FLUCTUATIONS OF THE FREE ENERGY IN THE REM AND THE p-SPIN SK MODELS

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Abstract: We consider the random fluctuations of the free energy in the p-spin version of the Sherrington-Kirkpatrick model in the high temperature regime. Using the martingale approach of Comets and Neveu as used in the standard SK model combined with truncation techniques inspired by a recent paper by Talagrand on the p-spin version, we prove that (for p even) the random corrections to the free energy are on a scale $N^{-(p-2)/4}$ only, and after proper rescaling converge to a standard Gaussian random variable. This is shown to hold for all values of the inverse temperature, β, smaller than a critical β_p. We also show that $\beta_p \to \sqrt{2\ln 2}$ as $p \uparrow +\infty$. Additionally we study the formal $p \uparrow +\infty$ limit of these models, the random energy model. Here we compute the precise limit theorem for the partition function at *all* temperatures. For $\beta < \sqrt{2\ln 2}$, fluctuations are found at an *exponentially small* scale, with two distinct limit laws above and below a second critical value $\sqrt{\ln 2/2}$: For β up to that value the rescaled fluctuations are Gaussian, while below that there are non-Gaussian fluctuations driven by the Poisson process of the extreme values of the random energies. For β larger than the critical $\sqrt{2\ln 2}$, the fluctuations of the logarithm of the partition function are on scale one and are expressed in terms of the Poisson process of extremes. At the critical temperature, the partition function divided by its expectation converges to $1/2$.

Keywords: spin glasses, Sherrington-Kirkpatrick model, p-spin model, random energy model, Central Limit Theorem, extreme values, martingales

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1. Introduction.

In recent years it has become increasingly clear that a problem of central importance for the understanding of disordered spin systems is the control of random fluctuations of thermodynamic quantities [AW,NS,BM,T1]. Unfortunately, a precise control of such quantities is very hard to come by. Concentration of measure techniques [T2] have been realized to be efficient tools to get *upper bounds* [BGP1,BG1], but lower bounds or exact limit theorems are scarce. One of these examples is the Sherrington-Kirkpatrick (SK) model in the high-temperature phase, where a central limit theorem for the free energy was proven first by Aizenman, Lebowitz and Ruelle [ALR], using cluster expansion techniques, and later by Comets and Neveu [CN], making use of martingale methods and stochastic calculus. Their methods have been extended to a few related cases [Tou,B1] later. In the present paper we want to continue this effort by investigating a large class of natural generalisation of the SK model, the so called p-spin SK models, and their $p \uparrow +\infty$ limit, the random energy model (REM).

For our present purposes it is natural to consider the class of models we study as Gaussian processes on the hypercube $\mathcal{S}_N = \{-1,1\}^N$. We will always denote the corner of $\mathcal{S}_N$ by σ; for historical reasons they are called *spin configurations.* A Gaussian process X on $\mathcal{S}_N$ is characterized completely by its mean and covariance function. The processes we consider will always be assumed to have mean zero and covariance

$$\mathbb{E}X_\sigma X_{\sigma'} \equiv f(R_N(\sigma,\sigma')), \tag{1.1}$$

where f depends on the so-called *overlap*[4], $R_N(\sigma,\sigma') \equiv N^{-1}(\sigma,\sigma') \equiv N^{-1}\sum_{i=1}^N \sigma_i {\sigma'}_i$, In this note we will concentrate on the case where $f(x) = f_p(x) := x^p$, with p even.[5] In this case, X_σ can be represented in the form

$$X_\sigma = N^{-p/2} \sum_{i_1,i_2,\dots,i_p} J_{i_1,i_2,\dots,i_p} \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_p} \tag{1.2}$$

with $J_{i_1,\dots,i_p}$ a family of i.i.d. normal random variables. Since for $p=2$ we obtain the classical SK model, this representation justifies the name *p-spin SK model.* Note that as p increases, the process gets more and more de-correlated, and in the limit $p \uparrow +\infty$ we arrive at the case where X_σ are i.i.d. normal random variables.

[4]The overlap is related to the Hamming distance d_{Ham} by $d_{Ham}(\sigma,\sigma') = N(1-R_N(\sigma,\sigma'))/2$.

[5]The case p odd can also be treated, but presents considerable additional computational problems.

Given such a Gaussian process, our main object of interest is the so called *partition function*,

$$Z_{\beta,N} \equiv \mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma} \equiv 2^{-N} \sum_{\sigma\in\mathcal{S}_N} e^{\beta\sqrt{N}X_\sigma}. \tag{1.3}$$

The quantities $e^{\beta\sqrt{N}X_\sigma}$ are called *Boltzmann weights* and the parameter $\beta \in \mathbb{R}_+$ is known as the *inverse temperature*, and $H_N(\sigma) \equiv \sqrt{N}X_\sigma$ as (minus) the *Hamiltonian* in statistical mechanics. $Z_{\beta,N}$ are random variables, and we are primarily interested in their behaviour as N tends to infinity. In statistical mechanics, it is customary to introduce the so-called *free energy*

$$F_{\beta,N} \equiv -\frac{1}{\beta N} \ln Z_{\beta,N}. \tag{1.4}$$

It is easy to prove in all the models we consider here, that for all values of β, $F_{\beta,N}$ is a *self-averaging* quantity, i.e. that

$$\lim_{N\uparrow+\infty} |F_{\beta,N} - \mathbb{E}F_{\beta,N}| = 0 \quad \text{a.s.} \tag{1.5}$$

It is, however, not known in general whether the so called *quenched* free energy $\mathbb{E}F_{\beta,N}$ converges to a limit as N tends to infinity. This has, however, been proven for sufficiently small values of β: more precisely, one knows that

Theorem 1.1. *Define* $\tilde\beta_2 = 1$, *and for* $p > 2$

$$\tilde\beta_p^2 \equiv \inf_{0<m<1} (1 + m^{-p})\phi(m) \tag{1.6}$$

where $\phi(m) \equiv [(1-m)\ln(1-m) + (1+m)\ln(1+m)]/2$. *Then for all* $\beta < \tilde\beta_p$

$$\lim_{N\uparrow+\infty} F_{\beta,N,p} = -\beta/2. \tag{1.7}$$

Remark: For $p = 2$ this result was first proven in [ALR]. A very simple proof has later been given by Talagrand [T]. Comets [C] has shown that the value $\beta = 1$ is optimal in the sense that (1.7) fails for $\beta > 1$. The result for $p \geq 3$ is due to Talagrand [T1]. It is clear that in all cases (1.7) will fail for $\beta \geq \sqrt{2\ln 2}$ which by a more elaborate computation can be improved to $\beta \geq \sqrt{2\ln 2}(1 - 2^{-c_p p})$ with $c_p < 5$, for p large [B2]. On the other hand, a simple calculation shows that $\tilde\beta_p \sim \sqrt{2\ln 2}(1 - 2^{-p}/2\ln 2)$. One should note that to get (1.7) up to a value so close to $\sqrt{2\ln 2}$ required a substantial modification of the original argument of [T2], namely the use of a "truncated" second moment method. Such a truncation will also be the main difficulty in obtaining our results[6].

[6] For similar reasons, slightly different truncations were also used by Toubol [Tou] (and probably first) in the study of the CLT for the SK model with vector valued spins.

In the case of the REM, it is well known that the critical inverse temperature $\tilde{\beta}_{REM} = \sqrt{2\ln 2}$ and that [D2]

$$\lim_{N\uparrow+\infty} F_{\beta,N,REM} = \begin{cases} -\beta/2, & \text{if } \beta \leq \sqrt{2\ln 2} \\ -\sqrt{2\ln 2} + \beta^{-1}\ln 2, & \text{if } \beta \geq \sqrt{2\ln 2}. \end{cases} \tag{1.8}$$

As a consequence one has that (this result is essentially contained in [D1], a rigorous proof follows easily from the results contained in [T1][7])

$$\lim_{p\uparrow+\infty} \limsup_{N\uparrow+\infty} F_{\beta,N,p} = \lim_{p\uparrow+\infty} \liminf_{N\uparrow+\infty} F_{\beta,N,p} = \lim_{N\uparrow+\infty} F_{\beta,N,REM}. \tag{1.9}$$

In this note we will control the fluctuations of the free energy in (essentially) all of the domain of parameters β, p (even) where the limit is known to exists, i.e. the high temperature regions of the p-spin models, and the *entire* temperature range in the REM. Although the REM is rather singular and the techniques used for that case are totally different from those we will use for the p-spin models, we felt it would be instructive to include this singular limiting case in this paper. Moreover, it turns out that in spite of the heavy investigation the REM has enjoyed over the years [D1,D2,OP,GMP,Ru], no precise fluctuation results for the free energy are available in the literature. Finally, we are convinced that the reader will be rather surprised by the rich structure the fluctuation behaviour this model exhibits.

Let us now state our results. We begin with the p-spin SK models.

Theorem 1.2 *Consider the p-spin SK-model with $p = 2k \geq 2$. There exists $\beta_p > 0$ such that for all $\beta < \beta_p$*

$$N^{(p-2)/4} \ln \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \xrightarrow{\mathcal{D}} M_\infty(\sqrt{\beta}) \tag{1.10}$$

in distribution as $N \uparrow +\infty$, where $M_\infty(t)$ is the centered Gaussian process with mean zero and covariance

$$\mathbb{E}(M_\infty(t) - M_\infty(s))^2 = (t-s)\big[(p-1)!!\big]. \tag{1.11}$$

The value of β_p can be estimated reasonably well. To state lower bound on β_p we need, however, some notation. We define the functions

$$\begin{aligned} I(m_1,m_2,m_3) = \frac{1}{4}\Big(&(1+m_1+m_2+m_3)\ln(1+m_1+m_2+m_3) \\ &+ (1-m_1-m_2+m_3)\ln(1-m_1-m_2+m_3) \\ &+ (1+m_1-m_2-m_3)\ln(1+m_1-m_2-m_3) \\ &+ (1-m_1+m_2-m_3)\ln(1-m_1+m_2-m_3)\Big), \end{aligned} \tag{1.12}$$

[7] Private communication by M. Talagrand.

$$S_p(m_1,m_2,m_3) = \Big[\Big(1+\frac{m_1^p - m_2^p m_3^p}{1-m_3^{2p}}\Big)^2 + \Big(1+\frac{m_2^p - m_1^p m_3^p}{1-m_3^{2p}}\Big)^2 + 2m_3^p\Big(1+\frac{m_1^p - m_2^p m_3^p}{1-m_3^{2p}}\Big)\Big(1+\frac{m_2^p - m_1^p m_3^p}{1-m_3^{2p}}\Big)\Big]^{1/2}, \tag{1.13}$$

$$R_p(m_1,m_2,m_3) = \frac{2m_1^p m_2^p m_3^p - m_1^{2p} - m_2^{2p}}{2(1-m_3^{2p})}, \tag{1.14}$$

and

$$U_p(m_1,m_2,m_3) = I(m_1,m_2,m_3)(1+m_3^p)\Big[S_p(m_1,m_2,m_3)\sqrt{2+2m_3^p} + R_p(m_1,m_2,m_3)(1+m_3^p) - (2+m_3^p)\Big]^{-1} \tag{1.15}$$

on the set

$$\mathcal{A} \equiv \big\{m_1,m_2,m_3 \in [-1,1]^3 \mid 1-m_1-m_2+m_3>0, 1-m_1+m_2-m_3>0, 1+m_1-m_2-m_3>0\big\}. \tag{1.16}$$

Note that the function $I(m_1,m_2,m_3)$ is symmetric in m_1, m_2 and m_3, and that $S_p(m_1,m_2,m_3)$, $R_p(m_1,m_2,m_3)$ and $U_p(m_1,m_2,m_3)$ are symmetric in m_1 and m_2. Let

$$Y_p(m_1,m_2,m_3) = \max\Big\{I(m_1,m_2,m_3)\left(\frac{2}{3}+\frac{1}{m_1^p+m_2^p+m_3^p}\right), U_p(m_1,m_2,m_3),\ U_p(m_1,m_3,m_2),\ U_p(m_2,m_3,m_1)\Big\}. \tag{1.17}$$

With this notation we have

Theorem 1.3 *Let $p=2k>2$. Then*

$$\inf_{m_1,m_2,m_3\in\mathcal{A}} Y_p(m_1,m_2,m_3) \le \beta_p^2 < 2\ln 2. \tag{1.18}$$

In particular

$$\lim_{p\uparrow+\infty} \beta_p^2 = 2\ln 2. \tag{1.19}$$

We see that the scale on which the partition functions fluctuate decreases rapidly as p increases. One might guess that the scale becomes exponentially small in N in the limiting random energy model. This is indeed true, but more surprising things happen, as the following theorem states:

Theorem 1.4 *The free energy of the REM has the following fluctuations:*

(i) If $\beta < \sqrt{\ln 2/2}$, then

$$e^{\frac{N}{2}(\ln 2-\beta^2)} \ln\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{1.20}$$

(ii) If $\beta = \sqrt{\ln 2/2}$, then

$$\sqrt{2} e^{\frac{N}{2}(\ln 2-\beta^2)} \ln \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \stackrel{\mathcal{D}}{\to} \mathcal{N}(0,1). \tag{1.21}$$

(iii) Let $\alpha \equiv \beta/\sqrt{2\ln 2}$. If $\sqrt{\ln 2/2} < \beta < \sqrt{2\ln 2}$, then

$$e^{\frac{N}{2}(\sqrt{2\ln 2}-\beta)^2+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} \ln \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \stackrel{\mathcal{D}}{\to} \int_{-\infty}^{\infty} e^{\alpha z}(\mathcal{P}(dz) - e^{-z}dz), \tag{1.22}$$

where $\mathcal{P}$ denotes the Poisson point process on $\mathcal{R}$ with intensity measure $e^{-x}dx$.

Theorem 1.3 covers the high temperature regime. However, in the REM we can also compute the fluctuations in the low temperature phase.

Theorem 1.5

(i) If $\beta = \sqrt{2\ln 2}$, then

$$e^{\frac{1}{2}[\ln(N\ln 2)+\ln 4\pi]}\left(\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} - \frac{1}{2} + \frac{\ln(N\ln 2)+\ln 4\pi}{4\sqrt{\pi N \ln 2}}\right) \stackrel{\mathcal{D}}{\to} \int_{-\infty}^{0} e^{\alpha z}(\mathcal{P}(dz) - e^{-z}dz) + \int_{0}^{\infty} e^{z}\mathcal{P}(dz). \tag{1.23}$$

(ii) If $\beta > \sqrt{2\ln 2}$, then

$$e^{-N[\beta\sqrt{2\ln 2}-\ln 2]+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} Z_{\beta,N} \stackrel{\mathcal{D}}{\to} \int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz) \tag{1.24}$$

and

$$\ln Z_{\beta,N} - \mathbb{E}\ln Z_{\beta,N} \stackrel{\mathcal{D}}{\to} \ln \int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz) - \mathbb{E}\ln \int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz). \tag{1.25}$$

Remark: Note that expressions like $\int_{-\infty}^{0} e^{\alpha z}(\mathcal{P}(dz) - e^{-z}dz)$ are always understood as $\lim_{y\downarrow -\infty} \int_{y}^{0} e^{\alpha z}(\mathcal{P}(dz) - e^{-z}dz)$. We will see that all the functionals of the Poisson point process appearing are almost surely finite random variables.

Remark: Note that the Poisson integral $\int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz)$ is the partition function of Ruelle's version of the REM [Ru]. Thus (1.25) affirms that above the critical temperature, the fluctuations of the free energy of the REM converge in distribution to those of Ruelle's model. While this connection was surely evident for Ruelle and motivated the introduction of his model, we have not been able to find a rigorous statement of this connection in the literature. In [GMP] the scale on which fluctuations take place has been established, but no actual limit theorem was proven.

Remark: It is interesting to observe that in the REM there is a second "phase transition" within the high-temperature phase at which the fluctuations become non-Gaussian. In fact, in the REM the main phase transition can be interpreted as a breakdown of the Law of Large Numbers, while the second transition corresponds to a breakdown of the Central Limit Theorem.

The remainder of this paper is organized as follows. In the next section we present the proofs of Theorems 1.2 and 1.3. They are based on an adaptation of the martingale method of Comets and Neveu. The essential new ingredient is the rather involved truncation procedure inspired by Talagrand's work. However, in the proof of the CLT, the computational aspects become even more involved and require the consideration of truncated third moment of the partition function. For this reason Section 2 is rather long and quite technical. However, the proof is organized in such a way that the CLT is first proven for "very high" temperatures where no truncations are necessary, while the more technical aspects needed to approach the critical temperature are dealt with separately later. Section 3 is devoted to proving Theorems 1.4 and 1.5 for the REM. It is technically completely different and independent from Section 2. It can therefore be read independently from the rest of the paper. In an appendix we explain some of the technical difficulties that appear in the case p odd and we explain the result to be expected in that case.

2. The CLT in the p-spin model

The proof of the central limit theorem in the p-spin SK model relies on a martingale central limit theorem which uses that fact that a Gaussian random variable can always be seen as the marginal distribution of a Brownian motion. Thus we follow Comets and Neveu and introduce the p-parameter family of independent standard Brownian motions $(J_{i_1,i_2,\dots,i_p}(t), t \in \mathbb{R}^+)_{i_1,i_2,\dots,i_p\in\mathbb{N}}$ with $\mathbb{E}J_{i_1,i_2,\dots,i_p}(t) = 0$ and $\mathbb{E}J^2_{i_1,i_2,\dots,i_p}(t) = t$. The Hamiltonian of the p-spin SK model can then be written as $H_N(\sigma,t) = \sqrt{N}X_\sigma(t)$, where

$$X_\sigma(t) = \frac{1}{\sqrt{N^p}} \sum_{1\leq i_1,i_2,\dots,i_p\leq N} J_{i_1,i_2,\dots,i_p}(t)\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_p}. \tag{2.1}$$

Note that we can also consider it as a Gaussian process on $\{-1,1\}^N \times \mathbb{R}^+$ with mean zero and correlation function

$$\operatorname{cov}\left(X_\sigma(t), X_{\sigma'}(s)\right) = (s\wedge t)\, f_p\Big(R_N(\sigma,\sigma')\Big), \tag{2.2}$$

where $f_p(x) = x^p$. In particular, we have $\mathbb{E}H^2_N(\sigma,t) = Nt$ and $\mathbb{E}\exp\{H_N(t,\sigma)\} = \exp\{Nt/2\}$ for all σ. For later convenience we introduce the *normalized* partition function

$$\bar{Z}_N(t) = \mathbb{E}_\sigma \exp\{H_N(t,\sigma) - Nt/2\}, \tag{2.3}$$

It is related to the partition function $Z_{\beta,N}$ of Section 1 by $\bar{Z}_N(\beta^2) = Z_{\beta,N}/\mathbb{E}Z_{\beta,N}$, with equality holding in law. The important point of this construction is that that for all fixed $N > 1$, $\bar{Z}_N(t)$ is a *continuous martingale* in the variable t with $\mathbb{E}\bar{Z}_N(t) = 1$.

We begin the proof with some preliminary steps along the lines of [CN]. Let us find the bracket $< \bar{Z}_N(t) >$ of the martingale $\bar{Z}_N(t)$, i. e. the unique increasing process vanishing at zero, such that $\bar{Z}_N^2(t) - < \bar{Z}_N(t) >$ is the continuous martingale (see [RY]). By Ito's formula, $\bar{Z}_N(t)$ satisfies the following stochastic differential equation:

$$\mathrm{d}\bar{Z}_N(t) = \mathbb{E}_\sigma \exp\{H_N(t,\sigma) - Nt/2\} \,\mathrm{d}H_N(t,\sigma). \tag{2.4}$$

Then due to well-known properties of martingale brackets

$$\begin{aligned} < \bar{Z}_N(t) > &= \mathbb{E}_{\sigma,\sigma'} < \int_0^t e^{H_N(s,\sigma)-Ns/2} \,\mathrm{d}H_N(s,\sigma), \int_0^t e^{H_N(s,\sigma')-Ns/2} \,\mathrm{d}H_N(s,\sigma') > \\ &= \mathbb{E}_{\sigma,\sigma'} \int_0^t e^{H_N(s,\sigma)+H_N(s,\sigma')-Ns} \,\mathrm{d} < H_N(s,\sigma), H_N(s,\sigma') > \\ &= \mathbb{E}_{\sigma,\sigma'} \int_0^t e^{H_N(s,\sigma)+H_N(s,\sigma')-Ns} \, N f_p\Big(R_N(\sigma,\sigma')\Big) \,\mathrm{d}s. \end{aligned} \tag{2.5}$$

Since

$$\mathbb{E}\int_0^t \bar{Z}_N^{-2}(s) \,\mathrm{d} < \bar{Z}_N(s) > = \mathbb{E}\int_0^t \frac{\mathbb{E}_{\sigma,\sigma'} e^{H_N(s,\sigma)+H_N(s,\sigma')-Ns} \, N f_p\Big(R_N(\sigma,\sigma')\Big)}{\mathbb{E}_{\sigma,\sigma'} e^{H_N(s,\sigma)+H_N(s,\sigma')-Ns}} \,\mathrm{d}s \le Nt < \infty, \tag{2.6}$$

we may introduce a continuous local martingale $M_N(t) = \int_0^t \bar{Z}_N^{-1}(s) \,\mathrm{d}\bar{Z}_N(s)$. Thus $\bar{Z}_N(t)$ solves the stochastic differential equation

$$\mathrm{d}\bar{Z}_N(t) = \bar{Z}_N(t) \,\mathrm{d}M_N(t)$$

and the following fundamental representation of $\bar{Z}_N(t)$ holds:

$$\bar{Z}_N(t) = \exp\{M_N(t) - 1/2 < M_N(t) >\}. \tag{2.7}$$

Here $< M_N(t) >$ is the bracket of $M_N(t)$ and $< M_N(t) > = \int_0^t \bar{Z}_N^{-2}(s) \,\mathrm{d} < \bar{Z}_N(s) >$. Let us note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} < M_N(t) > &= \bar{Z}_N^{-2}(t) \frac{d}{dt} < \bar{Z}_N(t) > \\ &= \bar{Z}_N^{-2}(t) \Big(\mathbb{E}_{\sigma,\sigma'} e^{H_N(t,\sigma)+H_N(t,\sigma')-Nt} \, N f_p\Big(R_N(\sigma,\sigma')\Big) \Big). \end{aligned} \tag{2.8}$$

Note also that $M_N(t)$ is locally square integrable. In fact, by (2.6)

$$\mathbb{E}M_N^2(t) = \mathbb{E} < M_N(t) >= \mathbb{E}\int_0^t \bar{Z}_N^{-2}(s)\,\mathrm{d} < \bar{Z}_N(s) >\leq Nt < \infty. \tag{2.9}$$

To prove Theorems 1.2 and 1.3, we will show that for all t satisfying

$$t < \inf_{m_1,m_2,m_3\in\mathcal{A}} Y_p(m_1,m_2,m_3). \tag{2.10}$$

the bracket of the local martingale $N^{(p-2)/4}M_N(t)$, which is $N^{(p-2)/2} < M_N(t) >$, converges to $t\mathbb{E}\xi^p$ in probability as $N \uparrow +\infty$. Here ξ is a Gaussian random variable with $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$. Then by the martingale convergence theorem (see Theorem 3.1.8 in [JS]) the local martingale $N^{(p-2)/4}M_N(t)$ converges to $M_\infty(t)$ in law as $N \uparrow +\infty$. This fact together with the representation (2.7) implies immediately the statement of Theorem 1.2.

Sketch of the proof of Theorems 1.2 and 1.3: We will now outline further steps of the proof. First, we show the convergence $N^{(p-2)/2} < M_N(t) >\to t\mathbb{E}\xi^p$ on a more restricted interval of t. Lemma 2.1 reduces this problem to the convergence of

$$N^{(p-2)/2}\mathbb{E}|V_N(t)| \to 0, \qquad \text{as } N \uparrow +\infty, \tag{2.11}$$

where

$$V_N(t) := N^{-\frac{p-2}{2}}\mathbb{E}_{\sigma,\sigma'}\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma')\Big) - \mathbb{E}\xi^p\Big)e^{H_N(t,\sigma)+H_N(t,\sigma')-Nt}.$$

The proof of this lemma is based on the fact that

$$N^{(p-2)/2}\frac{\mathrm{d}}{\mathrm{d}t} < M_N(t) > -\mathbb{E}\xi^p = N^{(p-2)/2}\frac{V_N(t)}{\bar{Z}_N^2(t)}, \tag{2.12}$$

and is performed via integration. It almost mimics the proof proposed in [CN]. In particular, we use the fact that $\bar{Z}_N^2(t)$ is not small on events of large probability. The convergence (2.11) is proved in Proposition 2.2. Let us give some intuition for it. One can write

$$\mathbb{E}V_N(t) = \sum_{m=0,\pm 1/N,\dots,\pm 1} (Nf_p(m) - N^{(2-p)/2}\mathbb{E}\xi^p)e^{tNf_p(m)}\mathbb{P}(\sigma\cdot\sigma' = mN). \tag{2.13}$$

By Stirling's formula

$$\mathbb{P}(\sigma\cdot\sigma' = mN) \sim \frac{2}{\sqrt{2\pi(1+m)(1-m)N}}e^{-N\phi(m)},$$

where $\phi(m) = [(1+m)\ln(1+m) + (1-m)\ln(1-m)]/2$. (here and in the sequel we use the symbol $\sim$ to denote asymptotic equivalence, i.e. $a_N \sim b_N \Leftrightarrow \lim_{N\uparrow+\infty} \frac{a_N}{b_N} = 1$). Note that $\phi(m) = -m^2/2(1+o(1))$ as $m \to 0$. Now split the right-hand side of (2.13) into two terms: the summation in the first one will be over m with $|m|$ "small enough" and in the second — over all other m. It is not difficult to treat the first term. Since $p \geq 3$, we have for any fixed t

$$tf_p(m) + \phi(m) = -m^2/2(1+o(1)), \qquad m \to 0. \tag{2.14}$$

Then putting $m\sqrt{N} = s$, the first term becomes

$$\frac{2}{\sqrt{2\pi N}} \sum_{s=m\sqrt{N}} (N^{(2-p)/2} s^p - N^{(2-p)/2}\mathbb{E}\xi^p) e^{-s^2/2} \sim \frac{2N^{(2-p)/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (s^p - \mathbb{E}\xi^p) e^{-s^2/2}\, \mathrm{d}s,$$

from where the normalisation $N^{(p-2)/2}$ is immediate. To ensure the convergence to zero of the second term (the one with correlations m not close to zero), the power of the exponent in it should be negative:

$$\sup_{m\in[0,1]} (tf_p(m) - \phi(m)) < 0.$$

Thus for all $t < \inf_{0<m<1} \phi(m) m^{-p}$, we get $N^{(p-2)/2}\mathbb{E}V_N(t) \to 0$. Note that, Proposition 2.2 states a stronger result (2.11). To get rid of the absolute value of $V_N(t)$ in (2.11), we follow an idea suggested in [CN] to apply the Cauchy-Schwartz inequality. Thus, instead of $\mathbb{E}|V_N(t)|$, we get $W_N(t)$ (see the proof of Proposition 2.2) which refers to the third moment of $\bar{Z}_N(t)$. This makes technical computations slightly tougher and leads to the bound on t (2.19) given in Lemma 2.1 below.

Note also that these arguments are valid only for $p \geq 3$. The case $p = 2$ of [CN] and [Tou] is different, since there, (2.14) does not hold. This case is treated in [CN] by the multi-dimensional Central Limit Theorem for N independent vectors $(\sigma_i\sigma_i', \sigma_i'\sigma_i'', \sigma_i\sigma_i'')$.

Next, we will extend the bound (2.19) to the full regime announced in (2.10). We have seen, that (2.19) was imposed by configurations of spins with rather big correlations m in the sum (2.13). We will reduce their contribution, using Talagrand's idea to truncate the Hamiltonian. Consider instead of $V_N(t)$

$$\begin{aligned}\widetilde{V}_N(t,\epsilon) = &\mathbb{E}_{\sigma,\sigma'} \Big(Nf_p\Big(R_N(\sigma,\sigma')\Big) - N^{(2-p)/2}\mathbb{E}\xi^p\Big) e^{H_N(t,\sigma)+H_N(t,\sigma')-Nt} \\ &\times \mathbb{1}_{\{H_N(t,\sigma)<(1+\epsilon)tN, H_N(t,\sigma')<(1+\epsilon)tN\}}\end{aligned}$$

for some $\epsilon > 0$. Then

$$\mathbb{E}\widetilde{V}_N(t,\epsilon) = \sum_{m=0,\pm 1/N,\ldots,\pm 1} (Nf_p(m) - tN^{(2-p)/2})\mathbb{P}(\sigma\cdot\sigma' = mN) \times \mathbb{E}e^{\sqrt{Nt}\xi_1+\sqrt{Nt}\xi_2-Nt}\mathbb{1}_{\{\xi_1<\sqrt{Nt}(1+\epsilon),\xi_2<\sqrt{Nt}(1+\epsilon)\}}, \tag{2.15}$$

where ξ_1,ξ_2 are standard Gaussians with cov $(\xi_1,\xi_2)=m$. Let us again split $\mathbb{E}\widetilde{V}_N(t,\epsilon)$ into two terms with "small" and "large" m in the sum (2.15). The analysis of the first term is completely analogous to the one in the case of $V_N(t)$. We can neglect the truncation here, since ξ_1 and ξ_2 are almost independent. In the second term, ξ_1 and ξ_2 are more correlated. But due to the truncation, the expectation of the exponent involved in this term is much smaller than e^{tm^pN}. In fact, by the elementary estimate (5.2) for Gaussian random variables

$$\begin{aligned}
&\mathbb{E}e^{\sqrt{Nt}\xi_1+\sqrt{Nt}\xi_2-Nt}\mathbb{1}_{\{\xi_1<\sqrt{Nt}(1+\epsilon),\xi_2<\sqrt{Nt}(1+\epsilon)\}}\\
&\quad\le \mathbb{E}e^{\sqrt{Nt(2+2m^p)}\xi-Nt}\mathbb{1}_{\{\xi<2\sqrt{Nt}(1+\epsilon)(2+2m^p)^{-1}\}}\\
&\quad\le \exp\{[-4Nt(1+\epsilon)^2][4+4m^p]^{-1}+2Nt(1+\epsilon)-Nt\}\\
&\quad= \exp\{[Ntm^p(1+2\epsilon)-Nt\epsilon^2][1+m^p]^{-1}\}.
\end{aligned}$$

Then for any

$$t < \inf_{0<m<1}(1+m^{-p})\phi(m) \tag{2.16}$$

and for an appropriate choice of ϵ all terms of the sum (2.15) with m not close to zero are exponentially small. This implies $N^{(p-2)/2}\mathbb{E}\widetilde{V}_N(t,\epsilon)\to 0$. The bound (2.16) is Talagrand's bound for the critical temperature in the p-spin SK model, see (1.6). It tends to $2\ln 2$ as $p\uparrow +\infty$.

In order to incorporate this idea into our proof, we reduce the problem of convergence $N^{(p-2)/2} < M_N(t) > \to t\mathbb{E}\xi^p$ to the following statements:

$$N^{(p-2)/2}\mathbb{E}|\widetilde{V}_N(t,\epsilon)|\to 0, \tag{2.17}$$

and

$$N^{(p-2)/2}\mathbb{E}|(V_N(t)-\widetilde{V}_N(t,\epsilon))\bar{Z}_N^{-2}(t)|\to 0, \tag{2.18}$$

for all $\epsilon>0$. This is derived in Lemma 2.3 again from (2.11). In Proposition 2.4 we show (2.17). Again, because of the absolute value, we must apply the Cauchy-Schwartz inequality and pass to the third moment of $\bar{Z}_N(t)$. This makes technical computations much harder. Namely, we get three standard Gaussian random variables ξ_1,ξ_2,ξ_3 with covariances m_1, m_2,

m_3. To benefit from the truncation for obtaining a good bound on t, we have to take into account four different cases: one when all m_1, m_2, m_3 are large and others when two of these correlations are large and the third is small. Then the analogue of (2.16) is the minimum of four estimates of this kind. Therefore, the bound (2.10) is the minimum of four functions. The convergence (2.18) is the subject of Proposition 2.5. Its proof uses ideas of M. Talagrand and a concentration of measure inequality.

Lemma 2.1: *Let*

$$T < \inf_{\mathcal{A}} \frac{I(m_1, m_2, m_3)}{m_1^p + m_2^p + m_3^p}. \tag{2.19}$$

Then

$$\sup_{0\le t\le T} |N^{(p-2)/2} < M_N(t) > -t\mathbb{E}\xi^p| \to 0 \tag{2.20}$$

in probability, where ξ is a Gaussian random variable with $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$.

Proof. Let us denote by

$$V_N(t) = \frac{\mathrm{d}}{\mathrm{d}t} < \bar{Z}_N(t) > -N^{(2-p)/2}\bar{Z}_N^2(t)\mathbb{E}\xi^p.$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}N^{(p-2)/2} < M_N(t) > -\mathbb{E}\xi^p =& N^{(p-2)/2}V_N(t)\bar{Z}_N^{-2}(t)\\ =& N^{(p-2)/2}V_N(t)\exp\{-2M_N(t) + < M_N(t) >\}.\end{aligned}$$

Let us introduce the events

$$A_{a,b}^N := \{-M_N(t) \le a + (b/2) < M_N(t) > \quad \text{for all } t \ge 0\}.$$

Note that by an appropriate choice of $a > 0$ and $b > 0$, their probabilities can be made arbitrarily close to 1. In fact, the process $B_N(t) = M_N(S_t)$, where $S_t = \min\{s \mid < M_N(s) >= t\}$, is a standard Brownian motion and $M_N(t) = B_N(< M_N(t) >)$. By the well-known fact for Brownian motion

$$\mathbb{P}\{A_{a,b}^N\} = \mathbb{P}\{-B_N(t) \le a + (b/2)t \quad \text{for all } t \ge 0\} \ge 1 - \exp\{-ab\}. \tag{2.21}$$

We have:

$$\begin{aligned}&\left|\left(N^{(p-2)/2}\frac{\mathrm{d}}{\mathrm{d}t} < M_N(t) > -\mathbb{E}\xi^p\right)\mathbb{1}_{\{A_{a,b}^N\}}\right|\\ &= N^{(p-2)/2}|V_N(t)|\exp\{-2M_N(t) + < M_N(t) >\}\mathbb{1}_{\{A_{a,b}^N\}}\\ &\le N^{(p-2)/2}\exp\{2a\}|V_N(t)|\exp\{(1+b) < M_N(t) >\}.\end{aligned} \tag{2.22}$$

Let us also introduce the function $\chi_b(x) := [1-\exp\{(1+b)x\}][1+b]^{-1}$. Then by (2.22) for all $t \le T$

$$
\begin{aligned}
&|N^{(p-2)/2}\chi_b(< M_N(t) > -tN^{(2-p)/2}\mathbb{E}\xi^p)\mathbb{1}_{\{A^N_{a,b}\}}| \\
&\quad = N^{(p-2)/2}\Big|\int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s} < M_N(s) > -N^{(2-p)/2}\mathbb{E}\xi^p\Big)\mathbb{1}_{\{A^N_{a,b}\}} \\
&\qquad\qquad \times \exp\{-(1+b)(< M_N(s) > -sN^{(2-p)/2}\mathbb{E}\xi^p)\}\mathrm{d}s\Big| \\
&\le N^{(p-2)/2}\int_0^t \Big|\frac{\mathrm{d}}{\mathrm{d}s} < M_N(s) > -N^{(2-p)/2}\mathbb{E}\xi^p\Big|\mathbb{1}_{\{A^N_{a,b}\}} \\
&\qquad\qquad \times \exp\{-(1+b)(< M_N(s) > -sN^{(2-p)/2}\mathbb{E}\xi^p)\}\,\mathrm{d}s \qquad (2.23) \\
&\le N^{(p-2)/2}\exp\{2a+TN^{(2-p)/2}(1+b)\}\int_0^t |V_N(s)|\,\mathrm{d}s.
\end{aligned}
$$

This yields

$$
\begin{aligned}
&N^{(p-2)/2}\sup_{0\le t\le T}|\chi_b(< M_N(t) > -tN^{(2-p)/2}\mathbb{E}\xi^p)|\mathbb{1}_{\{A^N_{a,b}\}} \\
&\qquad \le N^{(p-2)/2}\exp\{2a+TN^{(2-p)/2}(1+b)\}\int_0^T |V_N(s)|\,\mathrm{d}s.
\end{aligned} \tag{2.24}
$$

We will show in Proposition 2.2 that

$$\lim_{N\uparrow+\infty} N^{(p-2)/2}\mathbb{E}|V_N(t)| = 0$$

uniformly in $t \in [0,T]$. Consequently $\sup_{N>1,t\le T} N^{(p-2)/2}\mathbb{E}|V_N(t)| < \infty$. Then by the dominated convergence theorem

$$\lim_{N\uparrow+\infty}\mathbb{E}\big[N^{(p-2)/2}\sup_{0\le t\le T}|\chi_b(< M_N(t) > -tN^{(2-p)/2}\mathbb{E}\xi^p)|\mathbb{1}_{\{A^N_{a,b}\}}\big] = 0.$$

It follows that for all $a, b > 0$

$$N^{(p-2)/2}\sup_{0\le t\le T}|\chi_b(< M_N(t) > -tN^{(2-p)/2}\mathbb{E}\xi^p)|\mathbb{1}_{\{A^N_{a,b}\}} \to 0 \qquad \text{as } N\uparrow+\infty$$

in probability. Then also $N^{(p-2)/2}\sup_{0\le t\le T}|\chi_b(< M_N(t) > -tN^{(2-p)/2}\mathbb{E}\xi^p)| \to \infty$, as $N \to \infty$ since by (2.21) the probability of the events $A^N_{a,b}$ can be made arbitrarily close to 1. This last fact implies (2.20) and the lemma is proved.$\diamondsuit$

It remains to prove the following proposition.

Proposition 2.2: *Assume that T satisfies (2.19). Then*

$$\lim_{N\uparrow+\infty} N^{(p-2)/2}\mathbb{E}|V_N(t)| = 0 \tag{2.25}$$

uniformly in $[0,T]$.

Proof. It follows from (2.5) and the definition of $V_N(t)$ that

$$N^{(p-2)/2}V_N(t) = \mathbb{E}_{\sigma,\sigma'}\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma')\Big) - \mathbb{E}\xi^p\Big)e^{H_N(t,\sigma)+H_N(t,\sigma')-Nt}.$$

By the Cauchy-Schwartz inequality:

$$\begin{aligned} N^{(p-2)/2}\mathbb{E}|V_N(t)| =& \mathbb{E}\Big|\mathbb{E}_\sigma e^{H_N(t,\sigma)-Nt/2}\mathbb{E}_{\sigma'}\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma')\Big) - \mathbb{E}\xi^p\Big)e^{H_N(t,\sigma')-Nt/2}\Big| \\ \le& \Big[\mathbb{E}\mathbb{E}_\sigma e^{H_N(t,\sigma)-Nt/2}\Big]^{1/2} \\ &\times \Big[\mathbb{E}\mathbb{E}_\sigma e^{H_N(t,\sigma)-Nt/2}\Big[\mathbb{E}_{\sigma'}\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma')\Big) - \mathbb{E}\xi^p\Big)e^{H_N(t,\sigma')-Nt/2}\Big]^2\Big]^{1/2} \\ =& \Big[\mathbb{E}_{\sigma,\sigma',\sigma''}\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma')\Big) - \mathbb{E}\xi^p\Big)\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma'')\Big) - \mathbb{E}\xi^p\Big) \\ &\times \exp\Big\{Nt\Big(f_p\Big(R_N(\sigma,\sigma')\Big) + f_p\Big(R_N(\sigma,\sigma'')\Big) + f_p\Big(R_N(\sigma(\sigma',\sigma''))\Big)\Big)\Big\}\Big]^{1/2}. \end{aligned}$$

Then it suffices to prove that

$$\begin{aligned} W_N(t) =& \mathbb{E}_{\sigma,\sigma',\sigma''}\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma')\Big) - \mathbb{E}\xi^p\Big)\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma'')\Big) - \mathbb{E}\xi^p\Big) \\ &\times \exp\Big\{Nt\Big(f_p\Big(R_N(\sigma,\sigma')\Big) + f_p\Big(R_N(\sigma,\sigma'')\Big) + f_p\Big(R_N(\sigma(\sigma',\sigma''))\Big)\Big)\Big\} \end{aligned}$$

tends to zero uniformly in $[0,T]$ as $N\uparrow+\infty$. We represent it as

$$\begin{aligned} W_N(t) =& \sum_{m_1,m_2,m_3\in\mathcal{A}_N}\Big(N^{p/2}f_p(m_1) - \mathbb{E}\xi^p\Big)\Big(N^{p/2}f_p(m_2) - \mathbb{E}\xi^p\Big)e^{Nt(f_p(m_1)+f_p(m_2)+f_p(m_3))} \\ &\times \mathbb{P}\{\sigma\cdot\sigma' = m_1N,\ \sigma\cdot\sigma'' = m_2N,\ \sigma'\cdot\sigma'' = m_3N\} \end{aligned}$$

where the set $\mathcal{A}_N = \mathcal{A}\cap\{0,\pm1/N,\pm2/N,\dots,\pm1\}^3$. A standard combinatorial calculation yields

$$\begin{aligned} &\mathbb{P}\{\sigma\cdot\sigma' = m_1N, \sigma\cdot\sigma'' = m_2N, \sigma'\cdot\sigma'' = m_3N\} \\ &= 2^{-2N}\binom{N}{N(1+m_1)/2}\binom{N(1+m_1)/2}{N(1+m_1+m_2+m_3)/4}\binom{N(1-m_1)/2}{N(1+m_2-m_1-m_3)/4}. \end{aligned} \tag{2.26}$$

By Stirling's formula we obtain

$$\begin{aligned} &\mathbb{P}\{\sigma\cdot\sigma' = m_1N, \sigma\cdot\sigma'' = m_2N, \sigma'\cdot\sigma'' = m_3N\} \\ &= \frac{16\exp\{-NI(m_1,m_2,m_3)\}}{\sqrt{(2\pi)^3N^3}}[(1+m_1+m_2+m_3)(1-m_1-m_2+m_3)]^{-1/2} \\ &\quad\times [(1+m_1-m_2-m_2)(1-m_1+m_2-m_3)]^{-1/2}\Big(1+O\Big(\frac{1}{N}\Big)\Big) \text{ as } N\uparrow+\infty, \end{aligned} \tag{2.27}$$

for any given $m_1, m_2, m_3 \in \mathcal{A}_N$. Let us remark that

$$t(m_1^p + m_2^p + m_3^p) + (m_1^2 + m_2^2 + m_3^2)/2 - I(m_1, m_2, m_3) = O\big((|m_1| + |m_2| + |m_3|)^3\big) \quad (2.28)$$

as $m_1, m_2, m_3 \to 0$ uniformly in $[0, T]$. Then for all sufficiently small $\epsilon > 0$ there exists a constant $h > 0$ such that

$$\sup_{t \in [0,T]} \big[t(m_1^p + m_2^p + m_3^p) - I(m_1, m_2, m_3)\big] < -h(m_1^2 + m_2^2 + m_3^2)/2 \quad (2.29)$$

for all $m_1, m_2, m_3 \in \mathcal{A} \cap \{|m_1| + |m_2| + |m_3| < \epsilon\}$. Let us fix such a small $\epsilon > 0$ and an arbitrary constant $0 < \delta < 1/6$ and then split $W_N(t)$ into four terms:

$$W_N(t) = I_N^1 + I_N^2(t) + I_N^3(t) + I_N^4(t),$$

where

$$I_N^1 = \frac{16}{\sqrt{(2\pi N)^3}} \sum_{\substack{m_1, m_2, m_3 \in \mathcal{A}_N \\ |m_1|+|m_2|+|m_3| < N^{-1/3-\delta}}} \Big(N^{p/2} f_p(m_1) - \mathbb{E}\xi^p\Big)\Big(N^{p/2} f_p(m_2) - \mathbb{E}\xi^p\Big) e^{-N(m_1^2+m_2^2+m_3^2)/2}$$

$$I_N^2(t) = \sum_{\substack{m_1, m_2, m_3 \in \mathcal{A}_N \\ |m_1|+|m_2|+|m_3| < N^{-1/3-\delta}}} \Big(N^{p/2} f_p(m_1) - \mathbb{E}\xi^p\Big)\Big(N^{p/2} f_p(m_2) - \mathbb{E}\xi^p\Big)$$
$$\times \Big(e^{Nt(f_p(m_1)+f_p(m_2)+f_p(m_3))}\mathbb{P}(\sigma \cdot \sigma' = m_1 N,\ \sigma \cdot \sigma'' = m_2 N,\ \sigma' \cdot \sigma'' = m_3 N)$$
$$- \frac{16}{\sqrt{(2\pi N)^3}} e^{-N(m_1^2+m_2^2+m_3^2)/2}\Big),$$

$$I_N^3(t) = \sum_{\substack{m_1, m_2, m_3 \in \mathcal{A}_N \\ |m_1|+|m_2|+|m_3| > N^{-1/3-\delta} \\ |m_1|+|m_2|+|m_2| < \epsilon}} \Big(N^{p/2} f_p(m_1) - \mathbb{E}\xi^p\Big)\Big(N^{p/2} f_p(m_2) - \mathbb{E}\xi^p\Big) e^{Nt(f_p(m_1)+f_p(m_2)+f_p(m_3))}$$
$$\times \mathbb{P}(\sigma \cdot \sigma' = m_1 N,\ \sigma \cdot \sigma'' = m_2 N,\ \sigma' \cdot \sigma'' = m_3 N),$$

$$I_N^4(t) = \sum_{\substack{m_1, m_2, m_3 \in \mathcal{A}_N \\ |m_1|+|m_2|+|m_3| > \epsilon}} \Big(N^{p/2} f_p(m_1) - \mathbb{E}\xi^p\Big)\Big(N^{p/2} f_p(m_2) - \mathbb{E}\xi^p\Big) e^{Nt(f_p(m_1)+f_p(m_2)+f_p(m_3))}$$
$$\times \mathbb{P}(\sigma \cdot \sigma' = m_1 N,\ \sigma \cdot \sigma'' = m_2 N,\ \sigma' \cdot \sigma'' = m_3 N).$$

We will prove that all four terms $I_N^1, I_N^2(t), I_N^3(t), I_N^4(t)$ tend to zero uniformly in $[0, T]$ as $N \uparrow +\infty$.

To show this for I_N^1, let us put $m_1\sqrt{N} = s_1, m_2\sqrt{N} = s_2, m_3\sqrt{N} = s_3$. Then

$$\lim_{N \uparrow +\infty} I_N^1 = \lim_{N \uparrow +\infty} \frac{16}{\sqrt{(2\pi N)^3}} \sum_{\substack{s_1, s_2, s_3 \\ = 0, \pm 1/\sqrt{N}, \pm 2/\sqrt{N}, \ldots \\ |s_1|+|s_2|+|s_3| < N^{1/6-\delta}}} (s_1^p - \mathbb{E}\xi^p)(s_p^2 - \mathbb{E}\xi^p) e^{-(s_1^2+s_2^2+s_3^2)/2}$$
$$= \frac{16}{\sqrt{(2\pi)^3}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^p - \mathbb{E}\xi^p)(y^p - \mathbb{E}\xi^p) e^{-(x^2+y^2+z^2)/2} \mathrm{d}x \mathrm{d}y \mathrm{d}z = 0.$$

To treat $I_N^2(t)$, we rewrite it using (2.27) as

$$I_N^2(t) = \frac{16}{\sqrt{(2\pi N)^3}} \sum_{\substack{m_1,m_2,m_3 \in \mathcal{A}_N \\ |m_1|+|m_2|+|m_3|<N^{-1/3-\delta}}} \Big(N^{p/2} f_p(m_1) - \mathbb{E}\xi^p\Big)\Big(N^{p/2} f_p(m_2) - \mathbb{E}\xi^p\Big) e^{-N(m_1^2+m_2^2+m_3^2)/2}$$
$$\times \Big[e^{N[t(f_p(m_1)++f_p(m_2)+f_p(m_3))+(m_1^2+m_2^2+m_3^2)/2 - I(m_1,m_2,m_3)]}$$
$$\times [(1+m_1+m_2+m_3)(1-m_1-m_2+m_3)]^{-1/2}$$
$$\times [(1+m_1-m_2-m_3)(1-m_1+m_2-m_3)]^{-1/2}\Big(1+O\Big(\frac{1}{N}\Big)\Big) - 1\Big]. \tag{2.30}$$

Moreover, here O(1) is bounded uniformly in $\mathcal{A}_N \cap \{|m_1|+|m_2|+|m_3| < \epsilon\}$ by Stirling's formula. It follows from (2.28) that

$$\lim_{N\uparrow+\infty} \sup_{\substack{t\in[0,T] \\ |m_1|+|m_2|+|m_3|<N^{-1/3-\delta}}} N|t(m_1^p+m_2^p+m_3^p) + (m_1^2+m_2^2+m_3^2)/2 - I(m_1,m_2,m_3)| = 0.$$

Then

$$\lim_{N\uparrow+\infty} \sup_{\substack{t\in[0,T] \\ |m_1|+|m_2|+|m_3|<N^{-1/3-\delta}}} \Big| e^{N[t(f_p(m_1)+f_p(m_2)+f_p(m_3))+(m_1^2+m_2^2+m_3^2)/2 - I(m_1,m_2,m_3)]}$$
$$\times [(1+m_1+m_2+m_3)(1-m_1-m_2+m_3)]^{-1/2}$$
$$\times [(1+m_1-m_2-m_2)(1-m_1+m_2-m_3)]^{-1/2}\Big(1+O\Big(\frac{1}{N}\Big)\Big) - 1\Big| = 0,$$

while

$$\lim_{N\uparrow+\infty} \frac{16}{\sqrt{(2\pi N)^3}} \sum_{\substack{m_1,m_2,m_3 \in \mathcal{A}_N \\ |m_1|+|m_2|+|m_3|<N^{-1/3-\delta}}} \Big|\Big(N^{p/2} f_p(m_1) - \mathbb{E}\xi^p\Big)\Big(N^{p/2} f_p(m_2) - \mathbb{E}\xi^p\Big)\Big| e^{-N(m_1^2+m_2^2+m_3^2)/2}$$
$$= \lim_{N\uparrow+\infty} \frac{16}{\sqrt{(2\pi N)^3}} \sum_{\substack{s_1,s_2,s_3=0,\pm 1/\sqrt{N},\dots \\ |s_1|+|s_2|+|s_3|<N^{1/6-\delta}}} \Big|\Big(s_1^p - \mathbb{E}\xi^p\Big)\Big(s_2^p - \mathbb{E}\xi^p\Big)\Big| e^{-(s_1^2+s_2^2+s_3^2)/2}$$
$$= \frac{16}{\sqrt{(2\pi)^3}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(x^p - \mathbb{E}\xi^p)(y^p - \mathbb{E}\xi^p)| e^{-(x^2+y^2+z^2)/2} \mathrm{d}x\mathrm{d}y\mathrm{d}z < \infty.$$

Thus $I_N^2(t) \to 0$ uniformly in $[0,T]$ as $N \uparrow +\infty$.

To estimate $I_N^3(t)$, we rewrite it in the same way using (2.27):

$$I_N^3(t) = \frac{16}{\sqrt{(2\pi N)^3}} \sum_{\substack{m_1,m_2,m_3\in\mathcal{A}_N \\ |m_1|+|m_2|+|m_3|>N^{-1/3-\delta} \\ |m_1|+|m_2|+|m_3|<\epsilon}} \Big(N^{p/2} f_p(m_1) - \mathbb{E}\xi^p\Big)\Big(N^{p/2} f_p(m_2) - \mathbb{E}\xi^p\Big)$$
$$e^{N[t(f_p(m_1)+f_p(m_2)+f_p(m_3))-I(m_1,m_2,m_3)]}$$
$$\times [(1+m_1+m_2+m_3)(1-m_1-m_2+m_3)]^{-1/2}$$
$$\times [(1+m_1-m_2-m_2)(1-m_1+m_2-m_3)]^{-1/2}\Big(1+O\Big(\frac{1}{N}\Big)\Big) \tag{2.31}$$

Due to (2.29), there exists a constant $h' > 0$ such that for all sufficiently large N

$$\sup_{\substack{t\in[0,T] \\ |m_1|+|m_2|+|m_3|>N^{-1/3-\delta} \\ |m_1|+|m_2|+|m_3|<\epsilon}} \exp\{-N[t(m_1^p+m_2^p+m_3^p) - I(m_1,m_2,m_3)]\}$$
$$\le \sup_{\substack{|m_1|+|m_2|+|m_3|>N^{-1/3-\delta} \\ |m_1|+|m_2|+|m_3|<\epsilon}} \exp\{-Nh(m_1^2+m_2^2+m_3^2)/2\} \le \exp\{-h'N^{1/3-2\delta}\}.$$

The sum

$$\sum_{\substack{m_1,m_2,m_3\in\mathcal{A}_N \\ |m_1|+|m_2|+|m_3|>N^{-1/3-\delta} \\ |m_1|+|m_2|+|m_3|<\epsilon}} \Big(N^{p/2} f_p(m_1) - \mathbb{E}\xi^p\Big)\Big(N^{p/2} f_p(m_2) - \mathbb{E}\xi^p\Big)$$

has polynomial growth as $N \uparrow +\infty$ and the uniform convergence $I_N^3(t) \to 0$ in $[0,T]$ is proved.

Finally, let us consider $I_N^4(t)$. By Stirling's formula there exists a constant C such that for all $(m_1,m_2,m_3) \in \mathcal{A}_N \cap \{|m_1|+|m_2|+|m_3| > \epsilon\}$

$$\mathbb{P}\{\sigma\cdot\sigma' = m_1 N, \sigma\cdot\sigma'' = m_2 N, \sigma'\cdot\sigma'' = m_3 N\} \le C\sqrt{N}\exp\{-NI(m_1,m_2,m_3)\}. \tag{2.32}$$

Then by the assumption (2.19), for given T there exists a constant $h'' > 0$ such that

$$\sup_{\substack{t\in[0,T] \\ |m_1|+|m_2|+|m_3|>\epsilon}} \exp\{Nt(m_1^p+m_2^p+m_3^p)\}\mathbb{P}\{\sigma\cdot\sigma' = m_1 N, \sigma\cdot\sigma'' = m_2 N, \sigma'\cdot s'' = m_3 N\}$$
$$\le C\sqrt{N} \sup_{\substack{t\in[0,T] \\ |m_1|+|m_2|+|m_3|>\epsilon}} \exp\{-N[t(m_1^p+m_2^p+m_3^p) - I(m_1,m_2,m_3)]\} < C\sqrt{N}\exp\{-h''N\}.$$

The remaining sum in this term has again polynomial growth, whence $I_N^4(t) \to 0$ uniformly in $[0,T]$. The lemma is proved.$\diamondsuit$

Remark. Let us note that the restriction (2.19) on T was essential only for the analysis of the fourth term $I_N^4(t)$. This means that the convergence $N^{(p-2)/2}\mathbb{E}|V_N(t)| \to 0$ breaks down for larger T only because of the configurations of spins with rather big correlations $\sigma\cdot\sigma'=m_1$, $\sigma\cdot\sigma''=m_2$, $\sigma'\cdot\sigma''=m_3$. To extend our result to the whole interval (2.10) of admissible T, we need to reduce the contribution of these configurations into $W_N(t)$. For that purpose we will follow the idea of M. Talagrand [T] to truncate the Hamiltonian.

Now we prove the statement of the previous lemma for all T satisfying (2.10).

Lemma 2.3: *Let*

$$T<\inf_{m_1,m_2,m_3\in\mathcal{A}} Y(m_1,m_2,m_3).$$

Then

$$\sup_{0\le t\le T} |<N^{(p-2)/4}M_N(t)> -t\mathbb{E}\xi^p| \to 0 \tag{2.33}$$

in probability.

Proof. Let us fix $\epsilon>0$ such that for some constants $h_1,h_2>0$

$$\sup_{\substack{t\in[0,T]\\ m_1^p+m_2^p+m_3^p<3\epsilon}} \big[t(m_1^p+m_2^p+m_3^p)-I(m_1,m_2,m_3)\big] < -h_1(m_1^2+m_2^2+m_3^2) \tag{2.34}$$

and

$$\begin{aligned}\sup_{\substack{t\in[0,T]\\ m_1,m_2,m_3\in\mathcal{A}\\ m_1^p+m_2^p+m_3^p>3\epsilon}} \quad & t\big[\min\{Q_p(m_1,m_2,m_3,\epsilon),\ L_p(m_1,m_2,m_3,\epsilon),\\ & L_p(m_1,m_3,m_2,\epsilon),\ L_p(m_2,m_3,m_1,\epsilon)\}\big]-I(m_1,m_2,m_3)<-h_2\end{aligned} \tag{2.35}$$

where

$$\begin{aligned}Q_p(m_1,m_2,m_3,\epsilon)=&[-9\epsilon^2+6(1+2\epsilon)(m_1^p+m_2^p+m_3^p)][2(3+2m_1^p+2m_2^p+2m_3^p)]^{-1},\\ L_p(m_1,m_2,m_3,\epsilon)=&\Big[-1-m_3^p-(1+\epsilon)^2+(1+\epsilon)S_p(m_1,m_2,m_3)\sqrt{2+2m_3^p}\\ &+R_p(m_1,m_2,m_3)(1+m_3^p)\Big][1+m_3^p]^{-1}.\end{aligned}$$

Condition (2.34) is the same as (2.29) and, due to (2.28), for any given $T>0$ it is possible to find an appropriate $\epsilon>0$ such that (2.35) is satisfied. However, $\epsilon>0$ ensuring (2.35) does exist, if and only if T satisfies the assumption (2.10). The meaning of (2.35) will become clear in the proof of a further Proposition 2.4. Let us introduce

$$\begin{aligned}\widetilde{V}_N(t,\epsilon)=&\mathbb{E}_{\sigma,\sigma'}\Big(Nf_p\Big(R_N(\sigma,\sigma')\Big)-N^{(2-p)/2}\mathbb{E}\xi^p\Big)e^{H_N(t,\sigma)+H_N(t,\sigma')-Nt}\\ &\times 1\!\!1_{\{H_N(t,\sigma)<(1+\epsilon)tN,H_N(t,\sigma')<(1+\epsilon)tN\}}\\ \bar{V}_N(t,\epsilon)=&\mathbb{E}_{\sigma,\sigma'}\Big(Nf_p\Big(R_N(\sigma,\sigma')\Big)-N^{(2-p)/2}\mathbb{E}\xi^p\Big)e^{H_N(t,\sigma)+H_N(t,\sigma')-Nt}\\ &\times 1\!\!1_{\{H_N(t,\sigma)>(1+\epsilon)tN,\ \text{or}\ H_N(t,\sigma')>(1+\epsilon)tN\}}\\ =&V_N(t)-\widetilde{V}_N(t,\epsilon).\end{aligned}$$

Let us also fix some $T_0 > 0$ satisfying the assumption (2.19) of the previous lemma. Proceeding along the lines of the proof of Lemma 2.1, we get for all $t \in [T_0, T]$:

$$\begin{aligned}
&N^{(p-2)/2}|F_b(< M_N(t) > -tN^{(2-p)/2}\mathbb{E}\xi^p)|\mathbb{1}_{\{A^N_{a,b}\}} \\
&\le N^{(p-2)/2}\exp\{2a + T_0 N^{(2-p)/2}(1+b)\}\int_0^{T_0}|V_N(s)|\,\mathrm{d}s \\
&\quad + N^{(p-2)/2}\exp\{2a + tN^{(2-p)/2}(1+b)\}\int_{T_0}^{t}|\widetilde{V}_N(s,\epsilon)|\,\mathrm{d}s \\
&\quad + N^{(p-2)/2}\int_{T_0}^{t}|\bar{V}_N(s,\epsilon)|\bar{Z}_N^{-2}(s)\exp\{-(1+b)(< M_N(s) > -sN^{(2-p)/2})\}\mathbb{1}_{\{A^N_{a,\epsilon}\}}\,\mathrm{d}s.
\end{aligned}$$

Then

$$\begin{aligned}
&N^{(p-2)/2}\sup_{T_0 < t \le T}|F_b(< M_N(t) > -tN^{(2-p)/2}\mathbb{E}\xi^p)|\mathbb{1}_{\{A^N_{a,b}\}} \\
&\le N^{(p-2)/2}\exp\{2a + T_0 N^{(2-p)/2}(1+b)\}\int_0^{T_0}|V_N(s)|\,\mathrm{d}s \\
&\quad + N^{(p-2)/2}\exp\{2a + TN^{(2-p)/2}(1+b)\}\int_{T_0}^{T}|\widetilde{V}_N(s,\epsilon)|\,\mathrm{d}s \\
&\quad + N^{(p-2)/2}\exp\{TN^{(2-p)/2}(1+b)\}\int_{T_0}^{T}|\bar{V}_N(s,\epsilon)|\bar{Z}_N^{-2}(s)\mathrm{d}s.
\end{aligned}$$

It was proved in Lemma 2.1 that $N^{(p-2)/2}\mathbb{E}|V_N(t)| \to 0$ uniformly in $[0, T_0]$ as $N \uparrow +\infty$. Proposition 2.4 shows that for $\epsilon > 0$ satisfying (2.34) and (2.35), $N^{(p-2)/2}\mathbb{E}|\widetilde{V}(t,\epsilon)| \to 0$ uniformly in $t \in [T_0, T]$. Proposition 2.5 proves that $N^{(p-2)/2}\mathbb{E}|\bar{V}(t,\epsilon)Z_N^{-2}(t)| \to 0$ uniformly in $[T_0, T]$ for all $\epsilon > 0$. Then

$$\lim_{N\uparrow+\infty}\mathbb{E}[\sup_{T_0 \le t \le T}|N^{(p-2)/2}F_b(< M_N(t) > -tN^{(2-p)/2}\mathbb{E}\xi^p)|\mathbb{1}_{\{A^N_{a,b}\}}] = 0.$$

Then $\sup_{0\le t\le T}|N^{(p-2)/2}F_b(< M_N(t) > -tN^{(2-p)/2}\mathbb{E}\xi^p)|$ converges to zero in probability, since the probability of the events $A^N_{a,b}$ can be made arbitrarily close to 1 by (2.21). This implies (2.33) and the proof of the lemma is complete.

Proposition 2.4: *Assume that $T > 0$ satisfies (2.10). Let us fix $0 < \epsilon < 1/2$ such that (2.34) and (2.35) hold. Then for any $T_0 > 0$, $T_0 < T$:*

$$\lim_{N\uparrow+\infty}N^{(p-2)/2}\mathbb{E}|\widetilde{V}_N(t,\epsilon)| = 0 \tag{2.36}$$

uniformly in $t \in [T_0, T]$.

Proof. Let us estimate $N^{(p-2)/2}\mathbb{E}|\widetilde{V}_N(t,\epsilon)|$ by the Cauchy-Schwartz inequality as in the proof of Proposition 2.2 for $N^{(p-2)/2}\mathbb{E}|V_N(t)|$. After that we split it into four terms:

$$N^{(p-2)/2}\mathbb{E}|\widetilde{V}_N(t,\epsilon)| \leq [\mathbb{E}\widetilde{W}_N(t,\epsilon)]^{1/2} = [\widetilde{I}^1_N(t,\epsilon) - \widetilde{I}^2_N(t,\epsilon) + \widetilde{I}^3_N(t,\epsilon) + \widetilde{I}^4_N(t,\epsilon)]^{1/2},$$

where

$$\begin{aligned}\widetilde{W}_N(t,\epsilon) =& \mathbb{E}\mathbb{E}_{\sigma,\sigma',\sigma''}\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma')\Big) - \mathbb{E}\xi^p\Big)\Big(N^{p/2}f_p\Big(R_N(\sigma,\sigma'')\Big) - \mathbb{E}\xi^p\Big)\\ &\times e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3tN/2}\\ &\times \mathbb{1}_{\{H_N(t,\sigma)<Nt(1+\epsilon),H_N(t,\sigma')<Nt(1+\epsilon),H_N(t,\sigma'')<Nt(1+\epsilon)\}}\end{aligned}$$

$$\begin{aligned}\widetilde{I}^1_N(t,\epsilon) =& \sum_{\substack{m_1,m_2,m_3\in\mathcal{A}_N\\ m_1^p+m_2^p+m_3^p\leq\epsilon^2/4}}\Big(Nf_p(m_1) - N^{(2-p)/2}\mathbb{E}\xi^p\Big)\Big(Nf_p(m_2) - N^{(2-p)/2}\mathbb{E}\xi^p\Big)\\ &\times \mathbb{P}\{\sigma\cdot\sigma' = m_1N,\ \sigma\cdot\sigma'' = m_2N,\ \sigma'\cdot\sigma'' = m_3N\}\\ &\times \mathbb{E}e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3tN/2}\end{aligned}$$

$$\begin{aligned}\widetilde{I}^2_N(t,\epsilon) =& \sum_{\substack{m_1,m_2,m_3\in\mathcal{A}_N\\ m_1^p+m_2^p+m_3^p<\epsilon^2/4}}\Big(Nf_p(m_1) - N^{(p-2)/2}\mathbb{E}\xi^p\Big)\Big(Nf_p(m_2) - N^{(p-2)/2}\mathbb{E}\xi^p\Big)\\ &\times \mathbb{P}\{\sigma\cdot\sigma' = m_1N,\ \sigma\cdot\sigma'' = m_2N,\ \sigma'\cdot\sigma'' = m_3N\}\\ &\times \mathbb{E}[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}\\ &\times \mathbb{1}_{\{H_N(t,\sigma)\geq Nt(1+\epsilon)\text{ or }H_N(t,\sigma')\geq Nt(1+\epsilon),\text{ or }H_N(t,\sigma'')\geq Nt(1+\epsilon)\}}]\end{aligned}$$

$$\begin{aligned}\widetilde{I}^3_N(t,\epsilon) =& \sum_{\substack{m_1,m_2,m_3\in\mathcal{A}_N\\ \epsilon^2/4\leq m_1^p+m_2^p+m_3^p\leq 3\epsilon}}\Big(Nf_p(m_1) - N^{(2-p)/2}\mathbb{E}\xi^p\Big)\Big(Nf_p(m_2) - N^{(2-p)/2}\mathbb{E}\xi^p\Big)\\ &\times \mathbb{P}\{\sigma\cdot\sigma' = m_1N,\ \sigma\cdot\sigma'' = m_2N,\ \sigma'\cdot\sigma'' = m_3N\}\\ &\times \mathbb{E}[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}\\ &\times \mathbb{1}_{\{H_N(t,\sigma)<Nt(1+\epsilon),H_N(t,\sigma')<Nt(1+\epsilon),H_N(t,\sigma'')<Nt(1+\epsilon)\}}]\end{aligned}$$

$$\begin{aligned}\widetilde{I}^4_N(t,\epsilon) =& \sum_{\substack{m_1,m_2,m_3\in\mathcal{A}_N\\ m_1^p+m_2^p+m_3^p>3\epsilon}}\Big(Nf_p(m_1) - N^{(p-2)/2}\mathbb{E}\xi^p\Big)\Big(Nf_p(m_2) - N^{(p-2)/2}\mathbb{E}\xi^p\Big)\\ &\times \mathbb{P}\{\sigma\cdot\sigma' = m_1N,\ \sigma\cdot\sigma'' = m_2N,\ \sigma'\cdot\sigma'' = m_3N\}\\ &\times \mathbb{E}[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3tN/2}\\ &\times \mathbb{1}_{\{H_N(t,\sigma)\geq Nt(1+\epsilon)\text{ or }H_N(t,\sigma')\geq Nt(1+\epsilon)\text{ or }H_N(t,\sigma'')\geq Nt(1+\epsilon)\}}].\end{aligned}$$

We will prove the uniform convergence to zero in $[T_0, T]$ as $N \uparrow +\infty$ of all these four terms.

The first term $\widetilde{I}_N^1(t)$ is not truncated and it refers to the configurations of spins with small correlations m_1, m_2 and m_3. The proof of its uniform convergence to zero in $[T_0, T]$ relies on (2.34) and it is completely analogous to the proof of the uniform convergence to zero of the sum $I_N^1 + I_N^2(t) + I_N^3(t)$ in the proof of Proposition 1. Therefore, we omit the details.

The second term $\widetilde{I}_N^2(t)$ also contains only configurations of spins with very small correlations. If these correlations were zero, i. e. if $H_N(t,\sigma)$ $H_N(t,\sigma')$ and $H_N(t,\sigma'')$ were independent, then, indeed, the expectation involved in this term satisfies

$$\mathbb{E}[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}\mathbb{I}_{\{\cdot\}}] \leq 3\mathbb{E}[e^{\sqrt{Nt}\xi - Nt/2}\mathbb{I}_{\{\xi>\sqrt{Nt}(1+\epsilon)\}}] \leq \exp\{-Nt\epsilon^2/2\}$$

(ξ is a standard Gaussian) by a well-known estimate for Gaussian random variables (5.1). We show that very small correlations m_1, m_2, m_3 do not destroy the exponential convergence to zero of the corresponding expectation. Considering the third term $\widetilde{I}_N^3(t)$, we neglect the truncation and use the asymptotic expansion (2.27) and condition (2.34). So we prove that the expectation $\mathbb{E}e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}$ multiplied by the probability of any given correlations goes to zero exponentially fast. Finally $\widetilde{I}_N^4(t)$ refers to the configurations of spins with rather big correlations. Here, applying the estimate (5.1), we benefit from the truncation. The choice of $\epsilon > 0$ according to (2.35) plays a crucial role in the analysis of this term. (Remember that this choice was possible only for T satisfying (2.10)).

Now we proceed with the detailed proof. To treat the second term $\widetilde{I}_N^2(t,\epsilon)$, we write

$$\mathbb{E}[e^{H_N(t,\sigma)+H_N(t,\sigma)+H_N(t,\sigma'')-3Nt/2}\mathbb{I}_{\{H_N(t,\sigma)>Nt(1+\epsilon) \text{ or } H_N(t,\sigma')>Nt(1+\epsilon) \text{ or } H_N(t,\sigma'')>Nt(1+\epsilon)\}}]$$

$$= \mathbb{E}[e^{\sqrt{Nt}(\xi_1+\xi_2+\xi_3)-3Nt/2}\,\mathbb{I}_{\{\xi_1>\sqrt{Nt}(1+\epsilon) \text{ or } \xi_2>\sqrt{Nt}(1+\epsilon) \text{ or } \xi_3>\sqrt{Nt}(1+\epsilon)\}}],$$

where ξ_1, ξ_2 and ξ_3 are Gaussian random variables with zero mean, variance 1 and covariances $\operatorname{cov}(\xi_1,\xi_2) = f_p(R_N(\sigma,\sigma')) = m_1^p$, $\operatorname{cov}(\xi_1,\xi_3) = f_p(R_N(\sigma,\sigma'')) = m_2^p$, $\operatorname{cov}(\xi_2,\xi_3) = f_p(R_N(\sigma',\sigma'')) = m_3^p$, $m_1^p + m_2^p + m_3^p \leq \epsilon^2/4$. One gets

$$\begin{aligned}\mathbb{E}[e^{\sqrt{Nt}(\xi_1+\xi_2+\xi_3)-3Nt/2}\,\mathbb{I}_{\{\xi_1>\sqrt{Nt}(1+\epsilon)\}}] =& e^{-3Nt/2}\mathbb{E}\big[e^{\sqrt{Nt}\xi_1}\,\mathbb{I}_{\{\xi_1>\sqrt{Nt}(1+\epsilon)\}}\mathbb{E}(e^{\xi_2+\xi_3} \mid \xi_1)\big]\\ =& e^{Nt\gamma-3Nt/2}\mathbb{E}[e^{\sqrt{Nt}(1+\mu)\xi_1}\,\mathbb{I}_{\{\xi_1>\sqrt{Nt}(1+\epsilon)\}}],\end{aligned}$$

where $\gamma = 1 + m_3^p - (m_1^p + m_2^p)^2/2$, $\mu = m_1^p + m_2^p$. Since $m_1^p + m_2^p \leq \epsilon^2/4 < \epsilon$, we may use the estimate for standard Gaussian random variables (5.1). It implies

$$\begin{aligned}\mathbb{E}[e^{\sqrt{Nt}(\xi_1+\xi_2+\xi_3)-3Nt/2}\,\mathbb{I}_{\{\xi_1>\sqrt{Nt}(1+\epsilon)\}}] \leq& C_1\exp\{Nt(m_1^p + m_2^p + m_3^p - (\epsilon - m_1^p + m_2^p)^2/2)\}\\ \leq& C_1\exp\{-NT_0\epsilon^2/8\}\end{aligned}$$

for some constant $C_1 > 0$, all $t \in [T_0, T]$ and all $N > 0$, if $m_1^p + m_2^p + m_3^p < \epsilon^2/4$, $0 < \epsilon < 1/2$. Thus

$$\sup_{0 \le m_1^p + m_2^p + m_3^p \le \epsilon^2/4} \mathbb{E}[e^{H_N(t,\sigma)+H_N(t,\sigma)+H_N(t,\sigma'')-3Nt/2} \times \mathbb{I}_{\{H_N(t,\sigma)>Nt(1+\epsilon) \text{ or } H_N(t,\sigma')>Nt(1+\epsilon) \text{ or } H_N(t,\sigma'')>Nt(1+\epsilon)\}}] \le 3C_1 \exp\{-NT_0\epsilon^2/8\}$$

for all $t \in [T_0, T]$. Since the other terms in $\widetilde{I}_N^2(t,\epsilon)$ have polynomial growth, the uniform convergence $\widetilde{I}_N^2(t,\epsilon) \to 0$ in $[T_0, T]$ follows.

Let us turn to $\widetilde{I}_N^3(t,\epsilon)$. By the expansion (2.27) and condition (2.35)

$$\begin{aligned}
\sup_{\epsilon^2/4 \le m_1^p + m_2^p + m_3^p \le 3\epsilon} & \mathbb{E}\big[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2} \\
& \times \mathbb{I}_{\{H_N(t,\sigma)<Nt(1+\epsilon), H_N(t,\sigma')<Nt(1+\epsilon), H_N(t,\sigma'')<Nt(1+\epsilon)\}}\big] \\
& \times \mathbb{P}\{\sigma\cdot\sigma' = m_1 N, \sigma\cdot\sigma'' = m_2 N, \sigma\cdot\sigma'' = m_3 N\} \\
\le C_2 & \sup_{\epsilon^2/4 \le m_1^p + m_2^p + m_3^p \le 3\epsilon} \exp\{N[t(m_1^p + m_2^p + m_3^p) - I(m)]\} \\
\le C_2 & \sup_{m_1^p + m_2^p + m_3^p \ge \epsilon^2/4} \exp\{-h_1 N(m_1^2 + m_2^2 + m_3^2)\} \le C_2 \exp\{-h_1 \epsilon^{4/p} N/4\}
\end{aligned}$$

for all $t \in [T_0, T]$, where $C_2 > 0$, $h_1 > 0$ are constants. All other terms in $\widetilde{I}_N^3(t,\epsilon)$ have polynomial growth, hence $I_N^3(t,\epsilon) \to 0$ uniformly in $[T_0, T]$.

Finally, consider $\widetilde{I}_N^4(t,\epsilon)$. We have

$$\begin{aligned}
&\mathbb{E}\left[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}\, \mathbb{I}_{\{H_N(t,\sigma)<Nt(1+\epsilon), H_N(t,\sigma')<Nt(1+\epsilon), H_N(t,\sigma'')<(1+\epsilon)Nt\}}\right] \\
&\quad \le \mathbb{E}\left[e^{\sqrt{Nt(3+2m_1^p+2m_2^p+2m_3^p)}\,\xi - 3Nt/2}\, \mathbb{I}_{\{\sqrt{3+2m_1^p+2m_2^p+2m_3^p}\,\xi \le 3Nt(1+\epsilon)\}}\right],
\end{aligned}$$

where ξ is a standard Gaussian, $m_1 = f_p(R_N(\sigma,\sigma'))$, $m_2 = f_p(R_N(\sigma,\sigma''))$, $m_3 = f_p(R_N(\sigma',\sigma''))$. Since $m_1^p + m_2^p + m_3^p > 3\epsilon$, we may apply the estimate (5.2). It yields

$$\begin{aligned}
&\mathbb{E}\left[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}\, \mathbb{I}_{\{H_N(t,\sigma)<Nt(1+\epsilon), H_N(t,\sigma')<Nt(1+\epsilon), H_N(t,\sigma'')<(1+\epsilon)Nt\}}\right] \\
&\quad \le C_3 \exp\{NtQ_p(m_1, m_2, m_3, \epsilon)\},
\end{aligned}$$

for some constant $C_3 > 0$, all $t \in [T_0, T]$, $N > 0$ and $m_1^p + m_2^p + m_3^p > 3\epsilon$. On the other hand, we also have:

$$
\begin{aligned}
&\mathbb{E}\Big[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}\,\mathbb{1}_{\{H_N(t,\sigma)<Nt(1+\epsilon),H_N(t,\sigma')<Nt(1+\epsilon),H_N(t,\sigma'')<(1+\epsilon)Nt\}}\Big]\\
&\quad\le \mathbb{E}\Big[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}\,\mathbb{1}_{\{H_N(t,\sigma')<Nt(1+\epsilon),H_N(t,\sigma'')<Nt(1+\epsilon)\}}\Big]\\
&\quad= \mathbb{E}\Big[e^{\sqrt{Nt}\xi_2+\sqrt{Nt}\xi_3-3Nt/2}\mathbb{E}(e^{\sqrt{Nt}\xi_1}\mid \xi_2,\xi_3)\mathbb{1}_{\{\xi_2<\sqrt{Nt}(1+\epsilon),\xi_3<\sqrt{Nt}(1+\epsilon)\}}\Big]\\
&\quad= e^{-Nt+Nt\alpha}\mathbb{E}\Big[e^{\sqrt{Nt}(1+\mu_2)\xi_2+\sqrt{Nt}(1+\mu_3)\xi_3}\,\mathbb{1}_{\{\xi_2<\sqrt{Nt}(1+\epsilon),\xi_3<\sqrt{Nt}(1+\epsilon)\}}\Big]\\
&\quad\le e^{-Nt+Nt\alpha}\mathbb{E}\Big[e^{\sqrt{Nt((1+\mu_2)^2+(1+\mu_3)^2+2m_3^p(1+\mu_2)(1+\mu_3))}\xi}\,\mathbb{1}_{\{\sqrt{2+2m_3^p}\,\xi<2\sqrt{Nt}(1+\epsilon)\}}\Big],
\end{aligned}
$$

where ξ_1, ξ_2, ξ_3 are the same as in the analysis of the second term, ξ is standard Gaussian and

$$
\begin{aligned}
\alpha =&(2m_1^pm_2^pm_3^p - m_1^{2p} - m_2^{2p})/(2-2m_3^{2p})\\
\mu_2 =&(m_1^p - m_2^pm_3^p)/(1-m_3^{2p})\\
\mu_3 =&(m_2^p - m_1^pm_3^p)/(1-m_3^{2p}).
\end{aligned}
$$

One checks that

$$
\begin{aligned}
&\sqrt{(1+\mu_2)^2+(1+\mu_3)^2+2m_3^p(1+\mu_2)(1+\mu_3)}\\
&\qquad\ge \frac{2(1+m_3^p+(m_1^p+m_2^p)/2)}{\sqrt{2+2m_3^p}} \ge \frac{2(1+3\epsilon/2)}{\sqrt{2+2m_3^p}},
\end{aligned}
$$

when $m_1^p+m_2^p+m_3^p>3\epsilon$. So, we are again in the position to apply (5.2). This yields

$$
\begin{aligned}
&\mathbb{E}\left[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}\,\mathbb{1}_{\{H_N(t,\sigma)<Nt(1+\epsilon),H_N(t,\sigma')<Nt(1+\epsilon),H_N(t,\sigma'')<(1+\epsilon)Nt\}}\right]\\
&\quad\le C_4\exp\{tNL_p(m_1,m_2,m_3,\epsilon)\},
\end{aligned}
$$

where $C_4>0$ is a constant. Permuting m_1, m_2 and m_3, we can derive in the same way that the same expectation does not exceed $\exp\{tNL_p(m_1,m_3,m_2,\epsilon)\}$ and $\exp\{tNL_p(m_2,m_3,m_1,\epsilon)\}$ multiplied by some constant. Thus, taking into account (2.32), we obtain

$$
\begin{aligned}
&\sup_{m_1^p+m_2^p+m_3^p>3\epsilon}\mathbb{E}\big[e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')-3Nt/2}\\
&\qquad\qquad\times\mathbb{1}_{\{H_N(t,\sigma)<Nt(1+\epsilon),H_N(t,\sigma')<Nt(1+\epsilon),H_N(t,\sigma'')<Nt(1+\epsilon)\}}\big]\\
&\qquad\qquad\times\mathbb{P}\{\sigma\cdot\sigma'=m_1N,\sigma\cdot\sigma''=m_2N,\sigma\cdot\sigma''=m_3N\}\\
&\le \sup_{m_1^p+m_2^p+m_3^p>3\epsilon} C_5\sqrt{N}\exp\Big\{tN\min\big[Q_p(m_1,m_2,m_3,\epsilon),L_p(m_1,m_2,m_3,\epsilon),\\
&\qquad\qquad L_p(m_1,m_3,m_2,\epsilon),\ L_p(m_2,m_3,m_1,\epsilon)\big]-NI(m_1,m_2,m_2)\Big\}
\end{aligned}
\tag{2.37}
$$

for all $t\in[0,T_0]$, where $C_5>0$ is a constant. Now the relevance of the assumption (2.35) becomes clear. Due to (2.35), the right-hand side of (2.37) tends to zero exponentially fast, as one can estimate it by $C_5\sqrt{N}\exp\{-h_2 N\}$. The other terms in $\widetilde{I}_N^4(t,\epsilon)$ have polynomial growth. Thus $I_N^4(t,\epsilon)\uparrow +\infty$ uniformly in $[T_0,T]$. This concludes the proof of the proposition.

Proposition 2.5: *For all $T>0$ satisfying (2.10) and all $\epsilon>0$*

$$\lim_{N\uparrow+\infty} N^{(p-2)/2}\mathbb{E}|\bar{V}_N(t,\epsilon)\bar{Z}_N^{-2}(t)| = 0 \tag{2.38}$$

uniformly in any interval $[T_0,T]$, where $0<T_0<T$.

Proof. It follows from the definition of $\bar{V}_N(t)$ that

$$N^{(p-2)/2}\mathbb{E}|\bar{V}_N(t,\epsilon)\bar{Z}_N^{-2}(t)| \le \bar{C}N\mathbb{E}\frac{\mathbb{E}_\sigma e^{H_N(t,\sigma)}1\!\!1_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}}}{\mathbb{E}_\sigma e^{H_N(t,\sigma)}} \tag{2.39}$$

for all $t\ge 0$, where $\bar{C}>0$ is a constant. We will show that the expectation of this last fraction tends to zero exponentially fast. First of all, we observe that by (5.1)

$$\frac{\mathbb{E}\mathbb{E}_\sigma e^{H_N(t,\sigma)}1\!\!1_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}}}{\mathbb{E}\mathbb{E}_\sigma e^{H_N(t,\sigma)}} = \mathbb{E}\mathbb{E}_\sigma e^{H_N(t,\sigma)-Nt/2}1\!\!1_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}} \le e^{-Nt\epsilon^2/2}. \tag{2.40}$$

Let us represent the fraction in the right-hand side of (2.39) as

$$\begin{aligned}&\mathbb{E}\frac{\mathbb{E}_\sigma e^{H_N(t,\sigma)}1\!\!1_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}}}{\mathbb{E}_\sigma e^{H_N(t,\sigma)}}\\ &=\mathbb{E}\frac{\mathbb{E}_\sigma e^{H_N(t,\sigma)-Nt/2}1\!\!1_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}}}{\exp\{\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}-\mathbb{E}\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}+\mathbb{E}\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}-Nt/2\}}.\end{aligned} \tag{2.41}$$

To expand this formula, we will use the concentration of measure as in (5.3). The random variable $\mathbb{E}_\sigma e^{H_N(t,\sigma)}$ has the same distribution as $\phi(J_1,\dots,J_{N^p})$, where the function

$$\phi(x_1,\dots,x_{N^p}) = \ln\mathbb{E}_\sigma\exp\left\{\sqrt{tN^{1-p}}\sum_{i_1,\dots,i_p} x_{i_1,i_2,\dots,i_p}\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_p}\right\}$$

is defined on $\mathbb{Z}^{N^p}$, $J_1,\dots,J_{N^p}$ are standard Gaussian random variables. The Lipschitz constant of $\phi(x_1,\dots,x_{N^p})$ is at most $\sqrt{tN^{1-p}}\sqrt{N^p}=\sqrt{tN}$. Substituting this function and $u=Nt\epsilon^2/4$ into (5.3), we derive:

$$\mathbb{P}\{|\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}-\mathbb{E}\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}|>Nt\epsilon^2/4\}\le\exp\{-Nt\epsilon^4/32\}. \tag{2.42}$$

Let us introduce the events $O_{t,\epsilon}^N := \{|\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}-\mathbb{E}\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}|>Nt\epsilon^2/4\}$. Consequently by (2.41) and (2.42)

$$\begin{aligned}&\mathbb{E}\frac{\mathbb{E}_\sigma e^{H_N(t,\sigma)}1\!\!1_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}}}{\mathbb{E}_\sigma e^{H_N(t,\sigma)}}\\ &=\mathbb{E}\frac{1\!\!1_{\{O_{t,\epsilon}^N\}}\mathbb{E}_\sigma e^{H_N(t,\sigma)-Nt/2}1\!\!1_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}}}{\exp\{\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}-\mathbb{E}\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}+\mathbb{E}\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}-Nt/2\}}+\mathbb{P}\{O_{t,\epsilon}^N\}\\ &\le e^{Nt\epsilon^2/4}\mathbb{E}\frac{\mathbb{E}_\sigma e^{H_N(t,\sigma)-Nt/2}1\!\!1_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}}}{\exp\{\mathbb{E}\ln\mathbb{E}_\sigma e^{H_N(t,\sigma)}-Nt/2\}}+e^{-Nt\epsilon^4/32}\end{aligned} \tag{2.43}$$

Observe that for any T satisfying (2.10) and any $0 < T_0 < T$, there exists a constant $K > 0$ such that

$$-K\sqrt{N} < \mathbb{E}\ln \mathbb{E}_\sigma e^{H_N(t,\sigma)} - Nt/2 \leq 0 \tag{2.44}$$

for all $t \in [T_0, T]$. The upper bound in (2.44) is immediate by Jensen inequality. Whenever the second moment of $\bar{Z}_N(t)$ truncated is finite, the left-hand side of (2.44) was established by Talagrand [T1] in the analysis of the critical temperature. We will outline his proof in our situation. For given T satisfying (2.10), let us fix $\widetilde{\epsilon} > 0$, such that (2.34) and (2.35) hold. Let us define after

$$\bar{Z}_N(t,\widetilde{\epsilon}) = \mathbb{E}_\sigma e^{H_N(t,\sigma)-Nt/2} \mathbb{1}_{\{H_N(t,\sigma)<Nt(1+\widetilde{\epsilon})\}}$$

By (5.2) there exists a constant $K_1 < 0$ such that

$$\mathbb{E}\bar{Z}_N(t,\widetilde{\epsilon}) \geq K_1 \tag{2.45}$$

for all $t \in [T_0, T]$. Moreover, there exists a constant $K_2 > 0$ such that

$$\mathbb{E}\bar{Z}_N^3(t,\widetilde{\epsilon}) \leq K_2 \tag{2.46}$$

for all $t \in [T_0, T]$. The proof of (2.46) is analogous to the proof of the uniform convergence to zero of $\widetilde{W}_N(t,\epsilon)$ in Proposition 2. We decompose $\bar{Z}_N(t,\widetilde{\epsilon})$ into four terms like it was for $\widetilde{W}_N(t,\epsilon)$. The last three of them go to zero uniformly in $t \in [T_0, T]$ and exponentially fast by the same arguments as $\widetilde{I}_N^2(t)$, $\widetilde{I}_N^3(t)$ and $\widetilde{I}_N^4(t)$ do. We work out the first term similarly to the sum $I_1^N + I_2^N(t) + I_3^N(t)$ in Proposition 1. The only difference is that I_1^N tends to the integral along $\mathbb{R}^3$ of the density of three independent standard Gaussians, which equals 1. Thus, in fact, $\bar{Z}_N(t,\widetilde{\epsilon})$ converges to 1 uniformly in $[T_0, T]$ and (2.46) is obvious. Hence, for all $t \in [T_0, T]$

$$\frac{\mathbb{E}\bar{Z}_N^2(t,\widetilde{\epsilon})}{\big(\mathbb{E}\bar{Z}_N(t,\widetilde{\epsilon})\big)^2} \leq \frac{\big(\mathbb{E}\bar{Z}_N^3(t,\widetilde{\epsilon})\big)^{2/3}}{\big(\mathbb{E}\bar{Z}_N(t,\widetilde{\epsilon})\big)^2} \leq \frac{K_2^{2/3}}{K_1^2} := K_3. \tag{2.47}$$

Then starting from the Paley-Zygmund inequality and finally applying the concentration of measure inequality (5.3) with $u = Nt/2 - \mathbb{E}\ln \mathbb{E}_\sigma e^{H_N(t,\sigma)} + \ln(K_1/2)$, we deduce

$$\begin{aligned} 1/4K_3 &\leq \frac{\mathbb{E}\bar{Z}_N^2(t,\widetilde{\epsilon})}{4\big(\mathbb{E}\bar{Z}_N(t,\widetilde{\epsilon})\big)^2} \leq \mathbb{P}\{\bar{Z}_N(t,\widetilde{\epsilon}) > \mathbb{E}\bar{Z}_N(t,\widetilde{\epsilon})/2\} \leq \mathbb{P}\{\mathbb{E}_\sigma e^{H_N(t,\sigma)} > K_1 e^{Nt/2}/2\} \\ &= \mathbb{P}\{\ln \mathbb{E}_\sigma e^{H_N(t,\sigma)} - \mathbb{E}\ln \mathbb{E}_\sigma e^{H_N(t,\sigma)} > Nt/2 - \mathbb{E}\ln \mathbb{E}_\sigma e^{H_N(t,\sigma)} + \ln(K_1/2)\} \\ &\leq \exp\{[Nt/2 - \mathbb{E}\ln \mathbb{E}_\sigma e^{H_N(t,\sigma)} + \ln(K_1/2)]^2/2Nt\}, \end{aligned}$$

from where (2.44) follows. Finally, (2.40), (2.43) and (2.44) together imply

$$
\begin{aligned}
&\mathbb{E}\frac{\mathbb{E}_\sigma e^{H_N(t,\sigma)} 1\!\!\mathrm{I}_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}}}{\mathbb{E}_\sigma e^{H_N(t,\sigma)}}\\
\le & e^{Nt\epsilon^2/4+K\sqrt{N}}\mathbb{E}\mathbb{E}_\sigma e^{H_N(t,\sigma)-Nt/2} 1\!\!\mathrm{I}_{\{H_N(t,\sigma)>Nt(1+\epsilon)\}} + e^{-Nt\epsilon^4/32}\\
\le & e^{-Nt^2/4+K\sqrt{N}} + e^{-Nt\epsilon^4/32},
\end{aligned}
\tag{2.48}
$$

and the proposition is proved.◇

Proof of (1.19). To complete the proof of Theorem 1.3, it remains to show that

$$
\lim_{p\uparrow+\infty} \inf_{m_1,m_2,m_3\in\mathcal{A}} Y_p(m_1,m_2,m_3) = 2\ln 2. \tag{2.49}
$$

After elaborating the functions $S_p(m_1,m_2,m_3)$ and $R_p(m_1,m_2,m_3)$, we get:

$$
\begin{aligned}
U_p(m_1,m_2,m_3) =& I(m_1,m_2,m_3)(1+m_3^p)\\
&\times\Big[\Big(4\Big(1+m_3^p+\frac{m_1^p+m_2^p}{2}\Big)^2+\frac{(m_1^p-m_2^p)^2(1+m_3^p)}{(1-m_3^p)}\Big)^{1/2}\\
&\qquad -\frac{(m_1^p-m_2^p)^2}{2(1-m_3^p)}-m_1^p m_2^p-(2+m_3^p)\Big]^{-1}.
\end{aligned}
\tag{2.50}
$$

It follows from (2.50) that for any $p=2k>2$ and any sequence $(m_{1,n},m_{2,n},m_{3,n})\in\mathcal{A}$ such that $m_{1,n}\to 1$, $m_{2,n}\to 1$, $m_{3,n}\to 1$, as $n\to\infty$,

$$
\lim_{n\uparrow+\infty} Y_p(m_{1,n},m_{2,n},m_{3,n}) = 2\ln 2. \tag{2.51}
$$

(In fact, by the definition of $\mathcal{A}$ we have $||m_1|-|m_2||\le 1-|m_3|$ for all $(m_1,m_2,m_3)\in\mathcal{A}$, whence $(m_{1,n}^p-m_{2,n}^p)^2=o(1-m_{3,n}^p)$.) Thus

$$
\limsup_{p\uparrow+\infty} \inf_{m_1,m_2,m_3\in A} Y_p(m_1,m_2,m_3)\le 2\ln 2.
$$

This fact and the next Proposition 2.6 together imply (2.49). ◇

Proposition 2.6:*Let $\{p_n\}$ be a sequence of positive even numbers, $p_n\uparrow+\infty$. Assume that the sequence $(m_{1,n},m_{2,n},m_{3,n})\in\mathcal{A}$ satisfies one of the following conditions:*

(i) $|m_{1,n}|\to 1$, $|m_{2,n}|\to 1$, $|m_{3,n}|\to 1$;

(ii) there exist $\delta>0$ and a pair i and j, $i,j=1,2,3$, $i\ne j$, such that $|m_{i,n}|\to 1$ and $|m_{j,n}|\le 1-\delta$ for all sufficiently large n;

(iii) there exists $\delta > 0$ such that $|m_{1,n}| \le 1-\delta$, $|m_{2,n}| \le 1-\delta$, $|m_{3,n}| \le 1-\delta$ for all sufficiently large n. Then

$$\lim_{n\uparrow+\infty}\inf Y_{p_n}(m_{1,n}, m_{2,n}, m_{3,n}) \ge 2\ln 2. \tag{2.52}$$

Proof: In the cases (i) and (iii) it suffices to substitute the sequence $(m_{1,n}, m_{2,n}, m_{3,n})$ into the function $I(m_1, m_2, m_3)(2/3+(m_1^p+m_2^p+m_3^p)^{-1})$. In case (ii) assume that e. g. $|m_{3,n}| \to 1$ and $|m_{1,n}| \le 1-\delta$. Then $m_{1,n}^{p_n} = o(1)$. By definition of the set $\mathcal{A}$ we obtain $||m_{1,n}|-|m_{2,n}|| \le 1-|m_{3,n}| \to 0$ as $n \uparrow +\infty$, thus $m_{2,n}^{p_n} = o(1)$ and $(m_{1,n}^{p_n}-m_{2,n}^{p_n})^2/(1-m_{3,n}^{p_n}) = o(1)$. Moreover, if $m_{3,n} \to 1$, then $m_{1,n} - m_{2,n} \to 0$ and if $m_{3,n} \to -1$, then $m_{1,n} + m_{2,n} \to 0$ and therefore in both of these cases $\lim\inf_{n\uparrow+\infty} I(m_{1,n}, m_{2,n}, m_{3,n}) \ge \ln 2$. This yields

$$\begin{aligned} &\lim_{n\uparrow+\infty}\inf Y_{p_n}(m_{1,n}, m_{2,n}, m_{3,n}) \\ &\ge \lim_{n\uparrow+\infty}\inf U_{p_n}(m_{1,n}, m_{2,n}, m_{3,n}) \ge \lim_{n\uparrow+\infty}\inf \ln 2\frac{1+m_{3,n}^{p_n}}{m_{3,n}^{p_n}+o(1)} \ge 2\ln 2 \end{aligned}$$

and the proposition is proved. ◇

3. The fluctuations of the partition function in the REM.

Amazingly enough, the simplest of all our models, the REM, will be seen to offer in some sense the most interesting behaviour with regard to the fluctuations of the free energy. The main surprise here will be the existence of an intermediate region of temperatures where a CLT does not hold, but there a non-standard limit theorem will be proven.

We begin with the proof of (i) of Theorem 1.4.

Proposition 3.1: *Whenever $0 \le \beta < \sqrt{\ln 2/2}$,*

$$e^{\frac{N}{2}(\ln 2-\beta^2)} \ln\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.1}$$

Proof. This result will follow from the standard CLT for triangular arrays. Let us first write

$$\ln\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} = \ln\Big(1+\frac{Z_{\beta,N}-\mathbb{E}Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}}\Big). \tag{3.2}$$

We will show that the second term in the logarithm properly normalized will converge to a normal random variable. To see this, write

$$\frac{Z_{\beta,N}-\mathbb{E}Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} = \sum_{\sigma\in\mathcal{S}_N} e^{-N(\ln 2+\beta^2/2)}\left(e^{\beta\sqrt{N}X_\sigma} - e^{N\beta^2/2}\right) \equiv \sum_{\sigma\in\mathcal{S}_N} \mathcal{Y}_N(\sigma). \tag{3.3}$$

Note that $\mathbb{E}\mathcal{Y}_N(\sigma) = 0$ and $\mathbb{E}\mathcal{Y}_N^2(\sigma) = e^{-N(2\ln 2-\beta^2)}[1-e^{-N\beta^2}]$ and thus

$$\mathbb{E}\Big(\frac{Z_{\beta,N}-\mathbb{E}Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}}\Big)^2 = e^{-N(\ln 2-\beta^2)}[1-e^{-N\beta^2}]. \tag{3.4}$$

Therefore we can write

$$\frac{Z_{\beta,N}-\mathbb{E}Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} = e^{-\frac{N}{2}(\ln 2-\beta^2)}\sqrt{1-e^{-N\beta^2}}\,\frac{1}{2^{N/2}}\sum_{\sigma\in\mathcal{S}_N}\widetilde{\mathcal{Y}}_N(\sigma), \tag{3.5}$$

where $\widetilde{\mathcal{Y}}_N(\sigma) = e^{\frac{N}{2}(2\ln 2-\beta^2)}[1-e^{-N\beta^2}]^{-1/2}\mathcal{Y}_N(\sigma)$ has mean zero and variance one. By the CLT for triangular arrays (see [Shi]), it follows readily that

$$\frac{1}{2^{N/2}}\sum_{\sigma\in\mathcal{S}_N}\widetilde{\mathcal{Y}}_N(\sigma) \overset{\mathcal{D}}{\to} \mathcal{N}(0,1) \tag{3.6}$$

if the Lindeberg condition holds, that is in this case if for any $\epsilon > 0$,

$$\lim_{N\uparrow+0}\mathbb{E}\widetilde{\mathcal{Y}}_N^2(\sigma)\mathbb{1}_{\{|\widetilde{\mathcal{Y}}_N(\sigma)|\geq\epsilon 2^{N/2}\}} = 0. \tag{3.7}$$

But

$$\begin{aligned}\mathbb{E}\widetilde{\mathcal{Y}}_N^2(\sigma)\mathbb{1}_{\{|\widetilde{\mathcal{Y}}_N(\sigma)|\geq\epsilon 2^{N/2}\}} &= \frac{1}{\sqrt{2\pi}(1-e^{-N\beta^2})}e^{-2N\beta^2}\int_{\sqrt{N}(\frac{\ln 2}{2\beta}+\beta)+\frac{\ln\epsilon}{\sqrt{N}\beta}+o(\frac{1}{\sqrt{N}})}^{\infty} e^{2\sqrt{N}\beta z-\frac{z^2}{2}}dz + o(1)\\ &= \frac{1}{\sqrt{2\pi}(1-e^{-N\beta^2})}\int_{\sqrt{N}(\frac{\ln 2}{2\beta}-\beta)+\frac{\ln\epsilon}{\sqrt{N}\beta}+o(\frac{1}{\sqrt{N}})}^{\infty} e^{-\frac{z^2}{2}}dz + o(1).\end{aligned} \tag{3.8}$$

It is easy to check that the latter integral converges to zero if and only if $\beta^2 < \ln 2/2$. Using now the fact that $e^x = 1+x+o(x)$ as $x\to 0$, it is now a trivial matter to deduce the assertion of the proposition. $\diamondsuit$

Since the Lindeberg condition clearly fails for $2\beta^2 \geq \ln 2$, it is clear that we cannot expect a simple CLT beyond this regime. Such a failure of a CLT is always a problem related to "heavy tails", and results from the fact that extremal events begin to influence the fluctuations of the sum. It appears therefore reasonable to separate form the sum the terms where X_σ is anomalously large. For Gaussian r.v.'s it is well known that the right scale of separation is given by $u_N(x)$ defined by

$$2^N\int_{u_N(x)}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2} = e^{-x} \tag{3.9}$$

which (for $x > -\ln N/\ln 2$) is equal to (see e.g. [LLR])

$$u_N(x) = \sqrt{2N\ln 2} + \frac{x}{\sqrt{2N\ln 2}} - \frac{\ln(N\ln 2) + \ln 4\pi}{2\sqrt{2N\ln 2}} + o(1/\sqrt{N}), \tag{3.10}$$

$x \in \mathbb{R}$ is a parameter. Let us now define

$$Z^x_{N,\beta} \equiv \mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma} \mathbb{1}_{\{X_\sigma \leq u_N(x)\}}. \tag{3.11}$$

We may write

$$\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} = 1 + \frac{\mathbb{Z}^x_{\beta,N} - \mathbb{E}Z^x_{\beta,N}}{\mathbb{E}Z_{\beta,N}} + \frac{Z_{\beta,N} - Z^x_{\beta,N} - \mathbb{E}(Z_{\beta,N} - Z^x_{\beta,N})}{\mathbb{E}Z_{\beta,N}} \tag{3.12}$$

Let us first consider the last summand. We introduce the random variable

$$\mathcal{W}_N(x) = \frac{Z_{\beta,N} - Z^x_{\beta,N}}{\mathbb{E}Z_{\beta,N}} = e^{-N(\ln 2 + \beta^2/2)} \sum_{\sigma \in \mathcal{S}_N} e^{\beta\sqrt{N}X_\sigma} \mathbb{1}_{\{X_\sigma > u_N(x)\}} \tag{3.13}$$

It will be convenient to rewrite this as (we ignore the subleading corrections to $u_N(x)$ and only keep the explicit representation (3.10))

$$\begin{aligned} \mathcal{W}_N(x) &= e^{-N(\ln 2 + \beta^2/2)} \sum_{\sigma \in \mathcal{S}_N} e^{\beta\sqrt{N}u_N(u_N^{-1}(X_\sigma))} \mathbb{1}_{\{u_N^{-1}(X_\sigma) > x\}} \\ &= e^{-N(\ln 2 + \beta^2/2)} e^{\beta N\sqrt{2\ln 2} - \beta\frac{\ln(N\ln 2) + \ln 4\pi}{2\sqrt{2\ln 2}}} \sum_{\sigma \in \mathcal{S}_N} e^{\frac{\beta}{\sqrt{2\ln 2}} u_N^{-1}(X_\sigma)} \mathbb{1}_{\{u_N^{-1}(X_\sigma) > x\}}. \end{aligned} \tag{3.14}$$

Let us now introduce the point process on $\mathbb{R}$ given by

$$\mathcal{P}_N \equiv \sum_{\sigma \in \mathcal{S}_N} \delta_{u_N^{-1}(X_\sigma)}. \tag{3.15}$$

A classical result from the theory of extreme order statistics (see e.g. [LLR]) asserts that the point process $\mathcal{P}_N$ converges weakly to a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}dx$. We can, of course, write

$$\sum_{\sigma \in \mathcal{S}_N} e^{\frac{\beta}{\sqrt{2\ln 2}} u_N^{-1}(X_\sigma)} \mathbb{1}_{\{u_N^{-1}(X_\sigma) > x\}} = \int_x^\infty e^{\alpha z} \mathcal{P}_N(dz), \tag{3.16}$$

where we set $\alpha \equiv \beta/\sqrt{2\ln 2}$. Clearly, the weak convergence of $\mathcal{P}_N$ to $\mathcal{P}$ implies convergence in law of the right hand side of (3.16), provided that $e^{\alpha x}$ is integrable on $[x, \infty)$ w.r.t. the Poisson process with intensity e^{-x}. This is, in fact never a problem: the Poisson point process has almost surely support on a finite set, and therefore $e^{\alpha x}$ always a.s. integrable.

Note, however, that for $\beta \geq \sqrt{2\ln 2}$ the mean of the integral is infinite, indicating the passage to the low temperature regime. Note also that the variance of the integral is finite exactly if $\alpha < 1/2$, i.e. $\beta^2 < \ln 2/2$, i.e. when the CLT holds. On the other hand, the mean of the integral diverges if $x \downarrow \infty$; note that at minus infinity the points of the Poisson point process accumulate, and there is no finite support argument as before that would assure the existence if x is taken to $-\infty$. The following lemma provides the first step in the proof of part (ii) of Theorem 1.4 and of Theorem 1.5:

Lemma 3.2: *Let $\mathcal{W}_N(x), \alpha$ be defined above, and let $\mathcal{P}$ be the Poisson point process with intensity measure $e^{-z}dz$. Then*

$$e^{\frac{N}{2}(\sqrt{2\ln 2}-\beta)^2+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]}\mathcal{W}_N(x) \stackrel{\mathcal{D}}{\to} \int_x^\infty e^{\alpha z}\mathcal{P}(dz). \tag{3.17}$$

Remark: Note that the mean of the right hand side is finite if and only of $\beta < \sqrt{2\ln 2}$. Thus only in that case does this lemma also allow to deal with the centered variable appearing in (3.12).

We now need to turn to the remaining term,

$$\frac{Z^x_{\beta,N} - \mathbb{E}Z^x_{\beta,N}}{\mathbb{E}Z_{\beta,N}} = \frac{\mathcal{V}_N(x)}{\mathbb{E}Z_{\beta,N}}, \tag{3.18}$$

where

$$\mathcal{V}_N(x) \equiv Z^x_{\beta,N} - \mathbb{E}Z^x_{\beta,N}. \tag{3.19}$$

One might first hope that this term upon proper scaling would converge to a Gaussian; however, one can easily check that this is not the case (the Lindeberg condition will not be verified). However, it will not be hard to compute all moments of this term:

Lemma 3.3: *Let $\mathcal{V}_N(x)$ be defined by (3.19). Then for $\alpha > 1/2$ and any integer $k \geq 2$*

$$\lim_{N\uparrow+\infty} \frac{\mathbb{E}[\mathcal{V}_N(x)]^k}{\left[2^{-N}e^{N\beta\sqrt{2\ln 2}-\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]}\right]^k} = \sum_{i=1}^k \frac{1}{i!} \sum_{\substack{\ell_1\geq 2,\dots,\ell_i\geq 2\\ \sum_j \ell_j = k}} \frac{k!}{\ell_1!\dots\ell_i!} \frac{e^{(k\alpha-i)x}}{(\ell_1\alpha-1)\dots(\ell_i\alpha-1)}. \tag{3.20}$$

For $\alpha = 1/2$, we have for k even

$$\lim_{N\uparrow+\infty} \frac{\mathbb{E}[\mathcal{V}_N(x)]^k}{\left[2^{-N}e^{N\beta\sqrt{2\ln 2}}\right]^k} = \frac{k!}{(k/2)!\,2^k} = \frac{(k-1)!!}{2^{k/2}} \tag{3.21}$$

and for k odd

$$\lim_{N\uparrow+\infty} \frac{\mathbb{E}[\mathcal{V}_N(x)]^k}{\left[2^{-N}e^{N\beta\sqrt{2\ln 2}}\right]^k} = 0 \tag{3.22}$$

(which are the moments of the normal distribution with variance $1/2$).

Proof. This is a pure computation. Set $T_N(\sigma) \equiv e^{\beta\sqrt{N}X_\sigma}\mathbb{1}_{\{X_\sigma \le u_N(x)\}}$. Note that for $\beta < \sqrt{2\ln 2}$

$$\mathbb{E}T_N(\sigma) = \int_{-\infty}^{u_N(x)} \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2}+\beta\sqrt{N}z} = e^{N\beta^2/2}\Big(1 - \int_{u_N(x)-\beta\sqrt{N}}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\Big) \sim e^{\beta^2 N/2}. \tag{3.23}$$

while for $\beta > \sqrt{2\ln 2}$ and all $k \ge 1$, and for $\beta > \sqrt{\ln 2/2}$ and for $k \ge 2$,

$$\begin{aligned}\mathbb{E}[T_N(\sigma)]^k &= \int_{-\infty}^{u_N(x)} \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2}+k\beta\sqrt{N}z} = e^{Nk^2\beta^2/2} \int_{-\infty}^{u_N(x)-k\beta\sqrt{N}} \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \\ &\sim e^{Nk^2\beta^2/2} \frac{e^{-(u_N(x)-k\beta\sqrt{N})^2/2}}{\sqrt{2\pi}(k\beta\sqrt{N}-u_N(x))} \sim \frac{2^{-N}e^{-x}}{k\alpha-1} e^{k[\beta\sqrt{2\ln 2}N+\alpha x-\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]]}.\end{aligned} \tag{3.24}$$

Formula (3.24) is also valid for $\beta = \sqrt{2\ln 2}$ with $k > 1$ and for $\beta = \sqrt{\ln 2/2}$ with $k > 2$. It is easy to see from the computations above that for $\beta = \sqrt{2\ln 2}$ with $k = 1$ and also for $\beta = \sqrt{\ln 2/2}$ with $k = 2$ we have

$$\mathbb{E}[T_N(\sigma)]^k \sim \frac{e^{k^2\beta^2 N/2}}{2} = \frac{2^{-N}e^{-x}}{2} e^{k[\beta\sqrt{2\ln 2}N+\alpha x]}. \tag{3.25}$$

We set $\widetilde{T}_N(\sigma) \equiv 2^{-N}T_N(\sigma)$; by (3.24) we get for $\beta > \sqrt{\ln 2/2}$ with $k \ge 2$ and also for $\beta > \sqrt{2\ln 2}$ with $k \ge 1$

$$\mathbb{E}[\widetilde{T}_N(\sigma)]^k = \frac{2^{-N}e^{-x}}{k\alpha-1} e^{k[\beta\sqrt{2\ln 2}N-\ln 2+\alpha x-\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]]}. \tag{3.26}$$

This formula is also true for $\beta = \sqrt{\ln 2/2}$, $k > 2$ and $\beta = \sqrt{2\ln 2}$, $k > 1$. For $\beta = \sqrt{2\ln 2}$ and $k = 1$ and also for $\beta = \sqrt{\ln 2/2}$ and $k = 2$ by (3.25)

$$\mathbb{E}[\widetilde{T}_N(\sigma)]^k = \frac{2^{-N}e^{-x}}{2} e^{k[\beta\sqrt{2\ln 2}N-\ln 2+\alpha x]}. \tag{3.27}$$

Now

$$\begin{aligned}\mathbb{E}\left[\mathcal{V}_N(x)\right]^k &= \mathbb{E}\Big(\sum_{\sigma\in\mathcal{S}_N}[\widetilde{T}_N(\sigma) - \mathbb{E}\widetilde{T}_N(\sigma)]\Big)^k = \sum_{\sigma_1,\dots,\sigma_k\in\mathcal{S}_N} \mathbb{E}\prod_{i=1}^k [\widetilde{T}_N(\sigma_i) - \mathbb{E}\widetilde{T}_N(\sigma_i)] \\ &= \sum_{i=1}^k \sum_{\substack{\ell_1,\dots,\ell_i\ge 2 \\ \sum_j \ell_j = k}} \frac{k!}{\ell_1!\dots\ell_i!}\binom{2^N}{i} \mathbb{E}[\widetilde{T}_N(\sigma) - \mathbb{E}\widetilde{T}_N(\sigma)]^{\ell_1}\dots\mathbb{E}[\widetilde{T}_N(\sigma) - \mathbb{E}\widetilde{T}_N(\sigma)]^{\ell_i}.\end{aligned} \tag{3.28}$$

Note finally that for $l \geq 2$ and $\beta \geq \sqrt{\ln 2/2}$

$$\mathbb{E}\big[\widetilde{T}_N(\sigma) - \mathbb{E}\widetilde{T}_N(\sigma)\big]^{\ell} = \sum_{j=1}^{\ell} (-1)^j \binom{\ell}{j} \mathbb{E}\widetilde{T}_N(\sigma)^{\ell-j} [\mathbb{E}\widetilde{T}_N(\sigma)]^j \sim \mathbb{E}\widetilde{T}_N(\sigma)^{\ell}. \tag{3.29}$$

In fact, if $\sqrt{\ln 2/2} \leq \beta < \sqrt{2\ln 2}$, $l \geq 2$, $j \geq 1$, $j \neq l-1, l$, then by (3.23) and (3.26), (3.27)

$$\frac{\mathbb{E}[T_N^{l-j}(\sigma)][\mathbb{E}T_N(\sigma)]^j}{\mathbb{E}[T_N^l(\sigma)]} == e^{Nj(\beta^2/2 - \beta\sqrt{2\ln 2})} O\left(N^{\alpha j/2}\right) \tag{3.30}$$

For $l \geq 2$, $j = l-1, l$

$$\frac{\mathbb{E}[T_N^{l-j}(\sigma)][\mathbb{E}T_N(\sigma)]^j}{\mathbb{E}[T_N^l(\sigma)]} e^{Nl(\beta^2/2 - \beta\sqrt{2\ln 2}) + N\ln 2} O\left(N^{\alpha l/2}\right) \leq e^{-N\ln 2/2)} N^{\alpha} \tag{3.31}$$

For $\beta \geq \sqrt{2\ln 2}$, $l \geq 2$ and $j \geq 1$ by (3.26) and (3.27)

$$\frac{\mathbb{E}[T_N^{l-j}(\sigma)][\mathbb{E}T_N(\sigma)]^j}{\mathbb{E}[T_N^l(\sigma)]} = O(2^{-Nj}). \tag{3.32}$$

Thus for $l \geq 2$ and $\beta > \sqrt{\ln 2/2}$ and also for $l \geq 3$ and $\beta = \sqrt{\ln 2/2}$

$$\mathbb{E}\big[\widetilde{T}_N(\sigma) - \mathbb{E}\widetilde{T}_N(\sigma)\big]^{\ell} = \frac{2^{-N} e^{-x}}{k\alpha - 1} \big[2^{-N} e^{N\beta\sqrt{2\ln 2}} e^{\alpha x} e^{-\frac{\alpha}{2}[\ln(N\ln 2) + \ln 4\pi]}\big]^{\ell}. \tag{3.33}$$

Inserting this result into (3.28) gives the assertion of the lemma (3.20).

For $\beta = \sqrt{\ln 2/2}$ and $l = 2$ by (3.27) we have

$$\mathbb{E}\big[\widetilde{T}_N(\sigma) - \mathbb{E}\widetilde{T}_N(\sigma)\big]^2 = \frac{2^{-N} e^{-x}}{2} \big[2^{-N} e^{N\beta\sqrt{2\ln 2}} e^{\alpha x}\big]^2. \tag{3.34}$$

Inserting this formula into (3.28) we see, that the term with $l_1, \ldots, l_i = 2$, $i = k/2$ brings the main contribution to the sum, and all others are of smaller order, because of the polynomial terms $e^{-l\frac{\alpha}{2}\ln(N\ln 2)}$ in (3.33). This implies (3.21) and (3.22) and the lemma is proved. $\diamondsuit$

Remark: One sees that if we let $x \downarrow -\infty$, and rescale properly, the corresponding moments converge to that of a centered Gaussian r.v. This could alternatively be seen by checking that the Lindeberg condition holds for the truncated variables provided $x \leq -2\ln\ln 2^N$.

A standard consequence of Lemma 3.3 is the weak convergence of the normalized version of $\mathcal{V}_N(x)$:

Corollary 3.4: *For $\sqrt{\ln 2/2} < \beta$,*

$$e^{\frac{N}{2}(\sqrt{2\ln 2}-\beta)^2+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]}\frac{\mathcal{V}_N(x)}{\mathbb{E}Z_{\beta,N}} \xrightarrow{\mathcal{D}} \mathcal{V}(x,\alpha), \tag{3.35}$$

where $\mathcal{V}(x,\alpha)$ is the random variable with mean zero and kth moments given by the right hand side of (3.20). For $\beta = \sqrt{\ln 2/2}$

$$\sqrt{2}e^{\frac{N}{2}(\sqrt{2\ln 2}-\beta)^2}\frac{\mathcal{V}_N(x)}{\mathbb{E}Z_{\beta,N}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.36}$$

The next proposition will imply (ii) of Theorem 1.4.

Proposition 3.5: *Let $\sqrt{\ln 2/2} < \beta < \sqrt{2\ln 2}$. Then for $x \in \mathbb{R}$ chosen arbitrarily,*

$$e^{\frac{N}{2}(\sqrt{2\ln 2}-\beta)^2+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]}\ln\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \xrightarrow{\mathcal{D}} \mathcal{V}(x,\alpha) + \int_x^\infty e^{\alpha z}\mathcal{P}(dz) - \int_x^\infty e^{\alpha z}e^{-z}dz, \tag{3.37}$$

where $\mathcal{V}(x,\alpha)$ and $\mathcal{P}$ are independent random variables.

Proof. (3.37) would be immediate from Lemma 3.2 and Corollary 3.4, if $\mathcal{W}_N(x)$ and $\mathcal{V}_N(x)$ were independent. However, while this is not true, they are not far from independent. To see this, note that if we condition on the number of variables X_σ, $n_N(x)$, that exceed $u_N(x)$, the decomposition in (3.12) is independent. On the other hand, one readily verifies that Corollary 3.4 also holds under the conditional law $\mathbb{P}[\cdot|n_N(x) = n]$, for any finite n, with the same right hand side $\mathcal{V}(x,\alpha)$. But this implies that the limit can be written as the sum of two independent random variables, as desired. $\diamondsuit$

Since for $\beta^2 > \ln 2/2$, $\alpha > 1/2$, one sees that $\mathbb{E}\mathcal{V}(x,\alpha)^2 = e^{x(2\alpha-1)}/(2\alpha-1)$ tends to zero as $x \downarrow -\infty$. Therefore we see that

$$\mathcal{V}(x,\alpha) =_{\mathcal{D}} \lim_{y\uparrow+\infty}\int_{-y}^x e^{\alpha z}\mathcal{P}(dz) - \int_{-y}^x e^{\alpha z}e^{-z}dz \tag{3.38}$$

which means that we can give sense to the Poisson integral $\int_{-\infty}^\infty e^{\alpha z}(\mathcal{P}(dz) - e^{-z}dz)$ We see that Propositions 3.1 and 3.5 imply Theorem 1.4. $\diamondsuit\diamondsuit$

Remark: The appearance of the intermediate region with non-Gaussian fluctuations may appear surprising in view of the fact that in the p-spin models, we could prove the CLT up to a much higher value of β, in fact up to almost the critical value. The reason, however, lies in the fact that in the p-spin model the Gaussian part of the fluctuation is always on a polynomial scale in N, while the truncation error $((Z_{\beta,N} - Z^T_{\beta,N})/\mathbb{E}Z_{\beta,N})$ is exponentially small even when we truncate at $\beta(1+\epsilon)\sqrt{N}$, way below where we truncate in the REM. This means that the CLT contribution will always dominate the extremal fluctuations. In the REM everything is exponentially small, and while a sufficiently truncated partition function gives a Gaussian contribution, this is dominated by the larger extremal fluctuations in the intermediate regime. In other words, the extra correlations in the p-spin models strengthen the Gaussian fluctuations more than the extremal ones which sounds intuitive.

We now turn to the

Proof of Theorem 1.5. We will see that the computions above almost suffice to conclude the low temperature case as well. With the notations from above, we write

$$Z_{\beta,N} = Z^x_{\beta,N} + (Z_{\beta,N} - Z^x_{\beta,N}) \tag{3.39}$$

Clearly for $\beta \geq \sqrt{2\ln 2}$

$$Z_{\beta,N} - Z^x_{\beta,N} = e^{N\left[\beta\sqrt{2\ln 2} - \ln 2\right] - \frac{\alpha}{2}[\ln(N\ln 2) + \ln 4\pi]} \sum_{\sigma\in\mathcal{S}_N} \mathbb{1}_{\{u_N^{-1}(\sigma)>x\}} e^{\alpha u_N^{-1}(X_\sigma)} \tag{3.40}$$

so that for any $x\in\mathbb{R}$,

$$(Z_{\beta,N} - Z^x_{\beta,N}) e^{-N\left[\beta\sqrt{2\ln 2} - \ln 2\right] + \frac{\alpha}{2}[\ln(N\ln 2) + \ln 4\pi]} \xrightarrow{\mathcal{D}} \int_x^\infty e^{\alpha z}\mathcal{P}(dz). \tag{3.41}$$

Now write

$$Z^x_{\beta,N} = \mathbb{E}Z^x_{\beta,N}\left(1 + \frac{Z^x_{\beta,N} - \mathbb{E}Z^x_{\beta,N}}{\mathbb{E}Z^x_{\beta,N}}\right). \tag{3.42}$$

Let us first treat the case $\beta > \sqrt{2\ln 2}$. By (3.24) we have

$$\mathbb{E}Z^x_{\beta,N} \sim \frac{2^{-N}e^{-x}}{\alpha - 1} e^{\beta\sqrt{2\ln 2}N + \alpha x - \frac{\alpha}{2}[\ln(N\ln 2) + \ln 4\pi]}. \tag{3.43}$$

Thus

$$e^{-N\left[\beta\sqrt{2\ln 2} - \ln 2\right] + \frac{\alpha}{2}[\ln(N\ln 2) + \ln 4\pi]} Z^x_{\beta,N} = \frac{e^{x(\alpha-1)}}{\alpha - 1}\left(1 + \frac{Z^x_{\beta,N} - \mathbb{E}Z^x_{\beta,N}}{\mathbb{E}Z^x_{\beta,N}}\right)(1 + o(1)). \tag{3.44}$$

Using Lemma 3.3 we see that now $\frac{Z^x_{\beta,N} - \mathbb{E}Z^x_{\beta,N}}{\mathbb{E}Z^x_{\beta,N}} \frac{e^{x(\alpha-1)}}{\alpha-1}$ converges in distribution to a random variable with moments given by the right hand side of (3.20). Moreover, as $x \downarrow -\infty$, this variable converges to zero in probability. Since the same is true for the prefactor, the assertion of the theorem is now immediate.

Let us consider now the case $\beta = \sqrt{2\ln 2}$. Proceeding as in (3.24),

$$\mathbb{E}Z^0_{\beta,N} = \frac{2^N}{\sqrt{2\pi}} \int_{-\infty}^{u_N(0)-\sqrt{2N\ln 2}} e^{-z^2/2}dz = 2^N\Big(\frac{1}{2} - \frac{\ln(N\ln 2)+\ln 4\pi}{4\sqrt{N\pi\ln 2}} + O\Big(\frac{(\ln N)^2}{N}\Big)\Big). \tag{3.45}$$

We use the decomposition

$$Z_{\beta,N} = Z_{\beta,N} - Z^0_{\beta,N} + \mathbb{E}Z^0_{\beta,N} + (Z^0_{\beta,N} - \mathbb{E}Z^0_{\beta,N}). \tag{3.46}$$

By (3.45), $\mathbb{E}Z^0_{\beta,N}/\mathbb{E}Z_{\beta,N} \sim 1/2$. By (3.14), we see easily that

$$\frac{Z_{\beta,N} - Z^0_{\beta,N}}{\mathbb{E}Z_{\beta,N}} = \mathcal{W}_N(x) \to 0 \quad \text{a.s.} \tag{3.47}$$

even though $\mathbb{E}\mathcal{W}_N(0) = 1/2$! Thus the more precise statement consists in saying that

$$e^{\frac{1}{2}[\ln(N\ln 2)+\ln 4\pi]}\mathcal{W}_N(0) \stackrel{\mathcal{D}}{\to} \int_0^\infty e^z \mathcal{P}(dz). \tag{3.48}$$

Note that of course the limiting varaible has infinite mean, but is a.s. finite. Finally, by Corollary 3.4,

$$e^{\frac{1}{2}[\ln(N\ln 2)+\ln 4\pi]}\frac{Z^0_{\beta,N} - \mathbb{E}Z^0_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \stackrel{\mathcal{D}}{\to} \mathcal{V}(0,1) \tag{3.49}$$

The same arguments as those given after Proposition 3.5 allow us to identify $\mathcal{V}(0,1)$ with the Centered Poisson integral $\int_{-\infty}^0 e^z\left(\mathcal{P}(dz) - e^{-z}dz\right)$. This implies (1.24). (1.25) is an immediate corollary. This concludes the proof of Theorem 1.5.◇◇

Appendix 1. Some remarks on the case p odd

Conjecture 4.1:*Let $p = 2k+1$, $k \geq 1$. There exists $\beta_p > 0$ such that for all $\beta < \beta_p$*

$$N^{(p-2)/2}\ln\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \to M_\infty(\sqrt{\beta}) \tag{4.1}$$

in distribution as $N \uparrow +\infty$, where $M_\infty(t)$ is a centred Gaussian process with independent increments and

$$\mathbb{E}(M_\infty(t) - M_\infty(s))^2 = \frac{(t^2-s^2)[(2p-1)!!]}{2},$$

Moreover $\beta_p \to \sqrt{2\ln 2}$, as $p \uparrow +\infty$.

Discussion. Let us try to adapt the martingale method in this case. This leads to

$$V_N(t) = \mathbb{E}_{\sigma,\sigma'}\left(N\Big(R_N(\sigma,\sigma')\Big)^p - N^{2-p}t\mathbb{E}\xi^{2p}\right)e^{H_N(t,\sigma)+H_N(t,\sigma')-Nt}.$$

Then

$$N^{p-2}\mathbb{E}V_N(t) = \sum_{m=0,\pm 1/N,\dots,\pm 1} \Big(N^{p-1}m^p - t\mathbb{E}\xi^{2p}\Big) e^{tNm^p}\mathbb{P}(\sigma\cdot\sigma' = mN). \tag{4.2}$$

It is easy to show that $N^{p-2}\mathbb{E}V_N(t) \to 0$ as $N\uparrow+\infty$ for all t such that $t < \inf_{0<m<1}\phi(m)m^{-p}$. As in the proof for p even, we can concentrate only on configurations of spins with correlations m close to zero, since others bring an exponentially small contribution. Note that $\mathbb{P}(\sigma\cdot\sigma' = mN) = \mathbb{P}(\sigma\cdot\sigma' = -mN)$ and consequently $I(m) = I(-m) = -m^2/2(1+o(1))$, $m\to 0$. Summing up the terms in (4.2) with correlations m and $-m$, we get

$$\begin{aligned} N^{p-2}\mathbb{E}V_N(t) =& \frac{2}{\sqrt{2\pi N}} \sum_{\substack{m\geq 0,\\ |m|<N^{-1/3-\delta}}} N^{p-1}m^p(e^{tNm^p} - e^{-tNm^p})e^{-NI(m)} - 2t\mathbb{E}\xi^{2p} + o(1) \\ =& \frac{4}{\sqrt{2N\pi}} \sum_{\substack{m\geq 0,\\ |m|<N^{1/3-\delta}}} N^{p-1}m^p\big(tNm^p\big)(1+o(1))e^{-NI(m)} - 2t\mathbb{E}\xi^{2p} + o(1) \\ =& \frac{4t}{\sqrt{2\pi}}\int_0^\infty s^{2p}e^{-s^2/2}ds - 2t\mathbb{E}\xi^{2p} + o(1) \to 0, \quad N\uparrow+\infty. \end{aligned}$$

Moreover, as for p even, it is also not difficult to show that the truncated value $N^{(p-2)}\mathbb{E}\widetilde{V}_N(t,\epsilon)$ tends to zero for all t up to Talagrand's bound (2.16).

Let us now try to perform a rigorous proof of Conjecture 4.1. Proceeding along the lines of the proof for p even, we come to the problem of convergence $N^{p-2}\mathbb{E}|V_N(t)| \to 0$. To get rid of the absolute value of $V_N(t)$, let us first apply the Cauchy-Schwartz inequality in the same way as it was in the proof of Proposition 2.2. We obtain

$$\begin{aligned} \big[N^{(p-2)}\mathbb{E}|V_N(t)|\big]^2 \leq \sum_{m_1,m_2,m_3} & (N^{p-1}m_1^p - t\mathbb{E}\xi^p)(N^{p-1}m_2^p - t\mathbb{E}\xi^p)e^{Nt(m_1^p+m_2^p+m_3^p)} \\ & \times \mathbb{P}(\sigma\cdot\sigma' = m_1N, \sigma\cdot\sigma'' = m_2N, \sigma'\cdot\sigma'' = m_3N). \end{aligned} \tag{4.3}$$

Surprisingly, the right-hand side of (4.3) does not converge to zero. The problem arises from the fact that

$$I(m_1,m_2,m_3) = I(-m_1,-m_2,m_3) = I(m_1,-m_2,-m_3) = I(-m_1,m_2,-m_3),$$

but

$$I(m_1,m_2,m_3) \neq I(m_1,m_2,-m_3).$$

In fact, opening the brackets in $(N^{p-1}m_1^p - t\mathbb{E}\xi^{2p})(N^{p-1}m_2^p - t\mathbb{E}\xi^{2p})$ one can split the right-hand side of (4.3) into four terms. Let us elaborate the first one summing up together the terms with correlations, having the same absolute values $|m_1|, |m_2|, |m_3|$ and the same probability:

$$
\begin{aligned}
&(2\pi N)^{-3/2} \sum_{m_1>0,m_2>0,m_3>0} N^{(2p-2)} m_1^p m_2^p \, e^{-NI(m_1,m_2,m_3)} \\
&\qquad \big[e^{tN(m_1^p+m_2^p+m_3^p)} + e^{tN(-m_1^p-m_2^p+m_3^p)} - e^{tN(-m_1^p+m_2^p-m_3^p)} - e^{tN(m_1^p-m_2^p-m_3^p)}\big] \\
&+ (2\pi N)^{-3/2} \sum_{m_1>0,m_2>0,m_3>0} N^{(2p-2)} m_1^p m_2^p \, e^{-NI(m_1,m_2,-m_3)} \\
&\qquad \big[e^{tN(m_1^p+m_2^p-m_3^p)} + e^{tN(-m_1^p-m_2^p-m_3^p)} - e^{tN(-m_1^p+m_2^p+m_3^p)} - e^{tN(m_1^p-m_2^p+m_3^p)}\big] \\
= {} & 4(2\pi N)^{-3/2} t \sum_{m_1>0,m_2>0,m_3>0} N^{2p-1} m_1^p m_2^p m_3^p (1+o(1)) \, e^{-N(m_1^2+m_2^2+m_3^2)/2} \\
&\qquad [e^{-NI(m_1,m_2,m_3)+N(m_1^2+m_2^2+m_3^2)/2} - e^{-NI(m_1,m_2,-m_3)+N(m_1^2+m_2^2+m_3^2)/2}] \\
&+ t^2(\mathbb{E}\xi^{2p})^2 + o(1).
\end{aligned}
$$

This term is of order $N^{(p-3)/2} t (\mathbb{E}\xi^{p+1})^3 (1+o(1)) + t^2 \mathbb{E}\xi^{2p}$, since in the expansion of

$$
[e^{-NI(m_1,m_2,m_3)+N(m_1^2+m_2^2+m_3^2)/2} - e^{-NI(m_1,m_2,-m_3)+N(m_1^2+m_2^2+m_3^2)/2}],
$$

the main term is of order $N m_1 m_2 m_3$. The sum of all other three terms in (4.3) tends to $-t^2(\mathbb{E}\xi^{2p})^2$. Thus the right-hand side of (4.3) is of order $N^{(p-3)/2} t (\mathbb{E}\xi^{p+1})^3$ and it does not converge to zero for $N \uparrow +\infty$. Therefore the proof for p even is not suitable at this point for p odd.

A possible solution for this problem is to apply the Cauchy-Schwartz inequality in a different way passing to the fourth moment of $Z_N(t)$:

$$
\big[N^{(p-2)} \mathbb{E}|V_N(t)|\big]^2 \le \mathbb{E}\mathbb{E}_{\sigma,\sigma',\sigma'',\sigma'''} \Big(N^{p-1}\Big(R_N(\sigma,\sigma')\Big) - t\mathbb{E}\xi^{2p}\Big)\Big(\Big(\frac{\sigma''\cdot\sigma'''}{N}\Big) - t\mathbb{E}\xi^{2p}\Big) \times e^{H_N(t,\sigma)+H_N(t,\sigma')+H_N(t,\sigma'')+H_N(t,\sigma''')-2Nt}.
$$

It can be proved that the right-hand side of this last inequality tends to zero for all t up to some bound. But technical details are very tedious. We will only say that six parameters $m_1, \ldots, m_6$ have to be considered. The group of 64 correlations with fixed absolute values $|m_1|, \ldots, |m_6|$ splits into eight groups of correlations having the same probabilities.

Furthermore, it will be technically even much harder to extend the bound of t by the truncation of the Hamiltonian. We will have to take into account five different cases and their permutations where some of correlations are large and some are small. Each of these cases will require very tough computations.

Appendix 2. Two useful theorems

Proposition 5.1: *Let ξ be a Gaussian random variable with $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$. Then for all $a, b > 0$*

$$\mathbb{E}[\exp\{a\xi\}1\!\!1_{\{\xi > b\}}] \leq \frac{1}{\sqrt{2\pi(b-a)}} \exp\{-b^2/2 + ab\}, \quad \textit{if } b > a, \tag{5.1}$$

$$\mathbb{E}[\exp\{a\xi\}1\!\!1_{\{\xi < b\}}] \leq \frac{1}{\sqrt{2\pi(a-b)}} \exp\{-b^2/2 + ab\}, \quad \textit{if } b < a. \tag{5.2}$$

Theorem 5.2: *Assume that $f(x_1, \ldots, x_d)$ is a function on R^d with a Lipschitz constant L. Let $J_1, \ldots, J_d$ be independent standard Gaussian random variables. Then for any $u > 0$*

$$\mathbb{P}\{|f(J_1, \ldots, J_d) - \mathbb{E}f(J_1, \ldots, J_d)| > u\} \leq \exp\{-u^2/(2L^2)\}. \tag{5.3}$$

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