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Singularly perturbed elliptic problems in the case of exchange of stabilities

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Abstract

We consider the singularly perturbed boundary value problem (E_ε) $\varepsilon^2 \Delta u = f(u, x, \varepsilon)$ for $x \in D$, $\frac{\partial u}{\partial n} - \lambda(x)u = 0$ for $x \in \Gamma$ where $D \subset R^2$ is an open bounded simply connected region with smooth boundary Γ, ε is a small positive parameter and $\partial/\partial n$ is the derivative along the inner normal of Γ. We assume that the degenerate problem (E_0) $f(u, x, 0) = 0$ has two solutions $\varphi_1(x)$ and $\varphi_2(x)$ intersecting in an smooth Jordan curve $\mathcal{C}$ located in D such that $f_u(\varphi_i(x), x, 0)$ changes its sign on $\mathcal{C}$ for $i = 1, 2$ (exchange of stabilities). By means of the method of asymptotic lower and upper solutions we prove that for sufficiently small ε, problem (E_ε) has at least one solution $u(x, \varepsilon)$ satisfying $\alpha(x, \varepsilon) \leq u(x, \varepsilon) \leq \beta(x, \varepsilon)$ where the upper and lower solutions $\beta(x, \varepsilon)$ and $\alpha(x, \varepsilon)$ respectively fulfil $\beta(x, \varepsilon) - \alpha(x, \varepsilon) = O(\sqrt{\varepsilon})$ for x in a δ-neighborhood of $\mathcal{C}$ where δ is any fixed positive number sufficiently small, while $\beta(x, \varepsilon) - \alpha(x, \varepsilon) = O(\varepsilon)$ for $x \in \overline{D} \backslash D_\delta$. Applying this result to a special reaction system in a nonhomogeneous medium we prove that the reaction rate exhibits a spatial jumping behavior.

Key words. Singular perturbation, asymptotic methods, upper and lower solutions, jumping behavior of reaction rates

1 Introduction.

This paper is devoted to the study of a boundary value problem for the scalar singularly perturbed elliptic equation

$$\varepsilon^2 \Delta u = f(u, x, \varepsilon) \tag{1.1}$$

where $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, f is a sufficiently smooth function, x belongs to some bounded region D in R^2, and ε is a small positive parameter. We assume that the degenerate equation

$$f(u, x, 0) = 0 \tag{1.2}$$

has two intersecting solutions $u = \varphi_1(x)$ and $u = \varphi_2(x)$ defined for $x \in D$. This assumption which is related to the phenomenon of exchange of stabilities implies that the standard theory of singularly perturbed systems cannot be applied. A similar problem for an ordinary differential equation has been considered in [1], for systems of ordinary differential equations - in [2, 4], and for a parabolic equation - in [3].
A motivating example to study such problems comes from reaction kinetics [5]. The problem to model the steady state behavior of a fast pure bimolecular reaction in a nonhomogeneous medium leads to the following system of elliptic differential equations

$$\begin{aligned} \Delta u &= -I_a(x) + r(u,v)/\varepsilon^2, \\ \Delta v &= -I_b(x) + r(u,v)/\varepsilon^2. \end{aligned} \tag{1.3}$$

Here, u and v denote the concentrations of the reacting substances, $I_a(x)$ and $I_b(x)$ are nonnegative inputs, $r(u,v)/\varepsilon^2$ is the reaction rate where the small parameter $\varepsilon > 0$ is used to express that the reactions are very fast. Additionally we have some boundary conditions.
Multiplying the equations in (1.3) by ε^2 we obtain a singular singularly perturbed system. By means of the transformation $u = u,\ w = u - v$ we get from (1.3) the (regular) singularly perturbed system

$$\begin{aligned} \varepsilon^2 \Delta u &= -\varepsilon^2 I_a(x) + r(u, u-w) \equiv \tilde{f}(u,w,x,\varepsilon), \\ \Delta w &= I_b(x) - I_a(x). \end{aligned} \tag{1.4}$$

If we assume that the second equation in (1.4) and the corresponding boundary conditions determine a solution $w(x)$, then by substituting $w(x)$ into the first equation we get an equation of type (1.1). The case that the corresponding degenerate equation has intersecting solutions is typical for reaction kinetics.
By means of the intersecting solutions $u = \varphi_1(x)$ and $u = \varphi_2(x)$ we define the so-called composed stable solution. This solution is used to construct ordered lower and upper solutions for the boundary value problem under consideration which imply the existence of at least one solution $u(x,\varepsilon)$ of our problem, at the same time they can be used to characterize the asymptotic behavior of $u(x,\varepsilon)$ in ε. Finally, we apply our results to the fast pure bimolecular reaction mentioned above in order to give a mathematical explanation of the jumping behavior of the fast reaction rate.

2 Formulation of the problem. Assumptions.

Let $D \subset R^2$ be an open bounded simply connected region with a smooth boundary Γ, let I_1 be the interval $I_1 := \{\varepsilon \in R : 0 \ < \varepsilon \le \varepsilon_1\}$ with $\varepsilon_1 << 1$. We consider the singularly perturbed nonlinear boundary value problem

$$\begin{aligned} &\varepsilon^2 \Delta u = f(u,x,\varepsilon) \qquad \text{for} \quad x \subset D, \\ &\frac{\partial u}{\partial n} - \lambda(x) u = 0 \qquad \text{for} \quad x \in \Gamma \end{aligned} \tag{2.1}$$

where $\partial/\partial n$ denotes the derivative along the inner normal of Γ. To investigate existence and asymptotic behavior in ε of a solution to (2.1) we use the following equations closely related to (2.1), namely the degenerate equation

$$f(u,x,0) = 0, \tag{2.2}$$

and the so-called associated equation

$$\frac{d^2u}{d\xi^2} = f(u, x, 0) \tag{2.3}$$

in which x is considered as parameter.
We study the boundary value problem (2.1) under the following assumptions.

(A_0). $f \in C^2(R \times \overline{D} \times \overline{I}_1, R), \ \lambda \in C^2(\Gamma, R^+)$.

(A_1). *The degenerate equation* (2.2) *has two solutions* $u = \varphi_1(x)$ *and* $u = \varphi_2(x)$ *with* $\varphi_1, \varphi_2 \in C^2(\overline{D}, R)$, *and there exists a smooth closed Jordan curve* $\mathcal{C}$ *located in* D *such that*

$$\begin{aligned} \varphi_1(x) &= \varphi_2(x) \quad \textit{for} \quad x \in \mathcal{C}, \\ \varphi_1(x) &> \varphi_2(x) \quad \textit{for} \quad x \in D_1 \cup \Gamma, \\ \varphi_1(x) &< \varphi_2(x) \quad \textit{for} \quad x \in D_2 \end{aligned}$$

where $D_2 \subset D$ *is the simply connected region bounded by* $\mathcal{C}$, *and* $D_1 := D \setminus \overline{D}_2$ (see Fig. 1).

Assumption (A_1) says that the surfaces $u = \varphi_1(x)$ and $u = \varphi_2(x)$ intersect in a curve whose projection into the region D is the curve $\mathcal{C}$. This property implies that the standard theory of singularly perturbed systems cannot be applied, at least near $\mathcal{C}$. To describe the behavior of a solution of (2.1) near $\mathcal{C}$ it is convenient to introduce local coordinates near $\mathcal{C}$. To this end we fixe some point P on $\mathcal{C}$ and introduce the coordinate s as the arclength on $\mathcal{C}$ measured from P in mathematically positive direction. The coordinate r is introduced in such a way that $|r|$ is the distance on the normal to $\mathcal{C}$ where $r \equiv 0$ describes the curve $\mathcal{C}$, $r < 0$ characterizes points in D_1, and $r > 0$ represent points in D_2 (see Fig.1). By a δ-neighborhood of $\mathcal{C}$ we mean the set of all points satisfying $|r| \leq \delta$. It is obvious that if δ is sufficiently small then (s, r) represents a local coordinate system in a δ-neighborhood of $\mathcal{C}$.

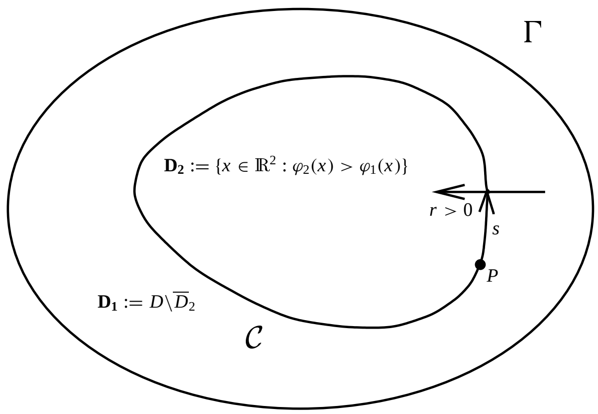

Fig. 1: Intersection of $u = \varphi_1(x)$ and $u = \varphi_2(x)$ at $\mathcal{C}$ in D

From (A_1) we get

$$\frac{\partial \varphi_2(x)}{\partial r} - \frac{\partial \varphi_1(x)}{\partial r} \geq 0 \quad \text{for } x \in \mathcal{C}. \tag{2.4}$$

Note that the surfaces $u = \varphi_1(x)$ and $u = \varphi_2(x)$ are families of equilibria of the associated equation (2.3). An equilibrium point $u = \tilde{u}(x)$ of (2.3) is called conditionally stable if the relation $f_u(\tilde{u}(x), x, 0) > 0$ holds. Assumption (A_2) describes an exchange of stabilities of the families $\varphi_1(x)$ and $\varphi_2(x)$ of equilibria at the curve $\mathcal{C}$.

(A_2).

$$\begin{aligned} f_u(\varphi_1(x), x, 0) > 0,\ f_u(\varphi_2(x), x, 0) < 0 \quad &\textit{for} \quad x \in D_1 \cup \Gamma, \\ f_u(\varphi_1(x), x, 0) < 0,\ f_u(\varphi_2(x), x, 0) > 0 \quad &\textit{for} \quad x \in D_2. \end{aligned}$$

Now we define the function $\hat{u}(x)$ by

$$\hat{u}(x) = \begin{cases} \varphi_1(x) & \text{for } x \in \overline{D}_1, \\ \varphi_2(x) & \text{for } x \in D_2. \end{cases} \tag{2.5}$$

It follows from assumption (A_1) that

$$\hat{f}(x) \equiv f(\hat{u}(x), x, 0) \equiv 0 \quad \text{for } x \in \overline{D}, \tag{2.6}$$

according to assumption (A_2) we have

$$\begin{aligned} \hat{f}_u(x) \equiv f_u(\hat{u}(x), x, 0) > 0 \quad &\text{for } x \in \overline{D} \backslash \mathcal{C}, \\ \hat{f}_u(x) \equiv 0 \quad &\text{for } x \in \mathcal{C}. \end{aligned} \tag{2.7}$$

Definition 2.1 *Under assumptions* (A_1), (A_2), *the function* $\hat{u}$ *defined by* (2.5) *is referred to as the composed stable solution to the degenerate equation* (2.2).

We will prove below that under some assumptions including (A_1) and (A_2) problem (2.1) has a solution $u(x, \varepsilon)$ which satisfies the relation

$$\lim_{\varepsilon \to 0} u(x, \varepsilon) = \hat{u}(x) \quad \text{for } x \in \overline{D}. \tag{2.8}$$

For this purpose we need assumption

(A_3).

$$\hat{f}_{uu}(x) \equiv f_{uu}(\hat{u}(x), x, 0) > 0 \quad \textit{for} \quad x \in \mathcal{C}.$$

The following assumption concerns the dependence of the function f on the parameter ε. The cases that f depends on ε and f is independent of ε require a separate treatment. In section 3.1 we consider the case that f depends on ε. In that case the sign of the derivative $\hat{f}_\varepsilon(x)$ for $x \in C$ plays an important role. We assume

(A_4)

$$\hat{f}_\varepsilon(x) \equiv f_\varepsilon(\hat{u}(x), x, 0) < 0 \quad \textit{for} \quad x \in \mathcal{C}.$$

If instead of (A_4) the inequality $\hat{f}_\varepsilon(x) > 0$ holds then the relation (2.8) may not be valid (see the example in the one-dimensional case in [1]).
In section 3.2 we investigate the case that f is independet of ε. Then hypothesis (A_4) does not hold and we use the following assumption

(A$_5$).

$$\hat{f}_u(x) \geq \kappa|r| \quad \textit{for} \quad x \in D_\delta$$

where κ is some positive number, and (s, r) are local coordinate in D_δ.

Note that assumption (A_5) corresponds to the relations in (2.7) which follow from assumption (A_2).

The concept of lower and upper solutions of problem (2.1) plays a central role in our approach.

Definition 2.2 *The functions $\alpha(x, \varepsilon)$ and $\beta(x, \varepsilon)$ which are defined in $\overline{D} \times I$ where I is some subset of I_1 are called lower and upper solutions respectively to the boundary value problem* (2.1) *if for all $\varepsilon \in I$ they satisfy the following conditions*

(i) *α and β are continuously differentiable with respect to x in $\overline{D}_1$ and twice continuously differentiable with respect to x in $D_1 \cup \mathcal{C}$ and in $\overline{D}_2$.*

(ii) $\dfrac{\partial \alpha}{\partial r}(x)\Big|_{+0} - \dfrac{\partial \alpha}{\partial r}(x)\Big|_{-0} \geq 0,\ \dfrac{\partial \beta}{\partial r}(x)\Big|_{+0} - \dfrac{\partial \beta}{\partial r}(x)\Big|_{-0} \leq 0$ *for $x \in \mathcal{C}$ where $\partial/\partial r$ denotes the differentiation with respect to the inner normal of $\mathcal{C}$.*

(iii) $L_\varepsilon \alpha(x, \varepsilon) := \Delta \alpha(x, \varepsilon) - f(\alpha(x, \varepsilon), x, \varepsilon) \geq 0,\ L_\varepsilon \beta(x, \varepsilon) \leq 0$ *for $x \in D_1 \cup \mathcal{C}$ and for $x \in \overline{D}_2$,*

(iv) $\dfrac{\partial \alpha}{\partial n} - \lambda(x)\alpha \geq 0,\ \dfrac{\partial \beta}{\partial n} - \lambda(x)\beta \leq 0$ *for $x \in \Gamma$.*

It is known (see, for example, [6]) that if there exist ordered lower and upper solutions to (2.1) i.e., they satisfy the inequality

$$\alpha(x, \varepsilon) \leq \beta(x, \varepsilon) \quad \text{for } (x, \varepsilon) \in \overline{D} \times I, \tag{2.9}$$

then problem (2.1) has a solution $u(x, \varepsilon)$ satisfying

$$\alpha(x, \varepsilon) \leq u(x, \varepsilon) \leq \beta(x, \varepsilon) \quad \text{for } (x, \varepsilon) \in \overline{D} \times I.$$

The goal of the following investigations is to prove the limit behavior (2.8) by constructing lower and upper solutions to the boundary value problem (2.1).

3 Existence and asymptotic behavior of the solution.

We consider the boundary value problem (2.1) and distinguish the cases that f depends on ε or not.

3.1 The case that f depends on ε.

Theorem 3.1. *Assume hypotheses* (A_0)–(A_4) *to be valid. Then, for sufficiently small* ε*, the boundary value problem* (2.1) *has a solution* $u(x,\varepsilon)$ *satisfying*

$$\lim_{\varepsilon\to 0} u(x,\varepsilon) = \hat{u}(x) \quad \textit{for } x \in \overline{D}. \tag{3.1}$$

Moreover, it holds

$$u(x,\varepsilon) - \hat{u}(x) \quad = \quad \begin{cases} O(\sqrt{\varepsilon}) & \textit{for } x \in D_\delta, \\ O(\varepsilon) & \textit{for } x \in \overline{D}\backslash D_\delta, \end{cases} \tag{3.2}$$

where D_δ *is a* δ*-neighborhoud of the curve* $\mathcal{C}$*, and* δ *is any fixed positive number sufficiently small.*

Proof. To prove our theorem we apply the technique of lower and upper solutions. For the construction of lower and upper solutions we use the composed stable solution $\hat{u}(x)$ defined in (2.5).
It follows from (2.4) that $\hat{u}(x)$ fulfills on $\mathcal{C}$ the condition *(ii)* of Definition 2.2 for the lower solution $\alpha(x,\varepsilon)$. But in case

$$\frac{\partial\varphi_2}{\partial r}(x) - \frac{\partial\varphi_1}{\partial r}(x) > 0 \quad \text{for } x \in \mathcal{C}$$

$\hat{u}(x)$ does not fulfill condition *(ii)* for $\beta(x,\varepsilon)$. Therefore, we construct an upper solution by using a smoothing procedure for $\hat{u}(x)$ as follows.

Let $\omega \in C^2(R,[0,1])$ be such that

$$\omega(\varrho) \quad = \quad \begin{cases} 0 & \text{for} \quad \varrho \le -1, \\ \in (0,1) & \text{for} \quad -1 < \varrho < 1, \\ 1 & \text{for} \quad \varrho \ge 1. \end{cases} \tag{3.3}$$

By means of $\omega(\varrho)$ we define the function $\tilde{u}(x,\varepsilon)$ for $(x,\varepsilon) \in \overline{D} \times I_1$ as follows:

$$\tilde{u}(x,\varepsilon) \quad := \quad \begin{cases} \varphi_1(x) + \omega(\frac{r}{\varepsilon})(\varphi_2(x) - \varphi_1(x)) & \text{for} \quad x \in D_\delta, \\ \varphi_1(x) & \text{for} \quad x \in \overline{D}_1\backslash D_\delta, \\ \varphi_2(x) & \text{for} \quad x \in D_2\backslash D_\delta, \end{cases} \tag{3.4}$$

where (s,r) are local coordinates in D_δ. It is obvious that $\tilde{u}(x,\varepsilon)$ is twice continuously differentiable in x. If we represent $\tilde{u}(x,\varepsilon)$ in the form

$$\tilde{u}(x,\varepsilon) \quad = \quad \hat{u}(x) + v(x,\varepsilon) \tag{3.5}$$

then, taking into account $\varphi_2(x) - \varphi_1(x) = O(|r|)$ in D_δ, it is easy to show that $v(x,\varepsilon)$ satisfies

$$v(x,\varepsilon) \;=\; \begin{cases} O(\varepsilon) & \text{for} \quad x \in D_\varepsilon := \{x \in R^2 : |r| < \varepsilon\}, \\ 0 & \text{for} \quad x \in \overline{D}\backslash D_\varepsilon, \end{cases} \tag{3.6}$$

moreover we have

$$\varepsilon^2 \Delta \tilde{u}(x,\varepsilon) \;=\; \begin{cases} O(\varepsilon) & \text{for} \quad x \in D_\varepsilon, \\ O(\varepsilon^2) & \text{for} \quad x \in \overline{D}\backslash D_\varepsilon. \end{cases} \tag{3.7}$$

In the sequel we construct an upper solution $\beta(x,\varepsilon)$ to (2.1) by using the smooth function $\tilde{u}(x,\varepsilon)$. To this end we introduce a local coordinate system (σ, n) in a sufficiently small δ-neighborhood Γ_δ of Γ, $\Gamma_\delta \subset D, \Gamma_\delta \cap D_\delta = \emptyset$, in the same way as we have introduced local coordinates (s,r) near $\mathcal{C}$. We use the twice continuously differentiable cut-off function $\kappa_a : R \to [0,1], a > 0$, satisfying

$$\kappa_a(\varrho) \;:=\; \begin{cases} 1 & \text{for} \quad |\varrho| \le a/2, \\ \in (0,1) & \text{for} \quad a/2 < |\varrho| < a, \\ 0 & \text{for} \quad |\varrho| \ge a \end{cases} \tag{3.8}$$

to define the following functions we need to construct upper and lower solutions to (2.1):

$$h(x,\varepsilon) \;:=\; \begin{cases} (\sqrt{\varepsilon} - \varepsilon)\kappa_\delta(r) + \varepsilon & \text{for} \quad x = (s,r) \in D_\delta, \\ \varepsilon & \text{for} \quad x \in \overline{D}\backslash D_\delta, \end{cases} \tag{3.9}$$

$$z(x,\varepsilon,k) := \begin{cases} \varepsilon \exp\left(-\frac{kn}{\varepsilon}\right) \kappa_\delta(n) & \text{for} \quad x = (\sigma, n) \in \Gamma_\delta, \\ 0 & \text{for} \quad x \in \overline{D}\backslash \Gamma_\delta \end{cases} \tag{3.10}$$

where k is some positive constant. From (3.9) we get

$$\varepsilon^2 \Delta h(x,\varepsilon) = o(\varepsilon^2) \text{ for } x \in \overline{D}, \tag{3.11}$$

and from (3.10)

$$0 \le z(x,\varepsilon,k) \le \varepsilon, \quad \varepsilon^2 |\Delta z(x,\varepsilon)| \le c_1 \varepsilon \quad \text{for} \quad x \in \overline{D}. \tag{3.12}$$

Here and in what follows we denote by $c_i, i = 1, 2...$ some appropriate positive constants which do not depend on ε.

Now we construct an upper solution $\beta(x,\varepsilon)$ to (2.1) as

$$\beta(x,\varepsilon) \;:=\; \tilde{u}(x,\varepsilon) + b_\beta h(x,\varepsilon) + z(x,\varepsilon,k_\beta) \tag{3.13}$$

where b_β and k_β are some positive numbers to be chosen in an appropriate way later.

Since $\tilde{u}, h$ and z are twice continuously differentiable with respect to x it follows from (3.13) that $\beta(x,\varepsilon)$ has the same smoothness property and therefore satisfies conditions (i) and (ii) in Definition 2.2 for an upper solution.

Now we check that $\beta(x,\varepsilon)$ satisfies the inequality (iii) in Definition 2.2. Using (3.13), (3.5), (2.6) we get

$$\begin{aligned} L_\varepsilon\beta(x,\varepsilon) \;\equiv\; & \varepsilon^2\Delta\beta(x,\varepsilon) - f(\beta(x,\varepsilon),x,\varepsilon) = \varepsilon^2\Delta\Big(\tilde{u}(x,\varepsilon) + b_\beta h(x,\varepsilon) + z(x,\varepsilon,k_\beta)\Big) \\ & -\hat{f}_u(x)\Big(b_\beta h(x,\varepsilon) + z(x,\varepsilon,k_\beta) + v(x,\varepsilon)\Big) \\ & -\frac{1}{2}\hat{f}_{uu}(x)\Big(b_\beta h(x,\varepsilon) + z(x,\varepsilon,k_\beta) + v(x,\varepsilon)\Big)^2 \\ & -\hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon). \end{aligned} \tag{3.14}$$

Our aim is to prove $L_\varepsilon\beta(x,\varepsilon) \le 0$ for $x \in D$ and for sufficiently small ε.

First we estimate $L_\varepsilon\beta(x,\varepsilon)$ in the region $D_{\delta/2}$. According to (3.8) –(3.10) we have $h(x,\varepsilon) \equiv \sqrt{\varepsilon}$, $z(x,\varepsilon,k_\beta) \equiv 0$ in $D_{\delta/2}$. Thus, we get from (3.14) for $x \in D_{\delta/2}$

$$\begin{aligned} L_\varepsilon\beta(x,\varepsilon) \;\equiv\; & \varepsilon^2\Delta\tilde{u}(x,\varepsilon) - \hat{f}_u(x)\Big(b_\beta\sqrt{\varepsilon} + v(x,\varepsilon)\Big) - \frac{1}{2}\hat{f}_{uu}(x)\Big(b_\beta\sqrt{\varepsilon} + v(x,\varepsilon)\Big)^2 \\ & -\hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon). \end{aligned} \tag{3.15}$$

From (3.7) it follows that

$$|\varepsilon^2\Delta\tilde{u}(x,\varepsilon)| \le c_2\varepsilon \quad \text{for} \quad x \in D_{\delta/2}. \tag{3.16}$$

Since b_β is positive we have by (3.6) for sufficiently small ε

$$b_\beta\sqrt{\varepsilon} + v(x,\varepsilon) \ge 0, \qquad (b_\beta\sqrt{\varepsilon} + v(x,\varepsilon))^2 = b_\beta^2\varepsilon + o(\varepsilon). \tag{3.17}$$

and hence, by (2.7) we obtain

$$-\hat{f}_u(x)(b_\beta\sqrt{\varepsilon} + v(x,\varepsilon)) \le 0.$$

From hypothesis (A_3) and from our smoothness asumption (A_0) it follows

$$\begin{aligned} & \hat{f}_{uu}(x) \ge c_3 && \text{for } x \in D_\delta,\ \delta \text{ sufficiently small}, \\ & |\hat{f}_\varepsilon(x)| \le c_4 && \text{for } x \in \overline{D}. \end{aligned} \tag{3.18}$$

By (3.16) – (3.18) we obtain from (3.15)

$$L_\varepsilon\beta(x,\varepsilon) \le (c_2 - \frac{1}{2}c_3 b_\beta^2 + c_4)\varepsilon + o(\varepsilon).$$

Therefore, for sufficiently large b_β we have $L_\varepsilon\beta(x,\varepsilon) \le 0$ for $x \in D_{\delta/2}$.

Next, we estimate $L_\varepsilon\beta(x,\varepsilon)$ in $D\backslash D_{\delta/2}$. According to (3.6) we have $v(x,\varepsilon)\equiv 0$ in $D\backslash D_{\delta/2}$. Therfore, $\varepsilon^2\Delta\tilde{u}(x,\varepsilon)=\varepsilon^2\Delta\hat{u}(x)=o(\varepsilon)$.

Taking into account (3.11) we get from (3.14) for $x\in D\backslash D_{\delta/2}$

$$\begin{aligned} L_\varepsilon\beta(x,\varepsilon) &= \varepsilon^2\Delta z(x,\varepsilon,k_\beta)-\hat{f}_u(x)\Big(b_\beta h(x,\varepsilon)+z(x,\varepsilon,k_\beta)\Big) \\ &- \frac{1}{2}\hat{f}_{uu}(x)\Big(b_\beta h(x,\varepsilon)+z(x,\varepsilon,k_\beta)\Big)^2-\hat{f}_\varepsilon(x)\varepsilon+o(\varepsilon). \end{aligned} \tag{3.19}$$

From (2.7) it follows that

$$\hat{f}_u(x)\geq c_5>0 \quad \text{for} \quad x\in D\backslash D_{\delta/2}. \tag{3.20}$$

Applying (3.12), (3.18) and the obvious inequality

$$|\frac{1}{2}\hat{f}_{uu}(x)|\leq c_6$$

we get from (3.19)

$$L_\varepsilon\beta(x,\varepsilon)\leq -c_5b_\beta h(x,\varepsilon)+c_6b_\beta^2h^2(x,\varepsilon)+(c_1+c_4)\varepsilon+o(\varepsilon). \tag{3.21}$$

Note that from (3.9) it follows

$$\varepsilon\leq h(x,\varepsilon)\leq\sqrt{\varepsilon} \quad \text{in} \quad D\backslash D_{\delta/2}.$$

Hence,

$$\frac{1}{2}c_5b_\beta h(x,\varepsilon)>(c_1+c_4)\varepsilon$$

for sufficiently large b_β and

$$\frac{1}{2}c_5b_\beta h(x,\varepsilon)>c_6b_\beta^2h^2(x,\varepsilon)$$

for any fixed b_β and suffciently small ε.
Therefore, from (3.21) we get for sufficiently large b_β and sufficiently small ε

$$L_\varepsilon\beta(x,\varepsilon)<0 \quad \text{fot} \quad x\in D\backslash D_{\delta/2}.$$

Consequently, the function $\beta(x,\varepsilon)$ satisfies condition (iii) in Definition 2.2 for an upper solution.

Taking into account that $\lambda(x),\varphi_1(x)$ and $\frac{\partial\varphi_1}{\partial x}(x)$ are bounded on $\mathcal{C}$ we get by (3.5), (3.6) (2.5) and (3.10) from (3.13) for $x\in\Gamma$ and for sufficiently large k_β

$$\frac{\partial\beta}{\partial n}(x,\varepsilon)-\lambda(x)\beta(x,\varepsilon)=\frac{\partial\varphi_1}{\partial n}(x)-k_\beta-\lambda(x)\Big(\varphi_1(x)+b_\beta\varepsilon+\varepsilon\Big)<0,$$

i.e. $\beta(x,\varepsilon)$ satisfies condition (iv) in Definition 2.2.
Consequently, the function $\beta(x,\varepsilon)$ defined in (3.13) satisfies the conditions (i) – (iv)

in Definition 2.2 and thus represents an upper solution to the boundary value problem (2.1).

Now we construct a lower solution $\alpha(x,\varepsilon)$ in the form

$$\alpha(x,\varepsilon) : \;=\; \hat{u}(x) - b_\alpha \varepsilon - z(x,\varepsilon,k_\alpha) \tag{3.22}$$

where the positive constants b_α and k_α have to be chosen in an appropriate way. Note that $\alpha(x,\varepsilon)$ may be non-smooth on the curve $\mathcal{C}$, but according to (2.4) it satisfies the condition (ii) in Definition 2.2. It is also obvious that $\alpha(x,\varepsilon)$ satisfies condition (i) in Definition 2.2.

For $L_\varepsilon \alpha(x,\varepsilon)$ we get analogously to (3.14)

$$\begin{aligned} L_\varepsilon \alpha(x,\varepsilon) &\equiv \varepsilon^2 \Delta \alpha(x,\varepsilon) - f(\alpha(x,\varepsilon),x,\varepsilon) = \\ &= \varepsilon^2 \Delta\Big(\hat{u}(x) - z(x,\varepsilon,k_\alpha)\Big) + \hat{f}_u(x)\Big(b_\alpha \varepsilon + z(x,\varepsilon,k_\alpha)\Big) - \hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon). \end{aligned} \tag{3.23}$$

First, we consider $L_\varepsilon \alpha(x,\varepsilon)$ in the region D_δ for sufficiently small δ. Taking into account (3.10), (2.7) and the boundedness of $\Delta \hat{u}(x)$ we get from (3.23)

$$L_\varepsilon \alpha(x,\varepsilon) \;=\; \varepsilon^2 \Delta \hat{u}(x) + \hat{f}_u(x) b_\alpha \varepsilon - \hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon) \geq -\hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon). \tag{3.24}$$

By assumption (A_4) it holds for sufficiently small δ

$$-\hat{f}_\varepsilon(x) \geq c_7 \quad \text{for } x \in D_\delta. \tag{3.25}$$

Thus, from (3.24) and (3.25) we get

$$L_\varepsilon \alpha(x,\varepsilon) \geq 0 \quad \text{for} \quad x \in D_\delta.$$

Finally, we study $L_\varepsilon \alpha(x,\varepsilon)$ in $D \setminus D_\delta$. By (3.12), (3.18), and (3.20) we get from (3.23)

$$L_\varepsilon \alpha(x,\varepsilon) \geq (-c_1 + c_5 b_\alpha - c_4)\varepsilon + o(\varepsilon).$$

Therefore, for sufficiently large b_α we obtain

$$L_\varepsilon \alpha(x,\varepsilon) \geq 0 \quad \text{for} \quad x \in D \setminus D_\delta.$$

Thus, the function $\alpha(x,\varepsilon)$ satisfies condition (iii) in Definition 2.2.

From (3.22), (2.5), and (3.10) we obtain for $x \in \Gamma$ and for sufficiently large k_α

$$\frac{\partial \alpha}{\partial n}(x,\varepsilon) - \lambda(x)\alpha(x,\varepsilon) = \frac{\partial \varphi_1}{\partial n}(x) + k_\alpha - \lambda(x)\Big(\varphi_1(x) - b_\alpha \varepsilon - \varepsilon\Big) > 0$$

i.e. $\alpha(x,\varepsilon)$ satisfies condition (iv) in Definition 2.2. Consequently, the function $\alpha(x,\varepsilon)$ defined in (3.22) is a lower solution to the boundary value problem (2.1).

From (3.13) and (3.22) it follows for sufficiently small ε that $\beta(x,\varepsilon) > \hat{u}(x)$ and $\alpha(x,\varepsilon) < \hat{u}(x)$ in $\overline{D}$. Hence, $\alpha(x,\varepsilon)$ and $\beta(x,\varepsilon)$ are ordered lower and upper solutions to (2.1). Therefore, we can conclude that for sufficiently small ε there exists a solution $u(x,\varepsilon)$ of (2.1) satisfying

$$\alpha(x,\varepsilon) \le u(x,\varepsilon) \le \beta(x,\varepsilon) \quad \text{for } x \in \overline{D}.$$

The relations (3.13), (3.9), (3.10), and (3.22) show that the relations (3.2) and consequently (3.1) for $u(x,\varepsilon)$ are fulfilled. This completes the proof of Theorem 3.1.

Remark 3.1. *In case of system* (1.4) *which models a fast pure bimolecular reaction we have* $\hat{f}_\varepsilon(x) \equiv 0$. *That means assumption* (A_4) *is not valid. In such cases we may replace hypothesis* (A_4) *by the following condition:*

($\tilde{A}_4$). *The composed stable solution* $\hat{u}(x)$ *of the degenerate equation* (2.2) *is a lower solution for* (2.1), *i.e.*

$$\begin{array}{ll} (i) & L_\varepsilon \hat{u}(x) \ge 0 \quad \text{for } x \in D_1 \cup \mathcal{C}, x \in \overline{D}_2, \varepsilon \in I_2 \subset I_1, \\ (ii) & \dfrac{\partial \hat{u}}{\partial n}(x) - \lambda(x)\hat{u}(x) \ge 0 \quad \text{for } x \in \Gamma. \end{array}$$

It is easy to verify that under the assumptions (A_0) - (A_3) *and* ($\tilde{A}_4$) *Theorem* 3.1 *remains true.*

Remark 3.2. *In the subsets* $\overline{D}_1 \backslash D_\delta$ *and* $D_2 \backslash D_\delta$ *we can derive an asymptotic expansion of any order in* ε *for the solution* $u(x,\varepsilon)$ *by means of standard theory for singularly perturbed problems provided the function* f *is sufficiently smooth [7].*
In $\overline{D}_1 \backslash D_\delta$ *the asymptotic expansion of* $u(x,\varepsilon)$ *reads*

$$\begin{aligned} u(x,\varepsilon) \;=\; & \varphi_1(x) + \varepsilon \overline{u}_1(x) + \ldots + \varepsilon^m \overline{u}_m(x) + \varepsilon \Pi_1\left(\sigma, \frac{n}{\varepsilon}\right) + \ldots + \varepsilon^m \Pi_m\left(\sigma, \frac{n}{\varepsilon}\right) + \\ & + O(\varepsilon^{m+1}) \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} &\overline{u}_1(x) = -\hat{f}_u^{-1}(x)\hat{f}_\varepsilon(x), \\ &\overline{u}_2(x) = \hat{f}_u^{-1}(x)\left[\Delta\varphi_1(x) - \frac{1}{2}\hat{f}_{\varepsilon\varepsilon}(x) - \hat{f}_{u\varepsilon}(x)\overline{u}_1(x) - \frac{1}{2}\hat{f}_{uu}(x)\overline{u}_1^2(x)\right], \\ &\ldots . \end{aligned} \tag{3.27}$$

$\Pi_i(\sigma, \frac{n}{\varepsilon}), i = 1,2,\ldots,$ *are boundary layer functions which can be constructed by means of the standard theory and which satisfy*

$$\left|\Pi_i\left(\sigma, \frac{n}{\varepsilon}\right)\right| \le c \exp\left(-\frac{\kappa n}{\varepsilon}\right), \; i = 0,1,\ldots,m, \tag{3.28}$$

where c and κ are some positive constants, σ and n are local coordinates near Γ. In $D_2 \setminus D_\delta$ the asymptotic expansion of $u(x,\varepsilon)$ has the form

$$u(x,\varepsilon) = \varphi_2(x) + \varepsilon\overline{u}_1(x) + \ldots + \varepsilon^m\overline{u}_m(x) + O(\varepsilon^{m+1}). \tag{3.29}$$

Here, the functions $\overline{u}_i(x)$ $(i = 1,\ldots,m)$ are defined as in (3.27) if we replace there $\varphi_1(x)$ by $\varphi_2(x)$.

From (3.26) and (3.29) we obtain the following corollary which we need to estimate the jumping behavior of the reaction rates (see example 4.2).

Corollary 3.1. *Under the assumptions of Theorem 3.1 or under the asumptions $(A_0)--(A_3)$ and $(\tilde{A}_4)$ we have*

$$\Delta u(x,\varepsilon) = \Delta\hat{u}(x) + O(\varepsilon) \quad \text{for } x \in D \setminus (\Gamma_\delta \cup D_\delta). \tag{3.30}$$

Proof. We prove (3.30) for $x \in D_1 \setminus (\Gamma_\delta \cup D_\delta)$. From (3.26) and (3.28) we get for $m = 2$

$$u(x,\varepsilon) = \varphi_1(x) + \varepsilon\overline{u}_1(x) + \varepsilon^2\overline{u}_2(x) + O(\varepsilon^3) \equiv U_2(x,\varepsilon) + O(\varepsilon^3).$$

Consequently,

$$\begin{aligned}&\Delta(u(x,\varepsilon) - U_2(x,\varepsilon)) = \frac{1}{\varepsilon^2} f(U_2(x,\varepsilon) + O(\varepsilon^3), x, \varepsilon) - \Delta U_2(x,\varepsilon) \\ &= \{f(U_2(x,\varepsilon) + O(\varepsilon^3), x, \varepsilon) - f(U_2(x,\varepsilon), x, \varepsilon) + f(U_2(x,\varepsilon), x, \varepsilon) - \varepsilon^2\Delta U_2(x,\varepsilon)\}/\varepsilon^2.\end{aligned} \tag{3.31}$$

Obviously we have

$$f(U_2(x,\varepsilon) + O(\varepsilon^3), x, \varepsilon) - f(U_2(x,\varepsilon), x, \varepsilon) = O(\varepsilon^3).$$

By means of (3.27) we get

$$f(U_2(x,\varepsilon), x, \varepsilon) - \varepsilon^2\Delta U_2(x,\varepsilon) = O(\varepsilon^3).$$

Therefore, we obtain from (3.31)

$$\Delta(u(x,\varepsilon) - U_2(x,\varepsilon)) = O(\varepsilon).$$

By using the obvious relation

$$\Delta U_2(x,\varepsilon) = \Delta\varphi_1(x) + O(\varepsilon)$$

we get $\Delta u(x,\varepsilon) = \Delta\varphi_1(x) + O(\varepsilon)$, i.e. the relation (3.30) holds for $x \in D_1 \setminus (\Gamma_\delta \cup D_\delta)$. For $x \in D_2 \setminus D_\delta$, relation (3.30) can be proved in a similar way.

3.2 The case that f does not depend of ε.

Consider now the boundary value problem (2.1) when f is independent of ε, i.e. $f = f(u,x)$. In this case, we preserve assumptions (A_0) - (A_3) and replace assumption (A_4) by assumption (A_5) (see section 2).

Theorem 3.2 *Assume hypotheses* (A_0) - (A_3) *and* (A_5) *to be valid. Then, for sufficiently small* ε *the boundary value problem* (2.1) *has a solution* $u(x,\varepsilon)$ *satisfying*

$$\lim_{\varepsilon \to 0} u(x,\varepsilon) \quad = \quad \hat{u}(x) \quad \textit{for} \quad x \in \overline{D}. \tag{3.32}$$

Moreover, it holds

$$u(x,\varepsilon) - \hat{u}(x) \quad = \quad \begin{cases} O(\varepsilon^{2/3}) & \textit{for} \quad x \in D_\delta, \\ O(\varepsilon) & \textit{for} \quad x \in \Gamma_\delta, \\ O(\varepsilon^2) & \textit{for} \quad x \in D \backslash (D_\delta \cup \Gamma_\delta), \end{cases} \tag{3.33}$$

where D_δ *and* Γ_δ *are* δ*-neighborhood of* $\mathcal{C}$ *and* Γ *respectively,* δ *is any fixed positive number sufficiently small.*

Proof. As in proof of Theorem 3.1 we use the technique of lower and upper solutions. We introduce the smooth function $\tilde{u}(x,\varepsilon)$ as in (3.4) by means of the function $\omega(\rho)$, defined in (3.3) but different to (3.4) we put now $\rho = r/\varepsilon^{2/3}$ such that we get

$$\tilde{u}(x,\varepsilon) \quad := \quad \begin{cases} \varphi_1(x) + \omega(\frac{r}{\varepsilon^{2/3}})(\varphi_2(x) - \varphi_1(x)) & \text{for} \quad x \in D_\delta, \\ \varphi_1(x) & \text{for} \quad x \in \overline{D}_1 \backslash D_\delta, \\ \varphi_2(x) & \text{for} \quad x \in D_2 \backslash D_\delta. \end{cases} \tag{3.34}$$

Hence, we have the representation

$$\tilde{u}(x,\varepsilon) = \hat{u}(x) + v(x,\varepsilon) \tag{3.35}$$

where

$$v(x,\varepsilon) = \begin{cases} O(\varepsilon^{2/3}) & \text{for} \quad x \in D_{\varepsilon^{2/3}}, \\ 0 & \text{for} \quad x \in \overline{D} \backslash D_{\varepsilon^{2/3}}, \end{cases} \tag{3.36}$$

and

$$\varepsilon^2 \Delta \tilde{u}(x,\varepsilon) \quad = \quad \begin{cases} O(\varepsilon^{4/3}) & \text{for} \quad x \in D_{\varepsilon^{2/3}}, \\ O(\varepsilon^2) & \text{for} \quad x \in \overline{D} \backslash D_{\varepsilon^{2/3}}. \end{cases} \tag{3.37}$$

We construct an upper solution $\beta(x,\varepsilon)$ to problem (2.1) in the form

$$\beta(x,\varepsilon) := \tilde{u}(x,\varepsilon) + \gamma h(x,\varepsilon) + z(x,\varepsilon,k_\beta) \tag{3.38}$$

where

$$h(x,\varepsilon) \;=\; \begin{cases} (\varepsilon^{2/3}-\varepsilon^2)\kappa_\delta(r)+\varepsilon^2 & \text{for} \quad x\in D_\delta, \\ (\varepsilon-\varepsilon^2)\kappa_\delta(n)+\varepsilon^2 & \text{for} \quad x\in \Gamma_\delta, \\ \varepsilon^2 & \text{for} \quad x\in \overline{D}\backslash(D_\delta\cup\Gamma_\delta), \end{cases} \tag{3.39}$$

$\kappa_a(\varrho)$ and $z(x,\varepsilon,k)$ are the same functions as in (3.8) and (3.10), respectively, γ and k_β in $z(x,\varepsilon,k_\beta)$ are some positive numbers to be chosen later in an appropriate way. Note that we have

$$\begin{aligned} h(x,\varepsilon) &= \varepsilon^{2/3} \quad \text{for} \quad x\in D_{\delta/2}, \\ h(x,\varepsilon) &= \varepsilon \quad \text{for} \quad x\in \Gamma_{\delta/2}, \\ h(x,\varepsilon) &= \varepsilon^2 \quad \text{for} \quad x\in (\overline{D}\backslash(D_\delta\cup\Gamma_\delta), \\ \varepsilon^2\Delta h(x,\varepsilon) &= o(\varepsilon^2) \quad \text{for} \quad x\in\overline{D}. \end{aligned} \tag{3.40}$$

Since $\tilde{u}, h$ and z are twice continuously differentiable with respect to x it follows from (3.38) that $\beta(x,\varepsilon)$ has the same smoothness property and therefore satisfies conditions (i) and (ii) in Definition 2.2 for an upper solution.
As in the proof of Theorem 3.1 we can establish that for sufficiently large k_β the function $\beta(x,\varepsilon)$ satisfies the inequality (iv) in Definition 2.2.

Now we check that $\beta(x,\varepsilon)$ satisfies inequality (iii) in Definition 2.2. Analogously to (3.14) we obtain by using (3.38)

$$\begin{aligned} L_\varepsilon\beta(x,\varepsilon) \;\equiv\; & \varepsilon^2\Delta\beta(x,\varepsilon)-f(\beta(x,\varepsilon),x)=\varepsilon^2\Delta(\tilde{u}(x,\varepsilon)+\gamma h(x,\varepsilon)+z(x,\varepsilon,k_\beta)) \\ & -\hat{f}_u(x)(\gamma h(x,\varepsilon)+z(x,\varepsilon,k_\beta)+v(x,\varepsilon))-\frac{1}{2}\hat{f}_{uu}(x)\Big(\gamma h(x,\varepsilon)+ \\ & +z(x,\varepsilon,k_\beta)+v(x,\varepsilon))^2+o((\gamma h(x,\varepsilon)+z(x,\varepsilon,k_\beta)+v(x,\varepsilon))^2). \end{aligned} \tag{3.41}$$

We want to prove $L_\varepsilon\beta(x,\varepsilon)\le 0$ for $x\in D$ and for sufficiently small ε. First we consider the neighborhood $D_{\delta/2}$ of the curve $\mathcal{C}$. Taking into account (3.10) and (3.40), relation (3.41) reads in $D_{\delta/2}$

$$\begin{aligned} L_\varepsilon\beta(x,\varepsilon) \;=\; & \varepsilon^2\Delta\tilde{u}(x,\varepsilon)-\hat{f}_u(x)(\gamma\varepsilon^{2/3}+v(x,\varepsilon)) \\ & -\frac{1}{2}\hat{f}_{uu}(x)(\gamma\varepsilon^{2/3}+v(x,\varepsilon))^2+o((\gamma\varepsilon^{2/3}+v(x,\varepsilon))^2). \end{aligned} \tag{3.42}$$

By (3.36) and (3.37) we have for $x\in D_{\delta/2}$

$$|v(x,\varepsilon)|\le c_8\varepsilon^{2/3},\; \varepsilon^2|\Delta\hat{u}(x)|\le c_9\varepsilon^{4/3}. \tag{3.43}$$

Thus, for $x\in D_{\delta/2}$ and sufficiently large γ we have

$$\gamma\varepsilon^{2/3}+v(x,\varepsilon)\ge 0. \tag{3.44}$$

Taking into account the relations (2.7), (3.18), (3.44) we obtain from (3.42) for $x \in D_{\delta/2}$

$$L_\varepsilon\beta(x,\varepsilon) \leq -\frac{1}{2}c_4(\gamma - c_8)^2\varepsilon^{4/3} + c_9\varepsilon^{4/3} + o(\varepsilon^{4/3}) < 0$$

for sufficiently large γ and sufficiently small ε.
Consider now the neighborhood $\Gamma_{\delta/2}$ of the boundary Γ. By (3.40) and (3.36) the expression (3.41) reads in $\Gamma_{\delta/2}$

$$\begin{aligned} L_\varepsilon\beta(x,\varepsilon) &= \varepsilon^2\Delta(\hat{u}(x,\varepsilon) + z(x,\varepsilon,k_\beta)) - \hat{f}_u(x)(\gamma\varepsilon + z(x,\varepsilon,k_\beta)) \\ &\quad -\frac{1}{2}\hat{f}_{uu}(x)(\gamma\varepsilon + z(x,\varepsilon,k_\beta))^2 + o((\gamma\varepsilon + z(x,\varepsilon,k_\beta))^2). \end{aligned} \tag{3.45}$$

By (3.12),(3.20) and by taking into account the boundedness of $\hat{f}_{uu}(x)$ and $\Delta\hat{u}(x)$ we obtain from (3.45)

$$L_\varepsilon\beta(x,\varepsilon) \leq (-c_6\gamma + c_1)\varepsilon + o(\varepsilon) < 0 \tag{3.46}$$

for sufficiently large γ and sufficiently small ε.
To estimate $L_\varepsilon\beta(x,\varepsilon)$ in $D\backslash(D_{\delta/2} \cup \Gamma_{\delta/2})$ we note that by (3.36) $v(x,\varepsilon)$ vanishes identically for $x \in D\backslash(D_{\delta/2} \cup \Gamma_{\delta/2})$ and for $\delta \geq \varepsilon^{2/3}$. Hence, we obtain from (3.41)

$$\begin{aligned} L_\varepsilon\beta(x,\varepsilon) &\equiv \varepsilon^2\Delta(\hat{u}(x) + \gamma h(x,\varepsilon) + z(x,\varepsilon,k_\beta)) - \hat{f}_u(x)(\gamma h(x,\varepsilon) + z(x,\varepsilon,k_\beta)) \\ &\quad -\frac{1}{2}\hat{f}_{uu}(x)(\gamma h(x,\varepsilon) + z(x,\varepsilon,k_\beta))^2 + o((\gamma h(x,\varepsilon) + z(x,\varepsilon,k_\beta))^2). \end{aligned} \tag{3.47}$$

Taking into account (3.40) and (3.10), we get

$$\varepsilon^2 \leq h(x,\varepsilon) \leq \varepsilon^{2/3},\ z(x,\varepsilon,k) = o(\varepsilon^N) \text{ and } \Delta z(x,\varepsilon,k) = o(\varepsilon^N) \text{ for any } N,$$

and hence, by (3.20) and the inequalities $|\Delta\hat{u}| \leq c_9, \frac{1}{2}|\hat{f}_{uu}(x)| \leq c_{10}$ we get from (3.47)

$$L_\varepsilon\beta(x,\varepsilon) \leq -c_6\gamma h(x,\varepsilon) + c_{10}\gamma^2h^2(x,\varepsilon) + c_9\varepsilon^2 + o(\varepsilon^2) < 0$$

for sufficiently large γ and sufficiently small ε.

Thus, the function $\beta(x,\varepsilon)$ defined by (3.38) satisfies all the conditions for an upper solution in Definition 2.2.

A lower solution cannot be constructed in the form (3.22) (as it was done in section 3.1) in the case when f does not depend of ε since that form of lower solution does not imply a positive sign for $L_\varepsilon\alpha$ near $\mathcal{C}$. Hence, in our case we construct a lower solution in the form

$$\alpha(x,\varepsilon) := \hat{u}(x) + w(x,\varepsilon) - \gamma_1 g(x,\varepsilon) - z(x,\varepsilon,k_\alpha) \tag{3.48}$$

where

$$w(x,\varepsilon) \;=\; \begin{cases} b_1\varepsilon^{4/3}\rho^2\exp(-|\varrho|)\kappa_\delta(r) & \text{for} \quad x \in D_\delta, \\ 0 & \text{for} \quad x \in (\overline{D}\backslash D_\delta), \end{cases}$$

$$g(x,\varepsilon) \;=\; \begin{cases} (\varepsilon^{4/3} - \varepsilon^2)\kappa_\delta(r) + \varepsilon^2 & \text{for} \quad x \in D_\delta, \\ (\varepsilon - \varepsilon^2)\kappa_\delta(n) + \varepsilon^2 & \text{for} \quad x \in \Gamma_\delta, \\ \varepsilon^2 & \text{for} \quad x \in \overline{D}\backslash(D_\delta \cup \Gamma_\delta), \end{cases}$$

here we have $\varrho = r\varepsilon^{-2/3}$, γ_1, b_1 and k_α in z are some positive numbers to be chosen later, in particular we suppose $\gamma_1 > b_1$.
Note that

$$0 \le w(x,\varepsilon) < b_1\varepsilon^{4/3} \quad \text{in} \quad \overline{D} \quad \text{and} \quad w(x,\varepsilon) \;=\; o(\varepsilon^N) \quad \text{in} \quad \overline{D}\backslash D_{\delta/2} \quad \text{for any } N,$$

$$\begin{aligned} g(x,\varepsilon) &= \varepsilon^{4/3} \quad \text{for} \quad x \in D_{\delta/2}, \\ g(x,\varepsilon) &= \varepsilon \quad \text{for} \quad x \in \Gamma_{\delta/2}, \\ g(x,\varepsilon) &= \varepsilon^2 \quad \text{for} \quad x \in \overline{D}\backslash(D_\delta \cup \Gamma_\delta). \end{aligned} \tag{3.49}$$

It can be easily checked that $\alpha(x,\varepsilon)$ satisfies the conditions (i), (ii), and for sufficiently large k_α condition (iv) in Definition 2.2.
Now we verify that $\alpha(x,\varepsilon)$ satisfies condition (iii) in Definition 2.2. Using (3.48) we get

$$\begin{aligned} L_\varepsilon\alpha(x,\varepsilon) \;\equiv\;& \varepsilon^2\Delta\alpha(x,\varepsilon) - f(\alpha(x,\varepsilon),x) = \varepsilon^2\Delta\Big(\hat{u}(x) + w(x,\varepsilon) - \gamma_1 g(x,\varepsilon) - z(x,\varepsilon,k_\alpha)\Big) \\ & -\hat{f}_u(x)\Big(w(x,\varepsilon) - \gamma_1 g(x,\varepsilon) - z(x,\varepsilon,k_\alpha)\Big) \\ & -\frac{1}{2}\hat{f}_{uu}(x)(w(x,\varepsilon) - \gamma_1 g(x,\varepsilon) - z(x,\varepsilon,z_\alpha))^2 \\ & +o((w(x,\varepsilon) - \gamma_1 g(x,\varepsilon) - z(x,\varepsilon,z_\alpha))^2). \end{aligned} \tag{3.50}$$

In the neighborhood $D_{\delta/2}$ of the curve $\mathcal{C}$ we have

$$g(x,\varepsilon) = \varepsilon^{4/3}, \;\; z(x,\varepsilon,k_\alpha) \equiv 0,$$

$$w(x,\varepsilon) - \gamma_1 g(x,\varepsilon) < (b_1 - \gamma_1)\varepsilon^{4/3} < 0 \;\; \text{for } b_1 < \gamma_1, \;\; |\Delta\hat{u}(x,\varepsilon)| \le c_{11}.$$

If we express the Laplacian in $D_{\delta/2}$ by means of the local coordinates (s,r) we get

$$\Delta w(x,\varepsilon) = b_1[(2 - 4|\rho| + \rho^2)\exp(-|\rho|) + O(\varepsilon^{2/3})].$$

Furthermore, we have $\hat{f}_u(x) \geq 0$ in $D_{\delta/2}$, and hence it holds

$$-\hat{f}_u(x)(w(x,\varepsilon) - \gamma_1 g(x,\varepsilon)) \geq 0,\ \frac{1}{2}\hat{f}_{uu}(x)(w(x,\varepsilon) - \gamma_1 g(x,\varepsilon))^2 = O(\varepsilon^{8/3}) = o(\varepsilon^2).$$

For $|r| < m\varepsilon^{2/3}$, i.e. $|\rho| < m$, where m be so small that we have

$$(2 - 4|\rho| + \rho^2)\exp(-|\rho|) \geq c_0 > 0 \quad \text{for } |\rho| \leq m$$

we get from (3.50)

$$L_\varepsilon \alpha(x,\varepsilon) \geq \varepsilon^2(b_1 c_0 - c_{11}) + o(\varepsilon^2) > 0$$

for sufficiently large b_1 and sufficiently small ε.
For $m\varepsilon^{2/3} \leq |r| < \delta/2$, i.e. in $D_{\delta/2}\backslash D_{m\varepsilon^{2/3}}$ we have

$$|\Delta w(x,\varepsilon)| \leq b_1 c_{12},$$

and according to (A_5)

$$\hat{f}_u(x) \geq \kappa |r| \geq \kappa m \varepsilon^{2/3}.$$

Hence,

$$-\hat{f}_u(x)(w(x,\varepsilon) - \gamma_1 g(x,\varepsilon)) \geq \kappa m(\gamma_1 - b_1)\varepsilon^2 \quad \text{in } D_{\delta/2}\backslash D_{m\varepsilon^{2/3}}$$

and from (3.50) we get

$$L_\varepsilon \alpha(x,\varepsilon) \geq \kappa m(\gamma_1 - b_1)\varepsilon^2 - (b_1 c_{12} + c_{11})\varepsilon^2 + o(\varepsilon^2) > 0$$

for sufficiently large γ_1 and sufficiently small ε.

Consider now the neighborhood $\Gamma_{\delta/2}$ of the boundary Γ. In this neighborhood we have

$$w \equiv 0,\ g(x,\varepsilon) = \varepsilon,\ z(x,\varepsilon,k_\alpha) = \varepsilon\exp(-\frac{k_\alpha n}{\varepsilon}),$$

and analogously to (3.46) we get

$$L_\varepsilon \alpha(x,\varepsilon) \geq (c_6\gamma_1 - c_1)\varepsilon + o(\varepsilon). \tag{3.51}$$

Thus, we have $L_\varepsilon \alpha > 0$ in $\Gamma_{\delta/2}$ for sufficiently large γ_1 and sufficiently small ε.
In $D\backslash(D_{\delta/2} \cup \Gamma_{\delta/2})$ it holds

$$w(x,\varepsilon) = o(\varepsilon^N),\ \Delta w(x,\varepsilon) = o(\varepsilon^N), z(x,\varepsilon,k_\alpha) = o(\varepsilon^N), \Delta z(x,\varepsilon,k) = o(\varepsilon^N) \text{ for any} N,$$

$$\varepsilon^2 \leq g(x,\varepsilon) \leq \varepsilon^{4/3}, |\Delta\hat{u}(x)| \leq c_{11},$$

$$\varepsilon^2 \Delta\gamma_1 g(x,\varepsilon) = o(\varepsilon^2),\ \hat{f}_u(x) \geq c_6 > 0,\ \left|\frac{1}{2}\hat{f}_{uu}(x)\right| \leq c_{10}$$

and hence, from (3.50) we get

$$L_\varepsilon \alpha(x,\varepsilon) \geq c_6\gamma_1 g(x,\varepsilon) - c_{10}\gamma_1^2 g^2(x,\varepsilon) - c_{11}\varepsilon^2 + o(\varepsilon^2) > 0$$

for sufficiently large γ_1 and sufficiently small ε.
Thus, the function $\alpha(x,\varepsilon)$ defined by (3.48) satisfies all the conditions for a lower solution in Definition 2.2.
From (3.38) and (3.48) it follows that $\beta(x,\varepsilon) > \hat{u}(x)$ and $\alpha(x,\varepsilon) < \hat{u}(x)$ in $\overline{D}$ and hence the inequality (2.9) is fulfilled, i.e. $\alpha(x,\varepsilon)$ and $\beta(x,\varepsilon)$ are ordered lower and upper solutions to (2.1). Therefore, we can conclude that for sufficiently small ε there exists a solution $u(x,\varepsilon)$ to the boundary problem (2.1) satisfying

$$\alpha(x,\varepsilon) \le u(x,\varepsilon) \le \beta(x,\varepsilon) \quad \text{for} \quad x \in \overline{D}.$$

The formulae (3.38), (3.40),(3.10), (3.48) and (3.49) show that the relations (3.33) and consequently (3.32) for $u(x,\varepsilon)$ are fulfilled. This completes the proof of Theorem 3.2.

4 Examples.

Example 4.1 We study the boundary value problem (2.1) with $f \equiv u(u - x_1^2 - x_2^2 + 1)$ in $D := \{x \in R^2 : x_1^2 + x_2^2 < 4\}$.
The degenerate equation

$$u(u - x_1^2 - x_2^2 + 1) = 0$$

has two solutions $u = \varphi_1(x) = x_1^2 + x_2^2 - 1$ and $u = \varphi_2(x) = 0$. These solutions intersect in the curve $\mathcal{C}$ defined by $x_1^2 + x_2^2 = 1$, i.e. the curve $\mathcal{C}$ is circle. The inequality $\varphi_1(x) < \varphi_2(x)$ holds in the subdomain $D_2 = \{x : x_1^2 + x_2^2 < 1\}$ and the inequality $\varphi_1(x) > \varphi_2(x)$ holds in the subdomain $D_1 = D\backslash D_2$ and on Γ, i.e. the assumption (A_1) is fulfilled.
Calculating f_u we get

$$f_u(\varphi_1(x), x) = x_1^2 + x_2^2 - 1, \;\; f_u(\varphi_2(x), x) = 1 - x_1^2 - x_2^2.$$

It is obviously that

$$\begin{array}{ll} f_u(\varphi_1(x), x) > 0, \;\; f_u(\varphi_2(x), x) < 0 & \text{in } (D_1 \cup \Gamma), \\ f_u(\varphi_1(x), x) < 0, \;\; f_u(\varphi_2(x), x) > 0 & \text{in } D_2, \end{array}$$

i.e. the assumption (A_2) holds.
The composed stable solution in our example has the form

$$\hat{u}(x) \;\; = \;\; \left\{ \begin{array}{lll} x_1^2 + x_2^2 - 1 & \text{for} & x \in \overline{D}_1 = (\overline{D}\backslash D_2), \\ 0 & \text{for} & x \in D_2. \end{array} \right. \qquad (4.1)$$

Since $f_{uu}(u,x) = 2 > 0$ the assumption (A_3) is fulfilled.
Finally, $\hat{f}_u(x)$ can be written in the form

$$\hat{f}_u(x) = |x_1^2 + x_2^2 - 1| = |(1+r)^2 - 1| = |r| \cdot |2 + r|,$$

where $|r|$ is distance from point (x_1, x_2) to the curve $\mathcal{C} = \{x : x_1^2 + x_2^2 = 1\}$. Taking into account that $r \geq -1$ ($r = -1$ for point $(0,0)$) we get

$$\hat{f}_u(x) \geq |r|,$$

i.e. the assumption (A_5) is satisfied with $\kappa = 1$.
Thus, all the assumptions (A_1) - (A_3) and (A_5) of the Theorem 3.2 are fulfilled. Therefore, problem (2.1) with $f = u(u - x_1^2 - x_2^2 + 1)$ has a solution $u(x, \varepsilon)$ satisfying

$$\lim_{\varepsilon \to 0} u(x, \varepsilon) = \hat{u}(x)$$

where $\hat{u}(x)$ is defined by (4.1).

Example 4.2 The fast purely bimolecular reaction.

We consider system (1.3) describing fast pure bimolecular reaction assuming that

$$r(u, v) \equiv kuv$$

where k is a positive constant. In this case the system (1.4) has the form

$$\begin{aligned} \varepsilon^2 \Delta u &= -\varepsilon^2 I_a(x) + ku(u - w), \\ \Delta w &= I_b(x) - I_a(x), \qquad x \in D. \end{aligned} \tag{4.2}$$

Let boundary conditions for (4.2) have the form

$$\frac{\partial u}{\partial n} - \lambda u = \frac{\partial w}{\partial n} - \lambda w = 0 \quad \text{for } x \in \Gamma. \tag{4.3}$$

Recall that $I_a(x), I_b(x)$ are nonnegative functions describing inputs. The function w can be determined independently of u ($w = w(x)$) and therefore we have to solve the first equation of (4.2) with $w = w(x)$ and prescribed boundary condition (4.3). Concerning $w(x)$ we assume that

$$\begin{aligned} w(x) = 0 &\quad \text{for} \quad x \in \mathcal{C}, \\ w(x) < 0 &\quad \text{for} \quad x \in D_1, \\ w(x) > 0 &\quad \text{for} \quad x \in D_2 \end{aligned}$$

where $\mathcal{C}$ is a closed smooth curve separating the domain D into two parts (D_1 outside $\mathcal{C}$ and D_2 inside $\mathcal{C}$). The assumptions (A_1) and (A_2) are fulfilled with $\varphi_1(x) \equiv 0$, $\varphi_2(x) \equiv w(x)$, hence the composed stable solution for this case reads

$$\hat{u}(x) = \begin{cases} 0 & \text{for} \quad x \in \overline{D}_1, \\ w(x) & \text{for} \quad x \in D_2 \end{cases} \tag{4.4}$$

(see Definition 2.1).
It is easily to check that $\hat{u}(x)$ is a lower solution of the problem for u. Indeed, we have

$$\varepsilon^2 \Delta \hat{u} + \varepsilon^2 I_a(x) - k\hat{u}(\hat{u} - w(x)) = \begin{cases} \varepsilon^2 I_a(x) \geq 0 & \text{in } D_1, \\ \varepsilon^2 I_b(x) \geq 0 & \text{in } D_2, \end{cases}$$

$$\frac{\partial \hat{u}}{\partial n}(x) - \lambda \hat{u}(x) = 0 \quad \text{for} \quad x \in \Gamma,$$

i.e. assumption $(\tilde{A}_4)$ is satisfied.
The assumption (A_3) also holds as $\hat{f}_{uu}(x) = 2k > 0$. Therefore by means of Theorem 3.1 (see Remark 3.1) we obtain that the problem for u has the solution $u(x, \varepsilon)$ satisfying

$$\lim_{\varepsilon \to 0} u(x, \varepsilon) = \hat{u}(x) \qquad \text{for } x \in \overline{D}.$$

In order to calculate important for application reaction rate

$$\tilde{r}(x, \varepsilon) = ku(x, \varepsilon)(u(x, \varepsilon) - w(x))/\varepsilon^2 = \Delta u(x, \varepsilon) + I_a(x) \tag{4.5}$$

we use the result of Corollary 3.1. According to (3.30) and (4.4) we have

$$\Delta u(x, \varepsilon) = \begin{cases} O(\varepsilon) & \text{for} \quad x \in D_1 \backslash (\Gamma_\delta \cup D_\delta), \\ \Delta w(x) + O(\varepsilon) & \text{for} \quad x \in D_2 \backslash D_\delta. \end{cases}$$

Therefore, using (4.2) and (4.5) we get

$$\tilde{r}(x, \varepsilon) = \begin{cases} I_a(x) + O(\varepsilon) & \text{for} \quad x \in D_1 \backslash (\Gamma_\delta \cup D_\delta), \\ I_b(x) + O(\varepsilon) & \text{for} \quad x \in D_2 \backslash D_\delta. \end{cases}$$

Thus, taking into account that δ is any small number we conclude that the reaction rate $\tilde{r}(x, \varepsilon)$ has a jump (transition layer) near the curve $\mathcal{C}$ of exchange of stabilities.

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