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An optimisation method in inverse acoustic scattering by an elastic obstacle

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Abstract

We consider the interaction between an elastic body and a compressible inviscid fluid, which occupies the unbounded exterior domain. The inverse problem of determining the shape of such an elastic scatterer from the measured far field pattern of the scattered fluid pressure field is of central importance in detecting and identifying submerged objects. Following a method proposed by Kirsch and Kress, we approximate the acoustic and elastodynamic wave by potentials over auxiliary surfaces, and we reformulate the inverse problem as an optimisation problem. The objective function to be minimised is the sum of three terms. The first is the deviation of the approximate far field pattern from the measured one, the second is a regularisation term, and the last a control term for the transmission condition. We prove that the optimisation problem has a solution and that, for the regularisation parameter tending to zero, the minimisers tend to a solution of the inverse problem. In contrast to a numerical method from a previous paper, the presented method does require neither a direct solution method nor an additional treatment of possible Jones modes.

1 Introduction

In this paper, we consider the interaction between an elastic body and a compressible inviscid fluid, which occupies the unbounded exterior domain. We suppose that a time-harmonic acoustic wave is incident upon the elastic target, and in the direct problem we are required to determine the incited elastic and the scattered acoustic wave. This leads to a transmission problem coupling the reduced elastodynamic (or Navier) equation inside the body with the Helmholtz equation in the exterior via the (smooth) interface Γ.

The inverse problem of determining the shape Γ of the elastic scatterer from a knowledge of the far field pattern of the scattered fluid pressure field is of central importance in detecting and identifying submerged objects. The efficient numerical solution of inverse problems of this type is challenging due to the fact that they are both nonlinear and severely ill-posed. We refer to [5, 3] for an overview on inverse problems and corresponding reconstruction methods for the Helmholtz and Maxwell equations. Recently, a first numerical method was investigated in [6] to solve the above-mentioned inverse fluid-solid interaction problem. The approach of [6] is based on the variational formulation of a modified forward problem, the reformulation of the inverse problem as an optimisation problem, and the use of finite element discretisations to find minimisers of the corresponding cost functional. Besides a regularisation term, the functional involves the least squares deviation of the measured data from the exact far field patterns corresponding to the optimal interface, and the numerical computation of minimisers is based on gradient formulas for the scattered field and a direct solver for the variational problem.

In this paper, we study an alternative reconstruction method, following an approach first developed by Kirsch and Kress [11] (see also [5], Chap. 5) for inverse acoustic scattering by a sound-soft obstacle, and later extended by Zinn [18] to the inverse acoustic transmission problem; see also Angell, Kleinman and Roach [2] for a closely related scheme. In this method, which does not require the solution of direct problems, the inverse problem is again reformulated as an optimisation problem. However, the scattered acoustic field is approximated by a potential with unknown density defined on an auxiliary surface, and the unknown interface is determined as a surface where the boundary conditions are fulfilled.

Our goal is to extend this method to the inverse fluid-solid interaction problem and to prove a corresponding convergence result. In Sect. 2 we recall basic solvability results for the direct scattering problem. In Sect. 3 we introduce the reconstruction method and state the convergence result. As in [11] and [18], the method splits the inverse problem into a linear ill-posed part to reconstruct the scattered pressure field and a nonlinear well-posed part to find the interface. The minimisation of the Tikhonov functional for the linear problem and the defect minimisation of the transmission conditions are then combined into one cost functional.

In Sect. 4 we discuss the details of the numerical discretisation of the optimisation problems. Note that, using Dirac δ-functionals as trial functions, no quadrature formula is needed. Furthermore, in contrast to the finite element based method of [6], the reconstruction method presented here is not affected by the possible occurrence of Jones modes (nontrivial solutions of the homogeneous forward problem) in the direct scattering problem, so that a regularisation with respect to the frequency can be avoided in such a case.

The remaining Sects. 5-7 are devoted to the proof of the convergence result (Theorem 3.2), which is complicated by the Jones modes. So, compared to the case of inverse boundary value problems for the Helmholtz equation, our convergence proof requires additional nontrivial solvability results on the direct interaction problem; see Sections 5 and 6. Note that the proofs for the 2D case and the 3D case are completely analogous. In order to simplify the notation we present the proofs in Sects. 5-7 for the 3D case only.

The implementation of the optimisation method and numerical results, including a comparison with the procedure of [6], will be reported in a forthcoming paper.

2 Direct scattering problem

Suppose $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary (i.e., $C^{1,\alpha}$). Either we suppose $d = 3$ and $\Omega \subset \mathbb{R}^3$ is a three-dimensional elastic body, or we suppose that $d = 2$ and $\Omega \subset \mathbb{R}^2$ is the cross section of a three-dimensional cylindrical body of infinite extension. This body is surrounded by a homogeneous compressible inviscid fluid filling the complementary exterior domain, i.e., either $d = 3$ and the fluid fills $\Omega^c = \mathbb{R}^3 \backslash cl(\Omega)$ or $d = 2$ and $\Omega^c = \mathbb{R}^2 \backslash cl(\Omega)$ is the cross section of the cylindrical fluid domain.

We denote by Γ the boundary of Ω and Ω^c. Assuming that the wave motion is time harmonic and that, for $d = 2$, the direction of the wave is contained in the cross-section plane, the direct scattering problem can be formulated in terms of the displacement field $u(x),\ x \in \Omega$, and the pressure function $p(x),\ x \in \Omega^c$, for the elastic structure and the fluid, respectively. The corresponding boundary value problem consists of the Navier and

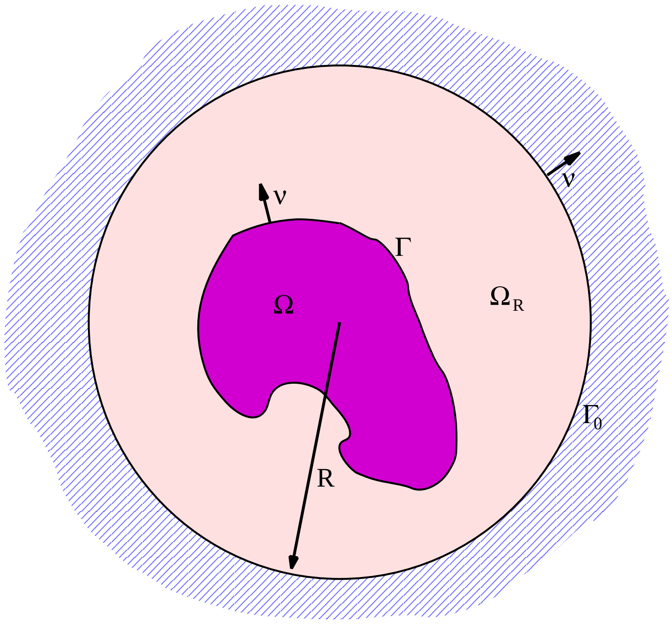

Figure 1: Domains.

Helmholtz equations

$$
\begin{aligned}
\mu\Delta u + (\lambda+\mu)\text{grad div } u + \varrho\omega^2 u = 0 \quad \text{in } \Omega\,, \\
\Delta p + k^2 p = 0 \quad \text{in } \Omega^c\,,
\end{aligned}
\tag{2.1}
$$

together with the transmission conditions

$$
\begin{aligned}
tu &= -p\nu \quad \text{on } \Gamma \\
u\cdot\nu &= \frac{1}{\varrho_f\omega^2}\,\partial_\nu p \quad \text{on } \Gamma\,,
\end{aligned}
\tag{2.2}
$$

and the Sommerfeld radiation condition for the scattered field $p^{sc} := p - p^{inc}$

$$
\partial_r p^{sc} - ikp^{sc} = o(r^{-(d-1)/2}) \quad \text{as } r\to\infty\,. \tag{2.3}
$$

Here μ and λ are the Lamé constants for the elastic material satisfying $\mu > 0$, $\lambda + \frac{2}{d}\mu > 0$; ϱ and ϱ_f are the densities of the elastic structure and the fluid; ω is the frequency, k the wave number defined by $k^2 = \omega^2/c^2$ with the sound speed c in the fluid; p^{inc} is the incident plane wave. The traction operator t on Γ is defined by

$$
tu := 2\mu\partial_\nu u + \lambda[\nabla\cdot u]\,n + \mu\left\{\begin{array}{ll}
\begin{pmatrix} n_2(\partial_{x_1}u_2 - \partial_{x_2}u_1) \\ n_1(\partial_{x_2}u_1 - \partial_{x_1}u_2)\end{pmatrix} & \text{if } d=2 \\
n\times[\nabla\times u] & \text{if } d=3,
\end{array}\right. \tag{2.4}
$$

where ν is the outward unit normal to Γ with respect to Ω.

There are various ways to reduce the transmission problem (2.1)-(2.3) to an equivalent nonlocal boundary problem on a bounded domain [13, 7, 6]. Here we follow [6] in spirit, using the Dirichlet-to-Neumann mapping for the Helmholtz equation on an artificial boundary Γ_0 and a strongly elliptic variational formulation of the problem (2.1)-(2.3) inside Γ_0.

In the following we assume that the origin lies in Ω, and Ω is contained in a ball resp. circle $\{x \in \mathbb{R}^d : |x| < R\}$ with boundary $\Gamma_0 := \{x \in \mathbb{R}^d : |x| = R\}$ (cf. Figure 1). Then the Helmholtz equation for p is solved in the annular domain $\Omega_R := \Omega^c \cap \{x \in \mathbb{R}^d : |x| < R\}$ with boundary $\Gamma \cup \Gamma_0$. Moreover, the radiation condition (2.3) can be written in the form

$$\partial_\nu p - Tp = h_0 \quad \text{on } \Gamma_0\,,\ h_0 := (\partial_\nu - T)p^{inc}\,, \tag{2.5}$$

where T denotes the Dirichlet-to-Neumann mapping $u|_{\Gamma_0} \mapsto \partial_\nu u|_{\Gamma_0}$ for the Helmholtz equation in the exterior of Γ_0; note that $\partial_\nu p^{sc} = Tp^{sc}$ on Γ_0. If $p|_{\Gamma_0}$ is given as a series in spherical harmonics, then (cf. [15], Sect. 2.6.3 and [4] for the details) Tp can be explicitly computed in terms of spherical harmonics (the eigenfunctions of T), and the linear operator

$$T : H^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0) \tag{2.6}$$

is continuous. Furthermore,

$$\Re\mathrm{e}\, \langle Tp, p\rangle_{\Gamma_0} \le 0 \quad \text{for all } p \in H^{1/2}(\Gamma_0)\,, \tag{2.7}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_0}$ denotes the usual duality pairing extending the L^2 scalar product on Γ_0, and $H^s(\Gamma_0)$ stands for the Sobolev space of order s on Γ_0. Introduce the energy space $\mathcal{H} := H^1(\Omega)^d \times H^1(\Omega_R)$ and its dual $\mathcal{H}' = H^{-1}(\Omega)^d \times H^{-1}(\Omega_R)$ with respect to the scalar product

$$\big\langle (u,p),(v,q)\big\rangle := \langle u,v\rangle_\Omega + \langle p,q\rangle_{\Omega_R} \tag{2.8}$$

in $L^2(\Omega)^d \times L^2(\Omega_R)$. Here, for a domain $D \subset \mathbb{R}^d$, we denote the dual of $H^1(D)$ by $H^{-1}(D)$, which differs from the standard notation used, e.g., in [8]. Integrating by parts and using (2.5) then leads to the variational formulation of problem (2.1)-(2.3):

Determine $(u,p) \in \mathcal{H}$ such that, for all $(v,q) \in \mathcal{H}$,

$$\begin{aligned} A(u,p;v,q) \;&:=\; a^*_\Omega(u,v) - \varrho\omega^2\langle u,v\rangle_\Omega + \langle p\nu, v\rangle_\Gamma \\ &\quad + a_{\Omega_R}(p,q) - k^2\langle p,q\rangle_{\Omega_R} + \varrho_f\omega^2\langle u\cdot\nu, q\rangle_\Gamma - \langle Tp,q\rangle_{\Gamma_0} \\ &=\; -\langle h_0, q\rangle_{\Gamma_0}\,. \end{aligned} \tag{2.9}$$

Here a^*_Ω and a_{Ω_R} denote the usual sesquilinear forms for the Lamé operator $\Delta^* := \mu\Delta + (\lambda+\mu)\mathrm{grad\,div}$ in Ω and the Laplace operator Δ in Ω_R, respectively. By (2.6), the sesquilinear form A generates a continuous linear operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}'$ via the formula $\langle \mathcal{A}(u,p),(v,q)\rangle = A(u,p;v,q)$ valid for all $(u,p),(v,q) \in \mathcal{H}$. Using Korn's inequality and (2.7), one obtains as in [6, Thm. 2.1] the strong ellipticity of the form A so that $\mathcal{A}$ is always a Fredholm operator with index zero. To ensure unique solvability of the variational equation

$$\mathcal{A}(u,p) = f\,,\ (u,p) \in \mathcal{H}\,,\ f \in \mathcal{H}' \tag{2.10}$$

for each right-hand side f, we need condition:

(C): There is no nontrivial solution of the problem
$$(\Delta^* + \varrho\omega^2)u = 0 \text{ in } \Omega\,, \quad tu = 0 \text{ and } u\cdot\nu = 0 \text{ on } \Gamma\,. \tag{2.11}$$

Nontrivial solutions of (2.11) are referred to as Jones modes, and the associated frequencies are called Jones frequencies. Jones modes may exist for balls and other axisymmetric bodies, but are "rare" in general (cf. [6] and the references therein). In the case that $\omega = \omega_0$ is a Jones frequency for the elastic obstacle Ω, we may pass to the subspaces

$$\widetilde{\mathcal{H}} := \Pi(\mathcal{H})\,,\ \widetilde{\mathcal{H}}' := \Pi(\mathcal{H}')$$

with the projection Π defined by

$$\Pi(g,h) := \big(g - \Sigma_j \langle u, u_j\rangle_\Omega u_j, h\big)\,,$$

where the sum is taken over the (finitely many) linearly independent normalised Jones modes u_j of $\mathcal{A}_{\omega_0}$, the operator of (2.10) corresponding to the frequency $\omega = \omega_0$. Then we have the invariance relations

$$(I - \Pi)\mathcal{A}_\omega \Pi = \Pi \mathcal{A}_\omega (I - \Pi) = 0\,,\ \forall \omega \in \mathbb{R} \tag{2.12}$$

and obtain the following invertibility results [6, Thms. 2.2 and 2.3].

Theorem 2.1 (i) *If condition* (C) *holds, then the operator* $\mathcal{A}_\omega : \mathcal{H} \to \mathcal{H}'$ *is invertible.*
(ii) *Suppose* ω_0 *is a Jones frequency. Then, for all* ω *with* $|\omega - \omega_0|$ *sufficiently small,* $\mathcal{A}_\omega : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}'$ *is invertible, and the inverse* $\mathcal{A}_\omega^{-1} : \widetilde{\mathcal{H}}' \to \mathcal{H}$ *is an analytic operator function in* ω *near* ω_0.

In particular, we observe that equation (2.9) has always a solution with unique pressure component p since its right-hand side is a special functional on $\mathcal{H}$ that is orthogonal to possible Jones mode solutions $(u_j, 0)$. We further note that the operators $\mathcal{A}_\omega$ in (ii) are even invertible as mappings of $\mathcal{H}$ onto $\mathcal{H}'$ for $\omega \neq \omega_0$ and $|\omega - \omega_0|$ sufficiently small. The proof of (ii) is based on the relations (2.12) and a Neumann series argument.

3 Inverse problem and reconstruction method

Let (u,p) be a solution of problem (2.9), the right-hand side of which is defined by the incident field p^{inc} via (2.5). Then the function p_∞ defined by the asymptotic relation (cp. with (2.3))

$$p^{sc}(x) = r^{-(d-1)/2} \exp(ikr)\big\{p_\infty(\hat{x}) + O(r^{-1})\big\}\,,\ \hat{x} = x/|x| \in \mathbb{S}^{d-1}\,, \quad \text{as } r \to \infty \tag{3.1}$$

is called the far field pattern of the scattered pressure field p^{sc}. Our goal in this paper is to study the inverse problem or the interface reconstruction problem.

(IP): Given the incident plane wave p^{inc}, determine the interface between the elastic body Ω and the fluid from a measured far field pattern $p_\infty^{\text{meas}} \in L^2(\mathbb{S}^{d-1})$.

Since (IP) is severely ill-posed and nonlinear, it is quite natural to apply regularisation and optimisation techniques. Suppose that we have the a priori information about our reconstruction problem that the unknown interface Γ lies between two closed smooth surfaces resp. curves $\Gamma_i \subset \Omega$ and $\Gamma_e \subset \Omega_R$, e.g., spheres resp. circles of centre O and radii

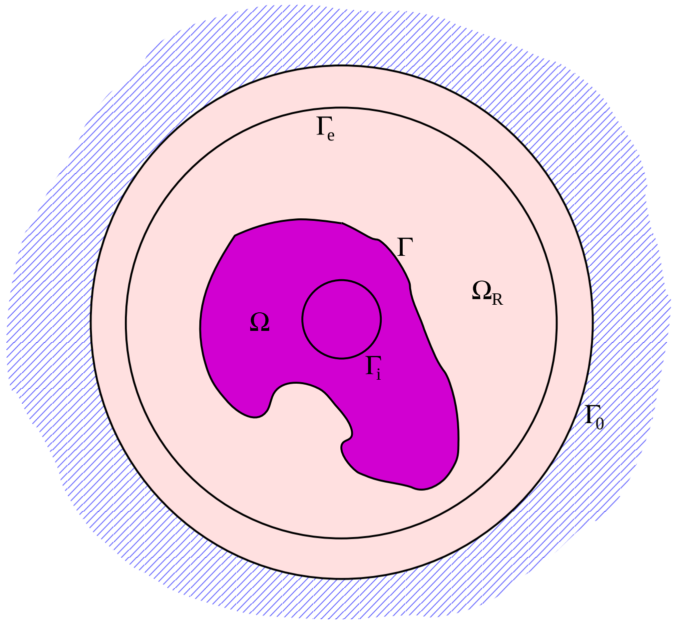

Figure 2: Auxiliary surfaces resp. curves.

r_i, r_e (cf. Figure 2). We will also need the following technical condition on the interior auxiliary surface resp. curve; see Sect. 5:

(D): k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Γ_i.

Note that this condition can be easily fulfilled, e.g., by slightly changing the radius r_i. Moreover, we fix a class of surfaces Γ in which a solution of (IP) is sought. We suppose that Γ is starlike, i.e., it can be represented as

$$\Gamma = \Gamma^{\mathbf{r}} = \left\{ \mathbf{r}(\hat{x})\hat{x} : \hat{x} \in \mathbb{S}^{d-1} \right\} \tag{3.2}$$

with $\mathbf{r} \in H^\delta(\mathbb{S}^{d-1})$ and a fixed order $\delta > (d+1)/2$. Furthermore, we also require that the interior and exterior auxiliary surfaces resp. curves Γ_i and Γ_e are starlike and given by the parameterisations $\mathbf{r}_i$, $\mathbf{r}_e \in H^\delta(\mathbb{S}^{d-1})$, respectively, and that

$$\mathbf{r}_i(\hat{x}) + \varepsilon \le \mathbf{r}(\hat{x}) \le \mathbf{r}_e(\hat{x}) - \varepsilon\,, \ \mathbf{r}_e(\hat{x}) < R\,, \tag{3.3}$$

for all $\hat{x} \in \mathbb{S}^{d-1}$ and some small $\varepsilon > 0$.

Finally, we choose a class $\mathcal{M}$ of admissible parameterisations to be the set of all $\mathbf{r} \in H^\delta(\mathbb{S}^{d-1})$ such that (3.3) holds and that, for some $c > 0$,

$$\|\mathbf{r}\|_{H^\delta(\mathbb{S}^{d-1})} \le c \tag{3.4}$$

is satisfied. Note that then $\mathcal{M}$ is weakly compact in $H^\delta(\mathbb{S}^{d-1})$, implying compactness in the norm of $C^{1,\beta}(\mathbb{S}^{d-1})$ for sufficiently small $\beta > 0$ because of $\delta > (d+1)/2$ and the compact imbedding $H^\delta(\mathbb{S}^{d-1}) \subset C^{1,\beta}(\mathbb{S}^{d-1})$. In the following, we shall simply write

$$\mathbf{r}_n \to \mathbf{r} \quad \text{or} \quad \Gamma_n = \Gamma^{\mathbf{r}_n} \to \Gamma^{\mathbf{r}} = \Gamma \quad \text{if} \quad \mathbf{r}_n \rightharpoonup \mathbf{r} \text{ weakly in } H^\delta(\mathbb{S}^{d-1})\,,\ n \to \infty\,. \tag{3.5}$$

Using the acoustic fundamental solution (cf. [5], [8])

$$G(x,y) := G(x,y;k) := \begin{cases} \dfrac{1}{4\pi}\,\dfrac{\exp(ik|x-y|)}{|x-y|} & \text{if } d=3 \\[2mm] \dfrac{i}{4}\,H_0^{(1)}(k|x-y|) & \text{if } d=2\,, \end{cases}$$

we define simple and double layer potentials on a closed $C^{1,\beta}$ surface resp. curve Λ by

$$(V_\Lambda p)(x) := \int_\Lambda p(y) G(x,y)\,\mathrm{d}s(y)\,,\ (K_\Lambda p)(x) := \int_\Lambda p(y)\frac{\partial G(x,y)}{\partial\nu(y)}\,\mathrm{d}s(y)\,. \tag{3.6}$$

For the elastic target, we use the fundamental Green's tensor (Kupradze matrix; cf. e.g. [10, 1, 9])

$$\left(G^{el}(x,y)\right)_{ij} := \frac{1}{\mu}\left(G(x,y;k_s)\delta_{ij} + \frac{1}{k_s^2}\,\frac{\partial^2}{\partial x_i \partial x_j}\left(G(x,y;k_s) - G(x,y;k_p)\right)\right)$$

where the wave numbers k_p and k_s are defined by $\varrho\omega^2 = (\lambda+2\mu)k_p^2 = \mu k_s^2$. Then we define elastic simple and double layer potentials on Λ by

$$\left(V_\Lambda^{el} u\right)(x) := \int_\Lambda G^{el}(y,x)u(y)\,\mathrm{d}s(y)\,,\ \left(K_\Lambda^{el} u\right)(x) := \int_\Lambda [t_y G^{el}(y,x)]^\top u(y)\,\mathrm{d}s(y)\,, \tag{3.7}$$

where t_y means that the traction operator (2.4) is applied at $y \in \Lambda$. We try to represent the elastic field u inside Γ_e respectively the scattered pressure field p^{sc} outside Γ_i as simple layer potentials

$$u(x) = (V_{\Gamma_e}^{el}\varphi_e)(x)\,, \quad p^{sc}(x) = (V_{\Gamma_i}\varphi_i)(x) \tag{3.8}$$

with unknown density functions $\varphi_e \in L^2(\Gamma_e)^d$, $\varphi_i \in L^2(\Gamma_i)$.

Next, we introduce the far field operator $F : L^2(\Gamma_i) \to L^2(\mathbb{S}^{d-1})$ by

$$\begin{aligned} (F\varphi)(\hat{x}) &:= c_{ff}\int_{\Gamma_i} \exp(-ik\,\hat{x}\cdot y)\varphi(y)\,\mathrm{d}s(y)\,,\ \hat{x}\in\mathbb{S}^{d-1}\,,\ \varphi \in L^2(\Gamma_i) \\ c_{ff} &:= \begin{cases} \dfrac{1}{4\pi} & \text{if } d=3 \\[2mm] -\dfrac{\exp(i\pi/4)}{\sqrt{8\pi k}} & \text{if } d=2 \end{cases} \end{aligned} \tag{3.9}$$

which has an analytic kernel. Note that $F\varphi$ is the far field pattern of the potential $V_{\Gamma_i}\varphi$ (cf. [5]). In other words, $F\varphi_i(\hat{x})\,,\ \hat{x}\in\mathbb{S}^{d-1}$ approximates the far field pattern of p^{sc}, whereas $(V_{\Gamma_i}\varphi_i)(x)\,,\ (V_{\Gamma_e}^{el}\varphi_e)(x)\,,\ x\in\Gamma$ represent approximations of the scattered pressure field p^{sc} and the elastic field u on $\Gamma = \Gamma^{\mathbf{r}}$, respectively; compare (3.2) and (3.8). Clearly, we can identify the spaces $L^2(\Gamma^{\mathbf{r}})$ with $L^2(\mathbb{S}^{d-1})$ via

$$\|u\|_{L^2(\Gamma^{\mathbf{r}})} = \|u\circ\hat{\mathbf{r}}\|_{L^2(\mathbb{S}^{d-1})}\,,\ u\in L^2(\Gamma^{\mathbf{r}})\,,$$

where $\hat{\mathbf{r}}$ denotes the diffeomorphism $\mathbb{S}^{d-1} \to \Gamma^{\mathbf{r}}$ defined by $\hat{\mathbf{r}}(\hat{x}) := \mathbf{r}(\hat{x})\hat{x}$, $\hat{x}\in\mathbb{S}^{d-1}$. Obviously, the norm defined above is uniformly equivalent to the standard L^2 norm when $\mathbf{r}$ varies in an admissible set of parameterisations.

Since the operator (3.9) is compact with exponentially decreasing singular values, the determination of the density φ_i from the first kind equation

$$F\varphi_i = p_\infty^{\text{meas}} \tag{3.10}$$

is a severely ill-posed problem. We may solve it by Tikhonov regularisation, and subsequently we could determine φ_e and the interface $\Gamma^{\mathbf{r}}$ by the transmission conditions (2.2). However, this is not satisfactory since the density φ_e does not appear in (3.10), but the far field p_∞^{meas} depends implicitly on the interior elastic field. Therefore, as in [18] it seems to be the right way to include φ_e into the regularisation procedure. So, for the approximate solution of our inverse problem (IP), we will formulate a nonlinear optimisation problem which incorporates these observations.

We define the cost functional $\mathcal{F} : L^2(\Gamma_i) \times L^2(\Gamma_e)^d \times \mathcal{M} \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{F}(\varphi_i, \varphi_e, \mathbf{r}; \alpha) \;:=\; & \|F\varphi_i - p_\infty^{\text{meas}}\|^2_{L^2(\mathbb{S}^{d-1})} + \alpha\Big(\|\varphi_i\|^2_{L^2(\Gamma_e)} + \|\varphi_e\|^2_{L^2(\Gamma_i)^d}\Big) \\ & + \varrho_1 \left\| \Big(tV^{el}_{\Gamma_e}\varphi_e + (p^{inc} + V_{\Gamma_i}\varphi_i)\nu\Big) \circ \hat{\mathbf{r}} \right\|^2_{L^2(\mathbb{S}^{d-1})^d} \\ & + \varrho_2 \left\| \Big(\nu \cdot V^{el}_{\Gamma_e}\varphi_e - \frac{1}{\varrho_f \omega^2}\partial_\nu(p^{inc} + V_{\Gamma_i}\varphi_i)\Big) \circ \hat{\mathbf{r}} \right\|^2_{L^2(\mathbb{S}^{d-1})},\end{aligned} \tag{3.11}$$

where $\mathcal{M}$ is an admissible class of parameterisations, $\alpha > 0$ is the regularisation parameter, and ϱ_1, $\varrho_2 > 0$ are coupling parameters which have to be chosen approximately for the numerical implementation. For theoretical purposes, we may assume $\varrho_1 = \varrho_2 = 1$ in the sequel. The first and second parts of (3.11) represent the Tikhonov regularisation of (3.10), where an additional regularisation term for φ_e is incorporated. The last two parts represent the defect minimisation of the transmission conditions on $\Gamma^{\mathbf{r}}$.

Our reconstruction method, which was first introduced by Kirsch and Kress [11, 5] in the case of acoustic scattering by a sound-soft obstacle (see also Zinn [18] who studied the inverse acoustic transmission problem), consists in solving the following optimisation problem:

(OP): Find $(\varphi_i, \varphi_e) \in L^2(\Gamma_i) \times L^2(\Gamma_e)^d$ and $\mathbf{r} \in \mathcal{M}$ such that $\mathcal{F}(\varphi_i, \varphi_e, \mathbf{r}; \alpha) = m(\alpha)$,
$m(\alpha) := \inf\big\{\mathcal{F}(\psi_i, \psi_e, \mathbf{r}'; \alpha) : (\psi_i, \psi_e) \in L^2(\Gamma_i) \times L^2(\Gamma_e)^d,\ \mathbf{r}' \in \mathcal{M}\big\}$.

The existence of a minimiser is guaranteed by the following result.

Theorem 3.1 *For each $\alpha > 0$, the problem* (OP) *has a solution.*

Since the proof is analogous to that of [5, Thm. 5.20], we will only present its main steps in Sect. 7 for the reader's convenience.

Theorem 3.2 *Assume condition* (D) *is satisfied. Let p_∞^{meas} be the exact far field pattern of the scattered field p^{sc} corresponding to some $\Gamma^{\mathbf{r}}$, $\mathbf{r} \in \mathcal{M}$. Then we have:*

(i) $\lim_{\alpha\to 0} m(\alpha) = 0$, *i.e., convergence of the cost functional.*

(ii) *Let (α_n) be a null sequence and let $(\varphi_i^{(n)}, \varphi_e^{(n)}, \mathbf{r}_n)$ be a corresponding sequence of minimisers of* (3.11) *with regularisation parameter α_n. Then there exists a convergent subsequence of $(\mathbf{r}_n)$ in the sense of* (3.5), *and every limit point $\mathbf{r}_*$ of $(\mathbf{r}_n)$ represents a solution $\Gamma^{\mathbf{r}_*}$ of problem* (IP).

The proof of this theorem will also be given in Sect. 7.

Remark 3.3 *If we have the a priori information that our inverse problem* (IP) *has at most one solution, then Theorem 3.2* (ii) *implies convergence of the whole sequence* $(\mathbf{r}_n)$ *to* $\mathbf{r}$*. However, uniqueness in the inverse problem is presently only known for infinitely many incident waves; see* [14].

One can try to achieve uniqueness and more accurate reconstructions by replacing the cost functional (3.11) *by a sum corresponding to several incident waves with different incident directions, and the preceding theorems carry over to this case.*

Moreover, Theorem 3.2 remains valid if, for any $\alpha > 0$*, we replace exact far field patterns* p_∞^{meas} *by noisy measurement data* $\tilde{p}_\infty^\alpha$ *such that* $\|\tilde{p}_\infty^\alpha - p_\infty^{\text{meas}}\|^2_{L^2(\mathbb{S}^{d-1})} \le c\alpha$, $\alpha \to 0$.

Remark 3.4 *Theorems 3.1 and 3.2 carry over to the case that the second term of the cost functional* (3.11) *is replaced, e.g., by*

$$\alpha\left(\|\varphi_i\|^2_{H^{-(d-1)}(\mathbb{S}^{d-1})} + \|\varphi_e\|^2_{H^{-(d-1)}(\mathbb{S}^{d-1})^d}\right) \tag{3.12}$$

An inspection of the corresponding proofs shows that only a somewhat higher smoothness, $r \in H^{\delta+d-1}(\mathbb{S}^{d-1})$*, is required in* (3.2), (3.4) *in order to prove* (7.6) *for the second resp. first order derivatives of* K_n *and thus estimate* (7.5) *in the* $H^{-(d-1)}$ *norm. The modified cost functional* (3.11), (3.12) *then allows the use of linear combinations of Dirac* δ*-functionals on the auxiliary surfaces resp. curves* Γ_i *and* Γ_e *for the approximation of the densities* φ_i *and* φ_e *and simplifies the discretisation of the optimisation problem* (OP).

Remark 3.5 *Of course, the accuracy of the field approximations* (3.8) *and the resulting reconstruction of the interface depends on the location of* Γ_e *and* Γ_i*. Choosing the inner and outer surfaces resp. curves closer to the unknown interface will surely enhance the convergence of our iterative method. Hence, it is natural to change* Γ_e *and* Γ_i *during the iterative process and to move them closer to the iterative solution (cf.* [17]*). However, we did not try to prove convergence for such a modification.*

4 Numerical discretisation of the optimisation problem

In this section we introduce a discretisation of the optimisation problem (OP) and give some remarks on its numerical solution. Recall that the unknown boundary $\Gamma = \Gamma^{\mathbf{r}}$ is sought in the class of all $\mathbf{r} \in H^{\delta+d-1}(\mathbb{S}^{d-1})$ such that $\mathbf{r}_i(\hat{x}) + \varepsilon \le \mathbf{r}(\hat{x}) \le \mathbf{r}_e(\hat{x}) - \varepsilon$ (cf. (3.3)) for a fixed small ε with $2\varepsilon < \sup_{\hat{x}\in\mathbb{S}^{d-1}}[\mathbf{r}_e(\hat{x}) - \mathbf{r}_i(\hat{x})]$. Hence, it is natural to seek an approximation $\mathbf{r}_N$ of $\mathbf{r}$ in the form

$$\mathbf{r}_N(\hat{x}) := \frac{[\mathbf{r}_e(\hat{x}) - \varepsilon] + [\mathbf{r}_i(\hat{x}) + \varepsilon]}{2} + \frac{[\mathbf{r}_e(\hat{x}) - \varepsilon] - [\mathbf{r}_i(\hat{x}) + \varepsilon]}{\pi} \arctan\left(\sum_{\iota\in I_N} a_\iota \psi_\iota(\hat{x})\right), \tag{4.1}$$

where $\psi_\iota \in H^{\delta+d-1}(\mathbb{S}^{d-1})$ are basis functions and $a_\iota \in \mathbb{R}$ are unknown coefficients. The index ι runs through an index set I_N which we shall specify below.

Next, for an index κ from a second index set K_M, we introduce the Dirac δ point functional $\delta_{i,\kappa}$ at a point $x_{i,\kappa} \in \Gamma_i$, which will be defined below. Similarly, let $\delta_{e,\kappa},\ \kappa \in K_M$ denote the Dirac δ point functional $\delta_{e,\kappa}$ at a point $x_{e,\kappa} \in \Gamma_e$. Then we can approximate the layer functions φ_i and φ_e by

$$\varphi_{i,M} := \sum_{\kappa \in K_M} b_\kappa \delta_{i,\kappa},\ b_\kappa \in \mathbb{C}, \quad \varphi_{e,M} := \sum_{\kappa \in K_M} c_\kappa \delta_{e,\kappa},\ c_\kappa \in \mathbb{C}^d. \tag{4.2}$$

In view of (3.8), we obtain the following approximations for the fields p^{sc} and u:

$$\begin{aligned} p^{sc}_M(x) &:= \sum_{\kappa \in K_M} b_\kappa G(x, x_{i,\kappa}), \quad p_{\infty,M}(\hat{x}) := c_{ff} \sum_{\kappa \in K_M} b_\kappa e^{ik\,\hat{x}\cdot x_{i,\kappa}}, \\ u_M(x) &:= \sum_{\kappa \in K_M} G^{el}(x_{e,\kappa}, x) c_\kappa. \end{aligned}$$

Using a set of appropriate points $\{x_{\mathbb{S}^{d-1},\kappa'},\ \kappa' \in K_{M'}\}$ on $\mathbb{S}^{d-1}$, we approximate the L^2 norms over $\mathbb{S}^{d-1}$ and $\Gamma^{\mathbf{r}}$ by

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{S}^{d-1})} &\sim \frac{1}{\#K_{M'}} \sum_{\kappa' \in K_{M'}} |f(x_{\mathbb{S}^{d-1},\kappa'})|^2, \quad \#K_{M'} := \sum_{\kappa \in K_{M'}} 1, \\ \|f \circ \mathbf{r}_N\|^2_{L^2(\mathbb{S}^{d-1})} &\sim \frac{1}{\#K_{M'}} \sum_{\kappa' \in K_{M'}} \left|f\left(x_{\mathbf{r},\kappa'}\right)\right|^2, \qquad x_{\mathbf{r},\kappa} := \mathbf{r}_N(x_{\mathbb{S}^{d-1},\kappa}). \end{aligned}$$

Finally, we approximate the norm of the approximate layer $\varphi_{i,M}$ by a coarse discretisation of $\|S\varphi_{i,M}\|_{L^2}$, where S is an integral operator, to be chosen, mapping $H^{-(d-1)}(\Gamma_i)$ into $L^2(\Gamma_i)$ isomorphically. Similarly, we treat the norm of the approximate layers $\varphi_{e,M}$ and get

$$\begin{aligned} \|\varphi_{i,M}\|^2_{H^{-(d-1)}(\Gamma_i)} &\sim \frac{1}{\#K_M} \sum_{\kappa' \in K_M} \Big| \sum_{\kappa \in K_M} \sigma_{\kappa',\kappa} b_\kappa \Big|^2, \\ \|\varphi_{e,M}\|^2_{H^{-(d-1)}(\Gamma_i)^d} &\sim \frac{1}{\#K_M} \sum_{\kappa' \in K_M} \Big\| \sum_{\kappa \in K_M} \sigma_{\kappa',\kappa} c_\kappa \Big\|^2_{\mathbb{C}^d}, \end{aligned}$$

where $\sigma_{\kappa',\kappa}$ is the kernel value of the integral operator S taken at the points $x_{i,\kappa}$ and $x_{i,\kappa'}$. Altogether we arrive at the discretised objective functional

$$\mathcal{F}_{N,M,M'}\Big(\varphi_{i,M}, \varphi_{e,M}, \mathbf{r}_N; \alpha\Big) := \mathcal{F}_{N,M,M'}\Big((b_\kappa)_{\kappa \in K_M}, (c_\kappa)_{\kappa \in K_M}, (a_\iota)_{\iota \in I_N}; \alpha\Big) \tag{4.3}$$

$$\begin{aligned} := \ & \frac{1}{\#K_{M'}} \sum_{\kappa' \in K_{M'}} \Big| c_{ff} \sum_{\kappa \in K_M} b_\kappa e^{ik\,x_{\mathbb{S}^{d-1},\kappa'}\cdot x_{i,\kappa}} - p^{\text{meas}}_\infty(x_{\mathbb{S}^{d-1},\kappa'}) \Big|^2 \\ & + \frac{\alpha}{\#K_M} \sum_{\kappa' \in K_M} \Big| \sum_{\kappa \in K_M} \sigma_{\kappa',\kappa} b_\kappa \Big|^2 + \frac{\alpha}{\#K_M} \sum_{\kappa' \in K_M} \Big\| \sum_{\kappa \in K_M} \sigma_{\kappa',\kappa} c_\kappa \Big\|^2_{\mathbb{C}^d} \end{aligned}$$

$$+\frac{1}{\#K_{M'}}\sum_{\kappa'\in K_{M'}}\Bigg\|\sum_{\kappa\in K_M} t_{x_{\mathbf{r},\kappa'}}[G^{el}\big(x_{e,\kappa},x_{\mathbf{r},\kappa'}\big)\,c_\kappa] \;-$$

$$\Bigg[p^{inc}\big(x_{\mathbf{r},\kappa'}\big)+\sum_{\kappa\in K_M} b_\kappa G\big(x_{\mathbf{r},\kappa'},x_{i,\kappa}\big)\Bigg]\,\nu\big(x_{\mathbf{r},\kappa'}\big)\Bigg\|^2_{\mathbb{C}^d}$$

$$+\frac{1}{\#K_{M'}}\sum_{\kappa'\in K_{M'}}\Bigg|\sum_{\kappa\in K_M}\nu(x_{\mathbf{r},\kappa'})\cdot G^{el}\big(x_{e,\kappa},x_{\mathbf{r},\kappa'}\big)\,c_\kappa \;-$$

$$\frac{1}{\varrho_f\omega^2}\Bigg[\partial_\nu p^{inc}\big(x_{\mathbf{r},\kappa'}\big)+\sum_{\kappa\in K_M} b_\kappa\partial_{\nu(x_{\mathbf{r},\kappa'})}G\big(x_{\mathbf{r},\kappa'},x_{i,\kappa}\big)\Bigg]\Bigg|^2$$

for the approximation of functional $\mathcal{F}$ defined in (3.11).

To define the missing functions, points, indices, and kernel values, we have to distinguish between the dimensions $d = 2$ and $d = 3$ of the problem. We begin with the simpler case $d = 2$. In this case, the basis functions ψ_ι are simply the trigonometric functions. More precisely, we set $I_N := \{\iota = (n,p):\ n = 1,2,\ldots,N,\ p = \pm\} \cup \{\iota = 0\}$ and define $\psi_0(\hat{x}) := 1$ as well as

$$\psi_{(n,p)}\left(e^{i2\pi t}\right) \;:=\; \begin{cases}\cos\left(2\pi nt\right) & \text{if } p=+\\ \sin\left(2\pi nt\right) & \text{if } p=-\end{cases}$$

The points $x_{e,m}$ and $x_{i,m}$ for the Dirac δ-functionals are nothing else than uniform grid points on Γ. In other words, we set $K_M := \{\kappa = m:\ m = 0,1,\ldots,M-1\}$ and

$$\begin{aligned} x_{i,\kappa} \;=\; x_{i,m} \;&:=\; \mathbf{r}_i\Big((\cos\frac{2\pi m}{M},\sin\frac{2\pi m}{M})\Big)\,(\cos\frac{2\pi m}{M},\sin\frac{2\pi m}{M}),\\ x_{e,\kappa} \;=x_{e,m} \;&:=\; \mathbf{r}_e\Big((\cos\frac{2\pi m}{M},\sin\frac{2\pi m}{M})\Big)\,(\cos\frac{2\pi m}{M},\sin\frac{2\pi m}{M}).\end{aligned}$$

Similarly, we introduce $K_{M'} := \{\kappa' = m:\ m = 0,1,\ldots,M'-1\}$ and define $x_{\mathbb{S}^1,\kappa'} = x_{\mathbb{S}^1,m} := (\cos(2\pi m/M'),\sin(2\pi m/M'))$. The integral operator $S:\ H^{-1}(\Gamma)\to L^2(\Gamma)$ is chosen as the harmonic single layer operator over the circle, i.e., we set $\sigma_{\kappa',\kappa} = \sigma_{m',m} := \sigma_{m'-m}$ with $\sigma_0 = 0$ and $\sigma_n := \log\sin^2(\pi\, n/M)$ for $n \neq 0$.

For the case $d = 3$, the trial functions ψ_ι will be spherical harmonics. Thus we choose $I_N := \{\iota = (n,m):\ n = 0,1,\ldots,N,\ m = -n,\ldots,n\}$ and set

$$\psi_{(n,m)}\left(\cos\varphi\sin\theta,\sin\varphi\sin\theta,\cos\theta\right) := \sqrt{\tfrac{2n+1}{4\pi}\tfrac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)\begin{cases}\sin\left(m\varphi\right) & \text{if } m>0\\ \cos\left(|m|\varphi\right) & \text{else,}\end{cases}$$

$$P_n^{|m|}(t) := (1-t^2)^{|m|/2}\frac{\mathrm{d}^{|m|}P_n(t)}{\mathrm{d}t^{|m|}},\qquad P_n(\cos\theta) \;:=\; \sin^n(\theta).$$

In order to define the points on the surfaces Γ_i, Γ_e, $\mathbb{S}^2$, we inscribe a cube into the ball, define uniform tensor product grids on the six faces of the cube, and map them to the sphere by stereographic projection. In other words, we introduce the index set

$K_M := \{\kappa = (m_1, m_2, n) : \ m_1, m_2 = 0, 1, \dots, M-1, \ n = 1, 2, \dots, 6\}$ and choose

$$\begin{aligned}
x_{\mathbb{S}^2,\kappa} &= x_{\mathbb{S}^2,(m_1,m_2,n)} := pr_n\Big(\frac{m_1+0.5}{M}, \frac{m_2+0.5}{M}\Big), \\
x_{i,\kappa} &= x_{i,(m_1,m_2,n)} := \mathbf{r}_i\left(pr_n\Big(\frac{m_1+0.5}{M}, \frac{m_2+0.5}{M}\Big)\right) pr_n\Big(\frac{m_1+0.5}{M}, \frac{m_2+0.5}{M}\Big), \\
x_{e,\kappa} &= x_{e,(m_1,m_2,n)} := \mathbf{r}_e\left(pr_n\Big(\frac{m_1+0.5}{M}, \frac{m_2+0.5}{M}\Big)\right) pr_n\Big(\frac{m_1+0.5}{M}, \frac{m_2+0.5}{M}\Big), \\
pr_n(s,t) &:= \begin{cases}
\frac{(-1+2s,-1+2t,-1)}{\sqrt{(-1+2s)^2+(-1+2t)^2+1}} & \text{if } n=1 \\
\frac{(-1+2s,-1+2t,1)}{\sqrt{(-1+2s)^2+(-1+2t)^2+1}} & \text{if } n=2 \\
\frac{(-1,-1+2t,-1+2s)}{\sqrt{1+(-1+2t)^2+(-1+2s)^2}} & \text{if } n=3 \\
\frac{(-1+2s,-1,-1+2t)}{\sqrt{(-1+2s)^2+1+(-1+2t)^2}} & \text{if } n=4 \\
\frac{(-1+2s,1,-1+2t)}{\sqrt{(-1+2s)^2+1+(-1+2t)^2}} & \text{if } n=5 \\
\frac{(1,-1+2t,-1+2s)}{\sqrt{1+(-1+2t)^2+(-1+2s)^2}} & \text{if } n=6\,.
\end{cases}
\end{aligned}$$

The integral operator $S : \ H^{-2}(\Gamma) \to L^2(\Gamma)$ is chosen as the operator over the sphere with logarithmic kernel, i.e., we set $\sigma_{\kappa',\kappa} = \log(|x_{\mathbb{S}^2,\kappa'} - x_{\mathbb{S}^2,\kappa}|/2)$.

Typically, one defines the objective functional $\mathcal{F}_{N,M,M'}$ in (4.3) with smaller numbers N and M and larger M'. Indeed a few basis functions in (4.3) and a few source points in (4.2) should be sufficient for a good approximation of the continuous functions, whereas the control terms for the transmission condition and the far field deviation should be discretised with higher resolutions. The discretisation of the optimisation problem (OP) takes the form

(DOP): Find coefficients $(b_\kappa)_{\kappa\in K_M}, \ (c_\kappa)_{\kappa\in K_M} \in \mathbb{C}^{\#K_M}, \ (a_\iota)_{\iota\in I_N} \in \mathbb{R}^{\#I_N}$ such that

$$\mathcal{F}_{N,M,M'}\Big((b_\kappa)_{\kappa\in K_M}, (c_\kappa)_{\kappa\in K_M}, (a_\iota)_{\iota\in I_N}; \alpha\Big) = m_{N,M,M'}(\alpha) \text{ with}$$

$$m_{N,M,M'}(\alpha) := \inf \Big\{ \mathcal{F}_{N,M,M'}\Big((\tilde{b}_\kappa)_{\kappa\in K_M}, (\tilde{c}_\kappa)_{\kappa\in K_M}, (\tilde{a}_\iota)_{\iota\in I_N}; \alpha\Big) : (\tilde{b}_\kappa)_{\kappa\in K_M} \in \mathbb{C}^{\#K_M}, (\tilde{c}_\kappa)_{\kappa\in K_M} \in \mathbb{C}^{\#K_M}, (\tilde{a}_\iota)_{\iota\in I_N} \in \mathbb{R}^{\#I_N} \Big\}.$$

The objective functional in (DOP) is like that of (OP) a non-linear smooth functional. Constraints have been avoided by the special representation of $\mathbf{r}_N$. A global solution of (DOP) could be computed by stochastic algorithms like simulated annealing (cf. e.g. [12]). We recommend to use faster local algorithms providing only local minima. In particular the Gauß-Newton algorithm or the Levenberg-Marquardt method (cf. e.g. [16]) are good candidates since the gradient of (4.3) is easy to determine. The regularisation parameter α is to be adapted by numerical experiments.

5 A denseness result

To prove Theorem 3.2, we need the following denseness result which justifies the ansatz (3.8) and the choice of the cost functional (3.11). Introduce the matrix boundary integral operator $\mathcal{B}$ defined by

$$\mathcal{B} := \begin{pmatrix} t_x V^{el}_{\Gamma_e} & \nu\, V_{\Gamma_i} \\ \nu \cdot V^{el}_{\Gamma_e} & -\dfrac{1}{\varrho_f \omega^2}\, \partial_\nu V_{\Gamma_i} \end{pmatrix}, \tag{5.1}$$

where we used the notation of Sections 2 and 3. Then

$$\mathcal{B} : L^2(\Gamma_e)^3 \times L^2(\Gamma_i) \to L^2(\Gamma)^3 \times L^2(\Gamma)$$

is a continuous mapping, and its transpose

$$\mathcal{B}' : L^2(\Gamma)^3 \times L^2(\Gamma) \to L^2(\Gamma_e)^3 \times L^2(\Gamma_i)$$

takes the form (cf. (3.6), (3.7))

$$\mathcal{B}' \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} (t_x V^{el}_{\Gamma_e})' & (\nu \cdot V^{el}_{\Gamma_e})' \\ (\nu V_{\Gamma_i})' & -\frac{1}{\varrho_f \omega^2}\, (\partial_\nu V_{\Gamma_i})' \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} K^{el}_{\Gamma} \varphi + V^{el}_{\Gamma}(\nu\psi) \\ V_\Gamma(\nu \cdot \varphi) - \frac{1}{\varrho_f \omega^2}\, K_\Gamma \psi \end{pmatrix} \tag{5.2}$$

Theorem 5.1 *Let* $\mathbf{r} \in \mathcal{M}$*. If condition* (C) *holds for* $\Gamma = \Gamma^{\mathbf{r}}$ *and* Γ_i *satisfies condition* (D)*, then the image space* $\operatorname{im}\mathcal{B}$ *of the operator* (5.1) *is dense in* $L^2(\Gamma)^3 \times L^2(\Gamma)$*.*

Proof. It is sufficient to verify that the relation

$$\mathcal{B}' \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0\,, \ \varphi \in L^2(\Gamma)^3\,, \ \psi \in L^2(\Gamma) \tag{5.3}$$

implies $\varphi = 0$, $\psi = 0$. To do this, for φ, ψ being any solution of (5.3), we define the functions u and p in $\mathbb{R}^3 \backslash \Gamma$ by setting

$$u := V^{el}_{\Gamma}(-\nu\psi) - K^{el}_{\Gamma}\varphi\,, \quad p := K_\Gamma \psi - \varrho_f \omega^2 V_\Gamma(\nu \cdot \varphi)\,, \tag{5.4}$$

which satisfy the Navier resp. Helmholtz equation in $\mathbb{R}^3 \backslash \Gamma$. By well known mapping properties of acoustic and elastic potentials we have $u \in H^1(\Omega \cup \Omega_R)^3$, $p \in H^1(\Omega \cup \Omega_R)$, and p satisfies the radiation condition (2.3), whereas u satisfies Kupradze's radiation condition (see [10]) for the elastic field. Moreover, using the jump relations for these potentials (see, e.g., [8] for the details), it follows from (5.4) that

$$[u]_\Gamma = -\varphi\,, \ [tu]_\Gamma = \nu\psi\,, \ [p]_\Gamma = \psi\,, \ [\partial_\nu p]_\Gamma = \varrho_f \omega^2 \nu \cdot \varphi\,, \tag{5.5}$$

where $[u]_\Gamma$ stands for the jump of u across Γ:

$$[u]_\Gamma(x) = u^-(x) - u^+(x) := \lim_{h \to 0} \{u(x + h\nu(x)) - u(x - h\nu(x))\}\,, \ x \in \Gamma\,.$$

Let Ω_i be the interior of Γ_i and Ω_e the exterior of Γ_e. From the relations (5.2)-(5.4) we get

$$u = 0 \text{ on } \Gamma_e\,, \quad p = 0 \text{ on } \Gamma_i\,. \tag{5.6}$$

Therefore, condition (D) implies $p = 0$ in Ω_i, hence $p = 0$ in Ω. Moreover, by (5.6) and the uniqueness of the exterior first boundary problem for the Navier equation in Ω_e (cf. [10, 8]), we get $u = 0$ in Ω_e, hence $u = 0$ in Ω^c. Thus, (5.5) can be rewritten as

$$u^+ = \varphi\,,\ tu^+ = -\nu\psi\,,\ p^- = \psi\,,\ \partial_\nu p^- = \varrho_f\omega^2\nu\cdot\varphi\,, \tag{5.7}$$

giving $tu^+ = -\nu p^-$ and $\nu\cdot u^+ = (\varrho_f\omega^2)^{-1}\partial_\nu p^-$, i.e., the transmission conditions (2.2). Thus (u, p) is a solution of the homogeneous scattering problem (2.10) (with $f = 0$), and Theorem 2.1 (i) implies that $u = 0$ in Ω, $p = 0$ in Ω^c. Applying the jump relations (5.5) again, we finally obtain $\varphi = 0$ and $\psi = 0$. □

If condition (C) is not satisfied for $\Gamma = \Gamma^{\mathbf{r}}$, then Theorem 5.1 does not hold. However, it is enough for our purposes to show that $\operatorname{im}\mathcal{B}$ is dense in an appropriate subspace defined by

$$\widetilde{L} := \{(\varphi,\psi)\in L^2(\Gamma)^3\times L^2(\Gamma) : \langle\varphi, u_j\rangle_\Gamma = 0\,,\ j = 1,\ldots,I\}\,, \tag{5.8}$$

where $\operatorname{span}\{u_j\}$ is the null space of problem (2.11).

Corollary 5.2 *Let $\mathbf{r}\in\mathcal{M}$ and assume that Γ_i satisfies condition* (D)*. Then the image space of* (5.1) *is dense in $\widetilde{L}$.*

Proof. Clearly, the inclusion $\widetilde{L} = \operatorname{span}\{(u_j|_\Gamma, 0)\}^\perp \subseteq cl(\operatorname{im}\mathcal{B})$ follows if we prove $\ker\mathcal{B}' \subseteq \widetilde{L}^\perp = \operatorname{span}\{(u_j|_\Gamma, 0)\}$. Thus we take $\chi_0 = (\varphi,\psi)\in L^2(\mathbb{S})^3\times L^2(\mathbb{S})$ and suppose $\mathcal{B}'\chi_0 = 0$. Repeating the proof of Theorem 5.1, we obtain relations (5.5)-(5.7) for the function (u, p) defined by (5.4). Moreover, from Theorem 2.1 (ii) we see that (u, p) must be a Jones mode solution, i.e., we have $p = 0$ and $u\in\operatorname{span}\{u_j\}$. From (5.7) we infer $\psi = 0$ and $\varphi\in\operatorname{span}\{u_j|_\Gamma\}$, i.e., $\chi_0\in\widetilde{L}^\perp$. □

6 Continuous dependence of direct solutions on the interface

Another crucial auxiliary result in the proof of Theorem 3.2 is the continuous dependence of the solutions to certain inhomogeneous transmission problems of the form (2.10) on the interface $\Gamma = \Gamma^{\mathbf{r}}$ if the parameterisation $\mathbf{r}$ varies in an admissible class $\mathcal{M}$.

Let ω be a fixed frequency and assume that $\Gamma_n := \Gamma^{\mathbf{r}_n} \to \Gamma := \Gamma^{\mathbf{r}}$ in the sense of (3.5), i.e., $\mathbf{r}_n \rightharpoonup \mathbf{r}$ in $H^\delta(\mathbb{S}^2)$. Introduce the operators

$$\mathcal{A}^{(n)}_\omega : \mathcal{H}_n \to \mathcal{H}'_n\,, \quad \mathcal{A}_\omega : \mathcal{H} \to \mathcal{H}' \tag{6.1}$$

generated by the variational problems (2.9) for the interfaces Γ_n and Γ respectively, where $\mathcal{H}_n$, $\mathcal{H}'_n$ denote the corresponding energy spaces respectively their duals

$$\mathcal{H}_n = H^1(\Omega_n)^3\times H^1(\Omega_{R,N})\,, \mathcal{H}'_n = H^{-1}(\Omega_n)^3\times H^{-1}(\Omega_{R,n})\,,$$

Ω_n denotes the interior of Γ_n, and $\Omega_{R,n} := \Omega_n^c \cap \{|x| < R\}$. We consider the inhomogeneous transmission problems

$$\mathcal{A}_\omega^{(n)}(u_n, p_n) = f_n \in \widetilde{\mathcal{H}}_n' , \quad \mathcal{A}_\omega(u,p) = f \in \widetilde{\mathcal{H}}' , \tag{6.2}$$

where the tilde spaces again denote the corresponding subspaces of elements that are L^2 orthogonal to the (possible) Jones modes associated with ω and Γ_n, Γ, respectively. By Theorem 2.1, there exist unique solutions $(u_n, p_n) \in \widetilde{\mathcal{H}}_n$, $(u,p) \in \widetilde{\mathcal{H}}$ of these equations, whereas the components u_n, u need not be unique in the energy spaces $\mathcal{H}_n$, $\mathcal{H}$.

In the following we are interested in the special case of right-hand sides in (6.2) that are defined by L^2 densities on the interfaces Γ_n, Γ:

$$f_n = (g_n \delta_{\Gamma_n}, h_n \delta_{\Gamma_n}) \ , \quad f = (g\delta_\Gamma, h\delta_\Gamma) \ , \quad \text{with}$$

$$g_n \in L^2(\Gamma_n)^3 \ , \ h_n \in L^2(\Gamma_n) \, ; \quad g \in L^2(\Gamma)^3 \ , \ h \in L^2(\Gamma) \, , \tag{6.3}$$

where δ_{Γ_n} and δ_Γ denote the δ-distributions with support on Γ_n and Γ, respectively. Note that the relations $f_n \in \widetilde{\mathcal{H}}_n'$, $f \in \widetilde{\mathcal{H}}'$ are then equivalent to the orthogonality relations

$$\langle g_n, u_{j,n} \rangle_{\Gamma_n} = 0 \, , \quad \langle g, u_j \rangle_\Gamma = 0 \, , \tag{6.4}$$

where $u_{j,n}$ respectively u_j run through the linearly independent Jones modes associated with ω and Γ_n respectively Γ. Of course, if there is no Jones mode for Γ_n, we have $\widetilde{\mathcal{H}}_n = \mathcal{H}_n$ and the condition (6.4) is void.

In the following we shall say that a sequence (h_n), $h_n \in L^2(\Gamma_n)$, is L^2 convergent to $h \in L^2(\Gamma)$ if

$$\lim_{n\to\infty} \int_{\mathbb{S}^2} \left| h_n(\mathbf{r}_n(\hat{x})\hat{x}) - h(\mathbf{r}(\hat{x})\hat{x}) \right|^2 \mathrm{d}s(\hat{x}) = 0 \, . \tag{6.5}$$

Now we can state our continuity result.

Theorem 6.1 *Assume that the right-hand sides of the transmission problems* (6.2) *satisfy* (6.3) *and* (6.4)*, and that the sequence* (g_n, h_n) *is* L^2 *convergent to* (g, h)*. Then we have*

$$\|p_n - p\|_{H^{1/2}(\Gamma_0)} \to 0 \quad as\ n \to \infty. \tag{6.6}$$

Remark 6.2 *Relation* (6.6) *also holds in case of the constant right-hand sides* $f_n = (0, -h_0\delta_{\Gamma_0})$*, with* h_0 *defined in* (2.5)*, where one has homogeneous transmission conditions on* Γ_n *respectively* Γ*. These transmission problems correspond to the direct scattering problems* (2.9) *with interfaces* Γ_n, Γ *and the incident wave* p^{inc}*. So our result is a modification of Lemma 3.1 in* [6] *to the case of different right-hand sides, and we will follow the arguments there in the proof below.*

Proof of Theorem 6.1. Following [6, Sect. 7.4], we choose reference domains

$$\widetilde{\Omega} := \{x \in \mathbb{R}^3 : |x| < R/2\} \, , \ \widetilde{\Omega}_R := \{x \in \mathbb{R}^3 : R/2 < |x| < R\}$$

and Lipschitz homeomorphisms

$$\tau_n : cl(\widetilde{\Omega} \cup \widetilde{\Omega}_R) \to cl(\Omega_n \cup \Omega_{R,n}) \,, \; \tau : cl(\widetilde{\Omega} \cup \widetilde{\Omega}_R) \to cl(\Omega \cup \Omega_R)$$

which map $\widetilde{\Gamma} := \{x \in \mathbb{R}^3 : |x| = R/2\}$ onto Γ, leave the artificial boundary Γ_0 invariant and have the following properties: For the operators (6.1), define the transformed operators

$$\widetilde{\mathcal{A}}_\omega^{(n)} \,, \; \widetilde{\mathcal{A}}_\omega : H^1(\widetilde{\Omega})^3 \times H^1(\widetilde{\Omega}_R) \to H^{-1}(\widetilde{\Omega})^3 \times H^{-1}(\widetilde{\Omega}_R) \tag{6.7}$$

via $\widetilde{\mathcal{A}}_\omega^{(n)}(u,p) \circ \tau_n = \widetilde{\mathcal{A}}_\omega^{(n)}((u,p) \circ \tau_n)$ etc. Then the operators (6.7) are bounded, and we have convergence in the corresponding operator norm:

$$\widetilde{\mathcal{A}}_\omega^{(n)} \to \widetilde{\mathcal{A}}_\omega \quad \text{as } n \to \infty \,, \quad \text{uniformly} \,. \tag{6.8}$$

This can be proved by substituting $x \mapsto \tau_n(x)$ into the sesquilinear forms of $\mathcal{A}_\omega^{(n)}$ and then discussing the transformed forms on the reference domains.

If ω is not a Jones frequency for Γ, then we obtain the result of the theorem in a standard way since the operators (6.8) are invertible for n sufficiently large, and we also have

$$\big(\widetilde{\mathcal{A}}_\omega^{(n)}\big)^{-1} \to (\widetilde{\mathcal{A}}_\omega)^{-1} \,, \; n \to \infty \,, \quad \text{uniformly} \,. \tag{6.9}$$

In the case that ω is a Jones frequency, we can select another frequency ω_* such that (6.8) and (6.9) hold with ω replaced by ω_*, hence

$$\widetilde{\mathcal{A}}_{\omega_*}^{(n)} : \mathcal{H}_n \to \mathcal{H}_n' \,, \; n \geq n_0 \,, \quad \mathcal{A}_{\omega_*} : \mathcal{H} \to \mathcal{H}'$$

are invertible, too. Furthermore, the first equation of (6.2) is equivalent to

$$\big(\mathcal{A}_{\omega_*}^{(n)} + (\omega^2 - \omega_*^2)\mathcal{D}_n\big)\,(u_n, p_n) = f_n \,,$$

where $\mathcal{D}_n$ denotes the operator generated by the sesquilinear form

$$d_n(u,p;v,q) = -\varrho\langle u, v\rangle_{\Omega_n} - c^{-2}\langle p, q\rangle_{\Omega_{R,n}} + \varrho_f\langle u\cdot\nu, q\rangle_{\Gamma_n} \,; \tag{6.10}$$

compare (2.9) and recall that $k^2 = \omega^2/c^2$. Setting $\lambda_0 := (\omega_*^2 - \omega^2)^{-1}$, we arrive at

$$\Big(\lambda_0 I - \big(\mathcal{A}_{\omega_*}^{(n)}\big)^{-1}\mathcal{D}_n\Big)\,(u_n,p_n) = \lambda_0\,\big(\mathcal{A}_{\omega_*}^{(n)}\big)^{-1} f_n \,. \tag{6.11}$$

We observe that each solution of the homogeneous equation (6.11) (with $f_n = 0$) in $\mathcal{H}_n$ is an eigenfunction of the compact operator

$$G_{\omega_*}^{(n)} := \big(\mathcal{A}_{\omega_*}^{(n)}\big)^{-1}\mathcal{D}_n : \mathcal{H}_n \to \mathcal{H}_n \,.$$

We choose a simple closed curve $\gamma \subset \mathbb{C}$ around λ_0 containing no eigenvalues of these operators for n sufficiently large. Now, we keep n fixed and apply the analyticity result of Theorem 2.1 (ii) to the operators $\mathcal{A}_{\omega(\lambda)}^{(n)}$ for $\omega(\lambda)$ in a vicinity of ω, with $\omega(\lambda)$ defined by $(\omega_*^2 - \omega(\lambda)^2)^{-1} = \lambda$ (or equivalently, $\omega(\lambda)^2 = \omega_*^2 - \lambda^{-1}$) and λ close to λ_0. Then, for any λ in a vicinity of λ_0, there exists a unique solution $(u_n^\lambda, p_n^\lambda) \in \widetilde{\mathcal{H}}_n$ given by

$(u_n^\lambda, p_n^\lambda) = (\mathcal{A}_{\omega(\lambda)}^{(n)})^{-1} f_n = \lambda(\lambda I - G_{\omega_*}^{(n)})^{-1}(\mathcal{A}_{\omega_*}^{(n)})^{-1} f_n$, and by the Cauchy integral formula we have

$$\left(u_n^{\lambda_0}, p_n^{\lambda_0}\right) = \frac{1}{2\pi i} \int_\gamma (\lambda - \lambda_0)^{-1} \left(\mathcal{A}_{\omega(\lambda)}^{(n)}\right)^{-1} f_n \, \mathrm{d}s(\lambda)\,. \tag{6.12}$$

We observe that the integrand in (6.12) is continuous in λ, uniformly bounded in n, and pointwise convergent on γ for $\mathbf{r}_n \to \mathbf{r}$, i.e.,

$$\left(\mathcal{A}_{\omega(\lambda)}^{(n)}\right)^{-1} f_n \circ \tau_n = \left(\widetilde{\mathcal{A}}_{\omega(\lambda)}^{(n)}\right)^{-1} (f_n \circ \tau_n) \to \widetilde{\mathcal{A}}_{\omega(\lambda)}^{-1} (f \circ \tau) = \mathcal{A}_{\omega(\lambda)}^{-1} f \circ \tau\,,$$

as $n \to \infty$. Here we have used (6.9) together with the fact that the L^2 convergence of (g_n, h_n) to (g, h) (see (6.5)) implies that

$$(g_n, h_n) \circ \tau_n \to (g, h) \circ \tau \quad \text{in } L^2(\widetilde{\Gamma})^4\,.$$

Therefore, letting $n \to \infty$ in (6.12) then gives

$$(u_n, p_n) \circ \tau_n = \left(u_n^{\lambda_0}, p_n^{\lambda_0}\right) \circ \tau_n \to (v, q) := \frac{1}{2\pi i} \int_\gamma (\lambda - \lambda_0)^{-1} \mathcal{A}_{\omega(\lambda)}^{-1} f \circ \tau \, \mathrm{d}s(\lambda)\,, \tag{6.13}$$

where the convergence takes place in the norm of $H^1(\widetilde{\Omega})^3 \times H^1(\widetilde{\Omega}_R)$. Noting that the diameter of γ can be chosen so small that $\omega(\lambda) \neq \omega$ is not a Jones mode for $\Gamma = \Gamma^{\mathbf{r}}$ inside and on γ and applying Theorem 2.1 (ii) to the operators $\mathcal{A}_{\omega(\lambda)}$ for $\omega(\lambda)$ near ω, we obtain analogously to (6.12):

$$(u, p) = \frac{1}{2\pi i} \int_\gamma (\lambda - \lambda_0)^{-1} \mathcal{A}_{\omega(\lambda)}^{-1} f \, \mathrm{d}s(\lambda)\,, \tag{6.14}$$

where $(u, p) \in \widetilde{\mathcal{H}}$ is the unique solution of the second equation of (6.2). Therefore, from (6.12)-(6.14) we obtain $(v, q) = (u, p) \circ \tau$ and finally

$$p_n|_{\Gamma_0} = p_n \circ \tau_n|_{\Gamma_0} \to p \circ \tau|_{\Gamma_0}\,, \; n \to \infty\,,$$

under the norm of $H^{1/2}(\Gamma_0)$. □

7 Proof of Theorems 3.1 and 3.2

Having Theorems 5.1 and 6.1 at hand, we are now in the position to prove the convergence of our reconstruction method, following the arguments of [5, Thms. 5.21 and 5.22] in the case of the inverse Dirichlet problem for the Helmholtz equation.

First, we give a sketch of the proof of Theorem 3.1, which is along the lines of [5, Thm. 5.20].

Proof of Theorem 3.1. Let $(\varphi_i^{(n)}, \varphi_e^{(n)}, \mathbf{r}_n)$ be a minimising sequence of the cost functional (3.11) in $L^2(\Gamma_i) \times L^2(\Gamma_e)^3 \times \mathcal{M}$, i.e.,

$$\lim_{n\to\infty} \mathcal{F}(\varphi_i^{(n)}, \varphi_e^{(n)}, \mathbf{r}_n; \alpha) = m(\alpha)\,. \tag{7.1}$$

By the compactness of $\mathcal{M}$, we can assume $\Gamma_n \to \Gamma$ in the sense of (3.5) where $\Gamma_n = \Gamma^{\mathbf{r}_n}$ and $\Gamma = \Gamma^{\mathbf{r}}$ for some $\mathbf{r} \in \mathcal{M}$, and from (7.1) and

$$\big\|\big(\varphi_i^{(n)}, \varphi_e^{(n)}\big)\big\|^2_{L^2(\Gamma_i)\times L^2(\Gamma_e)^3} \le \alpha^{-1}\mathcal{F}(\varphi_i^{(n)}, \varphi_e^{(n)}, \mathbf{r}_n; \alpha)$$

we obtain that the sequence $\big((\varphi_i^{(n)}, \varphi_e^{(n)})\big)$ is bounded in the L^2 norm. Therfore, we can assume that it converges weakly:

$$\big(\varphi_i^{(n)}, \varphi_e^{(n)}\big) \rightharpoonup (\varphi_i, \varphi_e) \in L^2(\Gamma_i) \times L^2(\Gamma_e)^3 \tag{7.2}$$

To verify that $(\varphi_i, \varphi_e, \mathbf{r})$ is a minimiser of $\mathcal{F}$, we have to show that

$$\lim_{n\to\infty} \mathcal{F}\big(\varphi_i^{(n)}, \varphi_e^{(n)}, \mathbf{r}_n; \alpha\big) = \mathcal{F}(\varphi_i, \varphi_e, \mathbf{r}; \alpha)\,. \tag{7.3}$$

We will prove the convergence in (7.3) for each term of $\mathcal{F}$ separately. Moreover, it is sufficient to prove convergence for the first and the last two terms. Indeed, once we have done this, from (7.1) we obtain

$$\alpha\big\|\big(\varphi_i^{(n)}, \varphi_e^{(n)}\big)\big\|^2 \to m(\alpha) - \mathcal{F}(\varphi_i, \varphi_e, \mathbf{r}; \alpha) + \alpha\|(\varphi_i, \varphi_e)\|^2 \le \alpha\|(\varphi_i, \varphi_e)\|^2\,,$$

where the norm is taken in $L^2(\Gamma_i) \times L^2(\Gamma_e)^3$. Together with the weak convergence (7.2), the latter relation then gives the strong convergence of $(\varphi_i^{(n)}, \varphi_e^{(n)})$, hence (7.3).

Since the far field operator $F : L^2(\Gamma_i) \to L^2(\mathbb{S}^2)$ is compact, we get the convergence in (7.3) for the first term of the functional (3.11). To study the last two terms, consider the operators

$$\hat{\mathcal{B}}_n, \hat{\mathcal{B}} : L^2(\Gamma_e)^3 \times L^2(\Gamma_i) \ \to L^2(\mathbb{S}^2)^4\,,\ \hat{\mathcal{B}}_n\chi := (\mathcal{B}_n\chi)\circ\hat{\mathbf{r}}_n\,,\ \hat{\mathcal{B}}\chi := (\mathcal{B}\chi)\circ\hat{\mathbf{r}}\,, \tag{7.4}$$

where $\mathcal{B}$ is the matrix potential operator introduced in (5.1) and $\mathcal{B}_n$ is the corresponding operator with Γ replaced by Γ_n. To prove the convergence in (7.3) for the last two terms of the functional $\mathcal{F}$, it is then sufficient to show that $\hat{\mathcal{B}}_n$ converges to $\hat{\mathcal{B}}$ uniformly, i.e., in the operator norm. Indeed, the differences of the corresponding last two terms in $\mathcal{F}\big(\varphi_i^{(n)}, \varphi_e^{(n)}, \mathbf{r}_n; \alpha\big)$ and $\mathcal{F}(\varphi_i, \varphi_e, \mathbf{r}; \alpha)$ can be estimated, uniformly in n, by

$$c\left\{\|(\hat{\mathcal{B}}_n - \hat{\mathcal{B}})\chi_n\|_{L^2(\mathbb{S}^2)^4} + \|\hat{\mathcal{B}}(\chi - \chi_n)\|_{L^2(\mathbb{S}^2)^4}\right\}, \tag{7.5}$$
$$\chi := (\varphi_e, \varphi_i)\,,\ \chi_n := \big(\varphi_e^{(n)}, \varphi_i^{(n)}\big)\,,$$

and the operator $\hat{\mathcal{B}}$ is compact.

Now it follows from our assumptions on the admissible class of parameterisations (cf. (3.2)-(3.4)) that $\hat{\mathcal{B}}_n - \hat{\mathcal{B}}$ are matrices of integral operators with sufficiently smooth kernels, say K_n, such that

$$K_n \to 0\,,\ n \to \infty\,, \text{in } C^{0,\beta}(\Gamma_i \times \mathbb{S}^2) \text{ resp. } C^{0,\beta}(\Gamma_e \times \mathbb{S}^2), \tag{7.6}$$

which implies the desired uniform convergence of the operators (7.4). □

Proof of Theorem 3.2. (i): Let $\Gamma = \Gamma^{\mathbf{r}}$, $\mathbf{r} \in \mathcal{M}$, be an interface, for which p_∞^{meas} is the exact far field of $p^{sc} = p - p^{inc}$, where p is the total pressure field corresponding to a solution

(u, p) of the forward problem (2.9). Recall that the component p is unique, whereas the displacement u is only uniquely defined modulo Jones modes.

First, we observe that in analogy to (7.5) the last two terms of the functional (3.11) can be estimated by

$$c\|\mathcal{B}\chi - f_0\|^2_{L^2(\Gamma)^4}\,, \quad \chi := (\varphi_e, \varphi_i)\,,\ f_0 := \left(-p^{inc}\nu\,, (\varrho_f\omega^2)^{-1}\partial_\nu p^{inc}\right)\,, \tag{7.7}$$

and that the first component of f_0 is orthogonal to the Jones modes (restricted to Γ). Hence, by Theorem 5.1 or rather Corollary 5.2, given any $\varepsilon > 0$ there exist densities

$$\chi^\varepsilon = (\varphi_e^\varepsilon, \varphi_i^\varepsilon) \in L^2(\Gamma_e)^3 \times L^2(\Gamma_i)$$

such that

$$\|\mathcal{B}\chi^\varepsilon - f_0\|_{L^2(\Gamma)^4} \le \varepsilon\,. \tag{7.8}$$

Next, we have to show that the first term in the cost functional $\mathcal{F}$ can be made arbitrarily small by choosing $\varphi_i = \varphi_i^\varepsilon$ and $\varphi_e = \varphi_e^\varepsilon$. Defining the functions w^ε and q^ε by

$$w^\varepsilon := V^{el}_{\Gamma_e}\varphi_e^\varepsilon - u \quad \text{in } \Omega\,, \quad q^\varepsilon := p^{inc} - p + V_{\Gamma_i}\varphi_i^\varepsilon \quad \text{in } \Omega_R\,, \tag{7.9}$$

we see that $(w^\varepsilon, q^\varepsilon) \in \mathcal{H}$ is a solution of the inhomogeneous transmission problem (2.10) with right-hand side $f^\varepsilon := (g^\varepsilon\delta_\Gamma, h^\varepsilon\delta_\Gamma)$, where

$$\begin{aligned} g^\varepsilon &:= tV^{el}_{\Gamma_e}\varphi_e^\varepsilon + (p^{inc} + V_{\Gamma_i}\varphi_i^\varepsilon)\nu\,, \\ h^\varepsilon &:= \nu \cdot V^{el}_{\Gamma_e}\varphi_e^\varepsilon - (\varrho_f\omega^2)^{-1}\partial_\nu(p^{inc} + V_{\Gamma_i}\varphi_i^\varepsilon)\,. \end{aligned} \tag{7.10}$$

Note that the functions (7.9) satisfy the transmission conditions

$$tw^\varepsilon + q^\varepsilon\nu = g^\varepsilon\,, \quad w^\varepsilon\cdot\nu - (\varrho_f\omega^2)^{-1}\partial_\nu q^\varepsilon = h^\varepsilon \quad \text{on } \Gamma\,, \tag{7.11}$$

and that $g^\varepsilon\partial_\Gamma$ is orthogonal to the Jones modes on Ω, i.e., $f^\varepsilon \in \widetilde{\mathcal{H}}'$. Using Theorem 2.1 (ii), (7.8) and the definition of $\mathcal{F}$, we then obtain the estimate

$$\begin{aligned} \|V_{\Gamma_i}\varphi_i^\varepsilon - p^{sc}\|^2_{H^{1/2}(\Gamma_0)} &= \|q^\varepsilon\|^2_{H^{1/2}(\Gamma_0)} \le c\|q^\varepsilon\|^2_{H^1(\Omega_R)} \\ &\le c\|f^\varepsilon\|^2_{\mathcal{H}'} \le c\left\{\|g^\varepsilon\|^2_{L^2(\Gamma)^3} + \|h^\varepsilon\|^2_{L^2(\Gamma)}\right\} \\ &\le c\|\mathcal{B}\chi^\varepsilon - f_0\|^2_{L^2(\Gamma)^4} \le c\varepsilon\,, \end{aligned} \tag{7.12}$$

where c does not depend on ε. To estimate the far field term of $\mathcal{F}$, we note that $\partial_\nu q^\varepsilon = Tq^\varepsilon$ on Γ_0 (with the Dirichlet-to-Neumann map T), that the far field q^ε_∞ of q^ε coincides with $F\varphi_i^\varepsilon - p^{\text{meas}}_\infty$ on $\mathbb{S}^2$, and that the following far field representation holds (see, e.g., [5]):

$$q^\varepsilon_\infty(\hat{x}) = \frac{1}{4\pi}\int_{\Gamma_0}\left\{q^\varepsilon(y)\,\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial\nu(y)} - \frac{\partial q^\varepsilon}{\partial\nu}(y)e^{-ik\hat{x}\cdot y}\right\}\mathrm{d}s(y)\,,\ \hat{x} \in \mathbb{S}^2\,. \tag{7.13}$$

Thus we have from (7.12) and (7.13) that

$$\|F\varphi_i^\varepsilon - p^{\text{meas}}_\infty\|^2_{L^2(\mathbb{S}^2)} \le c\|q^\varepsilon\|^2_{H^{1/2}(\Gamma_0)} \le c\varepsilon\,. \tag{7.14}$$

Finally, from (7.7), (7.8), (7.14) and the definition of $\mathcal{F}$,

$$\mathcal{F}(\varphi_i^\varepsilon, \varphi_e^\varepsilon, \mathbf{r}; \alpha) \leq c\varepsilon + \alpha \|(\varphi_i^\varepsilon, \varphi_e^\varepsilon)\|^2_{L^2(\Gamma_i)\times L^2(\Gamma_e)^3} \to c\varepsilon\,,\ \alpha \to 0$$

for any $\varepsilon > 0$, which completes the proof of assertion (i).

Proof of Theorem 3.2. (ii): Since $\mathcal{M}$ is compact, there is a convergent subsequence $\mathbf{r}_n \to \mathbf{r}^* \in \mathcal{M}$, $n \to \infty$, in the sense of (3.5). Let $(u^*, p^*) \in \mathcal{H}$ denote a solution of the direct scattering problem (2.9) with incident wave p^{inc} and interface $\Gamma^* = \Gamma^{\mathbf{r}^*}$. We need to show that Γ^* is a solution of the inverse problem (IP), i.e., the far field p^*_∞ of $p^* - p^{inc}$ coincides with p^{meas}_∞.

Consider the sequence of minimisers $(\varphi_i^{(n)}, \varphi_e^{(n)}, \mathbf{r}_n)$ of problem (OP) for the parameter α_n, so that by (i)

$$\mathcal{F}\big(\varphi_i^{(n)}, \varphi_e^{(n)}, \mathbf{r}_n; \alpha_n\big) = m(\alpha_n) \to 0\,,\ n \to \infty\,. \tag{7.15}$$

Let (u_n, p_n) be solutions of the forward problem (2.9) for the interfaces $\Gamma_n = \Gamma^{\mathbf{r}_n}$, and define the functions

$$w_n := V^{el}_{\Gamma_e}\varphi_e^{(n)} - u_n \quad \text{in } \Omega_n\,, \quad q_n := p^{inc} - p_n + V_{\Gamma_i}\varphi_i^{(n)} \quad \text{in } \Omega_{R,n}\,.$$

Then we observe that (w_n, q_n) is a solution of the inhomogeneous transmission problem (6.2) with right-hand side $f_n = (g_n\delta_{\Gamma_n}, h_n\delta_{\Gamma_n})$, where the functions

$$\begin{aligned} g_n &:= tV^{el}_{\Gamma_e}\varphi_e^{(n)} + \big(p^{inc} + V_{\Gamma_i}\varphi_i^{(n)}\big)\nu\,, \\ h_n &:= \nu \cdot V^{el}_{\Gamma_e}\varphi_e^{(n)} - \frac{1}{\varrho_f\omega^2}\partial_\nu\big(p^{inc} + V_{\Gamma_i}\varphi_i^{(n)}\big) \end{aligned}$$

indeed fulfill the conditions (6.3) and (6.4). By (7.15), f_n is L^2 convergent to 0 in the sense of (6.5). Therefore, from Theorem 6.1 we obtain

$$\|q_n\|^2_{H^{1/2}(\Gamma_0)} \longrightarrow 0,\ n \to \infty. \tag{7.16}$$

Moreover, by the same theorem or rather Remark 6.2, we have

$$\|p_n - p^*\|^2_{H^{1/2}(\Gamma_0)} \longrightarrow 0,\ n \to \infty,$$

and together with (7.16) this implies

$$\Big\|p^{inc} - p^* + V_{\Gamma_i}\varphi_i^{(n)}\Big\|^2_{H^{1/2}(\Gamma_0)} \longrightarrow 0,\ n \to \infty. \tag{7.17}$$

Arguing as at the end of (i), (7.17) leads to

$$\|F\varphi_i^{(n)} - p^*_\infty\|^2_{L^2(\mathbb{S}^2)} \longrightarrow 0,\ n \to \infty.$$

On the other hand, by (7.15) we also have

$$\|F\varphi_i^{(n)} - p^{meas}_\infty\|^2_{L^2(\mathbb{S}^2)} \longrightarrow 0,\ n \to \infty,$$

and combining the last two relations gives $p^*_\infty = p^{meas}_\infty$. □

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