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The Existence of Triangulations of Non-convex Polyhedra without New Vertices

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Abstract

It is well known that a simple three-dimensional non-convex polyhedron may not be triangulated without using new vertices (so-called *Steiner points*). In this paper, we prove a condition that guarantees the existence of a triangulation of a non-convex polyhedron (of any dimension) without Steiner points. We briefly discuss algorithms for efficiently triangulating three-dimensional polyhedra.

1 Introduction

Decomposing a geometric object into simpler parts is one of the most fundamental problems in computational geometry. It is well solved in two dimensions. For instances, it is known that every polygon can be triangulated without adding new vertices, so-called *Steiner points*, and Chazelle showed that a simple polygon can be triangulated in linear time [4]. However, this problem is known has many difficulties in three-dimensional space.

Without using Steiner points, Lennes [10] presented in 1911 the first simple three-dimensional non-convex polyhedron whose interior cannot be triangulated. The most famous example was given in 1927 by Schönhardt [14], which is a simple polyhedron with 6 vertices (see Fig. 1 left). Later on, Bagemihl [1] and Rambau [12] extended Schönhardt's example by showing that there exists an n-vertex simple polyhedron which can not be triangulated. Ruppert and Seidel [13] proved that the problem of determining whether or not a non-convex polyhedron can be triangulated is NP-complete.

If Steiner points are allowed, Chazelle [3] showed that any simple polyhedron of n vertices may need $O(n^2)$ Steiner points, and this bound is tight in the worst case (see Fig. 1 right). Chazelle and Palios [5] presented an algorithm to decompose a simple polyhedron P using $O(n+r^2)$ Steiner points, where r is the number of reflex edges (a quantitative measure of nonconvexity) of P. However, even for a simply shaped polyhedron, this algorithm will introduce an unnecessarily large number of Steiner points, see Fig. 2 (b). More practical approaches using conforming Delaunay triangulations [11, 6] and constrained Delaunay triangulations [16, 18] are proposed, see Fig. 2 (c) and (d). However, no polynomial upper bound on the number of Steiner

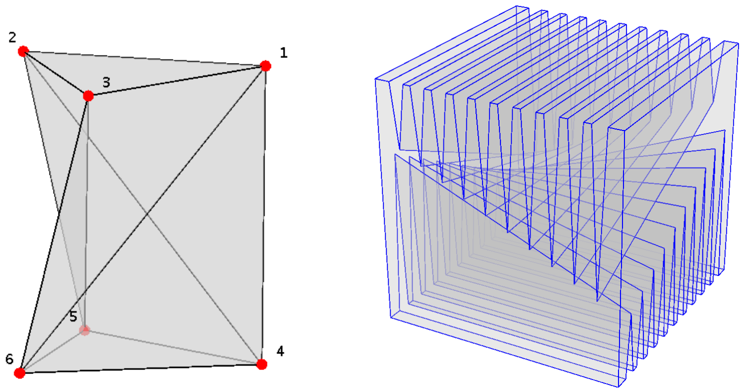


Figure 1: Two polyhedra which are not tetrahedralizable without using Steiner points. Left: The Schönhardt polyhedron [14] can be obtained by twisting the upper face around the axes of a parallel triangular prism by a small angle. Right: The Chazelle's polyhedron [3] which is formed by cutting wedges from a cube. In the middle of the polyhedron are two sets of orthogonal lines. The lower and upper lines lie on hyperbolic paraboloids $z = xy$, and $z = xy + \epsilon$, respectively.

points is known, see the 22th open problem in [8].

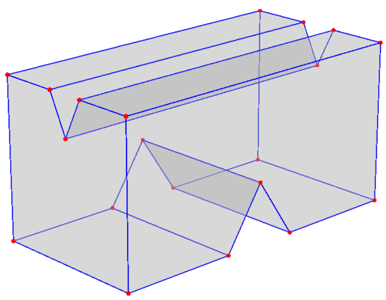
In this paper, we consider the problem of triangulating a non-convex polyhedron without using Steiner points. A right answer to this question is meaningful. For instances, it helps to decide what is the optimal number of Steiner points, and it helps to design efficient algorithms.

Previous work already showed that there are special classes of non-convex polyhedra (and a collection of polyhedra) that can be triangulated without Steiner points. They are summarized below.

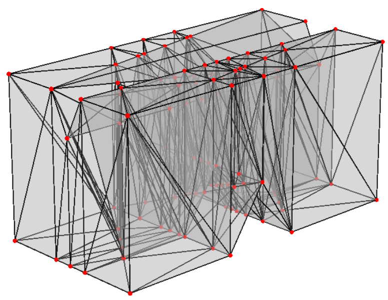
(1) Convex polyhedra can always be triangulated. A simple proof is given by Lennes (Problem 61 in [10]). Moreover, they can be triangulated in linear time [20].

(2) Simple non-convex polyhedra defined by $\mathrm{CH}(P \cup Q) - (P \cup Q)$, where CH denotes convex hull, P, Q are both convex polyhedra and $P \cap Q = \emptyset$, can be triangulated. It is proved by Goodman and Pach [9]. Their algorithm has $O(n^{\lfloor (d+1)/2 \rfloor})$ complexity, where d is the dimension of the polyhedron. This result generalizes to the region between any number of side-by-side convex polyhedra, as long as each P_i can be separated from $\mathrm{CH}(P_1 \cup P_2 \cdots \cup P_{i-1})$ by a hyperplane. Bern [2] showed that the separation condition is necessary.

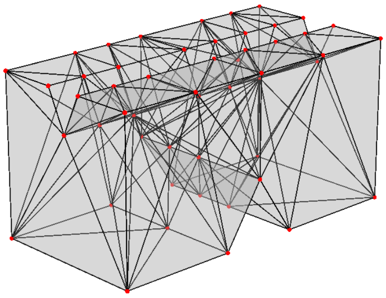
(3) Non-simple polyhedron defined by $P - Q$, where P, Q are both convex polyhedra and $Q \subset P$, can be triangulated. It is also proved by



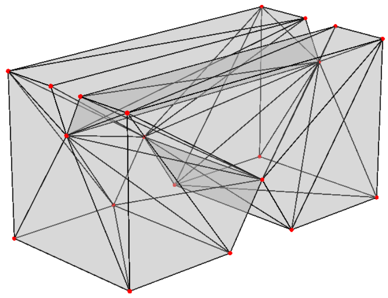
(a) The input polyhedron
20 nodes, 12 facets



(b) Convex decomposition
138 nodes, 280 tets



(c) Conforming Delaunay
51 nodes, 103 tets



(d) Constrained Delaunay
20 nodes, 29 tets

Figure 2: A comparison of meshing polyhedra by different approaches. (a): A simple polyhedron having 2 reflex edges. (b): Convex decomposition. (c): Conforming Delaunay tetrahedralization. (d) Constrained Delaunay tetrahedralization.

Goodman and Pach [9]. Bern [2] showed that it takes $O(n \log n)$ time to triangulate such a three-dimensional polyhedron. Note that this result trivially generalizes to triangulating a nested set of disjoint convex polyhedra.

(4) A *slab* (obtained by the translation of a two-dimensional polygon (possible with holes) by a fixed distance in an arbitrary direction can be tetrahedralized. Toussaint *et al* [19] gave an algorithm that takes $O(n \log n)$ time, where n is the number of vertices of the polygon.

(5) Two classes of three-dimensional rectilinear polyhedra formed by "digging" a set of pairwise non-intersecting rectilinear polygonal holes from the top of a rectangular box, see Fig. 3 for an example. If all the holes have the same depth h, the object is called a *type-1* box, otherwise, it is a *type-2* box. It is proved by Toussaint *et al* [19] that both type-1 and type-2 boxes can be tetrahedralized without using Steiner points with an additional assumption that the holes in type-2 box are linearly ordered.

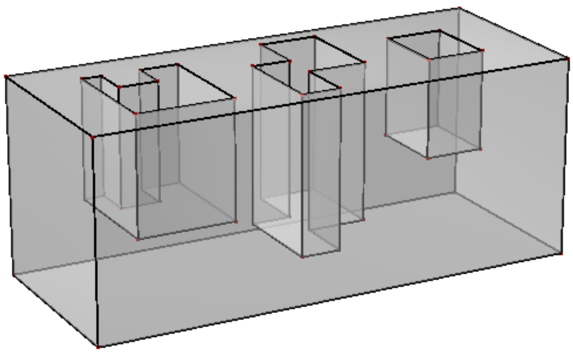


Figure 3: A rectilinear box formed by "digging" a set of pairwise non-intersecting rectilinear polygonal holes from the top of a rectangular box. According to [19], it is a type-2 box (the depths of the holes are vary). It can be tetrahedralized without Steiner points.

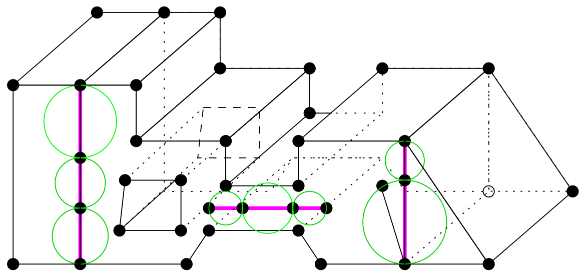


Figure 4: A three-dimensional piecewise linear complex $\mathcal{P}$. All edges of $\mathcal{P}$ are strongly Delaunay. $\mathcal{P}$ can be tetrahedralized without Steiner points.

(6) The union of up to three convex polyhedra can be tetrahedralized. It is proved by Toussaint *et al* [19].

(7) Define a three-dimensional polyhedral complex $\mathcal{P}$ to be a collection of polyhedra such that the intersection of any two elements of $\mathcal{P}$ is either empty or an element in $\mathcal{P}$. Call 0- and 1-dimensional polyhedra of $\mathcal{P}$ vertices and edges, respectively. An edge of $\mathcal{P}$ is *strongly Delaunay* if there is a sphere through its endpoints and all other vertices of $\mathcal{P}$ lie strictly outside that sphere. Shewchuk [15] proved if all edges of $\mathcal{P}$ are strongly Delaunay, then the *underlying space* of $\mathcal{P}$ (the union of all polyhedra in $\mathcal{P}$) can be tetrahedralized without Steiner points (see Fig. 4). This result generalizes to higher dimensions.

The objects from classes *(1)* to *(6)* are all in special cases, whereas Shewchuk's condition gives a more general class of polyhedra which can be triangulated without Steiner points. However, there are simple special cases which are not satisfied by Shewchuk's condition. For example, a prism whose vertices are on a common sphere is tetrahedralizable since it is in both classes *(1)* and *(3)*, while none of its edges is strongly Delaunay. Obviously, more general conditions need to be sought.

In this paper, we will prove a new condition which guarantees the existence of a triangulation of a non-convex polyhedron with no Steiner points. Differing to previous work, we will focus on finding subdivisions of polyhedra with no Steiner points. It can be shown that triangulations are just special cases of subdivisions. A central concept used in this paper is the "regular subdivision", which can be obtained from the projection of higher dimensional convex polytopes by "deleting the last coordinates". Our condition for three-dimensional cases can be described as follows:

Theorem 1.1 *Let P be a three-dimensional polyhedron. Call* 0- *and* 1-*faces of P vertices and edges of P. If the set of edges of P is contained in a regular subdivision of the convex hull of the set of vertices of P, then P can be triangulated without Steiner points.*

Note that all strongly Delaunay edges of P must be contained in the *Delaunay subdivision* $\mathcal{T}$ (the dual of the Voronoi diagram) of the set of vertices of P. $\mathcal{T}$ is a regular subdivision but may not be a triangulation. Our condition includes Shewchuk's condition as a special case. We will prove the above condition for polyhedra of any dimension in Section 5.

The rest of the paper is organized as follows: Section 2 reviews the definitions of (regular) subdivisions of point sets and some useful properties of them. We then define non-convex polyhedra and the (regular) subdivisions of them in Section 3. In Section 4 we prove a useful theorem on inserting an internal facet into a polyhedron. We then prove the new condition in Section 5. Efficient algorithms for triangulating three-dimensional polyhedra are discussed in Section 6.

2 Regular Subdivisions

In this section, we review the definition of regular subdivisions of point configurations and some properties of them. By a *point configuration* we essentially mean a finite set $\mathbf{A}$ of points, and we are interested in the convex hull $\mathrm{conv}(\mathbf{A})$ of $\mathbf{A}$, which is a topological subspace of $\mathbb{R}^d$. The majority part of this section are found in the book of De Loera, Rambau and Santos [7].

We brief recall some basic notions from convex polytopes. An excellent reference for this topic is in [22]. The *convex hull* of a (not necessary convex) point set $\mathbf{X} \subset \mathbb{R}^d$, denoted $\mathrm{conv}(\mathbf{X})$, is the intersection of all convex sets containing $\mathbf{X}$. A *convex polytope* P is the convex hull of a finite set of

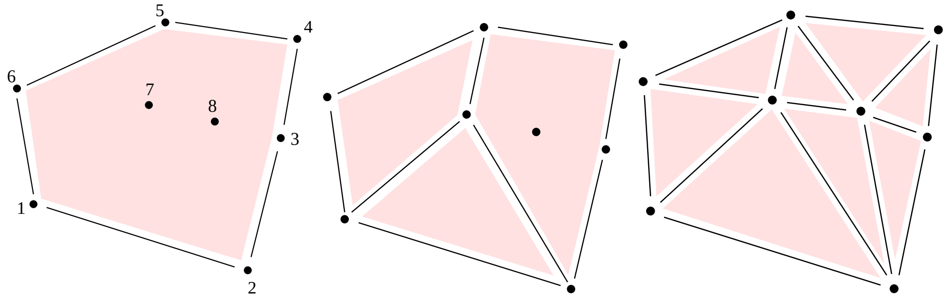


Figure 5: Subdivisions of a two-dimensional point configuration $\mathbf{A} = \{\mathbf{p}_1, ..., \mathbf{p}_8\}$. Left is a trivial subdivision (formed by conv($\mathbf{A}$) and all its faces), in the middle is a subdivision of $\mathbf{A}$ ($\mathbf{p}_8$ is not used), and right is a triangulation of $\mathbf{A}$ with all vertices of $\mathbf{A}$ are used.

points. The dimension $\dim(P)$ is the dimension of its *affine hull*, which is the smallest affine subspace containing P.

A *face* of a polytope $P \subset \mathbb{R}^d$ is any set of the form $F = P \cap \{\mathbf{x} \in \mathbb{R}^d \,|\, \mathbf{c}^T\mathbf{x} = c_0\}$, where $\mathbf{c}^T\mathbf{x} \leq c_0$ is a valid inequality for all points of P. Two trivial faces of P are P itself (obtained by $\mathbf{0}^T\mathbf{x} \leq 0$) and $\emptyset$ (obtained by $\mathbf{0}^T\mathbf{x} \leq 1$). All other faces of P are *proper* faces of P. We write $F \leq P$ (and $F < P$) to mean that F is a face (and proper face) of P.

A face of P is also a polytope. Faces of dimension 0, 1, $\dim(P) - 2$, and $\dim(P) - 1$ are called *vertices*, *edges*, *ridges*, and *facets* of P, respectively. The *boundary* $\mathrm{bd}(P)$ of P is the union of all proper faces of P, the *relative interior* $\mathrm{relint}(P)$ of P is: $\mathrm{relint}(P) = P - \mathrm{bd}(P)$. Note that P is the disjoint union of the relative interiors of all its faces. The relative interior of a vertex is the vertex itself. The set of all vertices of P is denoted as $\mathrm{vert}(P)$.

Definition 2.1 (Polyhedral Complex) *A polyhedral complex $\mathcal{C}$ is a finite collection of convex polytopes in $\mathbb{R}^d$ such that*

(i) $\emptyset \in \mathcal{C}$*,*

(ii) $P \in \mathcal{C} \Longrightarrow$ *all faces of* P *are in* $\mathcal{C}$*, and*

(iii) $P, Q \in \mathcal{C} \Longrightarrow P \cap Q$ *is a face of both* P *and* Q*.*

The *dimension* $\dim(\mathcal{C})$ is the largest dimension of a polytope in $\mathcal{C}$. The *underlying space* of $\mathcal{C}$ is the point set $|\mathcal{C}| = \bigcup_{P \in \mathcal{C}} P$. A *subcomplex* of $\mathcal{C}$ is a subset $\mathcal{C}' \subseteq \mathcal{C}$ that itself is a polyhedral complex. The set of all vertices of $\mathcal{C}$ is denoted as $\mathrm{vert}(\mathcal{C})$.

Subdivisions of point configurations are the fundamental objects used in this paper. Triangulations are nothing but particular cases of them. In the following, we assume that a point configuration $\mathbf{A}$ in $\mathbb{R}^d$ contains no *repeated* *points*, that is, no two points of $\mathbf{A}$ have exactly the same coordinates. Unless stated explicitly, we always assume that $\mathbf{A}$ is finite.

Definition 2.2 (Subdivision of Point Configuration) *Let $\mathbf{A}$ be a point configuration in $\mathbb{R}^d$. A subdivision of $\mathbf{A}$ is a polyhedral complex $\mathcal{T}$ such that*

(i) vert($\mathcal{T}$) $\subseteq \mathbf{A}$, and

(ii) $|\mathcal{T}| =$ conv($\mathbf{A}$).

The elements (convex polytopes) of a subdivision are called *cells*. Cells of dimension k are called k-cells. Cells of the same dimension as $\mathbf{A}$ are called *maximal cells*. A subdivision $\mathcal{T}$ of a point configuration $\mathbf{A}$ must use the vertices from $\mathbf{A}$ ($\mathcal{T}$ contains no Steiner point). While $\mathcal{T}$ may not use all points of $\mathbf{A}$. A *triangulation* of $\mathbf{A}$ is a subdivision whose elements are all simplices. See Fig. 5 for examples.

Now we introduce the central concept of this paper – regular subdivisions of point sets. There are many ways to define such objects. The natural way is to use the "lift and projection" processes as we will do below.

Definition 2.3 (Regular Subdivision of Point Configuration) *Let $\mathbf{A} \subset \mathbb{R}^d$ be a point configuration with n elements. Let $\omega : \mathbf{A} \to \mathbb{R}$ be a "height vector". We simply write ω_j to refer to the height given to $\mathbf{p}_j \in \mathbf{A}$. Let $\mathbf{p}_j^\omega \in \mathbb{R}^{d+1}$ be the lifted point from $\mathbf{p}_j \in \mathbf{A}$,*

$$\mathbf{p}_j^\omega = \begin{pmatrix} \mathbf{p}_j \\ \omega_j \end{pmatrix}.$$

Let $\mathbf{A}^\omega$ in $\mathbb{R}^{d+1}$ be the lifted point set,

$$\mathbf{A}^\omega = \{\mathbf{p}_j^\omega \,|\, \mathbf{p}_j \in \mathbf{A}\}.$$

A lower face of the polytope $P =$ conv($\mathbf{A}^\omega$) is the set

$$F = \{\mathbf{x} \in P \,|\, \mathbf{c}^T\mathbf{x} = c_0\},\ \mathbf{c}^T\mathbf{x} \le c_0 \text{ is valid for } P,\ c_{d+1} < 0.$$

(put simply, a face is "visible" from below.) We define the regular subdivision of $\mathbf{A}$ produced by ω, denoted as $\mathcal{T}(\mathbf{A}, \omega)$, to be the collection of all lower faces of conv($\mathbf{A}^\omega$), projected down to $\mathbb{R}^d$ through the canonical projection map $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ for $\pi(\mathbf{p}^\omega) = \mathbf{p}$, i.e.,

$$\mathcal{T}(\mathbf{A}, \omega) = \{\pi(F) \,|\, F \text{ is a lower face of conv}(\mathbf{A}^\omega)\}.$$

In other words, $\mathcal{T}(\mathbf{A}, \omega)$ is the collection of faces of *conv*($\mathbf{A}^\omega$) that can be "seen" from $-\lambda \mathbf{e}_{d+1}$, for $\lambda \longrightarrow \infty$ large enough. Fig. 6 shows an example of a regular subdivision. The following lemma shows that the object we've just defined is indeed a subdivision of $\mathbf{A}$ for any choice of ω.

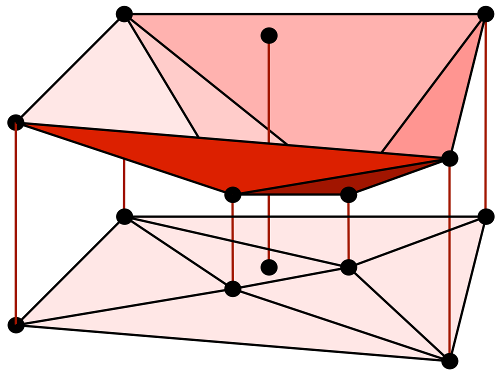


Figure 6: Regular subdivisions and triangulations. (De Loera *et al* [7])

Lemma 2.4 (De Loera *et al* [7]) *$\mathcal{T}(\mathbf{A}, \omega)$ is a subdivision of $\mathbf{A}$, for every ω.*

Proof See De Loera *et al* [7] Lemma 2.2.28. ∎

It is an active research topic to study the relations of the set of all polyhedral subdivisions of a point configuration. An important concept is the *refinement* of subdivisions, which roughly means that some pieces of that subdivision are subdivided further.

Definition 2.5 (Refinement) *Let $\mathcal{T}$ and $\mathcal{T}'$ be two subdivisions of a point configuration $\mathbf{A}$. Then $\mathcal{T}$ is a refinement of $\mathcal{T}'$, denoted $\mathcal{T} \preceq \mathcal{T}'$, if for each $P \in \mathcal{T}$ there is a $P' \in \mathcal{T}'$ with $P \subseteq P'$.*

By its definition, a refinement of a subdivision can use extra vertices which are not used in the subdivision it refines. However, a refinement can use at most as many vertices as the point set of the point configuration that all the subdivisons are based on. See Fig. 7 for examples.

Remark With the refinement relation, one can show that the set of all subdivisions of $\mathbf{A}$ is a partially ordered set (poset for short), and it has a maximal element, the trivial subdivision of $\mathbf{A}$, the minimal elements of the poset are triangulations of $\mathbf{A}$. See De Loera *et al* [7].

Lemma 2.6 (De Loera *et al* [7]) *Let $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega)$ be the regular subdivision of $\mathbf{A}$ produced by ω. Then there is an $\epsilon > 0$ such that for every height function $\omega' : \mathbf{A} \to \mathbb{R}$ that is ϵ-close to ω, i.e., $|\omega_j - \omega'_j| < \epsilon$ for all j, we have that $\mathcal{T}(\mathbf{A}, \omega') \preceq \mathcal{T}(\mathbf{A}, \omega)$.*

Proof See De Loera *et al* [7] Lemma 2.2.29, claim 3. ∎

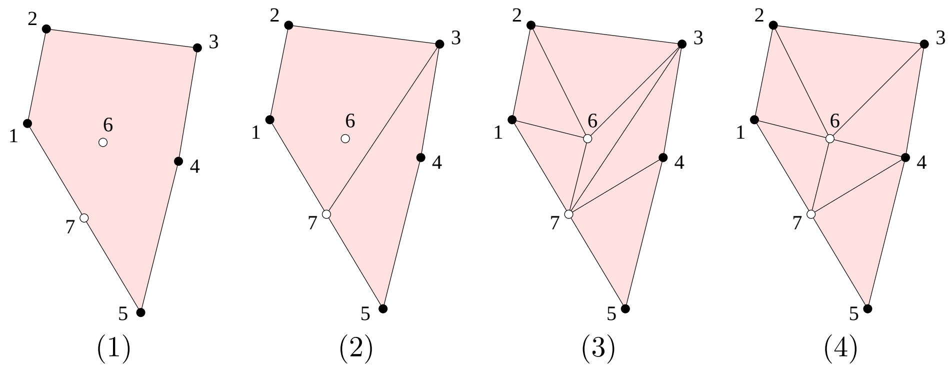


Figure 7: Refinements of a subdivision. (1) is a trivial subdivision of 7 points (the points $\mathbf{p}_1$, $\mathbf{p}_5$, and $\mathbf{p}_7$ are collinear), the subdivision in (2) is a refinement of the left, (3) is a refinement of both left and middle, and (4) is a refinement of (1) but not (2).

Lemma 2.7 (De Loera *et al* [7]) *Let $\mathcal{T}$ be a subdivision of $\mathbf{A}$. Let $\omega : \mathbf{A} \to \mathbb{R}$ be a height vector. Then the following is a subdivision of $\mathbf{A}$ and it refines $\mathcal{T}$:*

$$\mathcal{T}_\omega = \cup_{P\in\mathcal{T}}\mathcal{T}(\mathbf{A}|_P, \omega|_P).$$

Moreover, if $\mathcal{T}$ equals $\mathcal{T}(\mathbf{A}, \omega_0)$, then $\mathcal{T}_\omega$ is regular and equals to $\mathcal{T}(\mathbf{A}, \omega_0 + \epsilon\,\omega)$ for any positive and sufficiently small ϵ.

In above $\mathbf{A}|_P$ denotes the subset $\mathbf{A} \cap \mathrm{vert}(P)$, $\omega|_P$ means the function ω is restricted on $\mathrm{vert}(P)$. That is, $\mathcal{T}_\omega$ is obtained by refining each cell of $\mathcal{T}$ in the regular way given by the lifting vector ω. Fig. 8 shows an example.

Proof See De Loera *et al* [7] Lemma 2.2.30. ∎

The subdivision $\mathcal{T}_\omega$ of the previous lemma is called the *regular refinement* of $\mathcal{T}$ for the lifting vector ω.

Corollary 2.8 (De Loera *et al* [7]) *Every subdivision of $\mathbf{A}$ can be refined to a triangulation. Moreover, every regular subdivision of $\mathbf{A}$ can be refined to a regular triangulation.*

Proof For the first assertion, observe that if ω is "sufficiently generic", then $\mathcal{T}_\omega$ is a triangulation since every $\mathcal{T}(\mathbf{A}|_P, \omega|_P)$ is a triangulation. The second assertion follows from Lemma 2.7. ∎

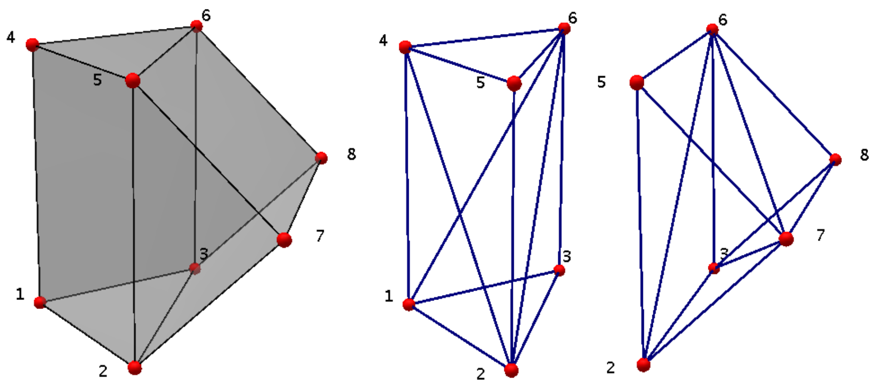


Figure 8: Refinement of a polyhedral subdivision. Left: the point set $\mathbf{A} = \{1, 2, ..., 8\}$. A polyhedral subdivision $\mathcal{T}$ of $\mathbf{A}$ contains two convex prisms, which are $\{1, 2, 3, 4, 5, 6\}$ and $\{2, 3, 5, 6, 7, 8\}$. Right, A refinement of two prisms using the way given in Lemma 2.7.

3 Subdivisions of Polyhedra

In this section, we first give the definitions of a general polyhedron (may be non-convex) and its faces. Then we define the (regular) subdivision of a polyhedron.

There exist many definitions for non-convex polyhedra, but less definitions about faces of non-convex polyhedra. The following definition of non-convex polyhedra and faces are close to those given by Edelsbrunner [8].

Definition 3.1 (Polyhedron) *A general and therefore not necessary convex polyhedron is the union of convex polyhedra, i.e.,* $P = \bigcup \mathcal{P}$*, where* $\mathcal{P}$ *is a finite set of convex polyhedra, and the space of* P *is connected.*

The space of P may not simply connected. See Fig. 9 for examples. The *dimension* of P, denoted as $\dim(P)$, is the largest dimension of a convex polyhedron in $\mathcal{P}$.

We define a *face* of P by the way suggested in [8]. Let $\mathbb{B}_\epsilon$ be the open ball of radius ϵ centered at the origin in $\mathbb{R}^d$. For a point $\mathbf{x} \in \mathbb{R}^d$ we consider a sufficiently small neighborhood $N_\epsilon(\mathbf{x}) = (\mathbf{x} + \mathbb{B}_\epsilon) \cap P$. The *face figure* of $\mathbf{x}$ is the enlarged version of this neighborhood within this polyhedron, i.e., $\mathbf{x} + \bigcup_{\lambda > 0} \lambda(N_\epsilon(\mathbf{x}) - \mathbf{x})$.

Definition 3.2 (Faces) *A face of* P *is the closure of a maximal connected set of points with identical face figures.*

By this definition, a face of P is always connected, but not necessarily simply connected, see Fig. 9 for examples.

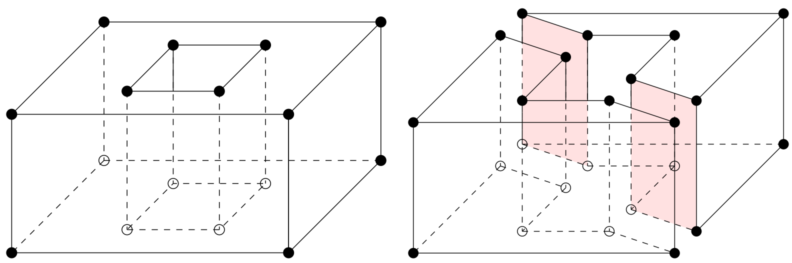


Figure 9: Polyhedra and faces. Left: A polyhedron (a torus) formed by the union of four convex polyhedra. It consists of 16 vertices (zero-faces), 24 edges (1-faces), 10 two-faces (the faces at top and bottom are not simply connected), and 1 three-face (which is itself). Right: Two polyhedra. Each has 12 vertices, 18 edges, 8 two-faces, and 1 three-faces. The shaded area highlights two 2-faces which are coplanar.

A face F of P is again a polyhedron. Particularly, $\emptyset$ is a face of P. If all convex polyhedra in $\mathcal{P}$ have the same dimension, then P itself is a face of P. All other faces of P are *proper* faces of P. We also write $F \leq P$ or $F < P$ if F is a face or a proper face of P. The faces of dimension 0, 1, $\dim(P) - 2$, and $\dim(P) - 1$ are called *vertices*, *edges*, *ridges*, and *facets*, respectively. The set of all vertices of P, the *vertex set*, will be denoted by $\mathrm{vert}(P)$. The union of all proper faces of P is called the *boundary* of P, denoted as $\mathrm{bd}(P)$. The *interior* $\mathrm{int}(P)$ is $P - \mathrm{bd}(P)$. The set of all proper faces of P is a polyhedral complex, called the *boundary complex* ∂P.

Recall that a subdivision (and regular subdivision) of a point configuration is a polyhedral complex whose underlying space equals to the convex hull of the point configuration. Moreover, it uses only vertices of the point configuration. Next we will introduce the definition of "subdivision of a polyhedron", which is also a polyhedral complex. The main difference between these two objects are: the latter may not be convex and it may contain Steiner points. However, they are closely related. To setup the connection between the two objects, we first introduce the notion of "subdivides".

Definition 3.3 (Subdivide) *Let $\mathcal{T}$ be a subdivision of a point configuration $\mathbf{A}$ in $\mathbb{R}^d$. Let P be a polyhedron. We say that $\mathcal{T}$ subdivides P if for each face F of P, there is a subcomplex $\mathcal{K}$ of $\mathcal{T}$ and $|\mathcal{K}| = F$. If $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega)$ is a regular subdivision of $\mathbf{A}$, then we say that $\mathcal{T}$ regularly subdivides P.*

If $\mathcal{T}$ (regularly) subdivides P, then there is a subcomplex $\mathcal{T}_P$ of $\mathcal{T}$ fills the interior of P. We say that $\mathcal{T}_P$ is a (regular) subdivision of P. see Fig. 10.

Now we formally give the definition of a (regular) subdivision of a polyhedron P.

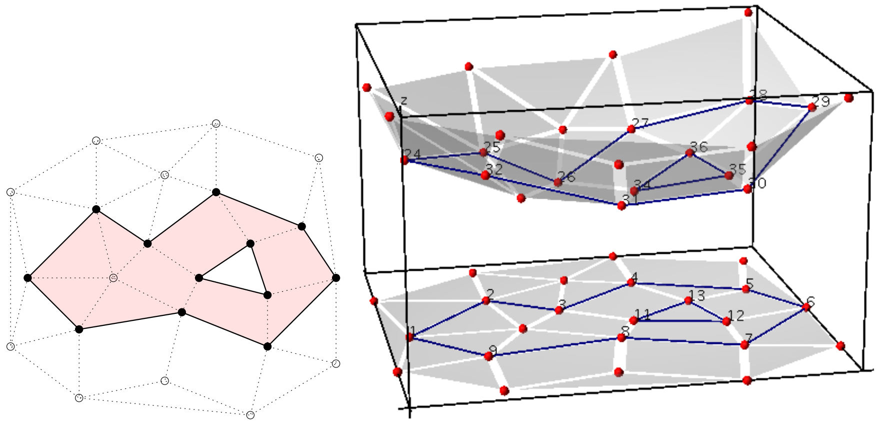


Figure 10: Left: A two-dimensional point configuration $\mathbf{A}$ and a polygon P (the shaded area). A subdivision $\mathcal{T}$ of $\mathbf{A}$ subdivides P. The subset $\mathcal{T}_P \subset \mathcal{T}$ in the shaded area is a subdivision of P. Right: the lifted point configuration $\mathbf{A}^\omega$ and the set of lower faces of $\text{conv}(\mathbf{A}^\omega)$. $\mathcal{T}_P$ is a regular subdivision of P.

Definition 3.4 (Subdivision of Polyhedron) *Let P be a polyhedron. A subdivision of P is a polyhedral complex $\mathcal{T}$ such that*

(i) $|\mathcal{T}| = P$, *and*

(ii) $\forall F \leq P \Longrightarrow \exists \mathcal{K} \subseteq \mathcal{T}$ *such that* $|\mathcal{K}| = F$.

A subdivision $\mathcal{T}$ of a polyhedron P is not necessarily convex (by property *(i)*). Moreover, every vertex of P must be in $\mathcal{T}$, and $\mathcal{T}$ may contain Steiner points (by property *(ii)*). If $\mathcal{T}$ contains no Steiner points, we say that $\mathcal{T}$ is a *pure* subdivision of P.

A subdivision $\mathcal{T}$ of a polyhedron P is a *triangulation* of P if all elements of $\mathcal{T}$ are simplices. The question of this paper is: Under which condition, P has a pure triangulation?

Definition 3.5 (Regular Subdivision of Polyhedron) *Let $\mathcal{T}$ be a subdivision of a polyhedron P. If there is a height function $\omega : \mathbf{A} \to \mathbb{R}$, where $\mathbf{A} = vert(\mathcal{T})$ such that $\mathcal{T}$ is the subset of the set of the projected lower faces of $conv(\boldsymbol{A}^\omega)$, i.e.,*

$$\mathcal{T} \subseteq \{\pi(F) \,|\, F \text{ is a lower face of } conv(\boldsymbol{A}^\omega)\},$$

then $\mathcal{T}$ is a regular subdivision of P, denoted as $\mathcal{T} = \mathcal{T}(P, \omega)$.

4 Inserting an Internal Facet

Let P be a polyhedron. Let $Q \subset P$ be another polyhedron formed by ridges of P of the same affine space such that $\dim(Q) = \dim(P) - 1$ and Q is

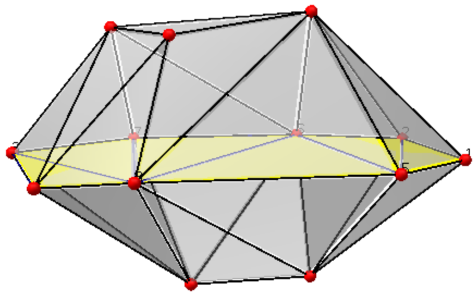


Figure 11: A three-dimensional polyhedron P. An internal facet Q (highlighted in yellow color) divides P into two disjoint parts.

not a facet of P. Such Q divides P into two polyhedra P_L and P_R, where $P = P_L \cup P_R$, $Q = P_L \cap P_R$, and $\mathrm{int}(P_L) \cap \mathrm{int}(P_R) = \emptyset$. We call Q an *internal facet* of P, see Fig. 11. In this section, we consider the problem of triangulating P and Q together without using Steiner points.

Theorem 4.1 *Let P be a polyhedron and Q be an internal facet of P such that Q divides P into two polyhedra P_L and P_R. If there is a regular subdivision $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega)$ subdivides P with no Steiner points on bd(P), then there are two regular subdivisions $\mathcal{T}_L$ and $\mathcal{T}_R$ that subdivide P_L and P_R without using Steiner points, respectively. $\mathcal{T}_L \cap \mathcal{T}_R = \mathcal{T}_Q$, where T_Q regularly subdivides Q.*

Proof Let $\mathbf{H}$ be the hyperplane passing through Q and separating P_L and P_Q. Let $\mathbb{H}_L \supset P_L$ and $\mathbb{H}_R \supset P_R$ be the two closed half spaces defined by $\mathbf{H}$ (see Fig. 12 top-left). Define two subsets $\mathbf{A}_L$ and $\mathbf{A}_R$ of $\mathbf{A}$ to be

$$\mathbf{A}_L = \{\mathbf{p} \,|\, \mathbf{p} \in \mathbf{A} \text{ and } \mathbf{p} \in \mathbb{H}_L\} \text{ and } \mathbf{A}_R = \{\mathbf{p} \,|\, \mathbf{p} \in \mathbf{A} \text{ and } \mathbf{p} \in \mathbb{H}_R\}.$$

Let $\mathcal{T}_L$ and $\mathcal{T}_R$ be the regular subdivisions of $\mathbf{A}_L$ and $\mathbf{A}_R$ obtained by restricting the height function ω on them, respectively (see Fig. 12 top-right).

$$\mathcal{T}_L = \mathcal{T}(\mathbf{A}_L, \omega|_{A_L}) \text{ and } \mathcal{T}_R = \mathcal{T}(\mathbf{A}_R, \omega|_{A_R}).$$

We show that $\mathcal{T}_L$ and $\mathcal{T}_R$ subdivide P_L and P_R, respectively.

Let F be a face of P_L and $F \neq Q$. There is a subcomplex $\mathcal{K}_F \subset \mathcal{T}$, such that $\mathcal{K}_F = \mathcal{T}(F, \omega|_F)$. We show that an arbitrary cell $U \in \mathcal{K}_F$ is also a cell of $\mathcal{T}_L$. If there is only one cell in $\mathcal{T}$ contains U, then U is on the convex hull of $\mathcal{T}$, hence $U \in \mathcal{T}_L$ as well. Otherwise let $W_1, W_2 \in \mathcal{T}$ be the two cells share at U. Since $U \subset \mathrm{bd}(P)$, only one of them can be inside P, w.o.l.g. we assume it is W_1. Then $W_2 \in \mathcal{T}_L$, since all other points in $\mathbf{A}^\omega$ are exactly above the hyperplane passing through the lifted point set vert$(W_2)^\omega$. In other words, W_2 belongs to any regular subdivision of a subset of $\mathbf{A}$. Hence $U \in \mathcal{T}_L$. It turns out, every cell of $\mathcal{K}_F$ is in $\mathcal{T}_L$. Hence $\mathcal{K}_F \subset \mathcal{T}_L$.

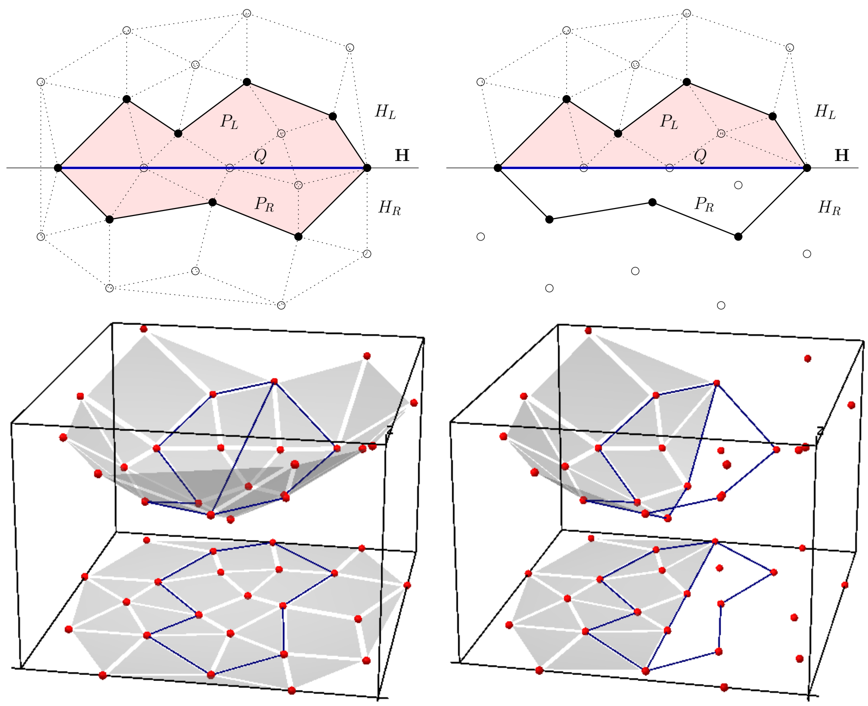


Figure 12: Top-left shows a two-dimensional polyhedron P (in the shaded area) with an internal facet Q (in blue) which divides P into two polyhedra P_L and P_R. $\mathbf{H}$ is the hyperplane passing through Q. $\mathbb{H}_L$ and $\mathbb{H}_R$ are closed half spaces defined by $\mathbf{H}$. A regular subdivision $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega)$ subdivides P. Bottom-left is a view of the lifted point set $\mathbf{A}^\omega$. Note that the lifted object from Q does not on the lower faces of $\text{conv}(\mathbf{A}^\omega)$. Top-right is the regular subdivision $\mathcal{T}_L = \mathcal{T}(\mathbf{A}_L, \omega|_{A_L})$. Bottom-right shows the lower faces of the convex hull of the lifted point set $\mathbf{A}_L^\omega$ and their projections in the plane which is the regular subdivision $\mathcal{T}_L$.

Now consider Q (which is a face of P_L). There is a subcomplex $\mathcal{K}_L \subset \mathcal{T}_L$, such that $\mathcal{K}_L$ is the regular subdivision of $\text{vert}(Q) = \mathbf{Q}$, i.e., $\mathcal{K}_L = \mathcal{T}(\mathbf{Q}, \omega|_Q)$. We show that $\mathcal{K}_L$ contains ∂Q as a subcomplex. It is equivalent to show that for an arbitrary maximal cell $U \in \mathcal{K}_L$, $\text{relint}(U) \cap \text{bd}(Q) = \emptyset$. To get a contradiction, let $U \in \mathcal{K}_L$ and $\text{relint}(U) \cap V \neq \emptyset$, $V \in \partial Q$. Let $W \in \mathcal{T}_L$ be the (maximal) cell containing U. (There is only one such W since Q is on the boundary of the closed half-space $\mathbb{H}_L$.) Then $\text{relint}(W) \cap V \neq \emptyset$, which implies that V can not belong to any projected lower face of $\text{conv}(\mathbf{A}^\omega)$ (said differently, it implies that V is not a "regular" cell with respect to ω). Since $V \in \mathcal{T}_L$ (which is a regular subdivision of $\mathbf{A}$), we arrive a contradiction. Hence $\partial Q \subset \mathcal{K}_L$, which implies that $\mathcal{K}_L$ regularly subdivides Q.

Since every face of P_L is subdivided by a subcomplex in $\mathcal{T}_L$. Hence $\mathcal{T}_L$ subdivides P_L. By the same way, we can show that $\mathcal{T}_R$ subdivides P_R.

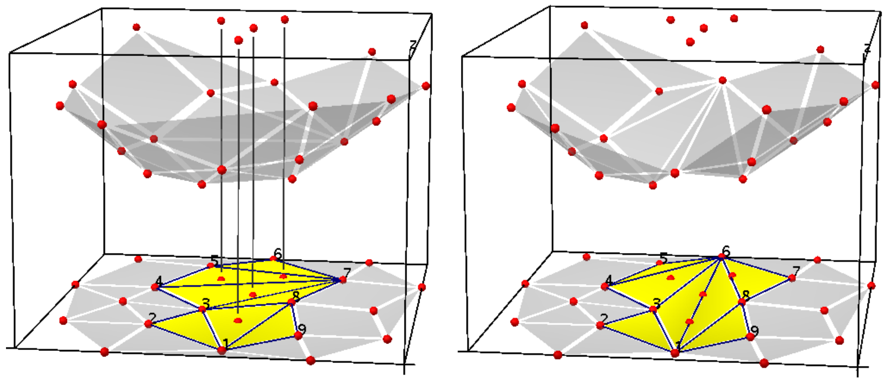


Figure 13: Left shows the regular subdivision $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega')$. The polyhedron P is highlighted in yellow. $\mathcal{T}$ subdivides P and contains no point insider and on P. Right shows the resulting $\mathcal{T}_L$ and $\mathcal{T}_R$.

It remains to show that $\mathcal{T}_L \cap \mathcal{T}_R = \mathcal{T}_Q$. We've show that $\mathcal{K}_L \subset \mathcal{T}_L$ regularly subdivides Q. The same, there is a $\mathcal{K}_R \subset \mathcal{T}_R$ regularly subdivides Q. It follows from the following equalities

$$\mathcal{K}_L = \mathcal{T}(\mathbf{Q}, \omega|_Q) = \mathcal{K}_R.$$

Hence we have $\mathcal{K}_L = \mathcal{K}_R = \mathcal{T}_Q$.

Note that both $\mathcal{T}_L$ and $\mathcal{T}_R$ may contain Steiner points, since we have no assumption on $\mathbf{A}$ such that it contains no Steiner points of P. In such case, we can form a new height function $\omega' : \mathbf{A} \to \mathbb{R}$ from ω to be,

$$\omega'(\mathbf{p}) = \begin{cases} \max(\omega) & \text{if } \mathbf{p} \in \mathrm{int}(P) \\ \omega(\mathbf{p}) & \text{otherwise.} \end{cases}$$

Then let $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega')$ (see Fig. 13 left). It can be shown that $\mathcal{T}$ subdivides P, and $\mathcal{T}$ contains no point that lies inside or on both P and Q. Hence the resulted $\mathcal{T}_L$ and $\mathcal{T}_R$ must contain no points inside or on P_L and P_R (see Fig. 13 right). By the same proof, $\mathcal{T}_L$ and $\mathcal{T}_R$ subdivides P_L and P_R with no Steiner points, respectively. ■

Remark. Theorem 4.1 shows that if a polyhedron P can be subdivided by a regular subdivision of a point set, then any internal facet Q of P which splits P into two polyhedra P_L and P_R can be recovered without using Steiner points. Furthermore, it shows that P_L and P_R can be again subdivided by regular subdivisions. Hence this theorem can be recursively applied to recover any internal facet of P_L and P_R.

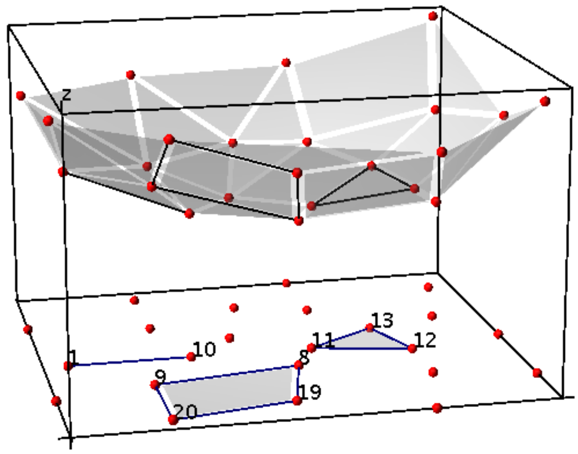


Figure 14: ω-regular cells. A two-dimensional point configuration $\mathbf{A}$ and the lower faces of the convex hull of $\mathbf{A}^\omega$ are shown. Cells $[\mathbf{p}_1, \mathbf{p}_{10}]$, $[\mathbf{p}_{11}, \mathbf{p}_{12}, \mathbf{p}_{13}]$, and $[\mathbf{p}_8, \mathbf{p}_9, \mathbf{p}_{20}, \mathbf{p}_{19}]$ are ω-regular. All vertices are ω-regular.

5 Triangulating Polyhedra

In this section, we will prove the main theorem of this paper. It states the following fact: If there is a regular subdivision containing the set of all ridges of a polyhedron P, then P can be triangulated without using Steiner points. Recall that a ridge is a $d-2$ face of P, where $d = \dim(P)$.

To prove the above fact, we're going to show that all facets of P can exist together in a subdivision $\mathcal{S}$ of $\mathbf{A}$ ($\mathcal{S}$ may not be regular), so that $\mathcal{S}$ subdivides P. Steiner points are not needed in $\mathcal{S}$.

First of all, we introduce a convenient notion "ω-regular" and some properties it has.

Definition 5.1 (ω-Regular) *Let $\mathbf{A} \subset \mathbb{R}^d$ be a point configuration, ω : $\mathbf{A} \to \mathbb{R}$ be a height function. A polytope $U \subset \mathbb{R}^d$ whose vert$(U) \subseteq \mathbf{A}$ is ω-regular in $\mathbf{A}$ if there is a hyperplane $\mathbf{H}$ in $\mathbb{R}^{d+1}$ containing the lifted points of vert$(U)^\omega \subseteq \boldsymbol{A}^\omega$ and no other point in $\mathbf{A}^\omega$ is below and on $\mathbf{H}$. Call $\mathbf{H}$ the supporting hyperplane of U.*

In other words, an ω-regular polytope in $\mathbf{A}$ must belong to the projection of a lower face of the convex hull of $\mathbf{A}^\omega$, see Fig. 14 for examples.

Proposition 5.2 *Let $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega)$. Then*

(1) Every cell $U \in \mathcal{T}$ is ω-regular in $\mathbf{A}$.

(2) $\mathcal{T}$ is exactly the collection of ω-regular cells in $\mathbf{A}$.

Proof *(1)* is directly followed by the definition of ω-regular cells. Hence $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega)$ is a collection of ω-regular cells in $\mathbf{A}$. To prove *(2)*, we still need to show that every ω-regular cell in $\mathbf{A}$ must also in $\mathcal{T}$. It is the case since every maximal ω-regular cell in $\mathbf{A}$ is in $\mathcal{T}$. ■

Next, we will prove a theorem which guarantees the existence of a subdivision of a polyhedron without using Steiner points.

Theorem 5.3 *Let P be a polyhedron. If there is a regular subdivision $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega)$ containing the set of all ridges of P, or equivalently, all ridges of P are ω-regular in $\mathbf{A}$ and vert$(P) \subseteq \mathbf{A}$, then P can be subdivided without using Steiner points.*

Proof First of all, note that if the point set $\mathbf{A}$ contains a point $\mathbf{p} \in \mathrm{int}(P)$, we can form a new regular subdivision $\mathcal{T}' = (\mathbf{A}, \omega')$ by letting $\omega' = \omega$ and $\omega'(\mathbf{p}) = \max(\omega)$. Hence $\mathbf{p}$ is not in $\mathcal{T}'$, and $\mathcal{T}'$ still contains the set of all ridges of P (see Fig. 13 left). In the following, we assume that $\mathbf{A}$ contains no point $\mathbf{p} \in \mathrm{int}(P)$. We show that every facet of P can be recovered in a polyhedral subdivision $\mathcal{S}$ of $\mathbf{A}$, so that $\mathcal{S}$ subdivides P. However $\mathcal{S}$ may not be a regular subdivision of $\mathbf{A}$.

Initially, Let $\mathcal{S} = \mathcal{T}$. Consider an arbitrary facet F of P. If F does not subdivided by $\mathcal{T}$, then F intersects with a set $\mathcal{C}$ of maximal cells of $\mathcal{S}$. Let $C = |\mathcal{C}|$. C is a polyhedron including F as an internal facet. Let C_L and C_R be the two polyhedra separated by F. Clearly C is subdivided by $\mathcal{S}$, i.e., there is a $\mathcal{K}_C \subset \mathcal{S}$ such that $|\mathcal{K}_C| = C$. We have two cases.

Case *1.* If F is the first facet of P to be recovered, i.e., $\mathcal{S}$ is still a regular subdivision, then $\mathcal{K}_C$ is a regular subdivision of C, then by Theorem 4.1 there are two regular subdivisions

$$\mathcal{K}_L \subseteq \mathcal{T}(C_L, \omega|_{C_L}) \text{ and } \mathcal{K}_R \subseteq \mathcal{T}(C_R, \omega|_{C_R}),$$

such that $\mathcal{K}_L$ and $\mathcal{K}_R$ are regular subdivisions of C_L and C_R, respectively. And $\mathcal{K}_L \cap \mathcal{K}_R$ is a regular subdivision of F. We then update $\mathcal{S}$ to be

$$\mathcal{S} = (\mathcal{S} \setminus \mathcal{K}_C) \cup (\mathcal{K}_L \cup \mathcal{K}_R).$$

Hence $\mathcal{S}$ becomes a subdivision containing a subdivision of F. We call a cell $U \in \mathcal{S}$ and $U \subseteq F$ *subfacet* to distinguish other cells of $\mathcal{S}$. If $\mathcal{S}$ contains subfacets, then $\mathcal{S}$ may not be regular anymore.

Case *2.* $\mathcal{S}$ already contains some recovered facets of P, hence $\mathcal{S}$ may not be a regular subdivision, see Fig. 15 left for an example. We now prove that $\mathcal{K}_C$ is still a regular subdivision of C.

We show that every maximal cell of $\mathcal{K}_C$ is ω-regular in $\mathbf{C} = \mathrm{vert}(\mathcal{C})$. To get a contradiction, let U be a maximal cell in $\mathcal{K}_C$ and U is not ω-regular

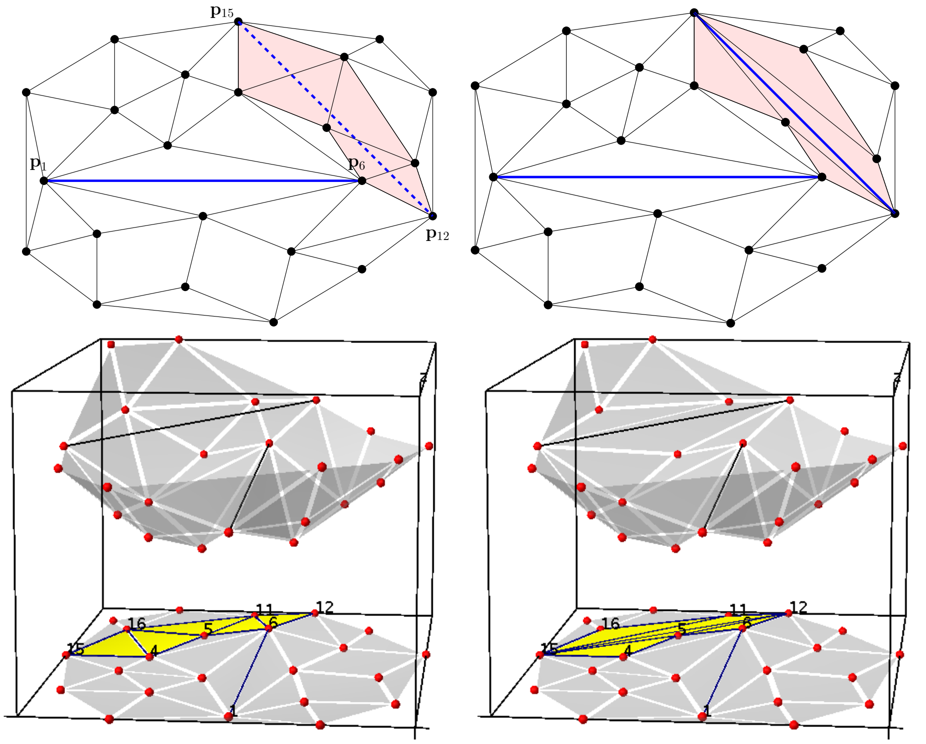


Figure 15: Facet recovery (illustrated in plane). Top-left is a two-dimensional subdivision $\mathcal{S}$ of a point configuration $\mathbf{A}$, $\mathbf{A}^\omega$ and the lifted cells of $\mathcal{S}$ are shown in bottom-left. $\mathcal{S}$ is non-regular since it includes the edge $[\mathbf{p}_1, \mathbf{p}_6]$ (shown in blue). The edge $[\mathbf{p}_{12}, \mathbf{p}_{15}]$ is going to be recovered in $\mathcal{S}$, it intersects with the marked triangles (shown in yellow). Top-right shows $\mathcal{S}$ after the edge $[\mathbf{p}_{12}, \mathbf{p}_{15}]$ is recovered, the marked triangles (shown in yellow) are new in $\mathcal{S}$. The lifted cells of $\mathcal{S}$ are shown in bottom-right.

in $\mathbf{C}$. Then there are lifted points in $\mathbf{C}^\omega$ below the supporting hyperplane $\mathbf{H}$ of U. Let W be the maximal cell in $\mathcal{K}_C$ and one of the lifted vertices $\mathbf{p}^\omega \in \text{vert}(W^\omega)$ is below the hyperplane. We can form a sequence of maximal cells of $\mathcal{K}_C$ starting from U and end at W (see Fig. 15 for an example),

$$\{U = U_0, U_1, ..., U_m = W\},$$

where U_i and U_{i+1} share a common maximal face. This sequence exists since all these cells are crossed by F. Now we "walk" from U towards W through this sequence. Let U_i, $0 \le i < m$ be the cell where we are now, and let $L_i < U_i$ be the face shared by its neighbor cell U_{i+1}. Let $\mathbf{p}_{i+1}$ be any vertex in $\text{vert}(U_{i+1}) \setminus \text{vert}(L_i)$. If $\mathbf{p}^\omega_{i+1}$ lies below the supporting hyperplane $\mathbf{H}_i$ of U_i, it implies that L_i must be a subfacet of $\mathcal{S}$ (since this is the only reason which causes the non-convexity between two hyperplanes in $\mathbb{R}^{d+1}$). It turns out there must be a facet F' of P intersects with F, where $F' \supset L_i$, which is impossible. Hence $\mathbf{p}^{\omega}_{i+1}$ must lie above $\mathbf{H}_i$, and it must lie above $\mathbf{H}$ as well. Inductively, we will find $\mathbf{p} \in \text{vert}(W)$ and $\mathbf{p}^{\omega}$ must lie above $\mathbf{H}$. We arrive a contradiction.

Hence every maximal cell of $\mathcal{K}_C$ is ω-regular in $\mathbf{C}$. It follows that $\mathcal{K}_C$ is a regular subdivision of C. Then we can apply the same operations in **Case** *1* to subdivide C_L and C_R and recover F in S, see Fig. 15 right for an example.

Since all facets of P can be recovered in $\mathcal{S}$ one after one. On finish, $\mathcal{S}$ is a subdivision of P with no Steiner points. ∎

The following corollary shows that the condition given in the above theorem is sufficient to guarantee the existence of a triangulation of a polyhedron without Steiner points.

Corollary 5.4 *Let P be a polyhedron. If there is a regular subdivision $\mathcal{T} = \mathcal{T}(\mathbf{A}, \omega)$ containing the set of all ridges of P, or equivalently, all ridges of P are ω-regular in $\mathbf{A} \supseteq$ vert(P), then P can be triangulated without using Steiner points.*

Proof By Theorem 5.3, there exist a subdivision $\mathcal{S}$ of P such that $\mathcal{S}$ contains no Steiner points of P. Obviously, $\mathcal{S}$ uses all vertices of P. By Corollary 2.8, $\mathcal{S}$ can be refined into a triangulation $\mathcal{S}'$ of P. The definition of refinement implies that $\mathcal{S}'$ must contain all vertices of P and no Steiner point. ∎

Discussion We now reconsider Shewchuk's condition and compare it to ours. Let P in $\mathbb{R}^d$ be a polyhedron whose all ridges are strongly Delaunay in vert(P). Let $\omega : \text{vert}(P) \to \mathbb{R}$ be the function $\|\mathbf{x}\|^2$ (where $\|\cdot\|$ denotes the Euclidean norm), i.e., ω lifts every vertex of P onto a paraboloid in $\mathbb{R}^{d+1}$. The set of ridges of P must be contained in the regular subdivision $\mathcal{T} = \mathcal{T}(\text{vert}(P), \omega)$. Hence it automatically satisfies our condition. On the other hand, a Delaunay but non-strongly Delaunay ridge may still exist in the Delaunay subdivision (which is a regular subdivision) of that point set. For example, this is the case for a prism whose all vertices share a common sphere and it is tetrahedralizable by our condition.

6 Algorithms

One of the main applications of our condition is to find triangulations for three-dimensional polyhedra or complex-like objects formed by a finite collection of polyhedra.

One of the important questions is: how to restrict the number of Steiner points? Although Chazelle showed that a quadratic number of Steiner points may be needed [3], it is not required to use that many Steiner points in most of cases. Our condition shows that if there is a function $\omega : \mathbf{A} \to \mathbb{R}$, and all edges of a polyhedron P is ω-regular in $\mathbf{A}$, then no Steiner point is needed to triangulate P. Hence to efficiently find an ω satisfying the condition would be helpful. However, it is generally hard to do so. A pre-requisite to this question is to find a tetrahedralization including the edge set of P, which is an NP-complete problem [21].

However, it is possible to modify the input edge set by inserting few Steiner points on some edges such that the resulting edge set is contained in a regular triangulation of the vertex set of P (including these Steiner points). In [18], a practical algorithm for recovering the edge set of P in a Delaunay triangulation of $\mathrm{vert}(P)$ is discussed. This algorithm runs in $O(m^2 \log m)$ time, where m is the total number of output points. However, the upper bound on the number of Steiner points is not yet available.

The next question is: Suppose there is an ω and all edges of P are ω-regular in $\mathrm{vert}(P)$, i.e., the existence of a triangulation of P with no Steiner points is known, how to efficiently generate the triangulation?

There are at least two algorithms which can be used for this purpose. Both run in polynomial time. Shewchuk proposed a flip-based facet insertion algorithm [17]. Starting from a Delaunay triangulation $\mathcal{D}$ of $\mathrm{vert}(P)$ where $\mathcal{D}$ contains all edges of P. He showed there is a sequence of elementary flips which can insert a facet F of P into $\mathcal{D}$ such that $\mathcal{D}$ is a constrained Delaunay triangulation of P. The correctness of this algorithm relies on the assumption that the vertex set of P is in *general position* (i.e., no 5 points share a common sphere), hence the flip sequence will not get stuck. This algorithm runs in $O(n^2 \log n)$ time, where n is the input size of P.

Si and Gärtner [18] proposed another algorithm for inserting the facets of P into $\mathcal{D}$. It first triangulates all facets into a set of triangles, called *subfaces*. Then it inserts each subface into $\mathcal{D}$. If a subface σ is missing in $\mathcal{D}$, one can form a missing region $G \supseteq \sigma$ of neighboring missing subfaces of σ, and collect the set of tetrahedra in $\mathcal{D}$ whose interiors intersect with G. This will result a cavity C in $\mathcal{D}$, C contains G as an internal facet which splits C into C_L and C_R. Then it triangulates C_L and C_R by forming the Delaunay tetrahedralizations of $\mathrm{vert}(C_L)$ and $\mathrm{vert}(C_R)$ respectively. The correctness of this algorithm also relies on the assumption that the vertex set of P is in *general position*. This algorithm runs in worst case $O(fn^2 \log n)$ time, where f is the number of facets of P. In average, it runs in $O(n^2 \log n)$ time.

7 Conclusion and Discussion

The problem of triangulating non-convex polyhedra without using Steiner points has been long discussed in discrete geometry and computational geometry. Although various special cases and useful conditions have been

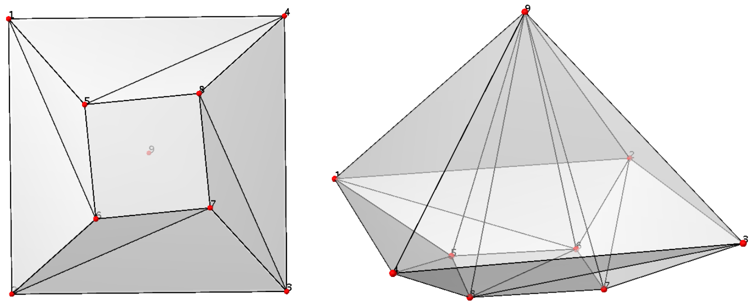


Figure 16: A polyhedron P of 9 vertices. It is constructed by the following steps: start with two congruent nested squares in the xy-plane; rotate the inner square around the center of the square by a small angle; translate the rotated square to z-axis by a small distance; connect the 8 side faces of the two squares as shown in the left; lastly mount a pyramid at the outer square. P can be tetrahedralized since the apex of the pyramid sees all the side faces from interior of P. A tetrahedralization of P is shown in the right. Indeed, it is the only tetrahedralization of P.

proved [9, 2, 19, 15], the question of asking for a general condition which decides the existence of a triangulation for a given polyhedron with no Steiner points is still open.

In this paper, we proved a new condition which towards the answer of the question. It states that if the set of ridges of a polyhedron P is contained in a regular subdivision, then there exists a triangulation of P with no Steiner points. In particular, this condition includes Shewchuk's condition [15] as a special case. It is still an question that the proved special cases [9, 19] can be reduced to this condition. However, our condition is not general. For example, our condition is not fulfilled by the three-dimensional polyhedron shown in Fig. 16 which is tetrahedralizable with no Steiner points.

A slightly improved condition could be the replacement of the requirement of (globally) ω-regular into the requirement of a locally ω-regular plus some additional requirements (need to be found). Hence the initial subdivision containing the set of ridges may not be a regular subdivision.

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