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Scaling limit and aging for directed trap models

Olivier Zindy[1]

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[1] Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstrasse 39
10117 Berlin
Germany
E-Mail:zindy@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider one-dimensional directed trap models and suppose that the trapping times are heavy-tailed. We obtain the inverse of a stable subordinator as scaling limit and prove an aging phenomenon expressed in terms of the generalized arcsine law. These results confirm the status of universality described by Ben Arous and Černý for a large class of graphs.

1 Introduction

What is usually called *aging* is a dynamical out-of-equilibrium physical phenomenon observed in disordered systems like spin-glasses at low temperature, defined by the existence of a limit of a given two-time (usually denoted by t_ω and $t_\omega + t$) correlation function of the system as both times diverge keeping a fixed ratio between them; the limit should be a non-trivial function of the ratio. It has been extensively studied in the physics literature, see [11] and therein references.

The *trap model* is a model of random walk that was first proposed by Bouchaud and Dean [10, 12] as a toy model for studying this aging phenomenon. In the mathematics litterature, much attention has recently been given to the trap model, and many aging result were derived from it, on $\mathbb{Z}$ in [16] and [4], on $\mathbb{Z}^2$ in [8], on $\mathbb{Z}^d$ ($d \geq 3$) in [6], or on the hypercube in [2, 3]. A comprehensive approach to obtaining aging results for the trap model in various settings was later developed in [7]. The striking fact is that these aging results are identical for $\mathbb{Z}^d$, $d \geq 2$ and the large complete graph, or the REM. In other terms, the mean-field results are valid from infinite dimension down to dimension 2.

The one-dimensional trap model has some specific features that distinguish it from all other cases. The most useful feature is that we can identify its scaling limit as an interesting one-dimensional singular diffusion in random environment, see [16], while the scaling limit for $d \geq 2$ is the *fractional kinetics* process, that is the time change of a d-dimensional Brownian motion by the inverse of an independent α-stable subordinator, see [6]. In fact, the universality of the aging phenomenon is a question about the transient part of relaxation to equilibrium and not necessarily related to equilibrium questions.

Here, we give an answer to a question of Ben Arous and Černý [5] by studying the influence of a drift in the one-dimensional trap model. We identify the scaling limit of the so-called *directed trap model* with the inverse of an α-stable subordinator and prove an aging result expressed in terms of the generalized arcsine law, so that it confirms the status of universality described by Ben Arous and Černý [7]. Moreover, this extends some results of Monthus [17], who studies the influence of a bias in the high disorder limit (i.e. when α tends to zero with our notations, see (2.2)) using renormalization arguments. Note that the ideas of the proof developed in this paper are deduced from a strong comparison with one-dimensional random walks in random environment in the sub-ballistic regime. Indeed, analogous results are obtained for this asymptotically equivalent model in [13] and [14].

The rest of the paper is organized as follows. The main results are stated in Section 2. In Section 3, we present some elementary result about the environment, the embedded random walk as well as preliminary estimates, which will be frequently used throughout the paper. Section 4 and Section 5 are respectively devoted to the proof of the scaling limit and to the proof of the aging result.

2 Notations and main results

Let us first fix $0 < \varepsilon \leq 1/2$. Then, the *directed trap model* is the nearest-neighbour continuous-time Markov process $X = (X_t)_{t\geq 0}$ given by $X_0 = 0$ and with jump rates

$$c(x,y) := \begin{cases} \left(\frac{1}{2}+\varepsilon\right)\tau_x^{-1} & \text{if } y = x+1, \\ \left(\frac{1}{2}-\varepsilon\right)\tau_x^{-1} & \text{if } y = x-1, \end{cases} \tag{2.1}$$

and zero otherwise, where $\tau = (\tau_x)_{x\in\mathbb{Z}}$ is a family of positive i.i.d. heavy-tailed random variables. More precisely, we suppose that there exists $\alpha \in (0,1)$ such that

$$\lim_{u\to\infty} u^\alpha\, \mathbb{P}(\tau_x \geq u) = 1. \tag{2.2}$$

In particular, this implies $\mathbb{E}[\tau_x] = +\infty$. Sometimes τ is called random environment of traps. The Markov process X_t spends at site x an exponentially distributed time of mean τ_x, and then jumps to the right with probability $p = p_\varepsilon := (\frac{1}{2}+\varepsilon)$ and to the left with probability $q = q_\varepsilon := (\frac{1}{2}-\varepsilon)$. Therefore, X is a time change of a discrete-time biased random walk on $\mathbb{Z}$. More precisely, we define the *clock process* and the *embedded random walk* associated with X as follows.

Definition 2.1. Let $S(0) := 0$ and let $S(k)$ be the time of the k-th jump of X, for $k \in \mathbb{N}^*$. For $s \in \mathbb{R}_+$, we define $S(s) := S(\lfloor s \rfloor)$ and call S the *clock process.* Define the embedded discrete-time random walk $(Y_n)_{n\geq 0}$ by $Y_n := X_t$ for $S(n) \leq t < S(n+1)$. Then obviously, $(Y_n)_{n\geq 0}$ is a biased random walk on $\mathbb{Z}$.

Observe that $(Y_n)_{n\geq 0}$ satisfies $P(Y_{n+1} = Y_n + 1) = \frac{1}{2} + \varepsilon = 1 - P(Y_{n+1} = Y_n - 1)$, for all $n \geq 0$. Therefore, $(Y_n)_{n\geq 0}$ is transient to $+\infty$ and the law of large numbers implies that, $\mathbb{P}$-almost surely,

$$\frac{Y_n}{n} \longrightarrow v_\varepsilon := 2\varepsilon > 0, \qquad n \to \infty. \tag{2.3}$$

Furthermore, it follows from the definition of X that the clock process can be written

$$S(k) = \sum_{i=0}^{k-1} \tau_{Y_i} \mathbf{e}_i, \qquad k \geq 1, \tag{2.4}$$

where $(\mathbf{e}_i)_{i\geq 0}$ is a family of i.i.d. mean-one exponentially distributed random variables. We always suppose that the $\mathbf{e}_i$'s are defined in this way. Then, the process $(X_t)_{t\geq 0}$ satisfies

$$X_t = Y_{S^{-1}(t)}, \qquad \forall\, t \in \mathbb{R}_+, \tag{2.5}$$

where the right-continuous inverse of an increasing function ϕ is defined by $\phi^{-1}(t) := \inf\{u \geq 0 :\ \phi(u) > t\}$.

Now, let us fix $T > 0$ and denote by $D([0,T])$ the space of càdlàg functions from $[0,T]$ to $\mathbb{R}$. Moreover, let $X_t^{(N)}$ be the sequence of elements of $D([0,T])$ defined by

$$X_t^{(N)} := \frac{X_{tN}}{N^\alpha}, \qquad 0 \leq t \leq T. \tag{2.6}$$

Then, the scaling limit result can be stated as follows.

Theorem 2.2. *The distribution of the process* $(X_t^{(N)};\, 0 \leq t \leq T)$ *converges weakly to the distribution of* $(v_\varepsilon^{\#} V_\alpha^{-1}(t);\, 0 \leq t \leq T)$ *on* $D([0,T])$ *equipped with the uniform topology, where* $(V_\alpha(t);\, t \geq 0)$ *is a standard* α*-stable subordinator and* $v_\varepsilon^{\#} := \frac{\sin(\alpha\pi)}{\alpha\pi} v_\varepsilon^\alpha = \frac{\sin(\alpha\pi)}{\alpha\pi}(2\varepsilon)^\alpha$.

Although this result can be compared with the limit in [6], we do not obtain the *fractional kinetics* process. This difference can be explained by recalling that the fractional kinetics process is the time change of a Brownian motion by the inverse of an independent α-stable subordinator while our embedded random walk satisfies the law of large numbers with positive speed, see (2.3). Furthermore, observe that the case $\varepsilon = 1/2$ is trivial; indeed Y is deterministic, $v_\varepsilon = 1$ and the clock process, which can be written $S(k) = \sum_{i=0}^{k-1} \tau_i \mathbf{e}_i$, is just a sum of i.i.d. heavy-tailed random variables. Now let us state the second main result, concerning the aging phenomenon.

Theorem 2.3. *For all* $h > 1$, *we have*

$$\lim_{t\to\infty} \mathbb{P}(X_{th} = X_t) = \frac{\sin(\alpha\pi)}{\pi} \int_0^{1/h} y^{\alpha-1}(1-y)^{-\alpha}\, \mathrm{d}y. \tag{2.7}$$

Remark. As in [8], we think that it is possible to prove a sub-aging result for the correlation function given by $\mathbb{P}(X_s = X_{t_\omega}; \forall\, t_\omega \leq s \leq t_\omega + t)$. Note that, in [9], Bertin and Bouchaud study the average position of the random walk at time $t_\omega + t$ given that a small bias h is applied at time t_ω. They found several scaling regime depending on the relative value of t, t_ω and h.

In the following, C denotes a constant large enough, whose value can change from line to line.

3 Preliminary estimates

In this section, we list some properties of the environment τ and of the embedded walk Y as well as preliminary results.

3.1 The environment

Let us define the critical depth for the first n traps of the environment by

$$g(n) := \frac{n^{1/\alpha}}{(\log n)^{\frac{2}{1-\alpha}}}. \tag{3.1}$$

Then, we can introduce the notion of *deep traps* as follows:

$$\delta_1 = \delta_1(n) := \inf\{x \geq 0 : \tau_x \geq g(n)\}, \tag{3.2}$$

$$\delta_j = \delta_j(n) := \inf\{x > \delta_{j-1} : \tau_x \geq g(n)\}, \qquad j \geq 2. \tag{3.3}$$

The number of such deep traps before site n will be denoted by θ_n and defined by

$$\theta_n := \sup\{j \geq 0 : \delta_j \leq n\}, \tag{3.4}$$

where $\delta_0 := 0$. Now, let us define $\varphi(n) := \mathbb{P}(\tau_1 \geq g(n))$. We introduce the following series of events, which will occur with high probability, when n goes to infinity:

$$\mathcal{E}_1(n) := \left\{ n\varphi(n)\Big(1 - \frac{1}{\log n}\Big) \leq \theta_n \leq n\varphi(n)\Big(1 + \frac{1}{\log n}\Big) \right\}, \tag{3.5}$$

$$\mathcal{E}_2(n) := \left\{ \delta_1 \wedge \min_{1 \leq j \leq \theta_n - 1} (\delta_{j+1} - \delta_j) \geq \rho(n) \right\}, \tag{3.6}$$

$$\mathcal{E}_3(n) := \left\{ \max_{-\nu(n) \leq x \leq 0} \tau_x < g(n) \right\}, \tag{3.7}$$

where $\rho(n) := n^\kappa$ with $0 < \kappa < 1/3$ and $\nu(n) := \lfloor (\log n)^{1+\gamma} \rfloor$ with $0 < \gamma < 1$.
In words, $\mathcal{E}_1(n)$ requires that the number of deep traps is not too large, $\mathcal{E}_2(n)$ requires that the distance between two deep traps is large enough and $\mathcal{E}_3(n)$ will ensure that the time spent by X on $\mathbb{Z}_-$ is negligible.

Lemma 3.1. *Let $\mathcal{E}(n) := \mathcal{E}_1(n) \cap \mathcal{E}_2(n) \cap \mathcal{E}_3(n)$, then we have*

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{E}(n)) = 1. \tag{3.8}$$

Proof. Note that the number of traps higher than $g(n)$ in the first n traps is a binomial with parameter $(n, \varphi(n))$. Then, recalling (2.2), the proof of Lemma 3.1 is easy and left to the reader. □

Since we want to consider intervals of size $2\nu(n)$ around the δ_j's that are disjoint, we introduce now a subsequence of the deep traps defined above. These $*$-*deep traps* are defined as follows:

$$\delta_1^* = \delta_1^*(n) := \inf\{x \geq \nu(n) : \tau_x \geq g(n)\}, \tag{3.9}$$

$$\delta_j^* = \delta_j^*(n) := \inf\{x > \delta_{j-1}^* + 2\nu(n) : \tau_x \geq g(n)\}, \qquad j \geq 2. \tag{3.10}$$

The number of such $*$-deep traps before site n will be denoted by θ_n^* and defined by

$$\theta_n^* := \sup\{j \geq 0 : \delta_j^* \leq n\}. \tag{3.11}$$

For any $\nu \in \mathbb{N}^*$ and any $x \in \mathbb{Z}$, let us denote by $B_\nu(x)$ the interval $[x-\nu, x+\nu]$. Observe that the intervals $(B_{\nu(n)}(\delta_j^*))_{1\le j\le \theta_n^*}$ will be made of independent and identically distributed portions of environment τ (up to some translation).

The following lemma tells us that the $*$-deep traps coincide with the sequence of deep traps with an overwhelming probability when n goes to infinity.

Lemma 3.2. *If $\mathcal{E}^*(n) := \{\theta_n = \theta_n^*\}$, then we have*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{E}^*(n)) = 1. \tag{3.12}$$

Proof. Recall first that the $*$-deep traps constitute a subsequence of the deep traps. Furthermore, we have $\mathcal{E}_2(n) \subset \mathcal{E}^*(n)$. Therefore, Lemma 3.1 implies Lemma 3.2. □

3.2 The embedded random walk

Let us first introduce $\zeta_n := \inf\{k \ge 0 : Y_k = n\}$, the hitting time of site $n \in \mathbb{N}$ for the embedded random walk Y. Observe that since Y is transient, we have $\zeta_n < \infty$, for all n almost surely. To control the behavior of Y, we consider the following fact, which is a classical result for biased random walks.

Fact 1. *Let $\mathcal{A}(n) := \{\min_{1\le i<j\le \zeta_n}(Y_j - Y_i) > -\nu(n)\}$, then we have*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{A}(n)) = 1. \tag{3.13}$$

Observe that, on $\mathcal{A}(n)$, each time X (or Y) hits a site x, it will necessarily exit $B_{\nu(n)}(x)$ on the right.

3.3 Between deep traps

Here, we prove that the time spent between deep trap is negligible.

Lemma 3.3. *Let us define $\mathcal{I}(n) := \left\{\sum_{i=0}^{\zeta_n} \tau_{Y_i} \mathbf{e}_i \mathbf{1}_{\{\tau_{Y_i} < g(n)\}} < \frac{n^{1/\alpha}}{\log n}\right\}$. Then, we have*

$$\mathbb{P}\left(\mathcal{I}(n)\right) \to 1, \qquad n \to \infty. \tag{3.14}$$

Proof. Observe first that, on $\mathcal{A}(n)$, we have $\inf_{i\le\zeta_n} Y_i \ge -\nu(n)$ and that Fact 3.2 implies $\mathbb{P}\left(\mathcal{I}(n)^c\right) = \mathbb{P}\left(\mathcal{I}(n)^c \cap \mathcal{A}(n)\right) + o(1)$. Therefore, using Markov inequality, we only have to prove that

$$\mathbb{E}\left[\sum_{i=0}^{\zeta_n} \tau_{Y_i} \mathbf{e}_i \mathbf{1}_{\{Y_i \ge -\nu(n)\}} \mathbf{1}_{\{\tau_{Y_i} < g(n)\}}\right] = o\Big(\frac{n^{1/\alpha}}{\log n}\Big), \qquad n \to \infty. \tag{3.15}$$

After reaching $x \in [-\nu(n), n]$ (if x is reached), the process Y visits x a geometrically distributed number of times before hitting n. The parameter of this geometrical variable is equal to $q + p\,\psi(x,n)$, where $\psi(x,n)$ denotes the probability that Y starting at $x+1$ hits x before n. An easy computation yields that

$$\psi(x,n) = r\frac{1-r^{n-x-1}}{1-r^{n-x}}, \tag{3.16}$$

where $r = r_\varepsilon := q_\varepsilon/p_\varepsilon < 1$. We will denote by $G(x,n)$ the mean of this geometrical random variable. Moreover, let us use respectively $\mathbb{P}_\tau(\cdot)$ and $\mathbb{E}_\tau[\cdot]$ to denote the conditional probability and the conditional expectation with respect to τ. Recalling that each visit takes an exponential time of mean τ_x, we obtain

$$\mathbb{E}_\tau\left[\sum_{i=0}^{\zeta_n} \tau_{Y_i}\mathbf{e}_i \mathbf{1}_{\{Y_i \geq -\nu(n)\}}\mathbf{1}_{\{\tau_{Y_i}<g(n)\}}\right] \leq \sum_{x=-\nu(n)}^{n} \tau_x(1+G(x,n))\mathbf{1}_{\{\tau_x<g(n)\}}. \tag{3.17}$$

Since $x \mapsto G(x,n)$ is decreasing and $G(-\nu(n),n) \to (1-v_\varepsilon)/v_\varepsilon$, when $n\to\infty$, we get that the expectation in (3.17) is, for all large n, less than $Cn\,\mathbb{E}\left[\tau_0\,;\,\tau_0 < g(n)\right] = Cn\,\mathbb{E}\left[\tau_0\,;\,1<\tau_0<g(n)\right]+O(n)$. Now, let us fix $0<\rho<1$ and introduce $\omega = \omega(n) := \inf\{j \geq 0:\ \rho \leq \rho^j g(n) < 1\}$. Then, we get

$$\begin{aligned}\mathbb{E}\left[\tau_0\,;\,1<\tau_0<g(n)\right] &\leq g(n)\sum_{j=0}^{\omega-1}\rho^j\mathbb{P}(\tau_0>\rho^{j+1}g(n)) \qquad (3.18)\\ &\leq Cg(n)^{1-\alpha}\sum_{j=0}^{\omega-1}\rho^{-\alpha j} \leq Cg(n)^{1-\alpha},\end{aligned}$$

where we used the fact that (2.2) yields that there exists $0<C<\infty$ such that $\mathbb{P}(\tau_x \geq u) \leq Cu^{-\alpha}$, for all $u>0$. Therefore, recalling (3.17), the fact that $ng(n)^{1-\alpha}$ is a $o(n^{1/\alpha}/\log n)$ concludes the proof of Lemma 3.3. □

3.4 Occupation time of a deep trap

Since $\zeta_y < \infty$ for all $y\in\mathbb{N}$, we can properly define for $x\in\mathbb{N}$,

$$T_x = T_x(n) := \sum_{0}^{\zeta_{x+\nu(n)}} \tau_{Y_i}\mathbf{e}_i\mathbf{1}_{\{Y_i=x\}}, \tag{3.19}$$

$$\overline{T}_x = \overline{T}_x(n) := \sum_{0}^{\zeta_{x+\nu(n)}} \tau_{Y_i}\mathbf{e}_i\mathbf{1}_{\{Y_i\in B_{\nu(n)}(x)\}}. \tag{3.20}$$

Moreover, let us introduce $\mathbb{P}^x$ and $\mathbb{E}^x$ the probability and the expectation associated with the process starting at site x. For convenience of notations, we write $\lambda_n := \lambda/n^{1/\alpha}$ for any $\lambda>0$. Then we have the following estimate for the Laplace transforms of T_x and $\overline{T}_x$.

Lemma 3.4. *For all $x\in\mathbb{N}$ and all $\lambda>0$, we have*

$$\mathbb{E}^x\left[1-e^{-\lambda_n T_x}|\tau_x\geq g(n)\right] \sim \frac{\mathbb{P}(\tau_x\geq g(n))^{-1}}{n}\frac{\alpha\pi}{\sin(\alpha\pi)}v_\varepsilon^{-\alpha}\lambda^\alpha, \qquad n\to\infty, \tag{3.21}$$

and the same result holds with T_x replaced by $\overline{T}_x$.

Proof. Let us first write

$$\mathbb{E}^x\left[(1-e^{-\lambda_n T_x})\mathbf{1}_{\{\tau_x\geq g(n)\}}\right] = \mathbb{E}\left[\mathbb{E}^x_\tau[1-e^{-\lambda_n T_x}]\mathbf{1}_{\{\tau_x\geq g(n)\}}\right]. \tag{3.22}$$

Starting at site x, the process Y visits x a geometrically distributed number of times before reaching $x+\nu(n)$. An easy computation yields that the mean of this geometrical variable, denoted by $G(x,x+\nu(n))$ satisfies $1+G(x,x+\nu(n)) \to \upsilon_\varepsilon^{-1}$, when $n\to\infty$. Therefore, recalling that each visit takes an exponential time of mean τ_x, we obtain

$$\mathbb{E}_\tau^x[e^{-\lambda_n T_x}] = \frac{1}{1+\lambda_n \upsilon_\varepsilon^{-1}\tau_x} + o(n^{-1/\alpha}), \qquad n\to\infty. \tag{3.23}$$

Now, using an integration by part, we get that $\mathbb{E}^x\Big[(1-e^{-\lambda_n T_x})\mathbf{1}_{\{\tau_x\ge g(n)\}}\Big]$ is equal to

$$\Big[-\frac{\lambda_n\upsilon_\varepsilon^{-1}z}{1+\lambda_n\upsilon_\varepsilon^{-1}z}\mathbb{P}(\tau_x\ge z)\Big]_{g(n)}^{\infty} + \int_{g(n)}^{\infty}\frac{\lambda_n\upsilon_\varepsilon^{-1}}{(1+\lambda_n\upsilon_\varepsilon^{-1}z)^2}\mathbb{P}(\tau_x\ge z)\,\mathrm{d}z + o(n^{-1/\alpha}). \tag{3.24}$$

The first term is lower than $C\lambda_n g(n)^{1-\alpha} = C\lambda_n^\alpha(\lambda_n g(n))^{1-\alpha} = o(n^{-1})$, since $\alpha<1$. For the second term, using (2.2), we can estimate $\mathbb{P}(\tau_x\ge z)$ by $(1-\eta)z^{-\alpha}\le\mathbb{P}(\tau_x\ge z)\le(1+\eta)z^{-\alpha}$, for any η, when n is sufficiently large (recall that $g(n)\to\infty$, when $n\to\infty$). Hence, we are lead to compute the integral

$$\int_{g(n)}^{\infty}\frac{\lambda_n\upsilon_\varepsilon^{-1}}{(1+\lambda_n\upsilon_\varepsilon^{-1}z)^2}z^{-\alpha}\,\mathrm{d}z = (\lambda_n\upsilon_\varepsilon^{-1})^\alpha\int_{\frac{\lambda_n\upsilon_\varepsilon^{-1}g(n)}{1+\lambda_n\upsilon_\varepsilon^{-1}g(n)}}^{1} y^{-\alpha}(1-y)^\alpha\,\mathrm{d}y, \tag{3.25}$$

(making the change of variables $y=\lambda_n\upsilon_\varepsilon^{-1}z/(1+\lambda_n\upsilon_\varepsilon^{-1}z)$). For $\alpha<1$ this integral converges, when $n\to\infty$, to $\Gamma(\alpha+1)\Gamma(-\alpha+1)=\frac{\pi\alpha}{\sin(\pi\alpha)}$, which concludes the proof of (3.21).

To prove that the result is true with $\overline{T}_x$ in place of T_x, observe first that $\mathbb{P}(\tau_x\ge g(n); \max_{y\in B_{\nu(n)}(x)\setminus\{x\}}\tau_y\ge g(n)) = o(n^{-1})$, when $n\to\infty$, which implies

$$\mathbb{E}^x\Big[(1-e^{-\lambda_n\overline{T}_x})\mathbf{1}_{\{\tau_x\ge g(n)\}}\Big] = \mathbb{E}^x\Big[(1-e^{-\lambda_n\overline{T}_x})\mathbf{1}_{\mathcal{E}_4(n)}\Big] + o(n^{-1}), \tag{3.26}$$

where $\mathcal{E}_4(n):=\{\tau_x\ge g(n)\}\cap\{\max_{y\in B_{\nu(n)}(x)\setminus\{x\}}\tau_y<g(n)\}$. Then, let us introduce $\tilde{T}_x := \sum_0^{\zeta_{x+\nu(n)}}\tau_{Y_i}\mathbf{e}_i\mathbf{1}_{\{Y_i\in B_{\nu(n)}(x)\setminus\{x\}\}} = \overline{T}_x - T_x$ and write

$$\mathbb{E}^x\Big[(e^{-\lambda_n T_x}-e^{-\lambda_n\overline{T}_x})\mathbf{1}_{\mathcal{E}_4(n)}\Big] \le \lambda_n\mathbb{E}^x\Big[\tilde{T}_x\mathbf{1}_{\mathcal{E}_4(n)}\Big], \tag{3.27}$$

where we used the fact that $1-\mathrm{e}^{-x}\le x$, for any $x\ge 0$. Using the same arguments as in the proof of Lemma 3.3, we can prove that

$$\mathbb{E}_\tau^x\Big[\tilde{T}_x\mathbf{1}_{\mathcal{E}_4(n)}\Big] \le \mathbf{1}_{\{\tau_x\ge g(n)\}}\sum_{y\in B_{\nu(n)}(x)\setminus\{x\}}\tau_y(1+G(y,x+\nu(n))\mathbf{1}_{\{\tau_y<g(n)\}}. \tag{3.28}$$

Using the fact that the previous sum depends only on site y in $B_{\nu(n)}(x)$ which are different from x, together with the same arguments as in the proof of Lemma 3.3, we get $\mathbb{E}^x\Big[\tilde{T}_x\mathbf{1}_{\mathcal{E}_4(n)}\Big] \le C\nu(n)g(n)^{1-\alpha}\mathbb{P}(\tau_x\ge g(n)) \le C\nu(n)g(n)^{1-2\alpha}$. Therefore, we obtain that the left-hand term in (3.27) is a $o(n^{-1})$, which together with (3.26) concludes the proof of Lemma 3.4. □

Remark. For any $t > 0$, let us first introduce $n_t := t^\kappa \log\log t$ and $\overline{\nu}(n_t) := C' \log\log n_t$. We consider

$$T^*(x) = T^*(x, n_t) \quad := \quad \sum_0^{\zeta_{x+\overline{\nu}(n_t)}} \tau_{Y_i} \mathbf{e}_i \mathbf{1}_{\{Y_i \in [x-\nu(n_t),\, x+\overline{\nu}(n_t)]\}}, \qquad x \in \mathbb{Z}. \tag{3.29}$$

Then, observe that the same arguments as in the proof of Lemma 3.4 yield that, for all $\lambda > 0$, we have

$$\mathbb{E}^x \Big[1 - e^{-\lambda \frac{T^*(x)}{t}} | \tau_x \geq g(n_t) \Big] \sim \frac{\mathbb{P}(\tau_x \geq g(n_t))^{-1}}{t^\alpha} \frac{\alpha\pi}{\sin(\alpha\pi)} v_\varepsilon^{-\alpha} \lambda^\alpha, \qquad t \to \infty. \tag{3.30}$$

4 Proof of Theorem 2.2

Let us first define $H_x := \inf\{t \geq 0 : X_t = x\}$, for any $x \in \mathbb{N}$. Now, fix $T > 0$, and let $H_t^{(N)}$ be the sequence of elements of $D([0, T])$ defined by

$$H_t^{(N)} := \frac{H_{\lfloor tN \rfloor}}{N^{1/\alpha}}, \qquad 0 \leq t \leq T. \tag{4.1}$$

Proposition 4.1. *The distribution of the process $(H_t^{(N)};\, 0 \leq t \leq T)$ converges weakly to the distribution of $(v_\varepsilon^{\#})^{-1/\alpha} V_\alpha(t);\, 0 \leq t \leq T)$ on $D([0, T])$ equipped with the M_1-Skorokhod topology, where $(V_\alpha(t);\, t \geq 0)$ is a standard α-stable subordinator.*

Proof. Let $0 = u_0 < u_1 < \cdots < u_K \leq T$ and $\beta_i > 0$ for $i \in \{1, \ldots, K\}$. We will check the convergence of the finite-dimensional distributions of H by proving the convergence of $\mathbb{E}[\exp\{-\sum_{i=1}^K \beta_i (H_{u_i}^{(N)} - H_{u_{i-1}}^{(N)})\}]$.

Observe first that since for any $x \in \mathbb{Z}$, we have $\mathbb{P}(\max_{y \in B_{\nu(TN)}(x)} \tau_y > g(TN)) = o(1)$, when $N \to \infty$, Lemma 3.3 yields

$$\mathbb{P}\Big(\sum_{i=0}^{\zeta_{\lfloor u_K N \rfloor}} \tau_{Y_i} \mathbf{e}_i \mathbf{1}_{\{Y_i \in B_{\nu(TN)}(\lfloor u_{K-1} N \rfloor)\}} < C N^{1/\alpha} (\log N)^{-1} \Big) \to 1, \qquad N \to \infty. \tag{4.2}$$

This implies that the time spent by X in $B_{\nu(TN)}(\lfloor u_{K-1} N \rfloor)$ is negligible. Recalling that on $\mathcal{A}(TN)$, the process never backtracks more than $\nu(TN)$, this allows us to decompose its trajectory in two main parts that are disjoint: the first between 0 and $H_{\lfloor u_{K-1} N \rfloor - \nu(TN)}$, the second between $H_{\lfloor u_{K-1} N \rfloor}$ and $H_{\lfloor u_K N \rfloor}$ (the time spent between $H_{\lfloor u_{K-1} N \rfloor - \nu(TN)}$ and $H_{\lfloor u_{K-1} N \rfloor}$ being negligible). More precisely, on $\mathcal{A}(TN)$ the process between $H_{\lfloor u_{K-1} N \rfloor}$ and $H_{\lfloor u_K N \rfloor}$ as the same law as the same process starting at site $\lfloor u_{K-1} N \rfloor$, reflected at $\lfloor u_{K-1} N \rfloor - \nu(TN)$ and independent of $(X_t; \leq t \leq H_{\lfloor u_{K-1} N \rfloor - \nu(TN)})$. Therefore, recalling Fact 3.2, the expectation $\mathbb{E}[\exp\{-\sum_{i=1}^K \beta_i (H_{u_i}^{(N)} - H_{u_{i-1}}^{(N)})\}]$ can be written

$$\mathbb{E}\Big[\exp\Big\{ -\sum_{i=1}^{K-1} \beta_i (H_{u_i}^{(N)} - H_{u_{i-1}}^{(N)}) \Big\} \Big] \mathbb{E}^{\lfloor u_{K-1} N \rfloor} \Big[\exp\Big\{ -\beta_K N^{-1/\alpha} H_{\lfloor u_K N \rfloor} \Big\} \Big] + o(1). \tag{4.3}$$

Using the strong markov property at $H_{\lfloor u_{K-1}N\rfloor}$ and the shift invariance of the environment, we just have to prove that

$$\mathbb{E}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha} H_{N'}}\right] \longrightarrow \exp\Big\{-\frac{\alpha\pi}{\sin(\alpha\pi)} v_\varepsilon^{-\alpha}\beta_K^\alpha(u_K-u_{K-1})\Big\}, \qquad N\to\infty, \tag{4.4}$$

where $N' := \lfloor u_K N\rfloor - \lfloor u_{K-1}N\rfloor \sim (u_K-u_{K-1})N$, when $N\to\infty$. Indeed, iterating this procedure $K-2$ times will give the convergence of the finite-dimensional distributions.

Recalling Lemma 3.1, Fact 3.2 and Lemma 3.3, we obtain

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha} H_{N'}}\right] &= \mathbb{E}\left[\mathbf{1}_{\mathcal{E}(N')\cap\mathcal{A}(N')\cap\mathcal{I}(N')}\mathrm{e}^{-\beta_K N^{-1/\alpha}H_{N'}}\right]+o(1)\\ &= \mathbb{E}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha}\sum_{i=1}^{\theta_{N'}} T_{\delta_i(N')}}\right]+o(1)\\ &= \mathbb{E}\left[\mathbf{1}_{\mathcal{E}^*(N')}\mathrm{e}^{-\beta_K N^{-1/\alpha}\sum_{i=1}^{\theta^*_{N'}} T_{\delta^*_i(N')}}\right]+o(1).\end{aligned} \tag{4.5}$$

Furthermore, since on $\mathcal{E}^*(N')\cap\mathcal{A}(N')$ the process never backtracks before $\delta_i^*-\nu(N')$ after hitting δ_i^* for $1\le i\le\theta^*_{N'}$, we get, by applying the strong markov property at the stopping times $H_{\delta_{\theta^*_{N'}}},\dots,H_{\delta^*_1}$,

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha} H_{N'}}\right] &= \mathbb{E}\left[\mathbf{1}_{\mathcal{E}^*(N')\cap\mathcal{A}(N')}\prod_{j=1}^{\theta^*_{N'}}\mathbb{E}^{\delta_i^*}_{\tau,|\delta_i^*-\nu}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha}T_{\delta_i^*}}\right]\right]+o(1)\\ &\le \mathbb{E}\left[\prod_{j=1}^{\underline{\theta}_{N'}}\mathbb{E}^{\delta_i^*}_{\tau,|\delta_i^*-\nu}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha}T_{\delta_i^*}}\right]\right]+o(1),\end{aligned} \tag{4.6}$$

where $\underline{\theta}_{N'} := N'\varphi(N')\big(1-\frac{1}{\log N'}\big)$ and with $\mathbb{E}^x_{\tau,|y}$ denoting the law of the process in the environment τ, starting at x and reflected at site y. Then, applying the Markov property (for the environment) successively at times $\delta_{\underline{\theta}_{N'}-1}+\nu(N'),\ \dots,\ \delta_1+\nu(N')$, and observing that the $\left(\mathbb{E}^{\delta_i^*}_{\tau,|\delta_i^*-\nu}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha}T_{\delta_i^*}}\right]\right)_{1\le j\le\underline{\theta}_{N'}}$ are i.i.d. random variables by definition, we obtain that

$$\mathbb{E}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha} H_{N'}}\right] \le \mathbb{E}\left[\mathbb{E}^{\delta_1^*}_{\tau,|\delta_1^*-\nu}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha}T_{\delta_1^*}}\right]\right]^{\underline{\theta}_{N'}}+o(1). \tag{4.7}$$

Since an easy computations yields that $\mathbb{P}(\delta_1^*\neq\delta_1)=\mathbb{P}(\max_{0\le y\le\nu(N')}\tau_y\ge g(N')) = o((N'\varphi(N'))^{-1})$ and $\mathbb{P}(H_{-\nu(N')}<H_{\nu(N')}) = o((N'\varphi(N'))^{-1})$ when $N'\to\infty$ (or equivalently when $N\to\infty$), we get

$$\mathbb{E}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha} H_{N'}}\right] \le \mathbb{E}^x\left[e^{-\beta_K N^{-1/\alpha}T_x}|\tau_x\ge g(N')\right]^{\underline{\theta}_{N'}}+o(1). \tag{4.8}$$

Now, using Lemma 3.4, this yields

$$\limsup_{N\to\infty}\mathbb{E}\left[\mathrm{e}^{-\beta_K N^{-1/\alpha} H_{N'}}\right] \le \exp\Big\{-\frac{\alpha\pi}{\sin(\alpha\pi)}\,v_\varepsilon^{-\alpha}\beta_K^\alpha(u_K-u_{K-1})\Big\}. \tag{4.9}$$

Moreover, we can similarly obtain the same lower bound, which implies (4.4) and concludes the proof of the convergence of the finite-dimensional distributions.

For the tightness, the arguments are exactly the same as in [1]. We refer to section 5 of [1] for a detailed discussion. □

Proof of Theorem 2.2. If we define $\overline{X}_t^{(N)} := \sup_{0\le s\le t} X_s^{(N)}$ for any $t \ge 0$, then Proposition 4.1 implies that the distribution of the process $(\overline{X}_t^{(N)};\ 0 \le t \le T)$ converges weakly to the distribution of $(v_\varepsilon^{\#} V_\alpha^{-1}(t);\ 0 \le t \le T)$ on $D([0,T])$ equipped with the uniform topology. Then, Theorem 2.2 will be a consequence of the fact that

$$\mathbb{P}\left(\sup\left\{|X_t^{(N)} - \overline{X}_t^{(N)}|;\ 0 \le t \le T\right\} > \gamma\right) \longrightarrow 0, \qquad N \to \infty, \tag{4.10}$$

for any $\gamma > 0$. To prove (4.10), recall first that Proposition 4.1 implies that $\mathbb{P}(H_{N^\alpha \log N} > TN) \to 1$, when $N \to \infty$, such that we can consider $\sup\{|X_t - \overline{X}_t|;\ 0 \le t \le H_{\lfloor N^\alpha \log N\rfloor}\}$, which by definition is bounded by $\max\{|Y_k - \overline{Y}_k|;\ 0 \le k \le \zeta_{\lfloor N^\alpha \log N\rfloor}\}$. Moreover, observe that on $\mathcal{A}(\lfloor N^\alpha \log N\rfloor)$, whose probability tends to 1 when N goes to infinity, this quantity is less than $\nu(\lfloor N^\alpha \log N\rfloor) = o(N^\alpha)$, when $N \to \infty$. This yields (4.10) and concludes the proof of Theorem 2.2. □

5 Proof of Theorem 2.3

To bound the number of traps the random walk can cross before time t let us consider $n_t := t^\kappa \log\log t$ and observe that Theorem 2.2 implies that $\mathbb{P}(\overline{X}_t \ge n_t) \to 0$, $t \to \infty$. Moreover, since we need more concentration properties for the random walk in the neighborhood of the δ_j's, we introduce $\overline{\nu} = \overline{\nu}(n_t) := C' \log\log n_t$, for some C' large enough which will be chosen later. For convenience of notations we will use ν, $\overline{\nu}$ and δ_j in place of $\nu(n_t)$, $\overline{\nu}(n_t)$ and $\delta_j(n_t)$ throughout this section.

Then, we define the sequence of random times $(T_j^*)_{j\ge1}$ as follows: conditioning on τ, $(T_j^*)_{j\ge1}$ is defined as an independent sequence of random variables with the law of $H_{\delta_j^*+\overline{\nu}}$ in the environment τ starting at site δ_j^* and reflected at $\delta_j^* - \nu$. Hence, under the annealed law $\mathbb{P}$, the T_j^*'s are are i.i.d. since the $B_\nu(\delta_j^*)$'s are i.i.d. by definition. Then, we give an analogous result to the extension of Dynkin's theorem proved in [14] (see Proposition 1 in [14]).

Proposition 5.1. *For any $t > 0$, let $\ell_t^* := \sup\{j \ge 0 :\ T_1^* + \cdots + T_j^* \le t\}$. Then, for all $0 \le x_1 < x_2 \le 1$, we have*

$$\lim_{t\to\infty} \mathbb{P}(t(1-x_2) \le T_1^* + \cdots + T_{\ell_t^*}^* \le t(1-x_1)) = \frac{\sin(\alpha\pi)}{\pi}\int_{x_1}^{x_2} \frac{x^{-\alpha}}{(1-x)^{\alpha-1}}\,\mathrm{d}x. \tag{5.1}$$

For all $0 \le x_1 < x_2$, we have

$$\lim_{t\to\infty} \mathbb{P}(t(1+x_1) \le T_1^* + \cdots + T_{\ell_t^*+1}^* \le t(1+x_2)) = \frac{\sin(\alpha\pi)}{\pi}\int_{x_1}^{x_2} \frac{\mathrm{d}x}{x^\alpha(1+x)}. \tag{5.2}$$

Proof. Observe first that an easy computation yields that $\mathbb{P}^x(H_{x-\nu} < \infty) = O(r_\varepsilon^\nu)$, when $t \to \infty$ (where $r_\varepsilon := q_\varepsilon/p_\varepsilon < 1$). Moreover, we have $r_\varepsilon^{\nu(n_t)} = o((t^\alpha \varphi(n_t))^{-1})$. Therefore, Remark 3.4 yields

$$\mathbb{E}\Big[1 - \mathrm{e}^{-\lambda\frac{T_1^*}{t}}\Big] \sim \frac{\mathbb{P}(\tau_x \ge g(n_t))^{-1}}{t^\alpha}\frac{\alpha\pi}{\sin(\alpha\pi)}\, v_\varepsilon^{-\alpha}\,\lambda^\alpha, \qquad t \to \infty. \tag{5.3}$$

Then, the arguments are exactly the same as in the proof of Proposition 1 in [14]. Observe that this result would exactly be Dynkin's theorem (see Feller, vol. II, [15], p. 472) if the sequence $(T_j^*)_{j\geq 1}$ was an independent sequence of random variables in the domain of attraction of a stable law of index α. Here, this sequence depends implicitly on the time t, since the $*$-deep traps are defined from the critical depth $g(n_t)$. $\square$

Recalling Lemma 3.3, we will now prove that the results of Proposition 5.1 are still true if we consider, in addition, the inter-arrival times between deep traps. Before, let us define the notion of inter-arrival times for any $0 \leq x < y$:

$$H(x,y) := \inf\{t \geq 0 : \ X_{H_x+t} = y\}. \tag{5.4}$$

Proposition 5.2. *For any $t > 0$, let $\ell_t := \sup\{j \geq 0 : \ H_{\delta_j} \leq t\}$. Then, we have*

$$\lim_{t\to\infty} \mathbb{P}(H_{\delta_{\ell_t}} \leq t < H_{\delta_{\ell_t}+\overline{\nu}}) = 1. \tag{5.5}$$

For all $0 \leq x_1 < x_2 \leq 1$, we have

$$\lim_{t\to\infty} \mathbb{P}(t(1-x_2) \leq H_{\delta_{\ell_t}} \leq t(1-x_1)) = \frac{\sin(\alpha\pi)}{\pi} \int_{x_1}^{x_2} \frac{x^{-\alpha}}{(1-x)^{\alpha-1}}\, \mathrm{d}x. \tag{5.6}$$

For all $0 \leq x_1 < x_2$, we have

$$\lim_{t\to\infty} \mathbb{P}(t(1+x_1) \leq H_{\delta_{\ell_t+1}} \leq t(1+x_2)) = \frac{\sin(\alpha\pi)}{\pi} \int_{x_1}^{x_2} \frac{\mathrm{d}x}{x^{\alpha}(1+x)}. \tag{5.7}$$

Proof. We first need to prove that after hitting $\delta_j + \overline{\nu}$, the particle does not backtrack more than $\overline{\nu}$. We detail this result with the following lemma.

Lemma 5.3. *Let us define $\mathcal{B}(n_t) := \mathcal{A}(n_t) \cap \bigcap_{j=1}^{\theta_{n_t}} \{H(\delta_j + \overline{\nu}, \delta_j + \nu) < H(\delta_j + \overline{\nu}, \delta_j)\}$. Then, we have*

$$\lim_{t\to\infty} \mathbb{P}\left(\mathcal{B}(n_t)\right) = 1. \tag{5.8}$$

Proof. Observe first that Fact 3.2 says that $\mathbb{P}\left(\mathcal{A}(n_t)\right)$ tends to one. Recalling that on $\mathcal{E}(n_t) \cap \mathcal{E}^*(n_t)$, whose probability tends to 1 when t tends to infinity (by Lemma 3.1 and Lemma 3.2), the intervals $B_\nu(\delta_j)$'s are i.i.d. and that the number of traps is bounded by $C(\log n_t)^{\frac{2\alpha}{1-\alpha}}$, it is sufficient to prove that

$$\mathbb{P}(\zeta_{-\overline{\nu}} < \infty) = o((\log n_t)^{-\frac{2\alpha}{1-\alpha}}), \qquad t \to \infty. \tag{5.9}$$

Since we have $\mathbb{P}(\zeta_{-\overline{\nu}} < \infty) \leq C r_\varepsilon^{\overline{\nu}}$, we obtain (5.9) and conclude the proof of Lemma 5.3 by choosing C' larger than $-2\alpha/(1-\alpha)\log r_\varepsilon$. $\square$

Let us introduce $\mathcal{C}(n_t) := \{\overline{X}_t \leq n_t\}$, whose probability tends to one (recall Theorem 2.2). Now, to prove Proposition 5.2, observe that on $\mathcal{E}^*(n_t) \cap \mathcal{A}(n_t)$, the random times $(H(\delta_j, \delta_j + \overline{\nu}))_{1\leq j\leq \theta^*_{n_t}}$ have the same law as the random times $(T_j^*)_{1\leq j \leq \theta^*_{n_t}}$ defined previously. If we define $\tilde{\ell}_t := \sup\{j \geq 0 : \ H(\delta_1, \delta_1 + \overline{\nu}) + \cdots + H(\delta_j, \delta_j + \overline{\nu}) \leq t\}$, then, using Proposition 5.1, Lemma 3.2 and Fact 3.2, we get that the result of Proposition

5.1 is true with $(H(\delta_j, \delta_j+\overline{\nu}))_{1\le j\le \theta^*_{n_t}}$ and $\tilde{\ell}_t$ in place of $(T^*_j)_{1\le j\le \theta^*_{n_t}}$ and ℓ^*_t. Now, recalling Lemma 3.3 and since $n_t^{1/\alpha}/\log n_t = o(t)$, when $t\to\infty$, we obtain that

$$\begin{aligned}
&\liminf_{t\to\infty} \mathbb{P}(\tilde{\ell}_t = \ell_t - 1\,;\, H_{\delta_{\ell_t}} \le t < H_{\delta_{\ell_t}+\overline{\nu}}) \\
\ge\ &\liminf_{t\to\infty} \mathbb{P}(\mathcal{I}(n_t)\,;\,\mathcal{B}(n_t)\,;\,\mathcal{C}(n_t)\,;\,|t-(H(\delta_1,\delta_1+\overline{\nu})+\cdots+H(\delta_{\tilde{\ell}_t},\delta_{\tilde{\ell}_t}+\overline{\nu}))|\ge \xi t),
\end{aligned}$$

for all $\xi > 0$. Thus, using Lemma 3.3, Lemma 5.3, Proposition 5.1 (for $\tilde{\ell}_t$ and $(H(\delta_j,\delta_j+\overline{\nu}))_{1\le j\le\theta^*_{n_t}}$) and letting ξ tends to 0, we get that

$$\lim_{t\to\infty} \mathbb{P}(\tilde{\ell}_t = \ell_t - 1\,;\, H_{\delta_{\ell_t}} \le t < H_{\delta_{\ell_t}+\overline{\nu}}) = 1. \tag{5.10}$$

We conclude the proof by the same type of arguments. □

To complete the proof of Theorem 2.3, we will prove the following *localization* result, which means that the particle is, with an overwhelming probability, in the last visited deep trap.

Proposition 5.4. *We have*

$$\lim_{t\to\infty} \mathbb{P}(X_t = \delta_{\ell_t}) = 1. \tag{5.11}$$

Proof. Now, for any deep trap δ_j, let us denote by μ_j the invariant measure associated with the random walk on $[\delta_j-\nu, \delta_j+\overline{\nu}]$ reflected at sites $\delta_j-\nu$ and $\delta_j+\overline{\nu}$ and normalized such that $\mu_j(\delta_j)=1$. Clearly, μ_j is the reversible measure given, for any $\delta_j-\nu < x < \delta_j+\overline{\nu}$, by

$$\mu_j(x) = r_\varepsilon^{\delta-x}\frac{\tau_x}{\tau_{\delta_j}}. \tag{5.12}$$

Since the random walk is reflected at sites $\delta_j-\nu$ and $\delta_j+\overline{\nu}$, we have $\mu_j(\delta_j-\nu)\le \tau_{\delta_j-\nu}/\tau_{\delta_j}$ and $\mu_j(\delta_j-\nu)\le r_\varepsilon^{\overline{\nu}}\tau_{\delta_j+\overline{\nu}}/\tau_{\delta_j}$. Moreover, since μ_j is an invariant measure and since $\mu_j(\delta_j)=1$, we have, for any $x\in[\delta_j-\nu,\delta_j+\overline{\nu}]$ and all $s\ge 0$,

$$\mathbb{P}^{\delta_j}_{\tau,|\delta_j-\nu,\delta_j+\overline{\nu}|}(X_s = x) \le \mu_j(x). \tag{5.13}$$

Furthermore, let us introduce the event

$$\mathcal{D}(n_t) := \bigcap_{j=1}^{\theta_{n_t}} \left\{ \max_{x\in B_\nu(\delta_j)\setminus\{\delta_j\}} \tau_x < (\log n_t)^\beta \right\} \tag{5.14}$$

with $\beta > \frac{1}{\alpha}(\frac{2\alpha}{1-\alpha}+1+\gamma)$. Observe that the probability of $\mathcal{D}(n_t)$ tends to one, when t tends to infinity. Indeed, since the number of traps is less than $C(\log n_t)^{\frac{2\alpha}{1-\alpha}}$, and recalling that the number of sites contained in the $B_\nu(\delta_j)$'s is less than 2ν (with $\nu=\nu(n_t)=(\log n_t)^{1+\gamma}$), this fact is just a consequence of (2.2). Recalling (5.12), observe that on $\mathcal{D}(n_t)$, we have

$$\mu_{j|[\delta_j-\nu,\delta_j+\overline{\nu}]\setminus\{\delta_j\}} \le C r_\varepsilon^{\overline{\nu}}(\log n_t)^{\beta+\frac{2}{1-\alpha}}\, n_t^{-\frac{1}{\alpha}} \le C n_t^{-\frac{1}{2\alpha}}, \tag{5.15}$$

for any $1 \le j \le \theta_{n_t}$. Hence, combining (5.13) and (5.15), we obtain on $\mathcal{D}(n_t)$

$$\mathbb{P}^{\delta_j}_{\tau,|\delta_j-\nu,\delta_j+\overline{\nu}|}(X_s \neq \delta_j) \le C n_t^{-\frac{1}{2\alpha}}, \qquad \forall\, s \ge 0. \tag{5.16}$$

Now, we fix $0 < \xi < 1$. Then, let us write that $\liminf_{t\to\infty} \mathbb{P}(X_t = \delta_{\ell_t})$ is larger than

$$\begin{aligned} &\liminf_{t\to\infty} \mathbb{P}(X_t = \delta_{\ell_t}\,;\, \ell_t = \ell_{(1+\xi)t}) \\ \ge\ &\liminf_{t\to\infty} \mathbb{P}(\ell_t = \ell_{(1+\xi)t}) - \limsup_{t\to\infty} \mathbb{P}(X_t \neq \delta_{\ell_t}\,;\, \ell_t = \ell_{(1+\xi)t}). \end{aligned} \tag{5.17}$$

Considering the first term, we get using Proposition 5.2 that it is equal to

$$\liminf_{t\to\infty} \mathbb{P}(H_{\delta_{\ell_t+1}} > (1+\xi)t) = \frac{\sin(\alpha\pi)}{\pi} \int_\xi^\infty \frac{\mathrm{d}x}{x^\alpha(1+x)}. \tag{5.18}$$

In order to estimate the second term, let us introduce the event

$$\mathcal{F}(n_t) := \mathcal{B}(n_t) \cap \mathcal{C}(n_t) \cap \mathcal{D}(n_t) \cap \mathcal{E}(n_t) \cap \mathcal{E}^*(n_t) \cap \mathcal{I}(n_t) \cap \left\{H_{\delta_{\ell_t}} \le t < H_{\delta_{\ell_t}+\overline{\nu}}\right\}.$$

Observe that the preliminary results obtained in Section 3 together with Theorem 2.2, Proposition 5.2 and Lemma 5.3 imply that $\mathbb{P}(\mathcal{F}(n_t)) \to 1$, when $t \to \infty$. Then, we have that $\limsup_{t\to\infty} \mathbb{P}(X_t \neq \delta_{\ell_t}\,;\, \ell_t = \ell_{t(1+\xi)})$ is less than

$$\begin{aligned} &\limsup_{t\to\infty} \mathbb{P}(\mathcal{F}(n_t)\,;\, X_t \neq \delta_{\ell_t}\,;\, \ell_t = \ell_{t(1+\xi)}) \\ \le\ &\limsup_{t\to\infty} \mathbb{E}\Big[\mathbf{1}_{\mathcal{F}(n_t)} \sum_{j=1}^{\theta_{n_t}} \mathbf{1}_{\{X_t \neq \delta_{\ell_t}\,;\, \ell_t = \ell_{t(1+\xi)} = j\}}\Big]. \end{aligned} \tag{5.19}$$

But on the event $\mathcal{F}(n_t) \cap \{\ell_t = \ell_{t(1+\xi)} = j\}$ we know that for all $s \in [H_{\delta_j}, t]$ the walk X_s is in the interval $[\delta_j - \nu, \delta_j + \overline{\nu}]$. Indeed, on the event $\mathcal{B}(n_t) \cap \mathcal{C}(n_t) \cap \mathcal{I}(n_t)$ we know that once the position $\delta_j + \overline{\nu}$ is reached then within a time $n_t^{1/\alpha}/\log n_t = o(t)$, when $t \to \infty$, the position δ_{j+1} is reached which would contradict the fact that $\ell_{t(1+\xi)} = j$. Hence, we obtain, for all $j \in \mathbb{N}$,

$$\begin{aligned} &\mathbb{P}\left(\mathcal{F}(n_t)\,;\, j \le \theta_{n_t}\,;\, X_t \neq \delta_{\ell_t}\,;\, \ell_t = \ell_{t(1+\xi)} = j\right) \\ \le\ &\mathbb{E}\Big[\mathbf{1}_{\{j \le \theta_{n_t}\}} \mathbf{1}_{\mathcal{D}(n_t) \cap \mathcal{E}(n_t)} \sup_{s \in [0,t]} \mathbb{P}^{\delta_j}_{\tau,|\delta_j-\nu,\delta_j+\overline{\nu}|}(X_s \neq \delta_j)\Big] \le C n_t^{-\frac{1}{2\alpha}}, \end{aligned} \tag{5.20}$$

where we used (5.16) on the event $\mathcal{D}(n_t)$. Considering now that, on the event $\mathcal{E}(n_t)$, the number θ_{n_t} of deep traps is smaller than $C(\log n_t)^{\frac{2\alpha}{1-\alpha}}$ we get that

$$\limsup_{t\to\infty} \mathbb{P}(X_t \neq \delta_{\ell_t}\,;\, \ell_t = \ell_{t(1+\xi)}) = 0. \tag{5.21}$$

Then, assembling (5.17), (5.18), (5.21) and letting ξ tends to 0 in (5.18) concludes the proof of Proposition 5.4. □

Proof of Theorem 2.3. let us fix $h > 1$ and introduce the event

$$\mathcal{G}(t,h) := \{X_t = \delta_{\ell_t}\} \cap \{X_{th} = \delta_{\ell_{th}}\}, \tag{5.22}$$

whose probability tends to 1, when t tends to infinity (it is a consequence of Proposition 5.4). Then, we easily have $\{X_{th} = X_t\} \cap \mathcal{G}(t,h) = \{\ell_{th} = \ell_t\} \cap \mathcal{G}(t,h)$. Therefore, since Proposition 5.2 implies that $\lim_{t\to\infty} \mathbb{P}(\ell_{th} = \ell_t)$ exists, we obtain

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}(X_{th} = X_t) &= \lim_{t\to\infty} \mathbb{P}(\ell_{th} = \ell_t) = \lim_{t\to\infty} \mathbb{P}(T_{\ell_t+1} \geq th) \\ &= \frac{\sin(\alpha\pi)}{\pi} \int_0^{1/h} y^{\kappa-1}(1-y)^{-\kappa}\,\mathrm{d}y, \end{aligned} \tag{5.23}$$

which concludes the proof of Theorem 2.3. □

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