence history.*

| $\sqrt{2}/h_2$ | 5 | 10 | 20 | 40 | 80 |
|---|---|---|---|---|---|
| #it | 4 | 8 | 20 | 20 | 32 |
| $\lVert u_h - u^* \rVert_{\mathbf{L}^2_h}$ | 1.1601 | 0.7982 | 0.4804 | 0.2572 | 0.1098 |

the algorithm does not appear to be mesh-independent. To illustrate the convergence behavior, we define the quantity

$$EOC_2(u) := \frac{\log(\|u_{h_1} - u^*\|_{L^2}) - \log(\|u_{h_2} - u^*\|_{L^2})}{\log(h_1) - \log(h_2)} \tag{7.1}$$

as the experimental order of convergence for the $L^2$-norm of $u$. Here, $h_1$ and $h_2$ denote two consecutive mesh sizes. In Table 7.2, $EOC_2(u)$ is evaluated for the current box constrained case. From Table

Table 7.2
*Example 1, experimental order of convergence.*

| $\sqrt{2}/h_2$ | 8 | 16 | 32 | 64 |
|---|---|---|---|---|
| $EOC_2(u)$ | 0.5394 | 0.7325 | 0.9013 | 1.2280 |

7.2, the coincidence between the theoretical and experimental convergence order can be inferred, since the experimental order of convergence order averages $1 - \varepsilon$. This observation confirms the theoretical predictions of Corollary 6.17.

**7.2. Example 2: constraint on the Euclidian norm of the velocity vector.** In this example, we consider the state constraint $v_1^2(x) + v_2^2(x) \leq 10^{-4}$ in the center of the driven cavity. With this

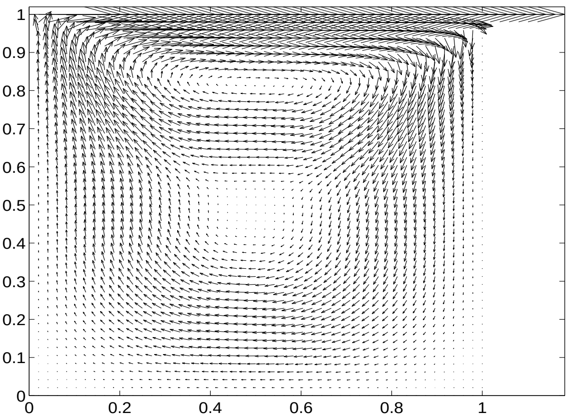


FIG. 7.3. *Example 2: velocity vector field;* $h = \sqrt{2}/24$.

constraint the norm of the velocity vector field is restricted pointwise in the domain $\Omega_s = [\frac{7}{16}, \frac{9}{16}]^2$. Hence, the example is covered by the setting of Corollary 6.18. The Tikhonov parameter is set to $\alpha = 0.1$, while we used $10^5$ for the penalization of the state constraints. The desired state is again given by $z_d \equiv 0$. The resulting velocity vector field is shown in Figure 7.3. The obstacle effect of the state constraint can be observed in the plot. The evolution of the finite element error and of the convergence rate as $h \to 0$ is registered in Table 7.3. In average, the order $1 - \epsilon$ for the $L^2$-norms of control can be observed also in this example. Thus, the theoretical error estimate of Corollary 6.18 can be seen to be experimentally verified.

TABLE 7.3
*Example 2, convergence history.*

| $\sqrt{2}/h_2$ | 5 | 10 | 20 | 40 | 80 |
|---|---|---|---|---|---|
| #it | 4 | 8 | 20 | 20 | 32 |
| $\|u_h - u^*\|_{\mathbf{L}^2_h}$ | 5.4043 | 3.3571 | 1.6865 | 1.1680 | 0.5171 |
| $EOC_2(u)$ | – | 0.6868 | 0.9931 | 0.5299 | 1.1755 |

REFERENCES

[1] D.N. ARNOLD, X. LIU, *Local error estimates for finite element discretizations of the Stokes equations*, M2AN, 29 (1995), pp. 367–389.

[2] C. BERNARDI, *Optimal finite-element interpolation on curved domains*, SIAM J. Numer. Anal., 25 (1989), pp. 1212–1240.

[3] P. BOCHEV AND M. GUNZBURGER, *Least-squares finite-element methods for optimization and control for the Stokes equations*, Comput. Math. Appl., 48 (2004), pp. 1035–1057.

[4] S.C. BRENNER AND L.R. SCOTT, *The Mathematical Theory of Finite Element Methods*, Springer, New York, Berlin, Heidelberg, 1994.

[5] R.M. BROWN, Z. SHEN, *Estimates for the Stokes operator in Lipschitz domains*, Indiana Univ. Math. J., 44 (1995), pp. 1183–1206.

[6] C. CARSTENSEN, *Quasi-interpolation and a posteriori error analysis in finite elements methods*, M2AN, 33 (1999), pp. 1187–1202.

[7] E. CASAS, *Boundary control of semilinear elliptic equations with pointwise state constraints*, SIAM J. Control Optim., 31 (1993), pp. 993–1006.

[8] E. CASAS, *Error estimates for the numerical approximation of semilinear elliptic control problems with finitely many state constraints*, ESAIM: COCV, 8 (2002), pp. 345–374.

[9] E. CASAS AND M. MATEOS, *Uniform convergence of the FEM. Applications to state constrained control problems*, Comp. Appl. Math, 21 (2002).

[10] H. CHEN, *Pointwise error estimates for finite element solutions of the Stokes problem*, SIAM J. Numer. Anal., 44 (2006), pp. 1–28.

[11] P. Clément, *Approximation by finite element functions using local regularization*, RAIRO Anal. Numer., R-2 (1975), pp. 77–84.

[12] M. Dauge, *Stationary Stokes and Navier-Stokes systems on two- and three-dimensional domains with corners. Part I: Linearized equations*, SIAM J. Math. Anal., 20 (1989), pp. 74–97.

[13] K. Deckelnick and M. Hinze, *Semidiscretization and error estimates for distributed control of the instationary Navier-Stokes equations*, Numer. Math., 97 (2004), pp. 297–320.

[14] K. Deckelnick and M. Hinze, *Convergence of a finite element approximation to a state constrained elliptic control problem*, SIAM J. Numer. Anal., 45 (2007), pp. 1937–1953.

[15] K. Deckelnick and M. Hinze, *Finite element approximations to elliptic control problems in the presence of control and state constraints*, Preprint HBAM 2007-01, Hamburger Beiträge zur Angewandten Mathematik, Universität Hamburg (2007),submitted.

[16] K. Deckelnick and M. Hinze, *Numerical analysis of a control and state constrained elliptic control problem with piecewise constant control approximations*, submitted.

[17] K. Deckelnick, A. Günther, and M. Hinze, *Finite element approximation of elliptic control problems with constraints on the gradient*, Priority Program 1253, German Research Foundation, Preprint SPP1253-08-02 (2007), submitted.

[18] J.-C. de los Reyes and K. Kunisch, *A Semi-Smooth Newton Method for Control Constrained Boundary Optimal Control of the Navier-Stokes Equations*, Nonlinear Analysis: Theory, Methods & Applications, 62 (2005), pp. 1289–1316.

[19] R.G. Duran and R.H. Nochetto, *Weighted inf-sup condition and pointwise error estimates for the Stokes problem*, Math. Comp., 54 (1990), pp. 63–79.

[20] R. S. Falk, *Approximation of a class of optimal control problems with order of convergence estimates*, J. Math. Anal. Appl., 44 (1973), pp. 28–47.

[21] G.P. Galdi, *Introduction to the mathematical theory of the Navier-Stokes equations, Vol. 1: Linearized steady problems*, Springer, New York, 1994.

[22] G.P. Galdi, C.G. Simander, and H. Sohr, *On the Stokes problem in Lipschitz domains*, Ann. Mat. pura appl., 167 (1994), pp. 147–163.

[23] M. Gunzburger, *Navier-Stokes equations for incompressible flows: finite-element methods*, in Handbook of Computational Fluid Mechanics, Academic Press, San Diego, pp. 99–158, 2000.

[24] M. Gunzburger, L. Hou, and T. Svobodny, *Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet controls*, Math. Model. Numer. Anal., 25 (1991), pp. 711–748.

[25] M. Gunzburger, L. Hou, and T. Svobodny, *Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with distributed and Neummann controls*, Math. Comp., 57 (1991), pp. 123–151.

[26] M. Hintermuller, K. Ito, and K. Kunisch, *The primal-dual active set strategy as a semismooth Newton method*, SIAM J. Optim., 13 (2002), pp. 865–888.

[27] R. B. Kellog and J. Osborn, *A regularity result for the Stokes problem in a convex polygon*, J. Func. Anal., 21 (1976), pp. 397–431.

[28] V. A. Kozlov, V. G. Maz'ya, and J. Rossmann *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*, AMS, Providence, 2001.

[29] D.G. Luenberger, *Optimization by Vector Space Methods*, Wiley, New York, 1969.

[30] C. Meyer, *Error estimates for the finite-element approximation of an elliptic control problem with pointwise state and control constraints*, submitted.

[31] F. Natterer, *Über punktweise Konvergenz Finiter Elemente*, Numer. Math., 25 (1975), pp. 67–77.

[32] A. Rösch and B. Vexler, *Optimal Control of the Stokes Equations: A Priori Error Analysis for Finite Element Discretization with Postprocessing*, SIAM J. Numer. Anal., 44 (2006), pp. 1903–1920.

[33] A. H. Schatz, *Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids. I: Global estimates*, Math. Comput., 67 (1998), pp. 877–899.

[34] R. Temam, *Navier-Stokes equations : theory and numerical analysis*, North-Holland Publ. Co., Amsterdam, 1977.

[35] J. Zowe and S. Kurcyusz, *Regularity and stability for the mathematical programming problem in Banach spaces*, Appl. Math. Optim., 5 (1979), pp. 49–62.