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Properties of stationary states of delay equations with large delay and applications to laser dynamics

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Abstract

We consider properties of periodic solutions of the differential-delay system, which models a laser with optical feedback. In particular, we describe a set of multipliers for these solutions in the limit of large delay. As a preliminary result, we obtain conditions for stability of an equilibrium of a generic differential-delay system with fixed large delay τ. We also show a connection between characteristic roots of the equilibrium and multipliers of the mapping obtained via the formal limit $\tau \to \infty$.

1 Introduction

Consider the differential-delay equation (DDE) with fixed delay

$$\frac{dy}{dt} = f(y(t), y(t-\tau)), \tag{1}$$

where $y \in \mathbb{R}^n$ and the function $f(s_1, s_2)$ is smooth valued and vanishes at zero $f(0,0) = 0$. We investigate the case when $\tau > 0$ is large. Such a choice is mainly motivated by applications from laser dynamics [11, 8].

After the rescaling $t' := t/\tau$ with the notation $x(t') := y(t'\tau)$, we obtain a singularly perturbed equation of the form

$$\frac{1}{\tau}\frac{dx}{dt} = f(x(t), x(t-1)), \tag{2}$$

where we drop the prime in t' for simplicity of notations. The limit $\tau \to \infty$ yields the system of difference equations

$$f(x(t), x(t-1)) = 0 \tag{3}$$

for which $x = 0$ is also a fixed point. In Sec. 2 we compare characteristic equations for the fixed point of (3) and (2). We will show that among the characteristic roots of the equilibrium of DDE system (2) there are two sets with different asymptotic properties for large τ: those roots, which approach characteristic roots of the corresponding difference system (3) as $\tau \to \infty$; and those approaching characteristic roots of the system

$$\frac{1}{\tau}\frac{dx}{dt} = f(x(t), 0) \tag{4}$$

of ordinary differential equations (ODE). This property can be useful to determine asymptotic properties of equilibriums of (2) for large τ. In Sec. 3 we consider similar approach applied to the Lang-Kobayashi model [8], which describes the dynamics of a single-mode semiconductor laser with optical feedback. The large delay regime apepars to be important in this model [11]. An additional property of Lang-Kobayashi system is $\mathbf{SO}(2)$ symmetry. Therefore, our attention will be paid to those periodic orbits, which are invariant under the action of the group S^1 [2, 3]. Since they are often called continuous-wave (cw) solutions (cf. [5, 15] and references therein), we adopt this notation here. In Sec. 4 we describe some stability properties of the cw solutions. Finally, we present numerical results that illustrate the theory. In parallel, throughout the paper, we discuss in Secs. 3 and 4.1 useful implications for coupled systems with delay.

2 Characteristic roots of the equilibrium

Let us obtain the characteristic equation for a fixed point of difference equation (3). The variational equation around $x(t) = 0$ has the form

$$\frac{\partial f}{\partial s_1}(0,0)\xi(t) + \frac{\partial f}{\partial s_2}(0,0)\xi(t-1) = 0. \tag{5}$$

Denote $A := \frac{\partial f}{\partial s_1}(0,0)$ and $B := \frac{\partial f}{\partial s_2}(0,0)$. Assuming that $\det A \neq 0$, we get the mapping

$$\xi(t) = -A^{-1}B\xi(t-1). \tag{6}$$

Multipliers of the zero fixed point of (6) are defined as roots of the following characteristic equation

$$\Delta(\mu) = \det(A\mu + B) = 0. \tag{7}$$

The characteristic equation for the equilibrium of DDE system (2) reads

$$\chi(\lambda,\tau) = \det\left(-\frac{\lambda}{\tau}I + A + Be^{-\lambda}\right) = 0, \tag{8}$$

where I is $n \times n$ identity matrix.

The following statement holds true:

Proposition 1 *Let μ_0 be a nonzero multiplier of the fixed point of the mapping (6), i.e. $\Delta(\mu_0) = 0$. Assume that the nondegeneracy condition*

$$\frac{d\Delta}{d\mu}(\mu_0) \neq 0 \tag{9}$$

is fulfilled. Then for any $k \in \mathbb{Z}$, there exists $\tau(k)$ such that for all $\tau > \tau(k)$ there exists a characteristic root $\lambda(k,\tau)$ of the equilibrium of DDE system (2), i.e. a solution of (8), such that
(i) $\lambda(k,\tau)$ is continuously differentiable[1] with respect to τ for $\tau(k) < \tau \leq \infty$.
(ii) $\lambda(k,\tau) \to \mathrm{Ln}\,\mu_0 + i2\pi k$ with $\tau \to \infty$,
where $\mathrm{Ln}\,\mu_0 = \ln|\mu_0| + i\mathrm{Arg}\mu_0$ is the principal branch of the logarithm and $i^2 = -1$.

[1]differentiability of $\lambda(k,\tau)$ at $\tau = \infty$ is understood in the sense of differentiability of $\lambda(k, 1/\mu)$ at $\mu = 0$.

Proof. The characteristic equation (8) can be written in the form

$$\chi(\lambda,\tau)=\chi_0(e^{-\lambda})+\frac{\lambda}{\tau}\chi_1(e^{-\lambda})+\cdots+\left(\frac{\lambda}{\tau}\right)^n\chi_n(e^{-\lambda})=0, \tag{10}$$

where n is the number of equations in (2), $\chi_l(\cdot)$, $l=0,\dots,n$ are polynomials. Note, that

$$\chi_0(e^{-\lambda})=\det(A+Be^{-\lambda})=0 \tag{11}$$

coincides with characteristic equation (7) if we denote $\mu=e^{\lambda}$. Therefore, given a solution μ_0 of (7), $\bar\lambda(k)=\mathrm{Ln}\,\mu_0+i2\pi k$, $k\in\mathbb{Z}$ is a solution of (11). Considering $\epsilon=1/\tau$ as a small parameter, equation (10) can be written as

$$\chi(\lambda,\epsilon)=\chi_0(e^{-\lambda})+\epsilon\bar\chi(\lambda,\epsilon)=0, \tag{12}$$

where $\bar\chi(\lambda,\epsilon)$ is smooth function in both variables. Taking into account (9), the conditions of the implicit function theorem are fulfilled for Eq. (12) at the point $\epsilon=0,\lambda=\bar\lambda(k)$. Hence, there exists a solution $\bar\lambda(k,\varepsilon)$ of (12) for small enough ε, which is continuously differentiable with respect to ε and $\bar\lambda(k,0)=\bar\lambda(k)$. ■

The above proposition implies that, for large τ, any multiplier μ_0 of mapping (6) corresponds to a large number of characteristic roots of (8) with real parts close to $\ln|\mu_0|$. Smallness of $\epsilon\bar\chi$ means $\tau\gg 2\pi k$. Therefore, for a fixed large delay τ, the number of such modes N satisfies $N\ll\tau/2\pi$.

Asymptotic behavior of characteristic roots $\lambda(k,\tau)$ can be analyzed in a perturbative way. Let us we rewrite Eq. (10) as

$$\chi(\lambda,\varepsilon)=\chi_0(e^{-\lambda})+\varepsilon\lambda\chi_1(e^{-\lambda})+\varepsilon^2\lambda^2\chi_2(e^{-\lambda})+\cdots, \tag{13}$$

where $\varepsilon=1/\tau$. We will look for the solutions of (13) of the form

$$\lambda(\varepsilon,k)=\mathrm{Ln}\mu_0+i2\pi k+\varepsilon\mathbf{X}+\varepsilon^2\mathbf{Y}+\cdots, \tag{14}$$

where $\chi_0(\mu_0^{-1})=0$. Substituting (14) into (13) and comparing terms of the same order of ε, we obtain

$$\begin{aligned}\mathbf{X}&=-(\mathrm{Ln}\mu_0+i2\pi k)\frac{\chi_1(\mu_0^{-1})\mu_0}{\chi_0'(\mu_0^{-1})},\\ \mathbf{Y}&=\frac{-\mu_0}{\chi_0'(\mu_0^{-1})}\Big[\frac{\mathbf{X}^2}{2\mu_0}(\chi_0'(\mu_0^{-1})+\chi_0''(\mu_0^{-1})\mu_0^{-1})+\chi_1(\mu_0^{-1})\mathbf{X}\\ &-\chi_1'(\mu_0^{-1})\mu_0^{-1}(\mathrm{Ln}\mu_0+i2\pi k)\mathbf{X}+\chi_2(\mu_0^{-1})(\mathrm{Ln}\mu_0+i2\pi k)^2\Big].\end{aligned} \tag{15}$$

It is not difficult to observe, that in the case of real $\mu_0>0$, we have $\Im(\lambda(\varepsilon,k))=2\pi k+C_1\varepsilon k+o(\varepsilon^2)$ and $\Re(\lambda(\varepsilon,k))=\ln\mu_0+C_2\varepsilon^2k^2+o(\varepsilon^2)$ in the lowest order of ε. Here C_1 and C_2 are some constants. Considering k as the parameter along the branch of characteristic roots $\lambda(\varepsilon,k)$, we see that this branch asymptotes tangentially to the line $\Re(\lambda)=\ln\mu_0$, cf. Fig. 1.

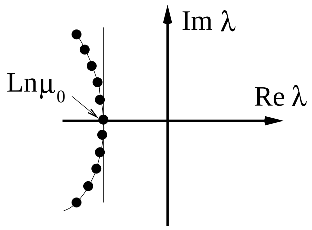

Figure 1: Asymptotic behavior of $\lambda(\tau,k)$ for the case of real positive μ_0 (schematic picture, the curvature can be opposite).

Proposition 1 describes solutions of (8), which are bounded as $\tau\to\infty$ or, more precisely, $\lambda/\tau\to 0$. In this case, the implicit function theorem can be applied to (12). Other solutions of (8), which grow with the increasing of τ, can also appear. The following result shows that these roots are connected with the characteristic equation for the ODE system $x'(t)=f(x(t),0)$, i.e.

$$\zeta_0(\lambda):=\det(-\lambda I+A)=0. \tag{16}$$

Proposition 2 *Let λ_0 be a solution of (16) with a positive real part. Assume that the nondegeneracy condition $\frac{d\zeta_0}{d\lambda}(\lambda_0)\neq 0$ is satisfied. Then there exists a root $\lambda(\tau)$ of (8) for large enough τ, such that $\lambda(\tau)/\tau\to\lambda_0$ as $\tau\to\infty$.*

Proof. Let us write Eq. (8) in the following form

$$\chi(\lambda,\tau)=\zeta_0\left(\frac{\lambda}{\tau}\right)+e^{-\lambda}\zeta_1\left(\frac{\lambda}{\tau}\right)+\cdots+e^{-\lambda n}\zeta_n\left(\frac{\lambda}{\tau}\right)=0, \tag{17}$$

where $\zeta_i(\cdot)$ are polynomials and $\zeta_0(\cdot)$ is given by (16). Substituting $\lambda=\lambda_1\tau$ into (17), we obtain

$$\chi(\lambda,\tau)=\zeta_0\left(\lambda_1\right)+e^{-\lambda_1\tau}\zeta_1\left(\lambda_1\right)+\cdots+e^{-\lambda_1\tau n}\zeta_n\left(\lambda_1\right)=0. \tag{18}$$

We are looking for solutions of (18) in the form $\lambda_1=\lambda_0+\mu$, i.e.

$$\zeta_0\left(\lambda_0+\mu\right)+e^{-\lambda_0\tau}\left(e^{-\mu\tau}\zeta_1(\lambda_0+\mu)+\cdots+e^{-(\lambda_0(n-1)+\mu n)\tau}\zeta_n(\lambda_0+\mu)\right)=0. \tag{19}$$

Denote $\gamma=\Re(\lambda_0)>0$, $\nu=\Im(\lambda_0)$, and $\varepsilon=e^{-\gamma\tau}$ be a small parameter. Then (19) can be rewritten as

$$\begin{aligned}G(\mu,\varepsilon)&:=\zeta_0\left(\lambda_0+\mu\right)+\varepsilon e^{i\nu\ln\varepsilon/\gamma}\left(e^{\mu\ln\varepsilon/\gamma}\zeta_1(\lambda_0+\mu)+\cdots+e^{(\lambda_0(n-1)+\mu n)\ln\varepsilon/\gamma}\zeta_n(\lambda_0+\mu)\right)\\&=\zeta_0\left(\lambda_0+\mu\right)+\varepsilon\left(\varepsilon^{\Re(\mu)/\gamma}f_1(\mu,\varepsilon)+\cdots\varepsilon^{n\Re(\mu)/\gamma}f_n(\mu,\varepsilon)\right),\end{aligned}$$

where $f_i(\mu,\varepsilon)$ are bounded functions of ε for each μ. One can see that for small enough $|\mu|$, there exists a limit $\lim_{\varepsilon\to 0}G(\mu,\varepsilon)=\zeta_0\left(\lambda_0+\mu\right)$. If we define $G(\mu,0):=\zeta_0\left(\lambda_0+\mu\right)$, then $G(\mu,\varepsilon)$ is continuous at $\varepsilon=0$ and $G(0,0)=0$. Its derivative with respect to μ is

$$G'_\mu(\mu,\varepsilon)=\zeta_0'(\lambda_0+\mu)+\varepsilon e^{i\nu\ln\varepsilon/\gamma}\Big[e^{\mu\ln\varepsilon/\gamma}\zeta_1'(\lambda_0+\mu)+\cdots+e^{(\lambda_0(n-1)+\mu n)\ln\varepsilon/\gamma}\zeta_n'(\lambda_0+\mu)$$

$$+\frac{\ln\varepsilon}{\gamma}\left(e^{\mu\ln\varepsilon/\gamma}\zeta_1(\lambda_0+\mu)+\cdots+ne^{(\lambda_0(n-1)+\mu n)\ln\varepsilon/\gamma}\zeta_n(\lambda_0+\mu)\right)\Big]$$
$$=\zeta_0'(\lambda_0+\mu)+\varepsilon\left(\varepsilon^{\Re(\mu)/\gamma}\bar{f}_1(\mu,\varepsilon)+\cdots\varepsilon^{n\Re(\mu)/\gamma}\bar{f}_n(\mu,\varepsilon)\right)$$
$$+\varepsilon\ln\varepsilon\left(\varepsilon^{\Re(\mu)/\gamma}g_1(\mu,\varepsilon)+\cdots\varepsilon^{n\Re(\mu)/\gamma}g_n(\mu,\varepsilon)\right),$$

where $\bar{f}_1,\ldots,\bar{f}_n$ and $g_1,\ldots,g_n$ are bounded functions of ε for each μ. One can check that $\lim_{\varepsilon\to 0}G_\mu'(\mu,\varepsilon)=\zeta_0'(\lambda_0+\mu)$ provided $|\mu|$ is small enough. Defining $G_\mu'(0,0):=\zeta_0'(\lambda_0)\neq 0$, the derivative G_μ' becomes a continuous function at $(\mu,\varepsilon)=(0,0)$. Therefore, conditions of the implicit function theorem [6] are satisfied and there exists a continuous branch $\mu(\varepsilon)$ of characteristic roots such that $\mu(\varepsilon)\to 0$ as $\varepsilon\to 0$. Taking into account the relation $\lambda=\tau(\lambda_0+\mu)$, we obtain the required statement. ■

We say [7] that the equilibrium $x=0$ of DDE system (2) is *stable* if, for any $\varepsilon>0$, there is a $\delta=\delta(\varepsilon)$ such that for any initial condition $\phi\in C$ and $\|\phi\|_{C[-\tau,0]}\leq\delta$ the solution $x_t(\varphi)$ of (2) satisfies $\|x_t(\phi)\|_{C[-\tau,0]}\leq\varepsilon$. Here we denote $\|\phi\|_{C[-\tau,0]}=\sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$.

The following corollary of Proposition 1 gives simple necessary conditions for stability of an equilibrium of DDE system (2) with large delay.

Corollary 3 *Assume that one of the following two conditions holds:*
(i) there exists a solution of the characteristic equation $\Delta(\mu_0)=0$*, such that* $\Delta'(\mu_0)\neq 0$ *and* $\ln|\mu_0|>0$*; or*
(ii) there exists a solution of (16) with positive real part.
Then the equilibrium $x=0$ *of system (2) is unstable for large enough* τ*.*

Proof. According to Propositions 1 and 2, each of the two assumptions implies the existence of a root of characteristic equation (8) with positive real part for large enough τ. Therefore, it follows from the stability theorem on the first approximation (cf. [7, 4]) that $x=0$ is unstable with respect to (2). ■

Thus, the asymptotic instability of an equilibrium of (3) or (4) implies also its instability with respect to (2) for large delay.

3 Two coupled differential delay equations

Two symmetrically coupled delay differential equations of the form

$$\begin{aligned}\frac{1}{\tau}\frac{dx}{dt}&=f(x(t))+g(y(t-1)),\\ \frac{1}{\tau}\frac{dy}{dt}&=f(y(t))+g(x(t-1)),\end{aligned}\tag{20}$$

naturally appear in many applications [12, 5, 15]. Here $x,y\in\mathbb{R}^n$ and f, g are smooth valued functions. Assume that $x=y=0$ is a symmetric equilibrium of (20). Because of $\mathbf{Z}_2$ symmetry, system (20) has the invariant subspace $x=y$ [10]. Using standard coordinate transformation $u = x - y$ and $v = x + y$, one can show (cf. [15]) that the characteristic equation for the symmetric equilibrium can be factorized into two parts $\chi(\lambda,\tau) = \chi_T(\lambda,\tau)\chi_L(\lambda,\tau)$. The first part corresponds to the perturbations δu transverse to the subspace $x = y$:

$$\chi_T(\lambda,\tau) = \det\left(\frac{\lambda}{\tau} - A_1 + B_1 e^{-\lambda}\right) = 0 \tag{21}$$

and the second one corresponds to the perturbations δv within the subspace

$$\chi_L(\lambda,\tau) = \det\left(\frac{\lambda}{\tau} - A_1 - B_1 e^{-\lambda}\right) = 0. \tag{22}$$

Here we denote $A_1 := \frac{df}{dx}(0)$ and $B_1 := \frac{dg}{dx}(0)$. In the limit $\tau \to \infty$, we obtain

$$\chi_{0T}(\lambda) = \det(A_1 - B_1 e^{-\lambda}) = 0 \tag{23}$$

and

$$\chi_{0L}(\lambda) = \det(A_1 + B_1 e^{-\lambda}) = 0. \tag{24}$$

Note that Eq. (23) can be obtained from Eq. (24) by transformation $\lambda \to \lambda + i\pi$. This observation, together with Proposition 1, leads us to the following corollary

Corollary 4 *Let μ_0 be a nonzero solution of the equation*

$$\Delta_1(\mu) := \det(A_1\mu + B_1) = 0.$$

Assume that the nondegeneracy condition $\frac{d\Delta_1}{d\mu}(\mu_0) \neq 0$ is fulfilled. Then for any $k \in \mathbb{Z}$, there exists $\tau(k)$ such that for all $\tau > \tau(k)$, there exist characteristic roots $\lambda_1(k,\tau)$ and $\lambda_2(k,\tau)$ of the symmetric equilibrium $(0,0)$ of system (20) such that:
(i) $\lambda_1(k,\tau)$ is a solution of (22) and corresponds to perturbations within the invariant subspace $x = y$.
(ii) $\lambda_2(k,\tau)$ is a solution of (21) and corresponds to perturbations transverse to the invariant subspace $x = y$.
(iii) $\lambda_{1,2}(k,\tau) \to \operatorname{Ln}\mu_0 + i2\pi k$ as $\tau \to \infty$.

The above Corollary implies, in particular, that if $\Delta_1(\mu_0) = 0$ and $|\mu_0| > 1$, then both directions: transverse and longitudinal are unstable for large τ. We remark that μ_0 is a multiplier of the mapping $\xi_n = A_1^{-1}B_1\xi_n$ in case when $\det A_1 \neq 0$.

Proposition 2 implies the following

Corollary 5 *Assume that λ_0 be a solution of the equation*

$$\zeta_0(\lambda) := \det(-\lambda I + A_1) = 0 \tag{25}$$

and $\Re(\lambda_0) > 0$. Assume that the nondegeneracy condition $\frac{d\zeta_0}{d\lambda}(\lambda_0) \neq 0$ is fulfilled. Then for large enough τ, there exist characteristic roots $\lambda_1(\tau)$ and $\lambda_2(\tau)$ of the symmetric equilibrium

$(0, 0)$ *of system (20) such that:*
(i) $\lambda_1(\tau)$ *is a solution of (22) and corresponds to perturbations within the invariant subspace* $x = y$.
(ii) $\lambda_2(\tau)$ *is a solution of (21) and corresponds to perturbations transverse to the invariant subspace* $x = y$.
(iii) $\lambda_i(\tau)/\tau \to \lambda_0$, $i = 1, 2$ *with* $\tau \to \infty$.

Note, that (25) is the characteristic equation for the equilibrum of the uncoupled ODE system $x' = f(x(t))$. Finally we remark, that characteristic roots of an equilibrium of system (1) differ from characteristic roots of the corresponding equilibrium of (2) by the factor $1/\tau$.

4 Continuous-wave solutions of Lang-Kobayashi model

In the following, we consider the Lang-Kobayashi model for single-mode semiconductor laser with optical feedback [8, 14]

$$\begin{aligned} \frac{1}{\tau}E'(t) &= (1 + i\alpha)N(t)E(t) + \eta e^{-i\varphi}E(t-1), \\ \frac{1}{\tau}N'(t) &= \varepsilon[J - N(t) - (N(t) + 0.5)|E(t)|^2] \end{aligned} \tag{26}$$

as well as the model for two mutually injecting lasers [12, 5, 15]

$$\begin{aligned} \frac{1}{\tau}E_1'(t) &= (1 + i\alpha)N_1(t)E_1(t) + \eta e^{-i\varphi}E_2(t-1), \\ \frac{1}{\tau}N_1'(t) &= \varepsilon[J - N_1(t) - (N_1(t) + 0.5)|E_1(t)|^2], \\ \frac{1}{\tau}E_2'(t) &= (1 + i\alpha)N_2(t)E_2(t) + \eta e^{-i\varphi}E_1(t-1), \\ \frac{1}{\tau}N_2'(t) &= \varepsilon[J - N_2(t) - (N_2(t) + 0.5)|E_2(t)|^2]. \end{aligned} \tag{27}$$

Physical meaning of the variables is as follows: $E, E_{1,2}$ denote dimensionless complex optical fields and $N, N_{1,2}$ carrier densities, respectively; ε is a small parameter, which is proportional to the ratio between photon and carrier lifetimes; J is proportional to the injection current; η and φ are coupling strength and coupling phase parameters for (27), and feedback strength and phase for (26), respectively. The delay τ in (27) appears due to the finite time, which is required for the signal to propagate from one laser to the other. Note that (27) and (26) are written already in the rescaled form (2). In many applications [11, 13], regimes with large τ are of interest.

Systems (26) and (27) possess $\mathbf{SO}(2)$ symmetry, since they are invariant with respect to the following transformations

$$\begin{aligned} &(E, N) \to (Ee^{i\psi}, N) && \text{for system (26)} \\ &(E_1, N_1, E_2, N_2) \to (E_1e^{i\psi}, N_1, E_2e^{i\psi}, N_2) && \text{for system (27).} \end{aligned} \tag{28}$$

This symmetry implies that for suitable parameter choice there exist continuous wave (cw) solutions, i.e. solutions of the type

$$E(t) = E_0 e^{i\omega t}, \quad N(t) = N_0 \quad (\omega, N_0 \in \mathbb{R}) \tag{29}$$

for (26) and the solutions

$$E_j(t) = E_{0_j} e^{i\omega_j t}, \quad N_j(t) = N_{0_j}, \quad j = 1, 2, \quad (\omega, N_j \in \mathbb{R}) \tag{30}$$

for system (27).

Let us first consider cw solutions of the Lang-Kobayashi model (26). Substituting (29) into (26), we obtain the following set of equations for determining unknown values of ω, N_0, and E_0 (cf. also [11, 15]):

$$N_0 = -\eta \cos\theta, \quad \omega - \alpha N_0 = -\eta \sin\theta, \quad E_0^2 = \frac{J - N_0}{N_0 + 0.5}, \tag{31}$$

where $\theta = \varphi + \omega\tau$.

Since cw solutions are periodic orbits of system (26), which are invariant with respect to symmetry (28), an appropriate coordinate transformation (cf. [14, 15]) brings them to the family of equilibriums $E = Ee^{i\psi}$, $N = N_0$, $0 \le \psi \le 2\pi$. The characteristic equation for these equilibriums was obtained in [14]:

$$\begin{aligned} \chi_L(\lambda) \quad = \quad & \left[\left(\frac{\lambda}{\tau}\right)^2 + 2\eta\cos\theta(1 - e^{-\lambda})\frac{\lambda}{\tau} + \eta^2(1 - e^{-\lambda})^2 \right] \left(\frac{\lambda}{\tau} + \varepsilon(1 + S) \right) \\ & + 2\varepsilon(J + \eta\cos\theta) \left[\frac{\lambda}{\tau} + \eta(\cos\theta - \alpha\sin\theta)(1 - e^{-\lambda}) \right] = 0, \end{aligned} \tag{32}$$

where

$$S := \frac{J + \eta\cos\theta}{0.5 - \eta\cos\theta}.$$

Formal limit $\tau \to \infty$ in (32) yields the following "approximate" equation

$$\chi_L^{(0)} = [1 - e^{-\lambda}] \left[\frac{\eta(J + 0.5)}{0.5 - \eta\cos\theta}(1 - e^{-\lambda}) + 2(J + \eta\cos\theta)(\cos\theta - \alpha\sin\theta) \right] = 0. \tag{33}$$

Note, that $\lambda_0(k) = i2\pi k$ are solutions of (33). Therefore, similar arguments as in the proof of Proposition 1 imply the following

Proposition 6 *For any $k \in \mathbb{Z}$ there exists $\tau(k) > 0$ such that any cw solution of (26) has a characteristic root $\lambda(k,\tau)$ for $\tau > \tau(k)$ such that*
(i) $\lambda(k,\tau)$ is a solution of (32).
(ii) $\lambda(k,\tau) \to i2\pi k$ with $\tau \to \infty$.

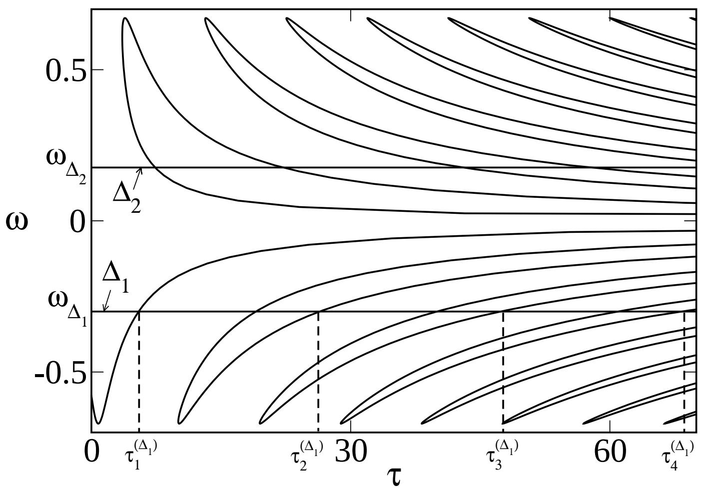

Figure 2: Frequencies ω of cw solutions for (26) versus τ for fixed $\alpha = 2$, $\varphi = 1$, and $\eta = 0.3$. The figure was obtained as the parametric plot of (34)-(35). The construction of the sequence τ_m^Δ is illustrated.

Note, that Proposition 6 was obtained by considering the term $[1 - e^{-\lambda}]$ in (33). The second term in (33) depends on τ via $\theta = \omega\tau + \varphi$ and possesses no limit as $\tau \to \infty$. We resolve this problem by choosing appropriate family of sequences $\{\tau_m^{(\Delta)}\}$, $m = 1, 2, \ldots$, $0 \le \Delta < 2\pi$ such that $\tau_m^{(\Delta)} \to \infty$ as $m \to \infty$ and $\chi_L^{(0)}$ is constant for each sequence with fixed Δ. The following Lemma determines the choice of $\{\tau_m^{(\Delta)}\}$.

Lemma 7 *Denote*

$$\tau_m^{(\Delta)} = \frac{\Delta + i2\pi m}{-\eta(\sin(\varphi + \Delta) + \alpha\cos(\varphi + \Delta))}, \quad m = 1, 2, \ldots, \quad 0 \le \Delta < 2\pi. \tag{34}$$

Then for each $\tau = \tau_m^{(\Delta)}$ there exists a cw solution of (26) characterized by

$$N_\Delta = -\eta\cos(\varphi + \Delta), \quad \omega_\Delta = -\eta(\sin(\varphi + \Delta) + \alpha\cos(\varphi + \Delta)), \quad E_\Delta^2 = \frac{J - N_\Delta}{N_\Delta + 0.5}. \tag{35}$$

Proof. The statement can be verified by substituting (35) and (34) into (31). ■

It is important, that N_Δ, ω_Δ, and E_Δ, evaluated at $\tau = \tau_m^{(\Delta)}$, do not depend on m. Figure 2 illustrates the sequence $\tau_m^{(\Delta)}$. In fact, the inverse statement holds true:

Lemma 8 *Any given cw solution of (26) can be represented in the form (34)-(35) with some $m \in \mathbb{Z}$ and $0 \le \Delta < 2\pi$.*

Proof. Any cw solution is uniquely determined by the values of N, ω, and E^2, satisfying (31) for the given parameters τ, φ, η, α, and J. Denoting $\bar{\Delta} = \omega\tau$, we see that (34) and (35) hold with $\Delta = \bar{\Delta} \mod 2\pi$ and m equals to the integer part of $\bar{\Delta}/2\pi$. ■

Lemmas 7 and 8 allow us to represent parametrically cw solutions versus τ. Namely, Eqs. (34) and (35) should be used with Δ as the free parameter. An example is shown in Fig. 2 where ω versus τ is plotted.

Let us confine now our attention to some sequence $\{\tau_m^{(\Delta)}\}$. At $\tau = \tau_m^{\Delta}$, Eq. (33) reads

$$\begin{aligned}\chi_L^{(0)}(\tau_m^{(\Delta)}) = [1 - e^{-\lambda}]\Big[&\frac{\eta(J + 0.5)}{0.5 - \eta\cos(\varphi + \Delta)}(1 - e^{-\lambda}) \\ +2(J + \eta\cos(\varphi + \Delta))&(\cos(\varphi + \Delta) - \alpha\sin(\varphi + \Delta))\Big] = 0.\end{aligned} \tag{36}$$

The second multiplier in (36) is now constant for all m and its root

$$\operatorname{Ln}\beta_\Delta + i2\pi k, \quad k \in \mathbb{Z}, \tag{37}$$

where

$$\beta_\Delta = 1 + \frac{2}{\eta(J + 0.5)}(0.5 - \eta\cos(\varphi + \Delta))(J + \eta\cos(\varphi + \Delta))(\cos(\varphi + \Delta) - \alpha\sin(\varphi + \Delta)) \tag{38}$$

does not depend on m. Since $1/\tau_m^{(\Delta)} \to 0$ as $m \to \infty$, system (36) can be considered as a perturbation of (32) with the small parameter $1/\tau_m^{(\Delta)}$. Thus, the following proposition holds

Proposition 9 *For any $0 \le \Delta < 2\pi$, $k \in \mathbb{Z}$, and $\tau = \tau_m^{(\Delta)}$ there exists a cw solution of (26) with N, ω, and E^2 given by (35). Moreover, for any given $k \in \mathbb{Z}$ there exists a large enough integer $M(k) > 0$ such that for all $m > M(k)$ this cw solution has a characteristic root $\lambda(k, \tau_m^{(\Delta)})$ such that*
(i) $\lambda(k, \tau_m^{(\Delta)})$ solves (32) for $\tau = \tau_m^{(\Delta)}$.
(ii) $\lambda(k, \tau_m^{(\Delta)}) \to \operatorname{Ln}\beta_\Delta + i2\pi k$ with $m \to \infty$, where β_Δ is determined by (38).

Now let us summarize the Propositions 6 and 9. With the increasing of τ, all cw solutions of (26) possess two branches of characteristic roots, which asymptote to the values $i2\pi k$ and $\operatorname{Ln}\beta_\Delta + i2\pi k$, respectively. In particular, the growing number of characteristic roots (their number is of order of τ) approach the imaginary axis as τ becomes large. This might be a source of additional problems for numerical integration of system (26) with large delay. According to the Proposition 2, additional roots may also arise, which are unbounded with the increasing of τ. These roots tend to those solutions of the polynomial

$$\left[\left(\frac{\lambda}{\tau}\right)^2 + 2\eta\cos\theta\frac{\lambda}{\tau} + \eta^2\right]\left(\frac{\lambda}{\tau} + \varepsilon(1 + S)\right) + 2\varepsilon(J + \eta\cos\theta)\left[\frac{\lambda}{\tau} + \eta(\cos\theta - \alpha\sin\theta)\right] = 0, \tag{39}$$

which have positive real parts. Eq. (39) was obtained from (32) by setting $e^{-\lambda} = 0$ (cf. Proposition 2 for details).

4.1 Two coupled lasers

Now let us consider synchronous cw solutions of system (27). Since the lasers are assumed to be identical, system (27) possesses $\mathbf{Z}_2$ symmetry $(E_1, N_1, E_2, N_2) \to (E_2, N_2, E_1, N_1)$ which is the permutation of subsystems. As a result (cf. [10, 9]), the diagonal subspace

$$\mathcal{S} = \{(E_1, N_1, E_2, N_2) : \; E_1 = E_2, \quad N_1 = N_2\} \tag{40}$$

is invariant with respect to the flow generated by (27). Moreover, the dynamics on $\mathcal{S}$ is governed by system (26). Solutions from $\mathcal{S}$ will be called *synchronous*. Hence, the synchronous cw solutions have the form

$$E_1(t) = E_2(t) = E_0 e^{i\omega t}, \quad N_1(t) = N_2(t) = N_0, \quad (\omega, N_0 \in \mathbb{R}) \tag{41}$$

and they satisfy (31) as in the case of Lang-Kobayashi model (26). The characteristic equation for the synchronous cw solutions was obtained in [14, 15]. As a result of the symmetry, this equation can be factorized as follows

$$\chi(\lambda) = \chi_L(\lambda) \cdot \chi_T(\lambda) = 0, \tag{42}$$

where $\chi_L(\lambda)$ is given by (32) and corresponds to the modes within the subspace $\mathcal{S}$. $\chi_T(\lambda)$ corresponds to the transverse modes

$$\begin{aligned} \chi_T(\lambda) \quad = \quad & \left[\left(\frac{\lambda}{\tau}\right)^2 + 2\eta\cos\theta(1+e^{-\lambda})\frac{\lambda}{\tau} + \eta^2(1+e^{-\lambda})^2\right]\left(\frac{\lambda}{\tau} + \varepsilon(1+S)\right) \\ & +2\varepsilon(J+\eta\cos\theta)\left[\frac{\lambda}{\tau} + \eta(\cos\theta - \alpha\sin\theta)(1+e^{-\lambda})\right]. \end{aligned} \tag{43}$$

The corresponding limit $\tau \to \infty$ gives

$$\chi_T^{(0)}(\lambda) = [1+e^{-\lambda}]\left[\frac{\eta(J+0.5)}{0.5-\eta\cos\theta}(1+e^{-\lambda}) + 2(J+\eta\cos\theta)(\cos\theta-\alpha\sin\theta)\right] = 0. \tag{44}$$

The equation $\chi_L(\lambda) = 0$ was the subject of the previous section. Observe that (44) can be obtained from (33) by substituting $e^{-\lambda} \to -e^{-\lambda}$ (it is a direct consequence of the coupling configuration of (27)). Hence, any solution λ_T of (44) can be expressed as $\lambda_T = \lambda_L + i\pi$, where λ_L is a solution of (33).

This immediately allows us to obtain analogs of Propositions 6 and 9 for the transverse modes.

Proposition 10 *The set of characteristic roots of symmetric cw solutions of coupled system (27) can be divided into two parts $\Lambda_L \cup \Lambda_T$. Λ_L corresponds to the modes within the invariant subspace $\mathcal{S}$ and solves (32). The statements of Propositions 6 and 9 hold for Λ_L. Λ_T corresponds to the perturbations transverse to the subspace $\mathcal{S}$ and possesses the following properties:*
(i) For $k \in \mathbb{Z}$, there exists a sufficiently large $\tau(k)$ such that for all $\tau > \tau(k)$ there exists

$\lambda_T(k,\tau) \in \Lambda_T$ and $\lambda_T(k,\tau) \to i\pi + i2\pi k, \quad$ as $\quad \tau \to \infty$.
(ii) Let $0 \le \Delta < 2\pi$ and $k \in \mathbb{Z}$. Then for all $\tau = \tau_m^{(\Delta)}$ there exists a synchronous cw solution of (27) with N, ω, and E^2 given by (35). For any $k \in \mathbb{Z}$ this cw solution has transverse characteristic root $\lambda_T(k, \tau_m^{(\Delta)}) \in \Lambda_T$ such that

$$\lambda(k, \tau_m^{(\Delta)}) \to \mathrm{Ln}\, \beta_\Delta + i\pi + i2\pi k, \tag{45}$$

where β_Δ is determined by (38).
(iii) If Eq. (39) possesses a root λ_0 with positive real part, then for sufficiently large τ there exists $\lambda_T(\tau) \in \Lambda_T$ and $\lambda_L(\tau) \in \Lambda_L$ such that $\lambda_T(\tau)/\tau \to \lambda_0$ and $\lambda_L(\tau)/\tau \to \lambda_0$ with $\tau \to \infty$.

The characteristic roots from Λ_L and Λ_T, described in Proposition 10 have asymptotically the same real parts: 0, $\ln|\beta_\Delta|$, or $\Re(\lambda_0)$. Therefore, if $\ln|\beta_\Delta| > 0$ or $\Re(\lambda_0) > 0$, then the synchronous cw solutions under considerations are unstable not only within the subspace $\mathcal{S}$, but also transversely. Figure 3 shows numerically obtained characteristic roots for some cw solutions of (26) for fixed $\alpha = 2$, $\varepsilon = 0.03$, $J = 1$, $\eta = 0.3$, $\varphi = 1.0$. We inspected two cases: $\tau = 100$ and $\tau = 200$. For our calculations, we used DDE-biftool software [1], where linear multi-step method is implemented. As it was expected, we observe in Fig. 3 two branches of roots. One of them asymptotes tangentially to the line $\Re(\lambda) = 0$. Additionally, we see the roots, which are unbounded with large τ, cf. rightmost points in Figs.3(d-f,j-l), and are the solutions of (39). In some cases, cf. Figs. 3(a,g), there are no growing roots. Figures 3a and 3g shows, that some cw solutions are still stable at these parameter values (one zero root appears because of the phase-shift invariance).

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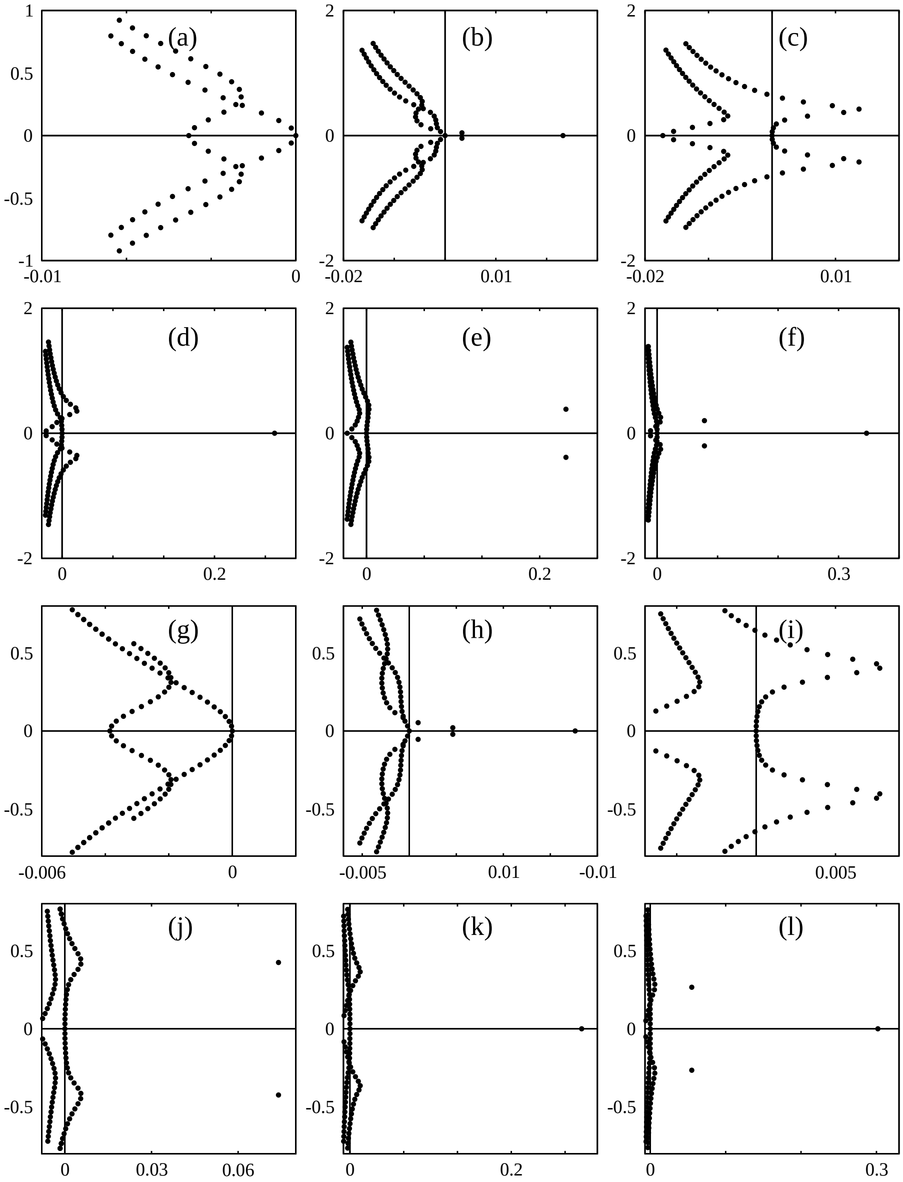

Figure 3: Numerically computed characteristic roots of cw solutions of the Lang-Kobayashi model. Shown are $\Re(\lambda)/\tau$ versus $\Im(\lambda)/\tau$. Parameter values, frequencies ω and inversion N of the chosen cw solutions. (a): $\tau = 100$, $\omega = -0.63$, $N = -0.3$; (b): $\tau = 100$, $\omega = -0.62$, $N = -0.2$; (c): $\tau = 100$, $\omega = -0.02$, $N = -0.14$; (d): $\tau = 100$, $\omega = -0.05$, $N = 0.11$; (e): $\tau = 100$, $\omega = 0.53$, $N = 0.13$; (f): $\tau = 100$, $\omega = 0.45$, $N = 0.28$; (g): $\tau = 200$, $\omega = -0.6$, $N = -0.3$; (h): $\tau = 200$, $\omega = -0.63$, $N = -0.2$; (i): $\tau = 200$, $\omega = -0.01$, $N = -0.13$; (j): $\tau = 200$, $\omega = 0.29$, $N = 0.0$; (k): $\tau = 200$, $\omega = -0.02$, $N = 0.12$; (l): $\tau = 200$, $\omega = 0.28$, $N = 0.236$.

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