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Optimal investment strategy under saving/borrowing rates spread with partial information

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Abstract

We study the optimal investment strategy for maximizing the expected utility of the terminal wealth with partial information. Under the assumption that the borrowing rate is higher than the saving rate and the utility function is $U(x) = \log x$, we develop a new method to solve such problem and derive the explicit solutions that are easy to implement.

1 Introduction

The study of continuous time portfolio optimization problem was initiated by Merton ([11], [12]). Under the hyperbolic absolute risk-avoiding (HARA) utility function, the closed form solution of the portfolio optimization problem is obtained for the constant coefficients model. With the same model and more general utility function, Karatzas et al [7] derived explicit solution which generalizes Merton's results. This problem is still of current research interest (cf. Browne ([1], [2], [3]), Davis and Norman [4], Yang and Ma [16] and Yang et al [15]). The common hypotheses to these papers are that the borrowing rate is the same as the saving rate, and the investor has access to the complete information.

In the real world situation, the borrowing rate is always higher than the saving rate. This spread of the rates will affect the decision an investor makes. The *aim* of this article is to see what is this effect. The investment strategy under various targets and under the interest rates spread have been studied for some cases (cf. Yang and Huang [14]). This article will consider the maximization of the utility functions under the interest rates spread which is not studied in the above mentioned papers.

On the other hand, the assumption of complete information does not fit the real world situation. Since the Brownian motion and the drift coefficient in the stochastic differential equation (SDE) satisfied by the price of the stock is usually not directly observable and cannot be estimated accurately, the information flow available to an investor is not the complete one. A reasonable assumption is that the investor has access to the information flow (partial information) generated by the past prices of the stock. The model with partial information is more reasonable, and its analysis becomes more complicated than the usual one with complete information. Based on partial information, Yang et al [15] and Yang and Ma [16] obtained the optimal strategy for maximizing the expected utility of the lifetime consumption and the optimal investment strategy, and the corresponding estimation formula for the information valuation. Yang and Xiong [17] studied the maximization of the expected utility for the terminal wealth and the related problems. An explicit solution is obtained under logarithmic utility. In this paper, we will consider the effect of both rate-spread and partial information. The utility function we take will be the logarithmic function.

Suppose a small investor can buy a stock whose price (per share) is a stochastic process S_t, can borrow money from a bank whose lending rate is R_t, and can deposit her money to a saving account with an interest rate r_t. For simplicity, we assume that $R_t > r_t > 0$

are deterministic. The goal of this investor is to choose a portfolio such that the expected utility of her wealth at terminal time T is maximized.

Suppose that the stock price S_t is governed by the following SDE:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t^1, \tag{1.1}$$

where the appreciation rate μ_t and the volatility σ_t of the stock are continuous stochastic processes, W^1 is a (standard) one-dimensional Brownian motion. All the processes considered in this paper are on a stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ and adapted to $\{\mathcal{F}_t\}$.

We assume that $\sigma_t = \sigma(\mu_t, \rho_t)$ where ρ_t (independent of (W^1, W^2)) is a stochastic process taking values in a measurable space E, σ is a real-valued function on $\mathbb{R} \times E$ and μ_t is a stochastic process governed by the following SDE:

$$d\mu_t = b_1(\mu_t)dW_t^1 + b_2(\mu_t)dW_t^2 + c(\mu_t)dt \tag{1.2}$$

where b_1, b_2, c are real functions on $\mathbb{R}$, W^2 is a one-dimensional Brownian motion independent of W^1. Note that the dependency of the volatility on the appreciation rate is a reasonable assumption. Usually, the higher the appreciation rate, the higher the volatility. ρ_t represents other random factors which may affect σ_t.

Assume that the investor can only get information from the movement of the stock price. Let

$$\mathcal{G}_t = \sigma(S_s : \ 0 \le s \le t).$$

Then, $\mathcal{G}_t$ is the information available to her at time t. Note that the quadratic variation process of S is

$$Var(S)_t = \int_0^t \sigma_r^2 S_r^2 dr.$$

Hence σ_t is $\mathcal{G}_t$-adapted. However, the appreciation rate μ_t is usually unobservable.

Let X_t be the wealth process. The portfolio at time t consists of three proportions $\pi_t \equiv (\xi_t, \eta_t, \zeta_t)$ of her total wealth X_t in the stock, her borrowing from the bank and in the saving deposit. Let $U(x) = \log x$ be the utility function. The *goal* of this investor is to choose $\{\pi_t : \ 0 \le t \le T\}$ such that $\mathbb{E}(U(X_T))$ is maximized.

However, her choice of the portfolio cannot be arbitrary. First of all, her decision must be based on the information available to her, i.e., π_t must be $\mathcal{G}_t$-adapted. We assume that the investor must live within her mean, i.e., $X_t \ge 0$ for all t.

In summary, we introduce the following

Definition 1.1. *An investment strategy $\{\pi_t\}_{0 \le t \le T}$ is* admissible *if*
i) The stochastic process π_t is $\mathcal{G}_t$-adapted;
ii) $\int_0^T \xi_t^2 \sigma_t^2 X_t^2 dt < \infty$, $X_t \ge 0$ a.s. for all $0 \le t \le T$;
iii) For all $t \in [0, T]$, we have

$$\xi_t - \eta_t + \zeta_t = 1, \quad \eta_t, \ \zeta_t \ge 0.$$

We denote the collection of all admissible strategies by $\mathcal{A}$.

Suppose that the investor has initial wealth X_0 and she will take self-financing trading strategy, namely, it generates neither positive nor negative dividends (cf. Duffie [5], p86). Then her wealth process $X = \{X_t\}_{0 \le t \le T}$ satisfies: $\forall \ 0 \le t \le T$,

$$\begin{aligned} dX_t &= \frac{\xi_t X_t}{S_t} dS_t - \eta_t X_t R_t dt + \zeta_t X_t r_t dt \\ &= (r_t + \xi_t(\mu_t - r_t) - \eta_t(R_t - r_t))X_t dt + \sigma_t \xi_t X_t dW_t^1. \end{aligned} \tag{1.3}$$

The investment portfolio problem is described as follows:

$$\max\left\{\mathbb{E}\left(U(X_T)\right) \quad \pi\in\mathcal{A}\right\} \tag{1.4}$$

subject to the constrain (1.3).

This paper is organized as follows: In the next section, we introduce some basic filtering techniques and derive a particle system representation of the optimal filtering of the appreciation rate μ_t. Note that the filtering model we encounter in this problem does not fall into the standard framework of the nonlinear filetring theory. *New technique* based on the particle system representation are developed in section 2. In section 3, we derive a solution for the optimal strategy based on dynamic programming principle.

2 Filtering setup

Since μ_t is not directly observable, it has to be estimated based on $\mathcal{G}_t$. We shall use the theory of nonlinear filtering to accomplish it. We refer the reader to the books of Kallianpur [6] and Liptser and Shiryayev [10] for an introduction on this topic.

Let U_t be the optimal filter of μ_t. Namely, U_t is a probability measure valued process such that for any $f\in C_b(\mathbb{R})$,

$$\langle U_t,f\rangle=\mathbb{E}(f(\mu_t)|\mathcal{G}_t),$$

here the notation $\langle U,f\rangle$ stands for the integral of a function f with respect to a measure U. Denote $m_t=\mathbb{E}(\mu_t|\mathcal{G}_t)$.

Applying Itô's formula to (1.1), we have

$$d\log S_t=\left(\mu_t-\frac{1}{2}\sigma_t^2\right)dt+\sigma_t dW_t^1. \tag{2.1}$$

The aim of this section is to solve the filtering problem with signal given by (1.2) and observation given by (2.1). Namely, we will derive the SDE satisfied by the optimal filter U_t. As $\sigma_t=\sigma(\mu_t,\rho_t)$, the coefficient before the observation noise, depends on the signal, this model is not covered by any known result of the nonlinear filtering theory. On the other hand, as we indicated in the introduction, σ_t can be regarded as a functional of $\{S_u:\ u\le t\}$. Therefore, the reader might think the model is covered by the classical theory of nonlinear filtering (cf. Theorem 8.1.1 in [10]). However, it is not known whether this functional is Lipschitz continuous. From this point of view, this model is again not covered by any existing result in the nonlinear filtering theory.

In this section, we derive the filtering equation based on the particle system representation idea of Kurtz and Xiong [8]. As a by-product of this approach, it provides a natural numerical scheme in solving the filtering equation (for the uncorrelated case, see Kurtz and Xiong [9]).

For fixed μ, let

$$\sigma_{+,t}(\mu)=\mathbb{E}\sigma(\mu,\rho_t)\text{ and }\sigma_{-,t}(\mu)=\mathbb{E}\sigma(\mu,\rho_t)^{-1}.$$

Theorem 2.1. *Suppose that b_1, b_2, c are bounded Lipschitz continuous functions. Suppose that $\sigma(\mu,\rho)$ is Lipschitz on μ with the Lipschitz constant independent of ρ. Then U_t is the unique solution to the following SDE:*

$$\begin{aligned} d\langle U_t,f\rangle &= \left(\left\langle U_t, b_1f'+\left(\sigma_{-,t}\iota-\frac{1}{2}\sigma_{+,t}\right)f\right\rangle-\langle U_t,f\rangle\left\langle U_t,\sigma_{-,t}\iota-\frac{1}{2}\sigma_{+,t}\right\rangle\right)d\tilde{\nu}_t \\ &\quad +\left\langle U_t, cf'+bf''\right\rangle dt \end{aligned} \tag{2.2}$$

where $b=\frac{b_1^2+b_2^2}{2}$, $\iota(\mu)=\mu$ *and* $\tilde{\nu}_t$ *is a Brownian motion.*

Proof: Let $d\nu_t \equiv \sigma_t^{-1} d\log S_t$. Since, $\sigma_t \in \mathcal{G}_t$, $\mathcal{F}_t^\nu \subset \mathcal{G}_t$. By Girsanov's formula, ν_t, independent of W^2, is a Brownian motion under the probability measure $\tilde{P}$ given by $d\tilde{P} = A_T dP$ where

$$A_t = \exp\left(\int_0^t \sigma_r^{-1}\left(\mu_r - \frac{1}{2}\sigma_r^2\right) d\nu_r - \frac{1}{2}\int_0^t \sigma_r^{-2}\left(\mu_r - \frac{1}{2}\sigma_r^2\right)^2 dr\right).$$

By Kallianpur-Striebel formula, we have

$$\langle U_t, f\rangle = \frac{\langle V_t, f\rangle}{\langle V_t, 1\rangle} \tag{2.3}$$

where

$$\langle V_t, f\rangle = \tilde{\mathbb{E}}\left(f(\mu_t)A_t \middle| \mathcal{G}_t\right)$$

is called the unnormalized filter.

Note that

$$dA_t = \sigma_t^{-1}\left(\mu_t - \frac{1}{2}\sigma_t^2\right) d\nu_t$$

and (1.2) can be rewritten as

$$d\mu_t = b_1(\mu_t)d\nu_t + b_2(\mu_t)dW_t^2 + \left(c - b_1\sigma^{-1}(\mu_t,\rho_t)\left(\mu_t - \frac{1}{2}\sigma^2(\mu_t,\rho_t)\right)\right)dt.$$

Now we derive a SDE for V_t. To this end, we consider an exchangeable particle system

$$\left\{\begin{array}{lll} d\mu_t^i &=& b_1(\mu_t^i)d\nu_t + b_2(\mu_t^i)dB_t^i + \left(c - b_1\sigma^{-1}(\mu_t^i,\rho_t^i)\left(\mu_t^i - \frac{1}{2}\sigma^2(\mu_t^i,\rho_t^i)\right)\right)dt \\ dA_t^i &=& \sigma^{-1}(\mu_t^i,\rho_t^i)\left(\mu_t^i - \frac{1}{2}\sigma^2(\mu_t^i,\rho_t^i)\right)A_t^i d\nu_t, \qquad i = 1,2,\cdot \end{array}\right. \tag{2.4}$$

and define

$$\left\langle \tilde{V}_t, f\right\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n A_t^i f(\mu_t^i)$$

where $\{B^i : i = 1,2,\cdots\}$ are independent copies of W^2. By the independence of ν and $\{B^i,\rho^i : i=1,2,\cdots\}$, we have

$$\left\langle \tilde{V}_t, f\right\rangle = \tilde{\mathbb{E}}\left(f(\mu_t)A_t \middle| \mathcal{F}_t^\nu\right). \tag{2.5}$$

On the other hand, (2.4) can be rewritten as

$$\left\{\begin{array}{lll} d\mu_t^i &=& b_1(\mu_t^i)\sigma^{-1}(\mu_t^i,\rho_t^i)d\log S_t + b_2(\mu_t^i)dB_t^i \\ && + \left(c - b_1\sigma^{-1}(\mu_t^i,\rho_t^i)\left(\mu_t^i - \frac{1}{2}\sigma^2(\mu_t^i,\rho_t^i)\right)\right)dt \\ dA_t^i &=& \sigma^{-1}(\mu_t^i,\rho_t^i)\left(\mu_t^i - \frac{1}{2}\sigma^2(\mu_t^i,\rho_t^i)\right)A_t^i\sigma^{-1}(\mu_t^i,\rho_t^i)d\log S_t, \qquad i=1,2,\cdot \end{array}\right. \tag{2.6}$$

which has strong solution $(\mu_t^i, A_t^i) = F_t(S, B^i, \rho^i)$. Let $\mathcal{I}_t$ be the invariant σ-field of the exchangeable sequence $\{(S,B^i,\rho^i) : i=1,2,\cdots\}$. Then $\mathcal{G}_t \subset \mathcal{I}_t$ and

$$\left\langle \tilde{V}_t, f\right\rangle = \tilde{\mathbb{E}}\left(f(\mu_t)A_t \middle| \mathcal{I}_t\right). \tag{2.7}$$

As $\mathcal{F}_t^\nu \subset \mathcal{G}_t \subset \mathcal{I}_t$, by (2.5) and (2.7), we get that $\tilde{V}_t = V_t$.

Applying Itô's formula to $A_t^i f(\mu_t^i)$, we have

$$\begin{aligned} d(A_t^i f(\mu_t^i)) &= A_t^i f'(\mu_t^i)\left(b_1(\mu_t^i)d\nu_t + b_2(\mu_t^i)dB_t^i + \left(c - b_1\sigma_t^{-1}\left(\mu_t^i - \frac{1}{2}\sigma_t^2\right)\right)dt\right) \\ &\quad + \frac{1}{2}A_t^i f''(\mu_t^i)b(\mu_t^i)dt + f(\mu_t^i)\sigma_t^{-1}\left(\mu_t^i - \frac{1}{2}\sigma_t^2\right)A_t^i d\nu_t \\ &\quad + b_1(\mu_t^i)f'(\mu_t^i)\sigma_t^{-1}\left(\mu_t^i - \frac{1}{2}\sigma_t^2\right)A_t^i dt. \end{aligned}$$

Take summation, divide both sides by n and let $n \to \infty$, we have

$$d\langle V_t, f\rangle = \left\langle V_t, b_1 f' + \left(\sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right)f\right\rangle d\nu_t + \langle V_t, cf' + bf''\rangle dt \tag{2.8}$$

here we used the fact that

$$\begin{aligned} &\lim_{n\to\infty}\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n\int_0^t A_s^i b_2(\mu_s^i)f'(\mu_s^i)dB_s^i\right|^2 \\ =\;& \lim_{n\to\infty}\mathbb{E}\int_0^t \mathbb{E}\left|A_s^i b_2(\mu_s^i)f'(\mu_s^i)\right|^2 ds \\ =\;& 0, \end{aligned}$$

$$\frac{1}{n}\sum_{i=1}^n A_t^i\sigma^{-1}(\mu_t^i,\rho_t^i)\mu_t^i f(\mu_t^i) \to \langle V_t, \sigma_{-,t}\iota f\rangle$$

and

$$\frac{1}{n}\sum_{i=1}^n A_t^i\sigma(\mu_t^i,\rho_t^i)f(\mu_t^i) \to \langle V_t, \sigma_{+,t} f\rangle.$$

Apply Itô's formula to (2.3), making use of (2.8), we have

$$\begin{aligned} d\langle U_t, f\rangle &= \langle V_t, 1\rangle^{-1} d\langle V_t, f\rangle - \langle V_t, f\rangle\langle V_t, 1\rangle^{-2} d\langle V_t, 1\rangle \\ &\quad + \langle V_t, f\rangle\langle V_t, 1\rangle^{-3}\left\langle V_t, \sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right\rangle^2 dt \\ &\quad - \langle V_t, 1\rangle^{-2}\left\langle V_t, b_1 f' + \left(\sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right)f\right\rangle\left\langle V_t, \sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right\rangle dt \\ &= \left\langle U_t, b_1 f' + \left(\sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right)f\right\rangle d\nu_t + \langle U_t, cf' + bf''\rangle dt \\ &\quad - \langle U_t, f\rangle\left\langle U_t, \sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right\rangle d\nu_t \\ &\quad + \langle U_t, f\rangle\left\langle U_t, \sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right\rangle^2 dt \\ &\quad - \left\langle U_t, b_1 f' + \left(\sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right)f\right\rangle\left\langle U_t, \sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right\rangle dt \\ &= \left\langle U_t, b_1 f' + \left(\sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right)f\right\rangle d\tilde{\nu}_t \\ &\quad - \langle U_t, f\rangle\left\langle U_t, \sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right\rangle d\tilde{\nu}_t + \langle U_t, cf' + bf''\rangle dt \end{aligned}$$

where $\tilde{\nu}_t$ is given by

$$d\tilde{\nu}_t = d\nu_t - \left\langle U_t, \sigma_{-,t}\iota - \frac{1}{2}\sigma_{+,t}\right\rangle dt.$$

This proves that (2.2) holds. The uniqueness follows from the same arguments as Theorem 3.3 in [8]. $\square$

$\tilde{\nu}_t$ given above is called the *innovation process* of the nonlinear filtering.

Denote $m_t = \langle U_t, \iota \rangle$. Note that

$$d\tilde{\nu}_t = \sigma_t^{-1} \left(\frac{dS_t}{S_t} - m_t dt \right).$$

The self-finance condition (1.3) can be rewritten as

$$dX_t = (r_t + (m_t - r_t)\xi_t - (R_t - r_t)\eta_t) X_t dt + \sigma_t \xi_t X_t d\tilde{\nu}_t. \tag{2.9}$$

When σ_t does not depend on μ_t, the classical filtering theory is applicable to the model (1.2, 2.1). The conclusion of the following remark is proved in [17].

Remark 2.2. *i) If b_1, b_2 are constants, $c(x) = ax$ and $0 < c_1 \le \sigma_t \le c_2 < \infty$ is deterministic, and U_0 is conditionally, given $\mathcal{G}_0$, normal with mean m_0 and variance γ_0 a.s., then U_t is conditionally, given $\mathcal{G}_t$, normal with mean m_t and variance γ_t a.s., where m_t and γ_t satisfies the following equations:*

$$\left\{ \begin{array}{rcl} dm_t & = & a m_t dt + \frac{b_1\sigma_t + \gamma_t}{\sigma_t^2} \left(\frac{dS_t}{S_t} - m_t dt \right), \\ \dot{\gamma}_t & = & 2a\gamma_t + 2b - \left(\frac{b_1 \sigma_t + \gamma_t}{\sigma_t} \right)^2, \end{array} \right. \qquad 0 \le t \le T. \tag{2.10}$$

ii) If, in addition, $\sigma_t = \sigma$ is a constant, then γ_t is solved explicitly as follows:

$$\gamma_t = \left\{ \begin{array}{ll} \gamma_- + 2\sqrt{\Lambda^2 + b_2^2\sigma^2} \left(1 - \frac{\gamma_0 - \gamma_+}{\gamma_0 - \gamma_-} \exp\left(-\frac{2\sqrt{\Lambda^2 + b_2^2 \sigma^2}}{\sigma^2} t \right) \right)^{-1} & \text{if } \Lambda^2 + b_2^2 \neq 0, \\ \left(\frac{1}{\gamma_0} + \frac{t}{\sigma^2} \right)^{-1} & \text{if } \Lambda = b_2 = 0, \end{array} \right.$$

where

$$\Lambda = a\sigma^2 - b_1 \sigma, \qquad \gamma_\pm = \Lambda \pm \sqrt{\Lambda^2 + b_2^2 \sigma^2}.$$

iii) In general, there is no explicit formula for U_t. A numerical solution can be derived based on the particle representation ((2.6), (2.5)). We refer the reader to Kurtz and Xiong [9] for a similar model.

3 Optimal strategy under partial information and logarithmic utility

In this section, we consider the optimization problem. Namely, we solve

$$\max_{\pi \in \mathcal{A}} \mathbb{E} \left[\log X_T \right], \tag{3.1}$$

subject to the constrain (2.9).

Applying Itô's formula to (2.9), we have

$$\begin{aligned} \log X_T &= \log X_0 + \int_0^T \sigma_t \xi_t d\tilde{\nu}_t \\ &\quad + \int_0^T \left(r_t + (m_t - r_t)\xi_t - (R_t - r_t)\eta_t - \frac{1}{2}\xi_t^2 \sigma_t^2 \right) dt. \end{aligned} \tag{3.2}$$

Taking expectation, we have

$$\mathbb{E}(\log X_T) = \log X_0 + \mathbb{E}\int_0^T \left(r_t + (m_t - r_t)\xi_t - (R_t - r_t)\eta_t - \frac{1}{2}\xi_t^2\sigma_t^2 \right) dt.$$

We now seek the solution to the following nonlinear dynamic programming problem (DPP):

$$\min_{\pi\in\mathcal{A}} \left\{ \frac{1}{2}\xi_t^2\sigma_t^2 - (m_t - r_t)\xi_t + (R_t - r_t)\eta_t - r_t \right\} \tag{3.3}$$

subjects to the constrains

$$\begin{cases} \eta_t - \xi_t + 1 & \geq & 0, \\ \eta_t & \geq & 0. \end{cases}$$

To this end, we define the Lagrange function

$$L(\xi_t, \eta_t, \lambda_1, \lambda_2) = \frac{1}{2}\xi_t^2\sigma_t^2 - (m_t - r_t)\xi_t + (R_t - r_t)\eta_t - r_t - \lambda_1(\eta_t - \xi_t + 1) - \lambda_2\eta_t.$$

By the principle of the nonlinear dynamic programming, we have

Lemma 3.1. *If $(\xi_t^*, \eta_t^*, \lambda_1, \lambda_2)$ solves*

$$\begin{cases} L_{\xi_t} = \xi_t\sigma_t^2 - m_t + r_t + \lambda_1 = 0 & (i) \\ L_{\eta_t} = R_t - r_t - \lambda_1 - \lambda_2 = 0 & (ii) \\ L_{\lambda_1} = -(\eta_t - \xi_t + 1) \leq 0 & (iii) \\ L_{\lambda_2} = -\eta_t \leq 0 & (iv) \\ \lambda_1 \geq 0, \quad \lambda_2 \geq 0 & (v) \\ \lambda_1(\eta_t - \xi_t + 1) + \lambda_2\eta_t = 0, & (vi) \end{cases} \tag{3.4}$$

then it is a solution to the nonlinear DPP (3.3).

To solve (3.4), we denote

$$\theta_t^- = \frac{m_t - R_t}{\sigma_t^2}, \qquad \theta_t^+ = \frac{m_t - r_t}{\sigma_t^2}.$$

It is obvious that $\theta_t^- < \theta_t^+$. By (iii) and (vi) in (3.4), we see that

$$(\xi_t - 1)\lambda_1 = (R_t - r_t)\eta_t, \qquad (\xi_t - 1)\lambda_2 = (R_t - r_t)(\xi_t - 1 - \eta_t). \tag{3.5}$$

Now we discuss the solution according to three cases.

Case 1: $(\xi_t > 1)$. By (iii) in (3.4), we have $\eta_t \geq \xi_t - 1 > 0$. On the other hand, (v) in (3.4) and (3.5) imply that $\xi_t - 1 \geq \eta_t$. Therefore $\eta_t = \xi_t - 1$. Plug it into (3.5), we see that $\lambda_1 = R_t - r_t$ and $\lambda_2 = 0$. (i) of (3.4) then implies $\xi_t = \theta_t^-$.

Case 2: $(\xi_t < 1)$. By (3.5) and (v) in (3.4), we have $\eta_t \leq 0$. Combine with (iv) in (3.4), we get $\eta_t = 0$. By (3.5), we see that $\lambda_1 = 0$. Therefore, (i) in (3.4) implies that $\xi_t = \theta_t^+$.

Case 3: $(\xi_t = 1)$. By (i) in (3.4),

$$\lambda_1 = m_t - r_t - \sigma_t^2. \tag{3.6}$$

By (v) in (3.4), it is clear that

$$m_t - r_t \geq \sigma_t^2.$$

Namely $\theta_t^+ \geq 1$ holds. Plug (3.6) into (ii) in (3.4), we arrive at

$$\lambda_2 = R_t - m_t + \sigma_t^2.$$

(v) in (3.4) then implies

$$m_t - R_t \leq \sigma_t^2.$$

Namely $\theta_t^- \leq 1$ holds. (3.5) clearly implies $\eta_t = 0$.

To summarize, we have

Theorem 3.2. *The solution $(\xi_t^*, \eta_t^*, \zeta_t^*)$ to the optimization problem (3.3) is given by*

$$(\xi_t^*, \eta_t^*, \zeta_t^*) = \begin{cases} (\theta_t^-, \theta_t^- - 1, 0) & \text{if } \theta_t^- > 1; \\ (\theta_t^+, 0, 1 - \theta_t^+) & \text{if } \theta_t^+ < 1; \\ (1, 0, 0) & \text{if } \theta_t^- \leq 1 \text{ and } \theta_t^+ \geq 1. \end{cases} \tag{3.7}$$

Proof: It is easy to calculate the Hessian matrix of L. In fact, its determinant is -1. Hence, the portfolio we obtained indeed maximizes the expected utility. $\square$

Remark 3.3. *(i) From the optimal strategy we see that the borrowing is zero if the saving is positive and vice versa. This coincides with the intuition.*

(ii) The optimal strategy formula shows that when the estimated appreciation rate is very high, then she should borrow some money and invest it in stock; when the estimated appreciation rate is very low, then she should make a certain cash reserve and buy a certain amount of the stock, but never borrow money from the bank; when the estimated appreciation rate is moderate, then she should put all her money in stock but no borrowing. This also conform with the intuition. However, the intuition does not give us the precise portfolio.

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