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PRINCIPLE OF LINEARIZED STABILITY AND SMOOTH CENTER MANIFOLD THEOREM FOR SEMILINEAR HYPERBOLIC SYSTEMS

MARK LICHTNER

Abstract. In this paper principle of linearized stability and smooth center manifold theorem are proven for a general class of semilinear hyperbolic systems (SH) in one space dimension. They are of the following form: For $0 < x < l$ and $t > 0$

$$\text{(SH)} \begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + H(x,u(t,x),v(t,x)) = 0, \\ \frac{d}{dt}\left[v(t,l) - Du(t,l)\right] = F(u(t,\cdot),v(t,\cdot)), \\ u(t,0) = E\,v(t,0), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), \end{cases}$$

where $u(t,x) \in \mathbb{R}^{n_1}$, $v(t,x) \in \mathbb{R}^{n_2}$, $K(x) = \operatorname{diag}\,(k_i(x))_{1\le i\le n}$ is a diagonal matrix of functions $k_i \in C^1([0,l],\mathbb{R})$, $k_i(x) > 0$ for $i = 1,\dots,n_1$ and $k_i(x) < 0$ for $i = n_1+1,\dots,n = n_1+n_2$, and D,E are matrices.

First we prove that weak solutions to (SH) define a smooth semiflow in a Banach space X of continuous functions under natural conditions on the nonlinearities H and F. Then we show a spectral gap mapping theorem for linearizations of (SH) in the complexification of X, which implies that growth and spectral bound coincide. Consequently we obtain principle of linearized stability for (SH). Moreover, the spectral gap mapping theorem characterizes exponential dichotomy in terms of a spectral gap of the infinitesimal generator for linearized hyperbolic systems. This resolves a key problem in applying invariant manifold theory to prove smooth center manifold theorem for (SH).

1. INTRODUCTION

The behaviour of an evolution equation near some stationary state can be determined by a decomposition into invariant (stable, unstable and center) manifolds. The stable and unstable manifolds are described by exponential decay and growth estimates, respectively. Of particular interest are center manifolds which completely determine the local dynamics and bifurcations near a stationary state.

This strategy has a long tradition for analysing the local dynamics of ordinary differential equations and in the last few decades significant progress has been made to extend the theory to infinite dimensional evolutionary equations, see for example [2, 3, 4, 29].

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For partial differential equations the relation between the linearization and the full nonlinear problem is more delicate. For a proof on existence and smoothness of invariant manifolds one usually assumes smoothness of the nonlinear Nemytskij operator or of the solution map [3, 4, 29] and exponential rates estimates (exponential dichotomy) for the spectral decompositions. As is well known the latter is equivalent to a spectral gap condition on the linearized semigroup (Theorem 4.3).

However, almost always one only has knowledge on the location of the spectrum of the infinitesimal generator, that is the equations and not the semigroup. Relating the spectrum of the infinitesimal generator to that of the semigroup is a spectral mapping problem which is often nontrivial [7, 13, 16].

For differentiablity of the nonlinear Nemytskij operator some regularity is required. Hence we seek to solve the spectral mapping problem in a suitable small function space. To have a natural description for the local dynamics of (SH) we work in a "small" Banach space of continuous functions X equipped with the supremum norm (not L^2 in contrast to [7, 16]). In X the equations (SH) generate a smooth semiflow (Theorem 2.5). For linearizations of (SH) we prove that an "open spectral gap mapping property holds" in the complexification $X^{\mathbb{C}}$ of the small Banach space X (Theorem 2.12). This implies that growth and spectral bound coincide in $X^{\mathbb{C}}$ and yields principle of linearized stability for (SH) (Theorem 2.8 and Theorem 2.16). Moreover, the spectral gap mapping theorem implies existence of dichotomic projections in X under the presence of a spectral gap for the infinitesimal generator of the linearization (Theorem 2.17). Thus we resolve a key problem in applying invariant manifold theory [3, 4] to prove existence of a smooth center manifold (Theorem 2.11) for the general class (SH). To prove the spectral gap mapping theorem in $X^{\mathbb{C}}$, according to theory of Kaashoek, Lunel and Latushkin [12, 14], we additionally estimate Fourier transforms of matrix elements of the resolvent on lines parallel to the imaginary axis. For this we use precise resolvent estimates, see Lemma 4.6 and [16].

The motivation of our work originated from applications in laser dynamics, where traveling wave models are used to describe the longitudinal dynamics of semiconductor lasers, see for example [5, 24, 31]. They belong to the class (SH) of semilinear hyperbolic systems. To understand and control the dynamics of these models has been of practical interest since the corresponding lasers are of technological importance for high speed signal generation and clock recovery in optoelectronic networks. For this a numerical bifurcation analysis of a center manifold reduced set of equations has been implemented, see for example [1, 5, 21, 24, 25, 26].

Yet, the theory of invariant manifolds has been mainly developed for semilinear parabolic PDEs or functional differential equations. There were no results for hyperbolic systems of PDEs belonging to the general class (SH). Moreover, due to this it has been unknown whether the stability analysis, performed in [9, 10] and [11] for Turing models with correlated random walk within a purely linear context only, implies stability and the occurrence of bifurcations on center manifolds near the homogeneous steady state of the nonlinear problem.

In this paper this situation is resolved for a large class of semilinear hyperbolic systems (SH) (including functional differential equations) in one space dimension which often appear in applications, see for example [10, 11, 15, 19, 20, 23].

2. Results

We consider the following class of semilinear hyperbolic systems

$$\text{(SH)}\qquad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x)\end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x)\end{pmatrix} + H(x,u(t,x),v(t,x)) = 0, \\ \dfrac{d}{dt}\left[v(t,l) - Du(t,l)\right] = F(u(t,\cdot),v(t,\cdot)), \\ u(t,0) = E\,v(t,0), \end{cases}$$

for $x \in]0,l[$ and $t > 0$ with the following assumptions:

(SHI) $K(x) = \operatorname{diag}\,(k_i(x))_{i=1,\ldots,n}$ is a diagonal $n \times n$ matrix of functions $k_i \in C^1([0,l],\mathbb{R})$ which satisfy $k_i(x) > 0$ for $i = 1,\ldots,n_1$ and $k_i(x) < 0$ for $i = n_1+1,\ldots,n = n_1+n_2$ $(x \in [0,l])$.

(SHII) The map $H :]0,l[\times \mathbb{R}^n \to \mathbb{R}^n$, $H = H(x,z)$, $x \in]0,l[$, $z = (u,v) \in \mathbb{R}^n$, satisfies a C^k Carathéodory condition, $k \geq 1$:
- For $a.a.$ $x \in]0,l[$ $H(x,\cdot) \in C^k(\mathbb{R}^n)$ and $H(\cdot,z)$ is measurable for all $z \in \mathbb{R}^n$.
- For all compact $K \subset \mathbb{R}^n$ there exists a constant $M > 0$ such that $\left\|\frac{\partial^i H(x,z)}{\partial z^i}\right\| \leq M$ for $0 \leq i \leq k$, all $z \in K$ and $a.a.$ $x \in]0,l[$.
- For all compact $K \subset \mathbb{R}^n$ and $\epsilon > 0$ there exists a $\delta > 0$ such that for all $z_1 \in K$, $z_2 \in \mathbb{R}^n$ with $\|z_1 - z_2\| < \delta$ and $a.a.$ $x \in]0,l[$ we have $\left\|\frac{\partial^k S(x,z_1)}{\partial z^k} - \frac{\partial^k S(x,z_2)}{\partial z^k}\right\| < \epsilon$.

(SHIII) $F : C([0,l],\mathbb{R}^n) \to \mathbb{R}^{n_2}$ is C^k and has bounded and uniformly continuous derivatives on bounded sets: for each $b > 0$ and $\epsilon > 0$ there exists $\delta > 0$ so that $\|\partial^k F(u_1,v_1) - \partial^k F(u_2,v_2)\| \leq \epsilon$ for $(u_1,v_1),(u_2,v_2) \in C([0,l],\mathbb{R}^{n_1+n_2})$ with $\|(u_1,v_1) - (u_2,v_2)\| \leq \delta$ and $\|(u_1,v_1)\| \leq b$.

(SHIV) $u(t,x) \in \mathbb{R}^{n_1}$, $v(t,x) \in \mathbb{R}^{n_2}$

(SHV) $D \in \mathbb{R}^{n_2 \times n_1}$, $E \in \mathbb{R}^{n_1 \times n_2}$

The function H generates a superposition operator via

$$\mathfrak{H}(u,v)(x) := H(x,u(x),v(x)) \quad \text{for almost all } x \in [0,l].$$

It follows that the map $\mathfrak{H}$ is C^k-smooth from $L^\infty(]0,l[;\mathbb{R}^n)$ into itself [8].

For (SH) we use the phase space

$$X := \{(u,v,d) \in C([0,l];\mathbb{R}^n) \times \mathbb{R}^{n_2} \mid u(0) = Ev(0),\ d = \Delta(u,v)\},$$

where X is equipped with the supremum norm $\|(u,v,d)\|_X := \|(u,v)\|_\infty + \|d\|$ and

$$\Delta(u,v) := v(l) - Du(l).$$

We denote the complexification of the real space X with

$$X^{\mathbb{C}} := \{(u,v,d) \in C([0,l];\mathbb{C}^n) \times \mathbb{C}^{n_2} \mid u(0) = Ev(0),\ d = \Delta(u,v)\}.$$

Let $T(t)$ denote the semigroup to

$$\text{(2.1)}\qquad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x)\end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x)\end{pmatrix} = 0, \\ \dfrac{d}{dt}\left[v(t,l) - Du(t,l)\right] = 0, \\ u(t,0) = E\,v(t,0). \end{cases}$$

For $T > 0$ denote

$$\mathcal{X}_T := C([0,T], X). \tag{2.2}$$

Definition 2.1. Let $T > 0$. The triplet $(u(\cdot), v(\cdot), \Delta(u(\cdot), v(\cdot))) \in \mathcal{X}_T$ is called a weak (or mild) solution of (SH) up to T for the initial data $(u_0, v_0, \Delta(u_0, v_0)) \in X$ if for all $t \in [0,T]$

$$(u(t), v(t), \Delta(u(t), v(t))) = \mathcal{G}(u, v, \Delta(u, v))(t),$$

where

$$\mathcal{G}(u, v, \Delta(u, v))(t) := T(t) \begin{pmatrix} u_0 \\ v_0 \\ \Delta(u_0, v_0) \end{pmatrix} + \int_0^t T(t-s) \begin{pmatrix} -\mathfrak{H}(u(s), v(s)) \\ F(u(s), v(s)) \end{pmatrix} ds. \tag{2.3}$$

Theorem 2.2 (Local existence). *For any $(u_0, v_0, \Delta(u_0, v_0)) \in X$ there exists a $\delta > 0$, depending only on $\|(u_0, v_0, \Delta(u_0, v_0))\|_X$, such that* (SH) *has a unique weak solution up to δ.*

Theorem 2.3 (Regularity). *Let $z = (u, v, \Delta(u, v)) \in \mathcal{X}_T$ be a weak solution of* (SH) *with initial data $z(0) = (u_0, v_0, \Delta(u_0, v_0)) \in X$. Suppose*

$$(u_0, v_0) \in W^{1,\infty}(]0, l[, \mathbb{R}^n).$$

Then for all $p \in]1, \infty[$

$$\begin{aligned} (u, v, \Delta(u, v)) &\in C([0,T], W^{1,p}(]0, l[, \mathbb{R}^n) \times \mathbb{R}^{n_2}) \\ &\cap C^1([0,T], L^p(]0, l[, \mathbb{R}^n) \times \mathbb{R}^{n_2}) \end{aligned} \tag{2.4}$$

and (SH) *holds in a classical sense.*

Theorem 2.4. *Let $z \in \mathcal{X}_T$ be a weak solution of* (SH) *up to T. Then there exists a neighborhood U of $z(0)$ in X such that for all $y_0 \in U$ there is a weak solution $y \in \mathcal{X}_T$ of* (SH) *up to T satisfying $y(0) = y_0$.*
There exists a constant $c > 0$ such that for all $y_0 \in U$

$$\|z(t) - y(t)\|_X \le c \|z(0) - y_0\|_X .$$

Suppose there exists a weak solution $z \in \mathcal{X}_T$ of (SH) up to T. Then according to Theorem 2.4 there exists an open neighborhood U of $z(0)$ in X so that we can define a solution map

$$S^t : U \to X, \quad S^t(y_0) := y(t) \quad (t \in [0,T]). \tag{2.5}$$

Theorem 2.5 (Smooth semiflow property). *For each $t \in [0,T]$ the map $S^t : U \to X$ is C^k smooth. The map $(t, u) \mapsto S^t u$ is continuous from $[0,T] \times U$ into X.*

Definition 2.6. We call $a \in X$ a stationary or equilibrium solution of (SH) if the constant function $z(t) := a$ is a weak solution of (SH) in the sense of Definition 2.1.

Proposition 2.7. A state $a = (a_u, a_v, \Delta(a_u, a_v)) \in X$ is an equilibrium solution if and only if there exists $p \in [1, \infty[$ so that $(a_u, a_v) \in W^{1,p}([0, l], \mathbb{R}^{n_1+n_2})$ and both $K\partial_x(a_u, a_v) + \mathfrak{H}(a_u, a_v) = 0$ and $F(a_u, a_v) = 0$ vanish. In this case $(a_u, a_v) \in \bigcap_{1 \le p < \infty} W^{1,p}([0, l], \mathbb{R}^{n_1+n_2})$.

Suppose a is an equilibrium. We formally linearize (SH) in a and obtain the following:

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} \\ +\partial_{(u,v)}H(x,a_u(x),a_v(x))\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ \frac{d}{dt}\left[v(t,l) - Du(t,l)\right] = \partial F(a_u,a_v)\begin{pmatrix} u(t,\cdot) \\ v(t,\cdot) \end{pmatrix}, \\ u(t,0) = E\,v(t,0). \end{array}\right. \tag{2.6}$$

Let A_a be the corresponding infinitesimal generator in the complexification $X^{\mathbb{C}}$ of X,

$$A_a\begin{pmatrix} u \\ v \\ \Delta(u,v) \end{pmatrix} := \begin{pmatrix} -K(\cdot)\frac{\partial}{\partial x} - \partial_{(u,v)}H(\cdot,a_u(\cdot),a_v(\cdot)) \\ \partial F(a_u,a_v) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

with domain

$$\mathcal{D}(A_a) = \left\{(u,v,\Delta(u,v)) \in X^{\mathbb{C}} \mid A_a(u,v,\Delta(u,v)) \in X^{\mathbb{C}}\right\}.$$

Then A_a generates a C_0 semigroup on $X^{\mathbb{C}}$.

Theorem 2.8 (Principle of linearized stability). *If* $\sup\{\Re\mathfrak{e}\,\lambda \mid \lambda \in \sigma(A_a)\} < 0$ *and the conditions* (HI)-(HIII) *printed below hold true for* (2.6)*, then a is exponentially stable: There exists a neighborhood U of a in X and constants $c > 0$, $\beta > 0$, such that if z is a weak solution of* (SH) *with $z(0) \in U$ then z exists for all $t \geq 0$, lies in U and*

$$\|z(t) - a\|_X \leq ce^{-\beta t}\|z(0) - a\|_X \quad \text{for} \quad t \geq 0.$$

Next we formulate center manifold theorem for (SH): Suppose

$$\sigma(A_a) \subset \{\lambda \in \mathbb{C} \mid \Re\mathfrak{e}\,\lambda \leq 0\} \quad \text{and} \quad E_c := \sigma \cap i\mathbb{R} \neq \emptyset.$$

Assume that (HI) $-$ (HIII) hold for (2.6). Moreover, suppose that E_c is finite and only contains eigenvalues of finite algebraic multiplicities [17] and that we have a spectral gap: There exists a $\delta > 0$ so that

$$\{\lambda \in \mathbb{C} \mid -\delta < \Re\mathfrak{e}\,\lambda < \delta\} \cap \sigma = E_c.$$

Remark 2.9. Comparing with [16] this means that we have $\gamma_+ < 0$ for A_a, where γ_+ is defined as the supremum of the real part of the eigenvalues of a reduced diagonal system (H_0) obtained from (2.6) by cancelling all non(block)diagonal entries in the differential equation (see section 4). We have shown in [16] that the spectrum is asymptotically close to the spectrum of the reduced system. It follows that for each $\gamma > \gamma_+$ there are only finitely many eigenvalues λ with $\Re\mathfrak{e}\,\lambda \geq \gamma$. Here the spectrum does not depend on the choice of the Banach space. Physically the condition $\gamma_+ < 0$ means that the system is dissipative and may only possess a finite number of critical modes, all others being (uniformly exponentially) damped.

Define the spectral projection

$$\pi_c := \int_\gamma (\lambda I - A_a)^{-1}\,d\lambda, \quad \pi_s := \mathrm{Id} - \pi_c,$$

where γ is a simple positive oriented loop in $\{\lambda \in \mathbb{C} \mid -\delta < \mathfrak{Re}\,\lambda < \delta\}$ around E_c. Denote

$$X_s^{\mathbb{C}} := \operatorname{Im}\pi_s, \quad X_c^{\mathbb{C}} := \operatorname{Im}\pi_c.$$

The linear spaces $X_c^{\mathbb{C}}$ and $X_s^{\mathbb{C}}$ decompose the complex space

$$X^{\mathbb{C}} = X_c^{\mathbb{C}} \oplus X_s^{\mathbb{C}}$$

into the direct sum of two closed linear subspaces which are invariant with respect to the semigroup $e^{A_a t}$ generated by A_a. The spaces $X_c^{\mathbb{C}}$ and $X_s^{\mathbb{C}}$ are invariant under complex conjugation. Hence

$$X_c := X_c^{\mathbb{C}} \cap X \quad \text{and} \quad X_s := X_s^{\mathbb{C}} \cap X$$

decompose the real space X

$$X = X_s \oplus X_c$$

into closed subspaces, which are invariant with respect to the a-linearized flow $\partial S^t(a) = e^{A_a t}_{|X}$.

Moreover, we need that F can be truncated in the following sense: Let

$$F(a_u + u, a_v + v) = F(a_u, a_v) + \partial F(a_u, a_v)(u, v) + r_F(u, v)$$

with $r_F(u,v) = o(\|(u,v)\|_\infty)$. We suppose that for any truncation parameter $\delta > 0$ there exists a C^k smooth map $r_{F\delta} : C([0,l], \mathbb{R}^{n_1+n_2}) \to \mathbb{R}^{n_2}$ having the following properties:
i) $r_{F\delta}(u,v) = r_F(u,v)$ for $(u,v) \in C([0,l], \mathbb{R}^{n_1+n_2})$ with $\|(u,v)\|_\infty \le \delta$
ii) there exists a positive function $\tilde{\delta} = \tilde{\delta}(\delta)$ with $\lim_{\delta \downarrow 0} \tilde{\delta}(\delta) = 0$ so that

$$\|r_{F\delta}(u,v)\|_\infty \le \tilde{\delta}(\delta)\delta \quad \text{and} \quad \|\partial r_{F\delta}(u,v)\|_\infty \le \tilde{\delta}(\delta)$$

for all $(u,v) \in C([0,l], \mathbb{R}^{n_1+n_2})$.

Example 2.10. Let $x_k \in [0,l]$, $1 \le k \le m$, and $F_k : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$ be C^k. Suppose F is of the form $F(\tilde{u}, \tilde{v}) = \sum_{k=1}^m F_k(\tilde{u}(x_k), \tilde{v}(x_k))$. Then F has the above truncation property.
Indeed, we have $r_F(u,v) = \sum_{k=1}^m r_k(u(x_k), v(x_k))$, where

$$\begin{aligned} r_k(u(x_k), v(x_k)) &= F_k(a_u(x_k) + u(x_k), a_v(x_k) + v(x_k)) - F_k(a_u(x_k), a_v(x_k)) \\ &\quad -\partial F_k(a_u(x_k), a_v(x_k))(u(x_k), v(x_k)). \end{aligned}$$

Then $r_{F\delta}(u,v) = \sum_{k=1}^m r_k(u(x_k), v(x_k))\chi_\delta(u(x_k), v(x_k))$ does it, where $\chi_\delta(\cdot) := \chi(\delta^{-1}\cdot)$ and $\chi : \mathbb{R}^{n_1+n_2} \to \mathbb{R}$ is a C^∞ cut off function so that $\chi(x) = 1$ for $\|x\| \le 1$, $\chi(x) = 0$ for $\|x\| \ge 2$ and $0 \le \chi(x) \le 1$ for $x \in \mathbb{R}^{n_1+n_2}$.

Theorem 2.11 (Center manifold theorem)**.** *Let $k \ge 1$. There exists an open neighborhood Ω of zero in X and a graph $\gamma \in C^k(\Omega \cap X_c, X_s)$ such that*
i) $\gamma(0) = 0$, $\partial\gamma(0) = 0$;
ii) the manifold

$$W := \{a + x_c + \gamma(x_c) \mid x_c \in \Omega \cap X_c\}$$

is locally invariant for (SH), *i.e. for $t \ge 0$ we have $S^t(W) \cap (a + \Omega) \subset W$;*
iii) if $z :]-\infty, 0] \to a + \Omega$ is a solution of (SH) *then $z(t) \in W$ for $t \in]-\infty, 0]$.*
iv) For $p \in [1, \infty[$ we have

$$\begin{aligned} \gamma(\Omega \cap X_c) &\subset X_s \cap \left(W^{1,p}(]0,l[, \mathbb{R}^{n_1+n_2}) \times \mathbb{R}^{n_2}\right), \\ W &\subset X \cap \left(W^{1,p}(]0,l[, \mathbb{R}^{n_1+n_2}) \times \mathbb{R}^{n_2}\right). \end{aligned}$$

If $z : [0,\delta] \to W$ $(\delta > 0)$ *is a solution of* (SH) *then*

$$z \in C^k([0,\delta], X).$$

The flow on W *is given by the ordinary differential equation*

$$\frac{d}{dt} x_c = A_a x_c + f(x_c),$$

where $f : X_c \to X_c$ *is* C^k *smooth,* $f(0) = 0$ *and* $\partial f(0) = 0$.

To prove the main Theorems 2.8 and 2.11 we show a spectral gap mapping theorem for A_a in the space $X^{\mathbb{C}}$.

For this we consider the following class of linear hyperbolic systems (compare with [16]):

$$\text{(H)}\quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ \dfrac{d}{dt}\left[v(t,l) - Du(t,l)\right] = Fu(t,\cdot) + Gv(t,\cdot), \\ u(t,0) = Ev(t,0). \end{cases}$$

The assumptions are:

(HI) K is a diagonal $n \times n$ matrix of the form

$$K = \begin{pmatrix} k_1 I_{d_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 I_{d_2} & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_\alpha I_{d_\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{\alpha+1} I_{d_{\alpha+1}} & 0 & 0 \\ & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{\alpha+\beta} I_{d_{\alpha+\beta}} \end{pmatrix},$$

where $d_i \in \mathbb{N}$, $d_i > 0$, $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$, $\sum_{i=1}^{\alpha} d_i = n_1$, $\sum_{i=1}^{\beta} d_{\alpha+i} = n_2$, I_{d_i} denotes the identity matrix in $\mathbb{C}^{d_i \times d_i}$ and $k_i \in C^1([0,l], \mathbb{R})$ satisfy for $x \in [0,l]$

$$\begin{aligned} k_i(x) &> 0 \text{ for } i = 1,\ldots,\alpha, \\ k_j(x) &< 0 \text{ for } j = \alpha+1,\ldots,\alpha+\beta. \end{aligned}$$

(HII) $C(x) = (C_{ij}(x))_{1 \le i,j \le \alpha+\beta} \in \mathbb{C}^{n \times n}$ with $C_{ij}(x) \in \mathbb{C}^{d_i \times d_j}$ and

$$\begin{aligned} C_{ii} &\in L^\infty\left(]0,l[, \mathbb{C}^{d_i \times d_i}\right), \quad i = 1,\ldots,\alpha+\beta, \\ C_{ij} &\in \mathrm{BV}\left([0,l], \mathbb{C}^{d_i \times d_j}\right), \quad i,j = 1,\ldots,\alpha+\beta \quad \text{with } i \ne j. \end{aligned}$$

(HIII) If $i \ne j$ and $k_i(x) = k_j(x)$ for some $x \in [0,l]$, then C_{ij} vanishes completely on $[0,l]$.

(HIV) $u(t,x) = (u_1(t,x),\ldots,u_{n_1}(t,x)) \in \mathbb{C}^{n_1}$ and $v(t,x) = (v_1(t,x),\ldots,v_{n_2}(t,x)) \in \mathbb{C}^{n_2}$.

(HV) $D \in \mathbb{C}^{n_2 \times n_1}$, $E \in \mathbb{C}^{n_1 \times n_2}$ and

$$F : C([0,l], \mathbb{C}^{n_1}) \to \mathbb{C}^{n_2}, \quad G : C([0,l], \mathbb{C}^{n_2}) \to \mathbb{C}^{n_2}$$

are linear continuous operators.

Let

$$A\begin{pmatrix}u\\v\\d\end{pmatrix} := \begin{pmatrix}-K\frac{\partial}{\partial x}\begin{pmatrix}u\\v\end{pmatrix} - C\begin{pmatrix}u\\v\end{pmatrix}\\ Fu+Gv\end{pmatrix} \tag{2.7}$$

denote the infinitesimal generator for (H) with domain

$$\mathcal{D}(A) := \left\{(u,v,d) \in X^{\mathbb{C}} \mid A(u,v,d) \in X^{\mathbb{C}}\right\}. \tag{2.8}$$

Then A generates a C_0 semigroup e^{At} on $X^{\mathbb{C}}$.

Theorem 2.12 (Spectral gap mapping theorem in Banach space $X^{\mathbb{C}}$)**.** *Consider A and the C_0 semigroup e^{At} in the complex Banach space $X^{\mathbb{C}}$. Let $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}$. Then*

$$\{\lambda \in \mathbb{C} \mid \alpha < \Re\mathfrak{e}\,\lambda < \beta\} \subset \rho(A) \quad \textit{if and only if} \quad \{\lambda \in \mathbb{C} \mid e^{\alpha t} < |\lambda| < e^{\beta t}\} \subset \rho(e^{At}).$$

We recall the notion of growth, spectral bound and exponential dichotomy:

Definition 2.13 (Growth and spectral bound)**.** The growth bound $\omega(A)$, also denoted $\omega(e^{At})$, is defined through

$$\omega(A) := \inf\Big\{\omega \in \mathbb{R} \mid \text{there exists a positive number } M = M(\omega) \text{ such that } \left\|e^{At}\right\| \le Me^{\omega t} \text{ for } t \ge 0\Big\}.$$

The spectral bound $s(A)$ is defined via

$$s(A) := \sup\{\Re\mathfrak{e}\,\lambda \mid \lambda \in \sigma(A)\}.$$

Definition 2.14 (Exponential dichotomy)**.** Suppose $\alpha < \beta$, then A has a (α, β) exponential dichotomy if there exists a projection $\pi : X^{\mathbb{C}} \to X^{\mathbb{C}}$ so that $\pi e^{At} = e^{At}\pi$ and for $T_1(t) := e^{At}_{|\pi(X^{\mathbb{C}})}$ and $T_2(t) := e^{At}_{|(I-\pi)(X^{\mathbb{C}})}$ one has $\omega(T_1(t)) \le \alpha$ and $T_2(t)$ extends to a group with $\omega(T_2(-t)) \le -\beta$.

Remark 2.15. It follows that π is unique. One calls π the splitting projection.

The next two theorems are a consequence or reformulation of Theorem 2.12:

Theorem 2.16 (Spectrum determined growth)**.** *Growth and spectral bound coincide:*

$$\omega(A) = s(A).$$

Theorem 2.17 (Spectrum determined exponential dichotomy)**.** *Suppose $\alpha < \beta$ and $\{\lambda \in \mathbb{C} \mid \alpha < \Re\mathfrak{e}\,\lambda < \beta\} \subset \rho(A)$. Then A has a (α, β) exponential dichotomy.*

Remark 2.18. Theorems 2.12, 2.16 and 2.17 also hold for the Banach space $X_p^{\mathbb{C}} := L^p([0,l], \mathbb{C}^n) \times \mathbb{C}^{n_2}$.

3. Proof of Theorems 2.2-2.5

For $1 \le p \le \infty$ let

$$X_p := L^p([0,l], \mathbb{R}^n) \times \mathbb{R}^{n_2}.$$

By integrating along characteristics we can write a formula for the semigroup $T(t)$ corresponding to (2.1). We do not require such a formula, we only need the following

Proposition 3.1. The semigroup $T(t)$ is strongly continuous on X_p for $1 \le p < \infty$ and X . For $T > 0$ there exists $c > 0$ such that for $(u_0, v_0) \in L^\infty([0,l], \mathbb{R}^n)$ and $d_0 \in \mathbb{R}^{n_2}$ we have

$$\|T(t)(u_0, v_0, d_0)\|_{X_\infty} \le c\,\|(u_0, v_0, d_0)\|_{X_\infty} \quad \text{for} \quad 0 \le t \le T.$$

In particular Proposition 3.1 states that $T(t)$ is a semigroup of bounded operators on X_∞. It is not C_0 (see also [18]), even not Bochner measurable according to the following:

Remark 3.2. The map

$$s \mapsto T(t-s)\begin{pmatrix} -\mathfrak{H}(u(s), v(s), w(s)) \\ F(u(s), v(s)) \end{pmatrix}$$

is not Bochner-measurable into X_∞. Hence the integral

$$\int_0^t T(t-s)\begin{pmatrix} -\mathfrak{H}(u(s), v(s), w(s)) \\ F(u(s), v(s)) \end{pmatrix} ds \tag{3.1}$$

can not be interpreted as a Bochner integral in the space X_∞.

Indeed, consider a real valued step function on $[0, l]$ which has a jump (shock) at $\frac{l}{2}$. Then translation of this function is not measurable on a time interval with values into the Banach space $L^\infty([0,l], \mathbb{R})$, because the image is not separable with respect to the strong L^∞ norm. Now the Nemytskij operator $\mathfrak{H}$ will not be compatible with boundary conditions (even if the generating function is linear with constant coefficients, in general), so that shocks will travel along the characteristics when the translation semigroup $T(t-s)$ is applied.

We propose two possibilities for defining (3.1): We can consider (3.1) as a Bochner integral in the Banach space X_p for $1 \le p < \infty$: Because T is a strongly continuous semigroup on X_p the integrand becomes Bochner measurable. Alternatively we can avoid Bochner integration and consider (3.1) in the weak star sense of X_∞ since the semigroup T is X_∞ weak star measurable. As is easily verified, we are allowed to estimate the X_∞ norm of the integral (3.1): Let $f : [0, T] \to X_\infty$ be Bochner measurable and bounded. Then we have

$$\left\| \int_0^t T(t-s) f(s)\, ds \right\|_{X_\infty} \le c \int_0^t \|f(s)\|_{X_\infty}\, ds. \tag{3.2}$$

We will need such L^∞ estimates several times.

Theorem 3.3. *Weak solutions of* (SH) *are unique.*

Proof. The proof is standard and uses Gronwall's Lemma and (3.2): Let

$$z_1 = (u_1, v_1, \Delta(u_1, v_1)),\ z_2 = (u_2, v_2, \Delta(u_2, v_2)) \in \mathcal{X}_T$$

be solutions of (SH) with $z_1(0) = z_2(0)$. By (SHII) $\mathfrak{H}$ is locally Lipschitz on $L^\infty(]0, l[\,, \mathbb{R}^n)$. Hence, by Proposition 3.1, there exist constants $c > 0$ and $L > 0$ so that for $t \in [0, T]$

$$\begin{aligned} \|z_1(t) - z_2(t)\|_X &\le \left\| \int_0^t T(t-s)\begin{pmatrix} -\mathfrak{H}(u_1(s), v_1(s)) + \mathfrak{H}(u_2(s), v_2(s)) \\ F(u_1(s), v_1(s)) - F(u_2(s), v_2(s)) \end{pmatrix} ds \right\|_{X_\infty} \\ &\le \int_0^t c \left\| \begin{pmatrix} \mathfrak{H}(u_1(s), v_1(s)) - \mathfrak{H}(u_2(s), v_2(s)) \\ F(u_1(s), v_1(s)) - F(u_2(s), v_2(s)) \end{pmatrix} \right\|_{X_\infty} ds \\ &\le cL \int_0^t \|z_1(s) - z_2(s)\|_X\, ds. \end{aligned}$$

Gronwall's inequality yields

$$\|(z_1 - z_2)(t)\|_X = 0 \quad \text{for } t \in [0,T].$$

□

Let $A_* = \begin{pmatrix} -K(\cdot)\partial_x \\ 0 \end{pmatrix}$ be the infinitesimal generator of the C_0 semigroup $T(t)$ on X_p, $1 \le p < \infty$, with domain

$$\mathcal{D}(A_*) = \left\{(u,v,d) \in W^{1,p}([0,l],\mathbb{R}^n) \times \mathbb{R}^{n_2} \mid u(0) = Ev(0),\ d = \Delta(u,v)\right\}.$$

We will use the following well known Proposition (see for example [6, Proposition 4.1.6, p.51])

Proposition 3.4. Let $f \in W^{1,1}(]0,T[\,,X_p)$ and

$$v(t) := \int_0^t T(t-s)f(s)\,ds.$$

Then

$$v \in C([0,T],\mathcal{D}(A_*)) \cap C^1([0,T],X_p)$$

and $\frac{d}{dt}v(t) = A_*v(t) + f(t)$.

Proposition 3.5. Suppose

$$\rho \in C\left([0,T], L^\infty(]0,l[\,,\mathbb{R}^n) \times \mathbb{R}^{n_2}\right).$$

Then

$$\int_0^\cdot T(\cdot - s)\rho(s)\,ds \in C([0,T],X).$$

Proof. By mollification there exists a sequence $\rho_k \in C^1([0,T],X_\infty)$ such that ρ_k converges uniformly to ρ in $C([0,T],X_\infty)$. Since $\rho_k \in C^1([0,T],X_p)$ Proposition 3.4 yields

$$\int_0^\cdot T(\cdot - s)\rho_k(s)\,ds \in C([0,T],D(A_*)) \hookrightarrow C([0,T],X).$$

For $t \in [0,T]$ by (3.2) and Proposition 3.1

$$\begin{aligned}\left\|\int_0^t T(t-s)(\rho(s)-\rho_k(s))\,ds\right\|_{X_\infty} &\le T \sup_{s\in[0,T]} \|T(s)\|_{\mathcal{L}(X_\infty)} \|\rho-\rho_k\|_{C([0,T],X_\infty)} \\ &\le c\,\|\rho-\rho_k\|_{C([0,T],X_\infty)}.\end{aligned}$$

Hence we have

$$\int_0^\cdot T(\cdot - s)\rho(s)\,ds \in C([0,T],X).$$

□

Corollary 3.6.

- If $(u_0,v_0,\Delta(u_0,v_0)) \in X$ and $(u,v,\Delta(u,v)) \in \mathcal{X}_T$, then

$$T(\cdot)\begin{pmatrix} u_0 \\ v_0 \\ \Delta(u_0,v_0) \end{pmatrix} + \int_0^\cdot T(\cdot - s)\begin{pmatrix} -\mathfrak{H}(u(s),v(s)) \\ F(u(s),v(s)) \end{pmatrix} ds \in C([0,T],X).$$

- If $(u, v, \Delta(u, v)) \in \mathcal{X}_T$ is a weak solution to (SH), then

$$\Delta(u, v) \in C^1([0, T], \mathbb{R}^{n_2}) \quad \text{and} \quad \frac{d}{dt}\Delta(u, v)(t) = F(u(t), v(t)).$$

Proof of Theorem 2.3. Let $h > 0$ and $0 \le t < t + h \le T$. Then

$$\begin{aligned} z(t+h) - z(t) =& (T(h) - I)T(t)z(0) \\ &+ \int_0^t T(t-s) \begin{pmatrix} -\mathfrak{H}((u,v)(h+s)) + \mathfrak{H}((u,v)(s)) \\ F(u(h+s), v(h+s)) - F(u(s), v(s)) \end{pmatrix} ds \\ &+ \int_0^h T(t+h-s) \begin{pmatrix} -\mathfrak{H}((u,v)(s)) \\ F(u(s), v(s)) \end{pmatrix} ds. \end{aligned}$$

By (SHII), (SHIII) and Proposition 3.1 there exists $c > 0$ so that

$$\begin{aligned} \|z(t+h) - z(t)\|_X \le& \|(T(h) - I)T(t)z(0)\|_{X_\infty} + ch \\ &+ c\int_0^t \|z(s+h) - z(s)\|_X \, ds. \end{aligned}$$

Moreover,

$$(T(h) - I)T(t)z(0) = \int_0^h T(s)T(t)(A_* z(0)) \, ds.$$

And because $A_* z(0) \in X_\infty$ by assumption we have (the constant c will differ from each line)

$$\|(T(h) - I)T(t)z(0)\|_{X_\infty} \le ch.$$

Hence

$$\|z(t+h) - z(t)\|_X \le ch + c\int_0^t \|z(s+h) - z(s)\|_X \, ds.$$

Gronwall's Lemma yields

$$\|z(t+h) - z(t)\|_X \le hc.$$

Hence $z : [0, T] \to X$ and

$$\begin{pmatrix} -\mathfrak{H}((u,v,w)(\cdot)) \\ F(u(\cdot), v(\cdot)) \end{pmatrix} : [0, T] \to X_\infty \subset X_p$$

are Lipschitz continuous. Because X_p is reflexive for $1 < p < \infty$ we have

$$\begin{pmatrix} -\mathfrak{H}((u,v,w)(\cdot)) \\ F(u(\cdot), v(\cdot)) \end{pmatrix} \in W^{1,\infty}([0, T], X_p)$$

and Proposition 3.4 yields the assertion. □

Proof of Theorem 2.2. By Corollary 3.6 $\mathcal{G}$ maps $\mathcal{X}_T$ into itself. Let $0 < \delta < 1$. Define the closed subspace of $\mathcal{X}_\delta$ (recall (2.2))

$$B_\delta := \Big\{ z = (u, v, \Delta(u, v)) \in \mathcal{X}_\delta \mid \text{for } t \in [0, \delta] \ \|z(t) - T(t)z_0\|_X \le 1 \Big\}.$$

By (SHII) and (SHIII) and (3.2) and Proposition 3.1 there exists $L > 0$, depending (essentially) only on $\|z_0\|_X$, such that if $z_1, z_2 \in B_\delta$ then

$$\|\mathcal{G}(z_1)(t) - \mathcal{G}(z_2)(t)\|_X \le \delta L \|z_1 - z_2\|_{\mathcal{X}_\delta}. \tag{3.3}$$

Moreover, since $\mathfrak{H}$ and F are locally bounded it follows from the definition of B_δ that there exists a bound $M > 0$, depending only on $\|z_0\|_X$, such that for $z \in B_\delta$

$$\|\mathcal{G}(z)(t) - T(t)(z_0)\|_X \leq \left\| \int_0^t T(t-s) \begin{pmatrix} -\mathfrak{H}(u(s), v(s)) \\ F(u(s), v(s)) \end{pmatrix} ds \right\|_{X_\infty} \leq M\delta \qquad \text{for } t \in [0, \delta]. \tag{3.4}$$

Therefore (3.3) and (3.4) imply that for sufficiently small $\delta > 0$ the operator $\mathcal{G}$ maps B_δ into itself and becomes a contraction. By Banachs contraction mapping theorem G has a fixed point in $B_\delta \subset \mathcal{X}_\delta$. □

For $z_0 \in X$ let $\omega = \omega(z_0) \in \,]0, \infty]$ denote the maximal time up to which the solution exists, i.e.

$$\omega(z_0) := \sup\{t \in \mathbb{R} \mid \text{there exists a weak solution up to } t \text{ with } z(0) = z_0\}.$$

We have the following standard consequence of Theorem 2.2

Corollary 3.7. For any $z_0 \in X$ either
i) $\omega(z_0) = \infty$
or
ii) $\omega(z_0) < \infty$ and $\lim_{t \uparrow \omega(z_0)} \|z(t)\|_X = \infty$, where $z : [0, \omega(z_0)[\, \to X$ denotes the weak solution with $z(0) = z_0$.

For a proof of Theorem 2.4 one can proceed as in [27, Theorem 11.15, p. 117].

Proof of Theorem 2.5. For $z = (\hat{z}, \Delta\hat{z}) \in \mathcal{X}_T$ and initial data $z_0 = (\hat{z}_0, \Delta\hat{z}_0) \in X$ the operator $\mathcal{G}(z)$ has been defined in formula (2.3) of Definition 2.1. To emphasize the dependence on z_0 we write $\mathcal{G}(z, z_0)$. Define

$$(\mathcal{F}(z, z_0))\,(t) := (\mathcal{G}(z, z_0))\,(t) - z(t).$$

By Corollary 3.6 $\mathcal{G}(\cdot, z_0)$ maps $\mathcal{X}_T$ into itself for each $z_0 \in X$. Thus $\mathcal{F} : \mathcal{X}_T \times X \to \mathcal{X}_T$. For each $z_0 \in U$ the equation $\mathcal{F}(z, z_0) = 0, z \in \mathcal{X}_T$, has a unique solution $z = \gamma(z_0)$.
It follows from (SHII), (SHIII) and definition of $\mathcal{G}$ that $\mathcal{G}$ is C^k from $\mathcal{X}_T \times X$ into $\mathcal{X}_T$ and that we have for $h_j = (\hat{h}_j, \Delta\hat{h}_j) \in \mathcal{X}_T$, $1 \leq j \leq k$, $t \in [0, T]$

$$\left(\frac{\partial^j \mathcal{G}}{\partial z^j}(z, z_0) h_1 \ldots h_j\right)(t) = \int_0^t T(t-s) \begin{pmatrix} -\partial^j \mathfrak{H}(\hat{z}(s))(\hat{h}_i(s))_{1 \leq i \leq j} \\ \partial^j F(\hat{z}(s))\,(\hat{h}_i(s))_{1 \leq i \leq j} \end{pmatrix} ds. \tag{3.5}$$

Indeed, for $j = 1$ we have

$$\begin{aligned} &\mathcal{G}(z + h_1, z_0)(t) - \mathcal{G}(z, z_0)(t) - \int_0^t T(t-s) \begin{pmatrix} -\partial\mathfrak{H}(\hat{z}(s))\hat{h}_1(s) \\ \partial F(\hat{z}(s))\,\hat{h}_1(s) \end{pmatrix} ds \\ &= \int_0^t T(t-s) \left[-\begin{pmatrix} -\partial\mathfrak{H}(\hat{z}(s))\hat{h}_1(s) \\ \partial F(\hat{z}(s))\hat{h}_1(s) \end{pmatrix} + \begin{pmatrix} -\mathfrak{H}(\hat{z}(s) + \hat{h}_1(s)) + \mathfrak{H}(\hat{z}(s)) \\ F(\hat{z}(s) + \hat{h}_1(s)) - F(\hat{z}(s)) \end{pmatrix} \right] ds \\ &= \int_0^t T(t-s) \int_0^1 \left[-\begin{pmatrix} -\partial\mathfrak{H}(\hat{z}(s)) \\ \partial F(\hat{z}(s)) \end{pmatrix} + \begin{pmatrix} -\partial\mathfrak{H}(\hat{z}(s) + \theta\hat{h}_1(s)) \\ \partial F(\hat{z}(s) + \theta\hat{h}_1(s)) \end{pmatrix} \right] \hat{h}_1(s)\, d\theta\, ds \end{aligned}$$

Therefore by (3.2), Proposition 3.1 and the uniform continuity of the derivative stated in conditions (SHII) and (SHIII) we have

$$\|h_1\|_{\mathcal{X}_T}^{-1} \left\| \mathcal{G}(z + h_1) - \mathcal{G}(z) - \int_0^{\cdot} T(\cdot - s) \begin{pmatrix} -\partial\mathfrak{H}(\hat{z}(s))\hat{h}_1(s) \\ \partial F(\hat{z}(s))\,\hat{h}_1(s) \end{pmatrix} ds \right\|_{\mathcal{X}_T} \overset{\|h_1\|_{\mathcal{X}_T} \downarrow 0}{\longrightarrow} 0.$$

By induction one obtains (3.5) for $1 \leq j \leq k$.
A generalization of Banachs fixed point theorem yields that $\frac{\partial \mathcal{F}}{\partial z}$ is an isomorphism from $\mathcal{X}_T$ onto itself: Indeed, assume $w \in \mathcal{X}_T$ is given. Then for $h = (\hat{h}, \Delta\hat{h}) \in \mathcal{X}_T$ the equation $\frac{\partial \mathcal{F}}{\partial z}(z, z_0)h = w$ is equivalent to $\mathcal{P}h = h$, where $\mathcal{P} : \mathcal{X}_T \to \mathcal{X}_T$,

$$(\mathcal{P}h)(t) = \int_0^t T(t-s) \begin{pmatrix} -\partial\mathfrak{H}(\hat{z}(s))\hat{h}(s) \\ \partial F(\hat{z}(s))\,\hat{h}(s) \end{pmatrix} ds - w(t).$$

There exists a constant $M > 0$, depending only on $T, \mathfrak{H}, F, z$, so that for $h_1, h_2 \in \mathcal{X}_T$

$$\|\mathcal{P}h_1(t) - \mathcal{P}h_2(t)\|_X \leq Mt\,\|h_1 - h_2\|_{\mathcal{X}_T}.$$

Proceeding with $\mathcal{P}^2 = \mathcal{P}\circ\mathcal{P}$ we get $\left\|(P^2h_1)(t) - (P^2h_2)(t)\right\|_X \leq \frac{(Mt)^2}{2}\|h_1 - h_2\|_{\mathcal{X}_T}$.
By induction

$$\left\|\mathcal{P}^i h_1 - \mathcal{P}^i h_2\right\|_{\mathcal{X}_T} \leq \frac{(MT)^i}{i!}\|h_1 - h_2\|_{\mathcal{X}_T}.$$

Thus for i sufficiently large $\mathcal{P}^i$ is a contraction on $\mathcal{X}_T$.
From the implicit function theorem it follows that γ is a C^k smooth map from U into $\mathcal{X}_T$. Hence $S^t : U \to X$ is C^k.

□

Remark 3.8. The map $S^{\cdot} : U \to \mathcal{X}_T, u \mapsto S^{\cdot}u$ is C^k smooth.

4. Proof of spectral gap mapping Theorem

In this section we prove Theorem 2.12 and the remaining consequences.

Let A denote the infinitesimal generator defined in (2.7) and (2.8).

First we need some general observations:

By Gelfand's theorem for the spectral radius one has the following [28, Proposition 1.2.2.]

Proposition 4.1. For all $t_0 > 0$

$$\omega(A) = \frac{\log r\left(e^{At_0}\right)}{t_0} = \lim_{t\to\infty} \frac{\log\left\|e^{At}\right\|}{t}.$$

Here $r\left(e^{At_0}\right)$ denotes spectral radius of e^{At_0}.

Remark 4.2. By Proposition 4.1 and the spectral inclusion Theorem $e^{t\sigma(A)} \subset \sigma(e^{At})$ one has

$$s(A) \leq \omega(A).$$

Theorem 4.3. *The following assertions are equivalent:*

ı) A is (α, β) exponentially dichotomous.

ıı) For all $t > 0$

$$\{\lambda \in \mathbb{C} \mid e^{\alpha t} < |\lambda| < e^{\beta t}\} \subset \rho(e^{At}).$$

ııı) There exists $t_0 > 0$ so that

$$\{\lambda \in \mathbb{C} \mid e^{\alpha t_0} < |\lambda| < e^{\beta t_0}\} \subset \rho(e^{At_0}).$$

If one of the conditions is true, then the splitting projection π is given by

$$\pi = \frac{1}{2\pi i}\int_{|z|=r} \left(zI - e^{At}\right)^{-1} dz,$$

where $r \in \left]e^{\alpha t}, e^{\beta t}\right[$.

Proof. Compare with [13, Lemma 2.15].
$\imath) \Rightarrow \imath\imath)$ By Proposition 4.1 we have $r(T_1^t) = \sup\{|z| \mid z \in \sigma(T_1^t)\} = e^{t\omega(T_1^t)} \leq e^{\alpha t}$ and $r(T_2^{-t}) = e^{t\omega(T_2^{-t})}$, which yields

$$\inf\{|z| \mid z \in \sigma(T_2^t)\} = \frac{1}{r(T_2^{-t})} = e^{-t\omega(T_2^{-t})} \geq e^{\beta t}.$$

Because $\rho(e^{At}) = \rho(T_1^t) \cap \rho(T_2^t)$ we have $\imath\imath)$.
$\imath\imath\imath) \Rightarrow \imath)$ Put $P := \frac{1}{2\pi i}\int_{|z|=r}\left(zI - e^{At_0}\right)^{-1} dz$, where $r \in \left]e^{\alpha t_0}, e^{\beta t_0}\right[$, and $Q := I - P$. Then for $t \geq 0$

$$Pe^{At} = \frac{1}{2\pi i}\int_{|z|=r}\left(zI - e^{At_0}\right)^{-1} e^{At}\, dz = \frac{1}{2\pi i}\int_{|z|=r} e^{At}\left(zI - e^{At_0}\right)^{-1} dz = e^{At}P.$$

Thus $X_1 := P(X^{\mathbb{C}})$ and $X_2 := Q(X^{\mathbb{C}})$ are e^{At} invariant and $r(T_1^{t_0}) \leq e^{\alpha t_0}$ if $P \neq 0$ and $r((T_2^{t_0})^{-1}) \leq e^{-\beta t_0}$ if $Q \neq 0$, where $T_1^t := e^{At}_{|X_1}$ and $T_2^t := e^{At}_{|X_2}$. From Proposition 4.1 we get $\omega((T_1^t)_{t\geq 0}) = \frac{\log r(T_1^{t_0})}{t_0} \leq \alpha$ and $\omega((T_2^{-t})_{t\geq 0}) = \frac{\log r((T_2^{t_0})^{-1})}{t_0} \leq -\beta$, if $(T_2^t)_{t\geq 0}$ extends to a C_0 group on X_2: For $\theta \in [0,1]$ we put $T_2^{-\theta t_0} := \left(T_2^{t_0}\right)^{-1} T_2^{t_0(1-\theta)}$. Then $T_2^{-\theta t_0}T_2^{\theta t_0} = T_2^{\theta t_0}T_2^{-\theta t_0} = I$, i.e. $T_2^{\theta t_0}$ is invertible. Thus for each $n \in \mathbb{N}$ and $\theta \in [0,1]$ the linear map $T_2^{n\theta t_0}$ is invertible which implies that T_2^t extends to a group on X_2. □

Assume $\alpha < \beta$ and

$$\mathbb{C}_{\alpha,\beta} = \{\lambda \in \mathbb{C} \mid \alpha < \mathfrak{Re}\,\lambda < \beta\} \subset \rho(A). \tag{4.1}$$

For Theorem 2.12 we need to prove that $\{\lambda \in \mathbb{C} \mid e^{\alpha t} < |\lambda| < e^{\beta t}\} \subset \rho(e^{At})$. By spectral inclusion Theorem the reverse statement is plain. In terms of Theorem 4.3 we will show that A has an (α,β) exponential dichotomy. For this we use the following important characterization of (α,β) exponential dichotomy obtained by Kaashoek, Lunel and Latushkin [12, 14]:

Theorem 4.4. *A is (α,β) exponentially dichotomous if and only if*

- $\imath)$ $\rho(A) \supset \mathbb{C}_{\alpha,\beta}$,
- $\imath\imath)$ *for all* $\delta > 0$ $\sup_{\lambda\in\mathbb{C}_{\alpha+\delta,\beta-\delta}} \|R(\lambda, A)\| < \infty$,
- $\imath\imath\imath)$ *for each* $\rho \in \left]\alpha,\beta\right[$ *there exists a constant* $K_\rho > 0$ *such that for all* $x \in X^{\mathbb{C}}$, $x^* \in \left(X^{\mathbb{C}}\right)^*$, *the function* $r(\cdot,\rho,x,x^*) : \mathbb{R} \to \mathbb{C}$, *defined by*

  $$r(\nu,\rho,x,x^*) = x^*R(\rho + i\nu, A)x,$$

  satisfies $\|\mathfrak{F}r(\cdot,\rho,x,x^*)\|_{L^\infty} \leq K_\rho\,\|x\|\,\|x^*\|$.

The symbol $\mathfrak{F}$ denotes Fourier transform in the sense of tempered distributions.

We will prove that conditions $\imath)$ to $\imath\imath\imath)$ of Theorem 4.4 are satisfied for A defined in (2.7) and (2.8).
For this we need to recall and refine some notions and results of [16]:
Corresponding to (H) we consider a reduced system (H_0).
Let C_{b0} be the block diagonal matrix containing the square matrices C_{ii}

$$C_{b0} := \text{blockdiag }(C_{ii})_{1\leq i\leq\alpha+\beta}\,.$$

The reduced system is per definitionem

$$(\mathrm{H}_0)\quad \begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C_{b0}(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ u(t,0) = Ev(t,0), \\ v(t,l) = Du(t,l), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x). \end{cases}$$

We denote the infinitesimal generator to (H_0)

$$A_0\begin{pmatrix} u \\ v \end{pmatrix} := -K\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} - C_{b0}\begin{pmatrix} u \\ v \end{pmatrix}$$

with domain

$$\mathcal{D}(A_0) := \left\{ (u,v) \in X_0 \mid A_0\begin{pmatrix} u \\ v \end{pmatrix} \in X_0 \right\},$$

where $X_0 := \{(u,v) \in C([0,l], \mathbb{C}^n) \mid u(0) = Ev(0),\ v(l) = Du(l)\}$. Then A_0 generates a C_0 semigroup on X_0.

Lemma 4.5. *There exists an exponential polynomial* h_0 *and an entire functions* h *with:*

- $\sigma(A) = \{\lambda \in \mathbb{C} \mid h(\lambda) = 0\}$,
- $\sigma(A_0) = \{\lambda \in \mathbb{C} \mid h_0(\lambda) = 0\}$.

A formula for h_0 is given below in (4.2). We need estimates for the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ similar to [16] but more precise including first order λ^{-1} approximations (in [16] we only needed a zero order approximation for the resolvent):

Lemma 4.6 (Estimates for resolvent). *Let* $U \subset \rho(A)$ *so that* $\sup_{\lambda\in U} |\mathfrak{Re}\,\lambda| < \infty$ *and* $\inf_{\lambda\in U} |h_0(\lambda)| > 0$. *Then there exist constants* $c, d > 0$ *so that for* $\lambda \in U$ *and* $|\mathfrak{Im}\,\lambda| \geq d$

-
$$R(\lambda, A)\begin{pmatrix} f \\ g \\ b \end{pmatrix} = \begin{pmatrix} u \\ v \\ (-D, I)\begin{pmatrix} u(l) \\ v(l) \end{pmatrix} \end{pmatrix},$$

  where

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} =& R(\lambda, A_0)\begin{pmatrix} f \\ g \end{pmatrix} + \frac{1}{\lambda}\Big(R_1(\lambda, A)\begin{pmatrix} f \\ g \end{pmatrix} + R_2(\lambda, A)\begin{pmatrix} f \\ g \end{pmatrix} \\ &+ R_3(\lambda, A)\begin{pmatrix} f \\ g \\ b \end{pmatrix} + R_4(\lambda, A)\begin{pmatrix} f \\ g \end{pmatrix}\Big) \\ &+ \frac{1}{\lambda^2}\mathcal{E}(\lambda, A)\begin{pmatrix} f \\ g \\ b \end{pmatrix} \end{aligned}$$

- $R(\lambda, A_0)$, $R_i(\lambda, A)$, $i = 1, \ldots, 4$, *and* $\mathcal{E}(\lambda, A)$ *are bounded by* c
- $R(\lambda, A)$ *is bounded by* c

- *$R_i(\lambda, A)$, $i = 1, \ldots, 4$, and $R(\lambda, A_0)$ are given by the following formulas:*

$$R(\lambda, A_0)\begin{pmatrix} f \\ g \end{pmatrix} = T_0(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}\beta_0(\lambda)(f, g) + \int_0^{\cdot} T_0(\cdot, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy,$$

where

$$\beta_0(\lambda)(f, g) := (D, -I)\int_0^l T_0(l, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy,$$

$$H_0(\lambda) := (-D, I)T_0(l, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix},$$

$$h_0(\lambda) := \det H_0(\lambda), \tag{4.2}$$

$$T_0(x, y, \lambda) := \exp\left(-\lambda\int_y^x K^{-1}(z)\,dz\right)F(x, y),$$

$$F := (\operatorname{blockdiag} F_i)_{1 \le i \le \alpha+\beta},$$

where F_i is the solution to

$$\frac{d}{dx}F_i(x, y) = -k_i^{-1}(x)C_{ii}(x)F_i(x, y), \quad F_i(y, y) = I_{d_i},$$

and

$$R_1(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} := F_1(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}\beta_0(\lambda)(f, g),$$

$$R_2(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} := -T_0(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}H_1(\lambda)H_0(\lambda)^{-1}\beta_0(\lambda)(f, g),$$

$$R_3(\lambda)\begin{pmatrix} f \\ g \\ b \end{pmatrix} := T_0(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}\beta_1(\lambda)(f, g, b),$$

$$R_4(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} := \int_0^{\cdot} F_1(\cdot, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy,$$

$$\beta_1(\lambda)(f, g, b) := b + (D, -I)\int_0^l F_1(l, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy + (F, G)\int_0^{\cdot} T_0(\cdot, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy,$$

where

$$H_1(\lambda) := -(F, G)T_0(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} - (D\delta_l, -I\delta_l)F_1(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix},$$

and F_1 is the matrix with the i-th blockdiagonal element, $1 \le i \le \alpha + \beta$,

$$(F_1(x, y, \lambda))_{ii} = -\exp\left(-\lambda\textstyle\int_y^x k_i^{-1}(u)\,du\right)F_i(x, y) \sum_{\substack{1\le\nu\le\alpha+\beta \\ \nu\ne i}}\int_y^x \frac{C_{i\nu}(z)}{k_i(z)}\rho_{\nu i}(z)F_i(z, y)\,dz,$$

where

$$\rho_{lm}(z) := \frac{C_{lm}(z)}{k_l(z)} \frac{1}{k_l^{-1}(z) - k_m^{-1}(z)}, \ z \in [0,l], \ 1 \le l, m \le \alpha+\beta, \ l \ne m,$$

and with the i***-th blockrow and*** j***-th blockcolumn,*** $1 \le i, j \le \alpha+\beta$, $i \ne j$,

$$\begin{aligned}(F_1(x,y,\lambda))_{ij} = &- \exp\left(-\lambda \int_y^x k_j^{-1}(u)\,du\right) \rho_{ij}(x) F_j(x,y) \\ &+ \exp\left(-\lambda \int_y^x k_i^{-1}(u)\,du\right) F_i(x,y)\rho_{ij}(y) \\ &+ \exp\left(-\lambda \int_y^x k_j^{-1}(u)\,du\right) F_i(x,y) \\ &\int_y^x \exp\left(\int_y^z \lambda \left(k_i^{-1}(u) - k_j^{-1}(u)\right)\,du\right) \\ &\frac{d}{dz}\left(F_i(y,z)\rho_{ij}(z)F_j(z,y)\right)\,dz.\end{aligned}$$

Now we are able to check conditions $\imath)$ to $\imath\imath\imath)$ of Theorem 4.4:

The first condition is assumption (4.1). It implies $h_0(\lambda) \ne 0$ for $\lambda \in \mathbb{C}_{\alpha,\beta}$. Indeed, if $h_0(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{C}_{\alpha,\beta}$, then there exist infinitely many zeros λ of h_0 with $\mathfrak{Re}\,\lambda$ close to $\mathfrak{Re}\,\lambda_0$ (this can be proven by applying [16, Lemma 3.8], for example). As in the proof of [16, Theorem 2.1] one deduces that there exists a $\lambda \in \mathbb{C}_{\alpha,\beta}$ with $h(\lambda) = 0$, which yields a contradiction to (4.1). Hence $h_0(\lambda) \ne 0$ for $\lambda \in \mathbb{C}_{\alpha,\beta}$. Applying [16, Lemma 3.8] we get for $\delta > 0$

$$\inf_{\lambda \in \mathbb{C}_{\alpha+\delta,\beta-\delta}} |h_0(\lambda)| > 0.$$

Thus Lemma 4.6 yields $\imath\imath)$. In the following we check the remaining condition $\imath\imath\imath)$. For this we need the precise approximation of the resolvent depicted in the formulas of Lemma 4.6.

Moreover we will apply a Cesaro/Fejér Fourier inversion formula [30] and a well known generalization of Wiener's $\frac{1}{f}$ theorem:

Theorem 4.7 (Cesaro/Fejér Fourier inversion formula)**.** *Let* $f \in L^1(\mathbb{R},\mathbb{C}) \cap L^\infty(\mathbb{R},\mathbb{C})$ *and* $t \in \mathbb{R}$ *be a point where both the limit from the right* $f(t+)$ *and left* $f(t-)$ *exist. Let* $\mathcal{F}^{-1}f(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau)\,d\tau$*. Then*

$$\frac{1}{2\pi} {}^{C_1}\!\!\int_{-\infty}^{\infty} e^{i\omega t}\mathcal{F}^{-1}f(\omega)\,d\omega = \frac{1}{2}\left(f(t+) + f(t-)\right). \tag{4.3}$$

The symbol ${}^{C_1}\!\int_{-\infty}^{\infty}$ denotes integration by Cesàro's means of order 1, i.e.

$${}^{C_1}\!\!\int_{-\infty}^{\infty} e^{i\omega t}\mathcal{F}^{-1}f(\omega)\,d\omega := \lim_{R\to\infty} \int_{-R}^{R} e^{i\omega t}\mathcal{F}^{-1}f(\omega)\left(1 - \frac{|\omega|}{R}\right)\,d\omega.$$

We will need the following

Proposition 4.8. The limit in (4.3) has $\|f\|_{L^\infty}$ as a uniform majorant, i.e.

$$\left|\frac{1}{2\pi}\int_{-R}^{R} e^{i\omega t}\mathcal{F}^{-1}f(\omega)\left(1 - \frac{|\omega|}{R}\right)\,d\omega\right| \le \|f\|_{L^\infty}.$$

Proof.

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{-R}^{R} e^{i\omega t}\left(1-\frac{|\omega|}{R}\right)\int_{-\infty}^{\infty} e^{-i\omega y} f(y)\,dy\,d\omega\right| \\
=&\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi Ri(t-y)}\left(\int_0^R e^{i\omega(t-y)}\,d\omega-\int_{-R}^{0} e^{i\omega(t-y)}\,d\omega\right) f(y)\,dy\right| \\
=&\left|\frac{2}{\pi R}\int_{-\infty}^{\infty}\frac{\sin^2((t-y)R/2)}{(t-y)^2} f(y)\,dy\right| \\
\leq&\frac{\|f\|_{L^\infty}}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 u}{u^2}\,du \\
=&\|f\|_{L^\infty}
\end{aligned}$$

□

Let

$$\mathfrak{A} := \left\{ f:\mathbb{R}\to\mathbb{C} \mid f(x)=\sum_{n=1}^{\infty} a_n e^{i\lambda_n x},\quad \sum_{n=1}^{\infty}|a_n|<\infty\right\}.$$

Theorem 4.9 (Wiener/Pitt $\frac{1}{f}$ Theorem [22])**.** *If* $f\in\mathfrak{A}$ *and* $\inf_{x\in\mathbb{R}}|f(x)|>0$*, then* $\frac{1}{f}\in\mathfrak{A}$*.*

In Lemma 4.6 we required that $|\mathfrak{Im}\,\lambda|$ was sufficiently large, but we need an estimate for the resolvent on the whole stripe $\mathbb{C}_{\alpha,\beta}$. Such is easily obtained: Let $-s<\alpha$. Then for $\lambda\in\mathbb{C}_{\alpha,\beta}$

$$R(\lambda,A)\begin{pmatrix} f\\ g\\ b\end{pmatrix} = \begin{pmatrix} u\\ v\\ (-D,I)\delta_l\begin{pmatrix}u\\ v\end{pmatrix}\end{pmatrix},$$

where

$$\begin{aligned}
\begin{pmatrix}u\\ v\end{pmatrix} =& R(\lambda,A_0)\begin{pmatrix}f\\ g\end{pmatrix} \\
&+\frac{1}{\lambda+s}\left(R_1(\lambda)\begin{pmatrix}f\\ g\end{pmatrix}+R_2(\lambda)\begin{pmatrix}f\\ g\end{pmatrix}+R_3(\lambda)\begin{pmatrix}f\\ g\\ b\end{pmatrix}+R_4(\lambda)\begin{pmatrix}f\\ g\end{pmatrix}\right) \\
&+\frac{1}{1+|\lambda|^2}\tilde{\mathcal{E}}(\lambda)(f,g,b),
\end{aligned}$$

$R(\lambda,A_0)$, $R_1(\lambda)$, $R_2(\lambda)$, $R_3(\lambda)$, $R_4(\lambda)$ are given by the formulas in Lemma 4.6 and $\tilde{\mathcal{E}}$ is bounded for $\lambda\in\mathbb{C}_{\alpha,\beta}$. Put

$$R_0(\lambda) := R(\lambda,A_0).$$

Denote for $(f,g,b)\in X^{\mathbb{C}}$, $x^*\in\left(X^{\mathbb{C}}\right)^*$, $\alpha<\rho<\beta$, $\nu\in\mathbb{R}$

$$r_0(\nu,\rho,(f,g),x^*) := \left\langle x^*, \left(R_0(\rho+i\nu)\begin{pmatrix}f\\ g\end{pmatrix}; (-D,I)\delta_l R_0(\rho+i\nu)\begin{pmatrix}f\\ g\end{pmatrix}\right)\right\rangle,$$

and for $j = 1, 2, 4$

$$r_j(\nu, \rho, (f, g), x^*) := \frac{1}{\rho+s+i\nu} \left\langle x^*, \left(R_j(\rho + i\nu) \begin{pmatrix} f \\ g \end{pmatrix}; (-D, I)\delta_l R_j(\rho + i\nu) \begin{pmatrix} f \\ g \end{pmatrix}\right)\right\rangle,$$

$$r_3(\nu, \rho, (f, g, b), x^*) := \frac{1}{\rho+s+i\nu} \left\langle x^*, \left(R_3(\rho + i\nu) \begin{pmatrix} f \\ g \\ b \end{pmatrix}; (-D, I)\delta_l R_3(\rho + i\nu) \begin{pmatrix} f \\ g \\ b \end{pmatrix}\right)\right\rangle.$$

Lemma 4.10. *Suppose* $\mathbb{C}_{\alpha,\beta} \subset \rho(A)$ *and* $\rho \in]\alpha, \beta[$. *Then there exists* $\kappa > 0$ *such that for* $x = (f, g, b) \in X^{\mathbb{C}}$, $x^* \in \left(X^{\mathbb{C}}\right)^*$, *we have* $\mathfrak{F}[r_i(\cdot, \rho, (f, g), x^*)] \in L^\infty(\mathbb{R}, \mathbb{C})$ *and*

$$\|\mathfrak{F}[r_i(\cdot, \rho, (f, g), x^*)]\|_{L^\infty} \leq \kappa \, \|(f, g, 0)\|_{X^{\mathbb{C}}} \, \|x^*\| \tag{4.4}$$

for $i = 0, 1, 2, 4$, *and* $\mathfrak{F}[r_3(\cdot, \rho, (f, g, b), x^*)] \in L^\infty(\mathbb{R})$ *with*

$$\|\mathfrak{F}[r_3(\cdot, \rho, (f, g, b), x^*)]\|_{L^\infty} \leq \kappa \, \|(f, g, b)\|_{X^{\mathbb{C}}} \, \|x^*\| . \tag{4.5}$$

Proof. To prepare the proof recall that the dual space C^* of $C = C([0, l], \mathbb{C}^n)$ is isometrically isomorphic to the space of countable additive $\mathbb{C}^n$ valued Radon measures on the Borel sigma algebra of $[0, l]$ with the finite total variation norm. That is for $x^* \in C^*$ there exists a Radon measure $\alpha = (\alpha_1, \ldots, \alpha_n) : \mathfrak{B} \to \mathbb{C}^n$ such that for $\varphi = (\varphi_1, \ldots, \varphi_n) \in C([0, l]; \mathbb{C}^n)$

$$\langle x^*, \varphi\rangle = \sum_{j=1}^{n} \int_{[0,l]} \varphi_j \, d\alpha_j.$$

For $\varphi = (\varphi_{ij})_{1\leq i\leq n, 1\leq j\leq m} \in C([0, l], \mathbb{C}^{n\times m})$ let $\int_{[0,l]} \varphi \, d\alpha$ denote the rowvector

$$\int_{[0,l]} \varphi \, d\alpha := \left(\sum_{i=1}^{n} \int_{[0,l]} \varphi_{ij} \, d\alpha_i\right)_{1\leq j\leq m}.$$

The dual of $L^p([0, l]; \mathbb{C}^n)$ is $L^q([0, l]; \mathbb{C}^n)$, where $q \in]1, \infty]$ satisfies $\frac{1}{q} + \frac{1}{p} = 1$: for $x^* \in (L^p([0, l]; \mathbb{C}^n))^*$ there exists a unique $f \in L^q([0, l]; \mathbb{C}^n)$ such that for $\varphi \in L^p([0, l]; \mathbb{C}^n)$ we have

$$\langle x^*, \varphi\rangle = \int_{[0,l]} \langle f, \varphi\rangle_{\mathbb{C}^n} d\lambda,$$

where λ denotes Lebesque's measure on $\mathbb{R}$.

Corresponding to $x^* \in X^*$ there exist bounded $\mathbb{C}^{d_i}$ valued Radon measures α_i, $1 \leq i \leq \alpha + \beta$, on $[0, l]$ and $x_i \in \mathbb{C}^{d_{\alpha+i}}$, $1 \leq i \leq \beta$ such that for $k = 0, 1, 2, 4$

$$r_k(\nu, \rho, (f, g), x^*) = \sum_{j=1}^{\alpha+\beta} r_{kj}(\nu, \rho, (f, g), \alpha_j) + \sum_{j=1}^{\beta} \tilde{r}_{kj}(\nu, \rho, (f, g), x_j),$$

where for $j = 1, \ldots, \alpha + \beta$

$$r_{kj}(\nu, \rho, (f, g), \alpha_j) := \int_0^l R_k^{(j)}(\rho + i\nu)(f, g) \, d\alpha_j. \tag{4.6}$$

and for $j = 1, \ldots, \beta$

$$\tilde{r}_{kj}(\nu, \rho, (f, g), x_j) := \left\langle ((-D, I)\delta_l R_k(\rho + i\nu)(f, g))_j \, , x_j\right\rangle_{\mathbb{C}^{d_{\alpha+j}}},$$

$R_k^{(j)} \in C([0,l],\mathbb{C}^{d_j})$ denotes the j-th component, $1 \le j \le \alpha+\beta$, of R_k, and $((-D,I)\delta_l R_k(\rho+i\nu)(f,g))_j \in \mathbb{C}^{d_{\alpha+j}}$ denotes the j-th component, $1 \le j \le \beta$, of the $\mathbb{C}^{n_2}$ vector $(-D,I)\delta_l R_k(\rho+i\nu)(f,g)$.

By Lemma 4.6

$$
\begin{aligned}
R_0^{(j)}(\rho+i\nu)(f,g)(y) =& e^{-(\rho+i\nu)\int_0^y k_j^{-1}(r)\,dr} F_j(y,0) \left(\sum_{m=1}^{\alpha+\beta} \tau_{jm}(\rho+i\nu) I_m(\nu) \right) \\
&+ \int_0^y e^{-(\rho+i\nu)\int_z^y k_j^{-1}(r)\,dr} F_j(y,z) k_j^{-1}(z) h_j(z)\,dz
\end{aligned} \tag{4.7}
$$

where $\tau_{jm}(\rho+i\nu) \in \mathbb{C}^{d_j \times d_m}$ denotes the j-th row and m-th blockcolumn of the matrix

$$
\begin{aligned}
(\tau_{jm})_{1 \le j,m \le \alpha+\beta} &:= \begin{pmatrix} E \\ I \end{pmatrix} H_0^{-1}(\rho+i\nu)(D,-I), \\
I_m(\nu) &:= \int_0^l e^{-(\rho+i\nu)\int_z^l k_m^{-1}(r)\,dr} F_m(l,z) k_m^{-1}(z) h_m(z)\,dz.
\end{aligned}
$$

We show (4.4) for r_{0j}, $1 \le j \le \alpha+\beta$, and omit the terms $\tilde{r}_{kj}$ because they are even simpler. By (4.7) we can assume $d_i = 1$, $1 \le i \le \alpha+\beta$, without loss of generality (otherwise we have linear combinations of such). Hence

$$
r_{0j} = \left(\sum_{m=1}^{\alpha+\beta} \tau_{jm}(\rho+i\nu) r_{0jm} \right) + r_{0j0},
$$

$$
r_{0jm} := \int_0^l e^{-(\rho+i\nu)\int_0^y k_j^{-1}(r)\,dr} F_j(y,0)\,d\alpha_j(y) \cdot I_m(\nu),
$$

and

$$
r_{0j0} := \int_0^l \left(\int_0^y e^{-(\rho+i\nu)\int_z^y k_j^{-1}(r)\,dr} F_j(y,z) k_j^{-1}(z) h_j(z)\,dz \right) d\alpha_j(y).
$$

By Fubini's Theorem, the inversion formula (4.3), Lebesgue's dominated convergence, Proposition 4.8 and the change of variables $x = \int_z^l k_m^{-1}(r)\,dr$ we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{C_1\infty} e^{i\omega\nu} r_{0jm}(\nu)d\nu \\
=&\int_0^l \left(\frac{1}{2\pi}\int_{-\infty}^{C_1\infty} \exp\left(i\nu\left(\omega - \int_0^y k_j^{-1}(r)\,dr\right)\right) I_m(\nu)\,d\nu\right) \\
&\exp\left(-\int_0^y \frac{\rho}{k_j(r)}\,dr\right) F_j(y,0)\,d\alpha_j(y) \\
=&\int_0^l \left(\frac{1}{2\pi}\int_{-\infty}^{C_1\infty} \exp\left(i\nu\left(\omega - \int_0^y k_j^{-1}(r)\,dr\right)\right) \int_0^{\int_0^l k_m^{-1}(r)\,dr} e^{-i\nu x}\right. \\
&\left.\exp\left(-\int_{z(x)}^l k_m^{-1}(r)\rho\,dr\right) F_m(l,z(x))h_m(z(x))\,dx\,d\nu\right) \\
&\exp\left(-\int_0^y k_j^{-1}(r)\rho\,dr\right) F_j(y,0)\,d\alpha_j(y) \\
=&\operatorname{sgn}(k_j)\int_0^l \left(\frac{1}{2\pi}\int_{-\infty}^{C_1\infty} \exp\left(i\nu\left(\omega - \int_0^y k_j^{-1}(r)\,dr\right)\right) \int_{-\infty}^{\infty} e^{-i\nu x}\tilde{\zeta}(x)\,dx\,d\nu\right) \\
&\exp\left(-\int_0^y k_j^{-1}(r)\rho\,dr\right) F_j(y,0)\,d\alpha_j(y) \\
=&\int_0^l \zeta\left(\omega - \int_0^y k_j^{-1}(r)\right) \exp\left(-\int_0^y k_j^{-1}(r)\rho\,dr\right) F_j(y,0)\,d\alpha_j(y)
\end{aligned}$$

where

$$\begin{aligned}
\chi(x) &:= \begin{cases} 1 & \text{if } x \in [0, \int_0^l k_m^{-1}(r)\,dr] \cup [\int_0^l k_m^{-1}(r)\,dr, 0] \\ 0 & \text{elsewhere} \end{cases}, \\
\tilde{\zeta}(x) &:= \chi(x)\exp\left(-\int_{z(x)}^l k_m^{-1}(r)\rho\,dr\right) F_m(l,z(x))h_m(z(x)), \\
\zeta(x) &:= \frac{1}{2}\left(\tilde{\zeta}(x+) + \tilde{\zeta}(x-)\right).
\end{aligned}$$

Since ζ has compact support we have proven

(4.8) $\quad \mathfrak{F}r_{0jm} \in L^\infty$ with compact support and (4.4) holds for r_{0jm}.

Because τ_{jm} belongs to the Algebra $\mathfrak{A}$ Theorem 4.9 yields that $\mathfrak{F}\tau_{jm}(\rho + i\cdot)$ is a measure of countable Dirac masses with finite total variation $\|\mathfrak{F}\tau_{jm}(\rho + i\cdot)\|_{Var} < \infty$.

Hence for $m = 1, \ldots, \alpha + \beta$

$$\begin{aligned}
&\mathfrak{F}\left(r_{0jm} \cdot \tau_{jm}(\rho + i\cdot)\right) = \mathfrak{F}r_{0jm} * \mathfrak{F}\left(\tau_{jm}(\rho + i\cdot)\right) \in L^\infty \\
\text{and}\quad &\|\mathfrak{F}\left(r_{0jm} \cdot \tau_{jm}(\rho + i\cdot)\right)\|_{L^\infty} \le \|\mathfrak{F}r_{0jm}\|_{L^\infty} \|\mathfrak{F}\tau_{jm}(\rho + i\cdot)\|_{Var}.
\end{aligned}$$

Instead of $X^{\mathbb{C}}$ consider $X_p^{\mathbb{C}}$. Then $h = (f,g) \in L^p([0,l];\mathbb{C}^n)$. Since $C([0,l];\mathbb{C}^n)$ is dense in $L^p([0,l];\mathbb{C}^n)$ $(1 \le p < \infty)$ we can choose a sequence $(h_i)_{i\in\mathbb{N}}$ in $C([0,l];\mathbb{C}^n)$ which converges in L^p to h. Then the above calculation is valid for h_i instead of h. The integration with respect to the bounded measure $d\alpha_j$ is replaced with Lebesgue integration with respect to some L^q density, where $q \in \,]1,\infty]$, $q^{-1} + p^{-1} = 1$, is the conjugated exponent to p. By Hölder's inequality (4.8) holds uniformly in i. Since $r_{0jm}(h_i) \to r_{0jm}(h)$ in $\mathcal{S}^*$ (even in L^∞) we have $\mathfrak{F}r_{0jm}(h) = \lim_{i\to\infty,\mathcal{S}^*} \mathfrak{F}r_{0jm}(h_i)$. Since $\mathfrak{F}r_{0jm}(h_i)$ is bounded in L^∞, by weak-$*$ compactness of L^∞, after possibly passing to a subsequence, we have that $\mathfrak{F}r_{0jm}(h) \in L^\infty$ and (4.8) holds for the limit also. This explains Remark 2.18. In the following we will only consider the space $X^{\mathbb{C}}$.

Using the change of variable $x = -\int_0^z k_j^{-1}(r)\,dr$ we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{C_1\infty} e^{i\omega\nu} r_{0j0}(\nu) d\nu \\
&= \int_0^l \exp\left(-\textstyle\int_0^y k_j^{-1}(r)\rho\,dr\right)\left(\frac{1}{2\pi}\int_{-\infty}^{C_1\infty} \exp\left(i\nu(\omega - \textstyle\int_0^y k_j^{-1}(r)\,dr)\right)\right.\\
&\left.\int_0^y \exp\left(i\nu \textstyle\int_0^z k_j^{-1}(r)\,dr\right)\exp\left(\textstyle\int_0^z k_j^{-1}(r)\rho\,dr\right) F_j(y,z)\frac{h_j(z)}{k_j(z)}\,dz\,d\nu\right) d\alpha_j(y) \\
&= \int_0^l \exp\left(-\textstyle\int_0^y k_j^{-1}(r)\rho\,dr\right) \\
&\left(\frac{1}{2\pi}\int_{-\infty}^{C_1\infty} \exp\left(i\nu(\omega - \textstyle\int_0^y k_j^{-1}(r)\,dr)\right)\int_{-\infty}^{\infty} e^{-i\nu x}\tilde{\zeta}(x,y)\,dx\,d\nu\right) d\alpha_j(y) \\
&= \int_0^l \exp\left(-\textstyle\int_0^y k_j^{-1}(r)\rho\,dr\right)\zeta\left(\omega - \textstyle\int_0^y k_j^{-1}(r)\,dr, y\right) d\alpha_j(y),
\end{aligned}$$

where $\tilde{\zeta}(x,y) = (-1)^{s(j)}\chi_y(x)\exp(-\rho x)\,F_j(y,z(x))h_j(z(x))$, $s(j) := 0$ if $1 \le j \le \alpha$, $s(j) := 1$ if $\alpha+1 \le j \le \alpha+\beta$, χ_y is the characteristic set function to $[0, -\int_0^y k_j^{-1}(r)\,dr] \cup [-\int_0^y k_j^{-1}(r)\,dr, 0]$ and $\zeta(x,y) := \frac{1}{2}\left(\tilde{\zeta}(x+,y) + \tilde{\zeta}(x-,y)\right)$. Thus we have

$$\mathfrak{F}r_{0j0} \in L^\infty \text{ with compact support and (4.4) holds for } r_{0j0}.$$

Hence (4.4) holds for $i = 0$.

By definition of R_1 and r_{1j} (Lemma 4.6 and (4.6)) we have

$$r_{1j} = \frac{1}{\rho + s + i\nu}\sum_{1\le p,q\le\alpha+\beta}\int_0^l F_1(\cdot, 0, \rho + i\nu)_{jp}\tau_{pq}(\rho + i\nu)I_q(\nu)\,d\alpha_j.$$

Assuming $d_i = 1$, $1 \le i \le \alpha+\beta$, without loss of generality, we have

$$\begin{aligned}
r_{1j} =& \frac{1}{\rho + s + i\nu}\left(\sum_{1\le q\le\alpha+\beta}\tau_{jq}(\rho + i\nu)r_{1jjq}(\nu, \rho, (f,g), \alpha_j)\right.\\
&\left.+ \sum_{\substack{1\le p,q\le\alpha+\beta\\ p\ne j}}\sum_{r=1}^{3}\tau_{pq}(\rho + i\nu)r_{1jpqr}(\nu, \rho, (f,g), \alpha_j)\right),
\end{aligned}\tag{4.9}$$

where (see the formulas for F_1 in Lemma 4.6)

$$\begin{aligned}
r_{1jjq} :=& -\int_0^l \exp\left(-(\rho + i\nu)\textstyle\int_0^x k_j^{-1}(u)\,du\right)F_j(x,0)\\
&\sum_{\substack{1\le\sigma\le\alpha+\beta\\ \sigma\ne j}}\int_0^x \frac{C_{j\sigma}(z)}{k_j(z)}\rho_{\sigma j}(z)F_j(z,0)\,dz\,d\alpha_j(x)\cdot I_q(\nu),
\end{aligned}$$

and for $p \neq j$

$$\begin{aligned}
r_{1jpq1} :=& -\int_0^l \exp\left(-(\rho+i\nu)\int_0^x k_p^{-1}(u)\,du\right)\rho_{jp}(x)F_p(x,0)\,d\alpha_j(x)\cdot I_q(\nu),\\
r_{1jpq2} :=& \int_0^l \exp\left(-(\rho+i\nu)\int_0^x k_j^{-1}(u)\,du\right)F_j(x,0)\rho_{jp}(0)\,d\alpha_j(x)\cdot I_q(\nu),\\
r_{1jpq3} :=& \int_0^l \exp\left(-(\rho+i\nu)\int_0^x k_p^{-1}(u)\,du\right)F_j(x,0)\\
& \int_0^x \exp\left((\rho+i\nu)\int_0^z (k_j^{-1}(u)-k_p^{-1}(u))\,du\right)\\
& \left\{\rho_{jp}(z)\frac{d}{dz}\left(F_j(0,z)F_p(z,0)\right)\,dz + F_j(0,z)F_p(z,0)d\rho_{jp}(z)\right\}d\alpha_j(x)\cdot I_q(\nu).
\end{aligned}$$

We calculate the Fourier transform of r_{1jpq3}. For $x,z\in[0,l]$ we have by the Fejér Fourier inversion Theorem 4.7 and the change of variable $w=\int_y^l k_q^{-1}(z)\,dz$:

$$\begin{aligned}
&\frac{1}{2\pi}\overset{C_1}{\int_{-\infty}^{\infty}} \exp\left(i\nu\left(\omega-\int_0^x k_p^{-1}(u)\,du+\int_0^z\left(k_j^{-1}(u)-k_p^{-1}(u)\right)\,du\right)\right)I_q(\nu)\,d\nu\\
=&\frac{1}{2\pi}\overset{C_1}{\int_{-\infty}^{\infty}} \exp\left(i\nu\left(\omega-\int_0^x k_p^{-1}(u)\,du+\int_0^z\left(k_j^{-1}(u)-k_p^{-1}(u)\right)\,du\right)\right)\\
&\int_0^{\int_0^l k_q^{-1}(z)\,dz} e^{-i\nu w}\exp\left(-\int_{y(w)}^l \rho k_q^{-1}(z)\,dz\right)F_q(l,y(w))h_q(y(w))\,dw\,d\nu\\
=&\frac{1}{2\pi}\overset{C_1}{\int_{-\infty}^{\infty}} \exp\left(i\nu\left(\omega-\int_0^x k_p^{-1}(u)\,du+\int_0^z\left(k_j^{-1}(u)-k_p^{-1}(u)\right)\,du\right)\right)\\
&\int_{-\infty}^{\infty} e^{-i\nu w}\tilde{\zeta}(w)\,dw\,d\nu\\
=&\zeta\left(\omega-\int_0^x k_p^{-1}(u)\,du+\int_x^z\left(k_j^{-1}(u)-k_p^{-1}(u)\right)\right),
\end{aligned}$$

where

$$\zeta:\mathbb{R}\to\mathbb{C},\zeta(w):=\frac{1}{2}\left(\tilde{\zeta}(w+)+\tilde{\zeta}(w-)\right)$$

is compactly supported,

$$\tilde{\zeta}(w):=(-1)^{s(q)}\chi(w)\exp\left(-\int_{y(w)}^l \frac{\rho}{k_q(z)}\,dz\right)F_q(l,y(w))h_q(y(w)),$$

χ is the characteristic function of the interval $[0,\int_0^l k_q^{-1}(z)\,dz]\cup[\int_0^l k_q^{-1}(z)\,dz,0]$ and $s(q):=0$, if $1\le q\le\alpha$, $s(q):=1$, if $\alpha+1\le q\le\alpha+\beta$.

Therefore by Fubini and Lebesgue's dominated convergence using Proposition 4.8

for passing to the limit we have

$$\frac{1}{2\pi}\overset{C_1}{\int_{-\infty}^{\infty}} e^{i\nu\omega} r_{1jpq3}(\nu,\rho,(f,g),\alpha_j)\,d\nu$$
$$= \int_0^l \exp\left(-\int_0^x \tfrac{\rho}{k_p(u)}\,du\right) F_j(x,0) \int_0^x \exp\left(\int_0^z \rho(k_j^{-1}(u)-k_p^{-1}(u))\,du\right)$$
$$\zeta\left(\omega - \int_0^x k_p^{-1}(u)\,du + \int_0^z \left(k_j^{-1}(u)-k_p^{-1}(u)\right)\right)$$
$$\left\{\rho_{jp}(z)\frac{d}{dz}\left(F_j(0,z)F_p(z,0)\right) dz + F_j(0,z)F_p(z,0)d\rho_{jp}(z)\right\} d\alpha_j(x).$$

Because the measure $d\rho_{jp}$ is bounded this shows the existence of a constant κ such that

$$\begin{aligned} &\mathfrak{F}r_{1jpq3} \in L^\infty \quad \text{with compact support and} \\ &\|\mathfrak{F}r_{1jpq3}\|_{L^\infty} \leq \kappa\,\|\alpha_j\|\,\|(f,g,0)\|_{X^{\mathbb{C}}}\,. \end{aligned} \tag{4.10}$$

The Fourier transforms of the simpler expressions r_{1jjq}, r_{1jpq1} and r_{1jpq2} are calculated similarly.

To verify (4.4) for r_{1j} it follows from (4.9), since $\mathfrak{F}(\tau_{jm}(\rho+i\cdot))$ is a bounded measure, that we only have to check that the Fourier transform of $\frac{1}{\rho+s+i\cdot}$ is in $L^1(\mathbb{R})$. For this let

$$\eta(x) := \begin{cases} e^{-x} & , \quad 0 \leq x < \infty \\ 0 & , \quad -\infty < x < 0 \end{cases} .$$

Then $\left(\mathfrak{F}^{-1}\eta\right)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\eta(x)\,dx = \frac{1}{1+i\omega}$. Hence Theorem 4.7 implies

$$\frac{1}{2\pi}\overset{C_1}{\int_{-\infty}^{\infty}} e^{i\omega x}\frac{1}{1+i\omega}\,d\omega = \begin{cases} e^{-x} & , \quad 0 < x < \infty \\ \frac{1}{2} & , \quad x = 0 \\ 0 & , \quad -\infty < x < 0 \end{cases} .$$

From this it follows easily that

$$\frac{1}{2\pi}\overset{C_1}{\int_{-\infty}^{\infty}} e^{i\omega x}\frac{1}{\rho+s+i\omega}\,d\omega = \begin{cases} e^{-(\rho+s)x} & , \quad 0 < x < \infty \\ \frac{1}{2} & , \quad x = 0 \\ 0 & , \quad -\infty < x < 0 \end{cases} ,$$

which is in $L^1(\mathbb{R})$.

Next we calculate the Fourier transform of r_4. Recall that in [16] for the expansion of the fundamental solution T we arrived in the first step to the matrix F_1 with nondiagonal entries $(i \neq j)$

$$\begin{aligned} (F_1(x,y,\lambda))_{ij} = &-\lambda\,\exp\left(-\lambda\int_y^x k_i^{-1}(u)\,du\right) F_i(x,y) \\ &\int_y^x \exp\left(\lambda\int_y^z (k_i^{-1}(u)-k_j^{-1}(u))\,du\right) \\ &F_i(y,z)\frac{C_{ij}(z)}{k_i(z)}F_j(z,y)\,dz. \end{aligned}$$

Therefore we have

$$
\begin{aligned}
r_4(\nu,\rho,(f,g),x^*) = & \frac{\rho+i\nu}{\rho+s+i\nu} \sum_{\substack{m,j=1\\ m\neq j}}^{\alpha+\beta} r_{4mj}(\nu,\rho,(f,g),\alpha_m) + \\
& \frac{1}{\rho+s+i\nu} \sum_{j=1}^{\alpha+\beta} r_{4jj}(\nu,\rho,(f,g),\alpha_j) + \\
& \tilde r_4\left(\nu,\rho,(f,g),(x_l)_{1\le l\le\beta}\right),
\end{aligned}
$$

where for $1\le m,j\le\alpha+\beta$, $m\neq j$,

$$
\begin{aligned}
r_{4mj} := & -\int_0^l \int_0^x \exp\left(-(\rho+i\nu)\textstyle\int_y^x k_m^{-1}(u)\,du\right) F_m(x,y) \\
& \int_y^x \exp\left(-(\rho+i\nu)\textstyle\int_z^y (k_m^{-1}(u)-k_j^{-1}(u))\,du\right) \\
& F_m(y,z)\frac{C_{mj}(z)}{k_m(z)} F_j(z,y)\,dz\, k_j^{-1}(y) h_j(y)\,dy\,d\alpha_m(x)
\end{aligned}
$$

and for $j=1,\dots,\alpha+\beta$

$$
\begin{aligned}
r_{4jj} := & -\int_0^l \int_0^x \exp\left(-(\rho+i\nu)\textstyle\int_y^x k_j^{-1}(u)\,du\right) F_j(x,y) \\
& \sum_{\substack{\nu=1\\ \nu\neq j}}^{\alpha+\beta} \int_y^x \frac{C_{j\nu}(z)}{k_j(z)} \rho_{\nu j}(z) F_j(z,y)\,dz\, k_j^{-1}(y) h_j(y)\,dy\,d\alpha_j(x).
\end{aligned}
$$

As for r_{0j0} the transform of r_{4jj} is in L^∞ with compact support and estimate (4.10) holds. Using the change of variable $r(z,y) := \int_z^y \left(k_m^{-1}(u)-k_j^{-1}(u)\right)\,du$ we can write for $m\neq j$ (recall the definition of ρ_{mj} in Lemma 4.6)

$$
\begin{aligned}
r_{4mj} \quad = \quad & -\int_0^l \int_0^x \exp\left(-i\nu \textstyle\int_y^x k_m^{-1}(u)\,du\right) \exp\left(-\textstyle\int_y^x \frac{\rho}{k_m(u)}\,du\right) F_m(x,y) \\
& \int_{-\infty}^{\infty} e^{-i\nu r}\tilde\zeta(y,x,r)\,dr\,\frac{h_j(y)}{k_j(y)}\,dy\,d\alpha_m(x),
\end{aligned}
$$

where

$$
\tilde\zeta(y,x,r) := -e^{-\rho r} F_m(y,z(r,y))\rho_{mj}(z(r,y))\chi(y,x,r)
$$

and $\chi(y,x,\cdot)$ is the characteristic function of the interval

$$
\left[\textstyle\int_x^y (k_m^{-1}(u)-k_j^{-1}(u))\,du, 0\right] \cup \left[0, \textstyle\int_x^y (k_m^{-1}(u)-k_j^{-1}(u))\,du\right].
$$

(if z is not a unique function on y and r, then by condition (HIII) ρ_{mj} vanishes completely on $[0,l]$).

Therefore for $m\neq j$

$$
\frac{1}{2\pi}\overset{C_1}{\int_{-\infty}^{\infty}} e^{i\nu\omega} r_{4mj}(\nu)\,d\nu =
$$

$$
\int_0^l \int_0^x \exp\left(-\textstyle\int_y^x \frac{\rho}{k_m(u)}\,du\right) F_m(x,y)\zeta\left(y,x,\omega-\textstyle\int_y^x k_m^{-1}(u)\right)\frac{h_j(y)}{k_j(y)}\,dy\,d\alpha_m(x),
$$

where $\zeta(y,x,r) := (\tilde\zeta(y,x,r+)+\tilde\zeta(y,x,r-))/2$.

Hence

$$\mathfrak{F}r_{4mj} \in L^\infty \quad \text{with compact support for } 1 \le m, j \le \alpha + \beta.$$

Considering the Fourier transform of r_2 it follows from the formula for F_1 and H_1 in Lemma 4.6 and the previous arguments that the transform of

$$S(\rho + i\nu) := -\begin{pmatrix} E \\ I \end{pmatrix} H_0(\rho + i\nu)^{-1} H_1(\rho + i\nu) H_0(\rho + i\nu)^{-1}(D, -I)$$

is a bounded measure. Since

$$r_2 = \frac{1}{\rho + s + i\nu} \int_0^l T_0(x, 0, \rho + i\nu) S(\rho + i\nu) \int_0^l T_0(l, y, \lambda) K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy\, d\alpha(x)$$

one sees as above that there exists a constant κ such that (4.4) is satisfied for $i = 2$.

Finally we check (4.5): Write

$$\begin{aligned} r_3(\nu, \rho, (f, g, b), x^*) = &\frac{1}{\rho + s + i\nu} \left(r_{31}(\nu, \rho, (f, g, b), \alpha) + r_{32}(\nu, \rho, (f, g), \alpha)\right) \\ &+ \tilde{r}_3(\nu, \rho, (f, g, b), (x_j)_{1 \le j \le \beta}), \end{aligned}$$

where

$$\begin{aligned} r_{31} := &\int_0^l T_0(x, 0, \rho + i\nu) \begin{pmatrix} E \\ I \end{pmatrix} H_0(\rho + i\nu)^{-1} \\ &\left(b + (D, -I) \int_0^l F_1(l, y, \lambda) K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy \right) d\alpha(x), \\ r_{32} := &\int_0^l T_0(x, 0, \rho + i\nu) \begin{pmatrix} E \\ I \end{pmatrix} H_0(\rho + i\nu)^{-1} \\ &(F, G) \int_0^{\cdot} T_0(\cdot, y, \lambda) K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy\, d\alpha(x) \end{aligned}$$

and $\tilde{r}_3$ is simpler than the preceding terms. One sees that r_{31} is composed of terms similar to the ones we have already discussed above. The term r_{32} differs slightly since it contains the $n_2 \times n_1$ matrix of measures (F, G). However, the previous arguments still work (only an additional integral with a bounded measure from (F, G) appears and one uses Fubini once more, the C_1-Fourier transform is taken in the first inner integrals as we did above). Thus we have checked that

$$\frac{1}{2\pi} {}^{C_1}\!\!\int_{-\infty}^{\infty} e^{i\nu\omega} r_{32}(\nu)\, d\nu \in L^\infty$$

so that (4.5) is true. □

We have proved Theorems 2.12, 2.16 and 2.17. The remaining Theorems 2.8 and 2.11 are straightforward consequences:

Proof of Theorems 2.8 and 2.11. To prove Theorem 2.8 first one applies Theorem 2.16 to obtain that the linearized system is exponentially stable with respect to the space X. Then the assertion follows by using Theorem 2.5 and a standard argument (see for example [27, Theorem 11.22, p. 121]).

For Theorem 2.11 first one uses Theorem 2.12 or 2.17 to find that the linearized system has an exponential dichotomy in X. Then one modifies the critical linear flow at the boarder of an ellipsoid to obtain an overflowing starting linear center manifold. Finally one truncates the nonlinearity close to the equilibrium so that the nonlinear problem can be regarded as a small smooth perturbation of the modified linearized problem and then one applies [4] and Theorem 2.5. For the details see [15]. □

The results are contained in the thesis [15] *of the author supervised by L. Recke. The author would like to thank L. Recke, A. Mielke, K. Lu and K. Schneider for helpful comments and support.*

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