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Stationary energy models for semiconductor devices with incompletely ionized impurities

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Dedicated to Professor Herbert Gajewski on his 65-th birthday

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Abstract. The paper deals with two-dimensional stationary energy models for semiconductor devices, which contain incompletely ionized impurities. We reduce the problem to a strongly coupled nonlinear system of four equations, which is elliptic in nondegenerated states. Heterostructures as well as mixed boundary conditions have to be taken into account. For boundary data which are compatible with thermodynamic equilibrium there exists a thermodynamic equilibrium. Using regularity results for systems of strongly coupled linear elliptic differential equations with mixed boundary conditions and nonsmooth data and applying the Implicit Function Theorem we prove that in a suitable neighbourhood of such a thermodynamic equilibrium there exists a unique stationary solution, too.

1. Model equations.

The charge transport in semiconductor devices is described by the van Roosbroeck equations (see [17] and e.g. [4, 11, 14]). They consist of two continuity equations for the densities n and p of electrons e and holes h, respectively, and a Poisson equation for the electrostatic potential φ. Physical parameters occurring in these equations depend on the device temperature T. Therefore, under nonisothermal conditions a balance equation for the density of total energy must be added, and a so called energy model arises (see [2, 18]) . Finally, if incompletely ionized impurities (for example radiation induced traps or other deep recombination centers) are taken into account, we have to consider further continuity equations for the densities of (in general immobile) species X_j, $j = 1, \dots, k$. These species exist in different charge states which are transformed into each other by ionization reactions. For the sake of simplicity we assume that each reaction is a binary one. Let X_j be an acceptor-like impurity which can accept an electron e or deliver a hole h and let X_j^- be its ion. Then we have to consider the reactions

$$\mathrm{e}^- + \mathrm{X}_j \rightleftharpoons \mathrm{X}_j^-, \quad \mathrm{h}^+ + \mathrm{X}_j^- \rightleftharpoons \mathrm{X}_j. \tag{1}$$

If X_j is a donor-like impurity which can deliver an electron e or accept a hole h and X_j^+ denotes its ion, then the reactions are

$$\mathrm{e}^- + \mathrm{X}_j^+ \rightleftharpoons \mathrm{X}_j, \quad \mathrm{h}^+ + \mathrm{X}_j \rightleftharpoons \mathrm{X}_j^+. \tag{2}$$

If X_j is a donor (an acceptor) we denote by u_{2j-1} the density of X_j (of X_j^-) and by u_{2j} the density of X_j^+ (of X_j). Furthermore, we define charge numbers as follows:

$$q_{2j-1} := \begin{cases} 0 & \text{if } \mathrm{X}_j \text{ is a donor} \\ -1 & \text{if } \mathrm{X}_j \text{ is an acceptor} \end{cases}, \qquad q_{2j} := 1 + q_{2j-1}, \quad j = 1, \dots, k.$$

Then the continuity equations have the form

$$\frac{\partial n}{\partial t} + \nabla \cdot j_n = R_0 + \sum_{j=1}^{k} R_{j1}, \quad \frac{\partial p}{\partial t} + \nabla \cdot j_p = R_0 + \sum_{j=1}^{k} R_{j2}, \tag{3}$$

$$\frac{\partial u_{2j-1}}{\partial t} = -R_{j1} + R_{j2}, \quad \frac{\partial u_{2j}}{\partial t} = R_{j1} - R_{j2}, \quad j = 1, \dots, k, \tag{4}$$

while the Poisson equation reads as

$$-\nabla\cdot(\varepsilon\nabla\varphi) = f_0 - n + p + \sum_{i=1}^{2k} q_i u_i. \tag{5}$$

Here j_n, j_p denote the particle flux densities of electrons and holes, R_{j1}, R_{j2} denote the reaction rates of the first and second reaction in (1) or in (2), respectively, while R_0 is the reaction rate of a (direct) electron-hole generation-recombination

$$\mathrm{e}^- + \mathrm{h}^+ \rightleftharpoons 0.$$

Finally, ε is the dielectric permittivity and f_0 is a given charge density arising from other completely ionized impurities. Adding both equations in (4) we find

$$\frac{\partial(u_{2j-1}+u_{2j})}{\partial t} = 0,$$

in other words $u_{2j-1}(t,x) + u_{2j}(t,x) = f_j(x)$ for all $t \ge 0$ such that f_j is a prescribed (local) invariant of the instationary reaction system (4) . This invariant must be taken into account in the stationary case, too. Therefore, in this case the equations in (4) have to be replaced by the equations

$$R_{j1} - R_{j2} = 0, \quad u_{2j-1} + u_{2j} = f_j, \quad j = 1,\dots,k.$$

Special isothermal models of the form (3) – (5) are presented in [15]. There also results of simulations with `WIAS-TeSCA` [5] are compared with experimental results.

In this paper we consider the stationary, but nonisothermal situation. Let Ω_0 be the domain which is occupied by the semiconductor device. We assume that each impurity X_j and its corresponding ion live only on some subset $\Omega_j \subset \Omega_0$. In order to simplify the notation we formally set $u_{2j-1} = u_{2j} = 0$ and $R_{j1} = R_{j2} = 0$ on $\Omega_0 \setminus \Omega_j$, $j = 1,\dots,k$. The basic equations are

$$-\nabla\cdot(\varepsilon\nabla\varphi) = f_0 - n + p + \sum_{i=1}^{2k} q_i u_i, \quad \nabla\cdot j_e = 0, \quad \nabla\cdot j_n = R_1, \quad \nabla\cdot j_p = R_2 \quad \text{on } \Omega_0, \tag{6}$$

$$R_{j1} - R_{j2} = 0, \qquad u_{2j-1} + u_{2j} = f_j \qquad \text{on } \Omega_j, \quad j = 1,\dots,k. \tag{7}$$

Here j_e denotes the flux density of the total energy, and R_1, R_2 are given by

$$R_l = R_0 + \sum_{j=1}^{k} R_{jl}, \quad l = 1,2. \tag{8}$$

In (6) – (8) we have to specify the underlying kinetic relations. For these purposes we introduce the electrochemical potentials ζ_n of electrons, ζ_p of holes, as well as ζ_{2j-1} and ζ_{2j} of the j-th impurity and its ion, respectively. These quantities are implicitly defined by state equations which we suppose to have the form

$$n = F_n\Big(x, T, \frac{\zeta_n + \varphi}{T}\Big), \quad p = F_p\Big(x, T, \frac{\zeta_p - \varphi}{T}\Big) \quad \text{on } \Omega_0, \tag{9}$$

$$u_{2j+l} = F_{2j+l}\Big(x, T, \frac{\zeta_{2j+l} - q_{2j+l}\varphi}{T}\Big) \quad \text{on } \Omega_j, \quad l = -1, 0, \quad j = 1,\dots,k. \tag{10}$$

For the flux densities j_e, j_n, and j_p we make the ansatz (see [2, 18])

$$
\begin{aligned}
j_e &= -\kappa\nabla T + \sum_{i=n,p}(\zeta_i + P_i T)j_i,\\
j_n &= -(\sigma_n+\sigma_{np})(\nabla\zeta_n + P_n\nabla T) - \sigma_{np}(\nabla\zeta_p + P_p\nabla T),\\
j_p &= -\sigma_{np}(\nabla\zeta_n + P_n\nabla T) - (\sigma_p+\sigma_{np})(\nabla\zeta_p + P_p\nabla T)
\end{aligned}
\tag{11}
$$

with conductivities $\kappa > 0$, σ_n, $\sigma_p > 0$, $\sigma_{np} \geq 0$, and transported entropies P_n, P_p. All kinetic coefficients κ, σ_n, σ_p, σ_{np}, P_n, P_p depend on x, T, n and p. Let us note, that the strong inequalities $\kappa > 0$, σ_n, $\sigma_p > 0$ are valid only for nondegenerated states $0 < T$, n, $p < +\infty$. Finally, according to the mass action law the reaction rates R_0 and R_{j1}, R_{j2} are given by

$$
\begin{aligned}
R_0 &= r_0\,(x,\varphi,T,n,p)\left(1 - \mathrm{e}^{(\zeta_n+\zeta_p)/T}\right) \quad \text{on } \Omega_0,\\
R_{j1} &= r_{j1}(x,\varphi,T,n,p)\left(\mathrm{e}^{\zeta_{2j-1}/T} - \mathrm{e}^{(\zeta_{2j}+\zeta_n)/T}\right),\\
R_{j2} &= r_{j2}(x,\varphi,T,n,p)\left(\mathrm{e}^{\zeta_{2j}/T} - \mathrm{e}^{(\zeta_{2j-1}+\zeta_p)/T}\right) \quad \text{on } \Omega_j, \quad j=1,\dots,k,
\end{aligned}
\tag{12}
$$

where the kinetic coefficients r_0, r_{j1}, and r_{j2} are positive for nondegenerated states.

We supplement the differential equations (6) by mixed boundary conditions. Let Γ be the boundary of Ω_0, and let Γ_D and Γ_N denote disjoint, relatively open parts of Γ with $\mathrm{mes}(\Gamma \setminus (\Gamma_D \cup \Gamma_N)) = 0$. We suppose boundary conditions of the form

$$
\begin{aligned}
\varphi &= \varphi_D, & T &= T_D, & \zeta_n &= \zeta_{nD}, & \zeta_p &= \zeta_{pD} & &\text{on } \Gamma_D,\\
\nu\cdot(\varepsilon\nabla\varphi) &= g_1, & -\nu\cdot j_e &= g_2, & -\nu\cdot j_n &= g_3, & -\nu\cdot j_p &= g_4 & &\text{on } \Gamma_N.
\end{aligned}
\tag{13}
$$

In summary, the stationary energy model which we are interested in consists of the equations (6) – (12) and of boundary conditions as in (13).

2. Basic assumptions.

Definition 1. Let $\mathcal{V} \subset \mathbb{R}^m$ be an open set. Let $\Omega \subset \mathbb{R}^2$ be a measurable set and $\Sigma \subset \mathbb{R}^2$ be a set of measure zero. We say that a function $b\colon \Omega\times\mathcal{V} \to \mathbb{R}$ is of the class $D(\Omega,\Sigma,\mathcal{V})$ iff it fulfills the following properties:

$x \mapsto b(x,z)$ is measurable for all $z \in \mathcal{V}$,
$z \mapsto b(x,z)$ is continuously differentiable for all $x \in \Omega\setminus\Sigma$.

For every compact subset $K \subset \mathcal{V}$ there exists an $M > 0$ such that
$|b(x,z)| \leq M$ and $\|\partial_z b(x,z)\| \leq M$ for $x \in \Omega\setminus\Sigma$ and $z \in K$.

For every compact subset $K \subset \mathcal{V}$ and $\tau > 0$ there exists a $\delta > 0$ such that
$|b(x,z) - b(x,\overline{z})| < \tau$ and $|\partial_z b(x,z) - \partial_z b(x,\overline{z})| < \tau$
for $x \in \Omega\setminus\Sigma$ and $z, \overline{z} \in K$ with $|z-\overline{z}| < \delta$.

In the paper we make use of the following special open sets $\mathcal{V}$:

$$
\begin{aligned}
&\mathcal{V}_* = \mathbb{R}\times(0,\infty)^3, \quad \widetilde{\mathcal{V}}_* = \mathbb{R}\times(0,\infty)\times\mathbb{R}^2,\\
&\mathcal{V}_0 = \widetilde{\mathcal{V}}_*\times\mathbb{R}, \quad \mathcal{V}_j = \widetilde{\mathcal{V}}_*\times(0,\infty), \quad j=1,\dots,k.
\end{aligned}
\tag{14}
$$

Concerning the problem (6) – (13) we suppose:

(A1) Ω_0 is a bounded Lipschitzian domain in $\mathbb{R}^2$, $\Gamma = \partial\Omega_0$,
Γ_D, Γ_N are disjoint open subsets of Γ, $\Gamma = \Gamma_D \cup \Gamma_N \cup (\overline{\Gamma_D} \cap \overline{\Gamma_N})$,
$\operatorname{mes}\Gamma_D > 0$, $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ consists of finitely many points,
$\Omega_j \subset \Omega_0$ are measurable subsets, $j = 1, \ldots, k$, $\Sigma \subset \Omega_0$ with $\operatorname{mes}\Sigma = 0$.

(A2) σ_n, σ_p, σ_{np}, κ, P_n, $P_p\colon \Omega_0 \times \mathcal{V} \to \mathbb{R}$ are of the class $D(\Omega_0, \Sigma, \mathcal{V})$ with $\mathcal{V} = (0,\infty)^3$.
For all $K > 1$ there exists a $c_K > 1$ such that
$\sigma_n(x, T, n, p)$, $\sigma_p(x, T, n, p)$, $\kappa(x, T, n, p) \in [1/c_K, c_K]$
for $x \in \Omega_0 \setminus \Sigma$, $(T, n, p) \in [1/K, K]^3$;
$\sigma_{np}(x, T, n, p) \geq 0$ for $x \in \Omega_0 \setminus \Sigma$, $(T, n, p) \in (0,\infty)^3$.

(A3) $\varepsilon \in L^\infty(\Omega_0)$, $0 < \varepsilon_0 \leq \varepsilon(x) \leq \varepsilon^0 < \infty$ in $\Omega_0 \setminus \Sigma$.

(A4) $F_i\colon \Omega_0 \times \mathcal{V} \to \mathbb{R}_+$ are of the class $D(\Omega_0, \Sigma, \mathcal{V})$ with $\mathcal{V} = (0,\infty) \times \mathbb{R}$.
For all $K > 1$ there exist $\widehat{c}_K > 0$, $c_K > 1$ such that $\frac{\partial F_i}{\partial y}(x, T, y) \geq \widehat{c}_K$,
$F_i(x, T, y) \in [1/c_K, c_K]$ for $x \in \Omega_j \setminus \Sigma$, $(T, y) \in [1/K, K] \times [-K, K]$,
$F_i(x, T, y) \leq c_K \operatorname{e}^{c_K|y|}$ for $x \in \Omega_0 \setminus \Sigma$, $(T, y) \in [1/K, K] \times \mathbb{R}$.
$\lim_{y\to-\infty} F_i(x, T, y) = 0$, $\lim_{y\to+\infty} F_i(x, T, y) = +\infty$
for $x \in \Omega_j \setminus \Sigma$, $T \in (0,\infty)$, $i = n, p$.
$F_{2j+l}\colon \Omega_j \times \mathcal{V} \to \mathbb{R}_+$ are of the class $D(\Omega_j, \Sigma, \mathcal{V})$ with $\mathcal{V} = (0,\infty) \times \mathbb{R}$.
For all $K > 1$ there exists $\widehat{c}_K > 0$, $c_K > 1$ such that
$F_{2j+l}(x, T, y) \in [1/c_K, c_K]$, $\frac{\partial F_{2j+l}}{\partial y}(x, T, y) \geq \widehat{c}_K$
for $x \in \Omega_j \setminus \Sigma$, $T \in [1/K, K]$, $y \in [-K, K]$.
$\lim_{y\to-\infty} F_{2j+l}(x, T, y) = 0$, $\lim_{y\to+\infty} F_{2j+l}(x, T, y) = +\infty$
for $x \in \Omega_j \setminus \Sigma$, $T \in (0,\infty)$, $j = 1, \ldots, k$, $l = -1, 0$.

(A5) $r_0\colon \Omega_0 \times \mathcal{V}_* \to \mathbb{R}_+$ is of the class $D(\Omega_0, \Sigma, \mathcal{V}_*)$ (see (14)).
$r_{ji}\colon \Omega_j \times \mathcal{V}_* \to \mathbb{R}_+$ are of the class $D(\Omega_j, \Sigma, \mathcal{V}_*)$ (see (14)).
For all $K > 1$ there exists a $c_K > 1$ such that
$r_{ji}(x, \varphi, T, n, p) \in [1/c_K, c_K]$ for $x \in \Omega_j \setminus \Sigma$,
$(\varphi, T, n, p) \in [-K, K] \times [1/K, K]^3$, $j = 1, \ldots, k$, $i = 1, 2$.

We use the notation $\zeta_{\text{imp}} = (\zeta_1, \ldots, \zeta_{2k})$, $v = (\varphi, T, \zeta_n, \zeta_p)$, $v_D = (\varphi_D, T_D, \zeta_{nD}, \zeta_{pD})$, $g = (g_1, \ldots, g_4)$ and $f = (f_0, f_1, \ldots, f_k)$. With respect to the data we assume that

(D) $v_D \in W^{1-1/p,p}(\Gamma_D)^4$ for some $p \in (2, p_0]$, where p_0 is specified in Lemma 5,
$g \in L^\infty(\Gamma_N)^4$, $f \in L^\infty(\Omega_0) \times \prod_{j=1}^k \Big\{h \in L^\infty(\Omega_j)\colon \operatorname{ess\,inf}_{x\in\Omega_j} h > 0\Big\}$.

We look for v in the form

$$v = V + v^D, \quad v_i^D = Lv_{Di}, \quad i = 1, \ldots, 4,$$

where L denotes the solution operator for the Laplace equation (36) with homogeneous Neumann boundary conditions on Γ_N and inhomogeneous Dirichlet boundary conditions on Γ_D. Shortly we will write Lv_D for the vector $(Lv_{D1}, \ldots, Lv_{D4})$.

3. Weak formulation $(\widetilde{\mathbf{P}})$.

Using the state equations (9) for n and p we can write the kinetic coefficients r_0, r_{ji} as functions $\widetilde{r}_0$, $\widetilde{r}_{ji}$ of the variables x and $v = (\varphi, T, \zeta_n, \zeta_p)$,

$$\begin{array}{rcccl} r_0(x,\varphi,T,n,p) &=& r_0(x,\varphi,T,F_n(x,T,\frac{\zeta_n+\varphi}{T}),F_p(x,T,\frac{\zeta_p-\varphi}{T})) &=& \widetilde{r}_0(x,v), \\ r_{ji}(x,\varphi,T,n,p) &=& r_{ji}(x,\varphi,T,F_n(x,T,\frac{\zeta_n+\varphi}{T}),F_p(x,T,\frac{\zeta_p-\varphi}{T})) &=& \widetilde{r}_{ji}(x,v), \\ && i=1,2,\ j=1,\dots,k. && \end{array}$$

REMARK 1. Assumption (A5) and properties of F_n, F_p in (A4) ensure that the function $\widetilde{r}_0 \colon \Omega_0 \times \widetilde{\mathcal{V}}_* \to \mathbb{R}_+$ is of the class $D(\Omega_0, \Sigma, \widetilde{\mathcal{V}}_*)$, and that the functions $\widetilde{r}_{ji} \colon \Omega_j \times \widetilde{\mathcal{V}}_* \to \mathbb{R}_+$ are of the class $D(\Omega_j, \Sigma, \widetilde{\mathcal{V}}_*)$. For all $K > 1$ there exists a $c_K > 1$ such that $\widetilde{r}_{ji}(x,v) \in [1/c_K, c_K]$ for $x \in \Omega_j \setminus \Sigma$, $v \in [-K,K] \times [1/K,K] \times [-K,K]^2$, $i = 1,2$, $j = 1,\dots,k$.

Moreover, we write

$$\sigma_i(x,T,n,p) = \sigma_i(x,T,F_n(x,T,\frac{\zeta_n+\varphi}{T}),F_p(x,T,\frac{\zeta_p-\varphi}{T})) = \widetilde{\sigma}_i(x,v),\ \ i = n,p,$$

and analogously $\sigma_{np}(x,T,n,p) = \widetilde{\sigma}_{np}(x,v)$, $\kappa(x,T,n,p) = \widetilde{\kappa}(x,v)$, $P_i(x,T,n,p) = \widetilde{P}_i(x,v)$, $i = n,p$. Next, we define the matrix function (see (11))

$$b(\cdot,v) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \kappa + \widehat{\omega}_0 & \omega_1 & \omega_2 \\ 0 & \widehat{\omega}_1 & \widetilde{\sigma}_n + \widetilde{\sigma}_{np} & \widetilde{\sigma}_{np} \\ 0 & \widehat{\omega}_2 & \widetilde{\sigma}_{np} & \widetilde{\sigma}_p + \widetilde{\sigma}_{np} \end{pmatrix}, \tag{15}$$

where

$$\widehat{\omega}_0 = (v_3 + \widetilde{P}_n v_2)\,\widehat{\omega}_1 + (v_4 + \widetilde{P}_p v_2)\,\widehat{\omega}_2,$$

$$\begin{pmatrix} \widehat{\omega}_1 \\ \widehat{\omega}_2 \end{pmatrix} = \begin{pmatrix} \widetilde{\sigma}_n + \widetilde{\sigma}_{np} & \widetilde{\sigma}_{np} \\ \widetilde{\sigma}_{np} & \widetilde{\sigma}_p + \widetilde{\sigma}_{np} \end{pmatrix} \begin{pmatrix} \widetilde{P}_n \\ \widetilde{P}_p \end{pmatrix},$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \widetilde{\sigma}_n + \widetilde{\sigma}_{np} & \widetilde{\sigma}_{np} \\ \widetilde{\sigma}_{np} & \widetilde{\sigma}_p + \widetilde{\sigma}_{np} \end{pmatrix} \begin{pmatrix} v_3 + \widetilde{P}_n v_2 \\ v_4 + \widetilde{P}_p v_2 \end{pmatrix}.$$

REMARK 2. Due to (A2), the functions b_{ij}, $i,j = 1,\dots,4$, are of the class $D(\Omega_0,\Sigma,\widetilde{\mathcal{V}}_*)$. In nondegenerated states the matrix $b_{ij}(\cdot,v)$ is regular, but not symmetric. Note that there is a change of the generalized forces $(\nabla T, \nabla\zeta_n, \nabla\zeta_p)$ to the new generalized forces $(\nabla(-\frac{1}{T}), \nabla\frac{\zeta_n}{T}, \nabla\frac{\zeta_p}{T})$ leading to a matrix, which is symmetric and positive definite for nondegenerated states. Thus the Onsager relations are fulfilled for the fluxes (j_e, j_n, j_p) and the new generalized forces. But in this paper we will not make use of this transformation.

In our analytical investigations we use the following function spaces and subsets

$$X_s = (W_0^{1,s}(\Omega_0 \cup \Gamma_N))^4, \quad Y_s = (W^{1,s}(\Omega_0))^4, \quad \mathcal{H}_s = (W^{1-1/s,s}(\Gamma_D))^4, \quad s \in (1,\infty),$$

$$\mathcal{H}_* = L^\infty(\Gamma_N)^4 \times L^\infty(\Omega_0), \quad \mathcal{H} = \mathcal{H}_* \times \prod_{j=1}^{k} \Big\{ h \in L^\infty(\Omega_j) \colon \operatorname{ess\,inf}_{x\in\Omega_j} h > 0 \Big\}.$$

$\mathcal{H}$ is open in $L^\infty(\Gamma_N)^4 \times L^\infty(\Omega_0) \times \prod_{j=1}^k L^\infty(\Omega_j)$. Moreover, for $q \in (2,p]$ and $\tau > 1$, we introduce the sets

$$N_{q,\tau} = \Big\{v \in Y_q \colon |v_i| < \tau,\ i=1,3,4,\ \frac{1}{\tau} < v_2 + < \tau \ \text{ on } \overline{\Omega_0}\Big\},$$
$$M_{q,\tau} = \Big\{(V, v_D) \in X_q \times \mathcal{H}_p \colon V + Lv_D \in N_{q,\tau}\Big\}.$$

Because of the continuous embeddings $W_0^{1,q}(\Omega_0) \hookrightarrow W^{1,q}(\Omega_0) \hookrightarrow C(\overline{\Omega_0})$ the set $N_{q,\tau}$ is open in Y_q, and the set $M_{q,\tau}$ is open in $X_q \times \mathcal{H}_p$. Clearly, if $q_2 > q_1$ then $N_{q_2,\tau} \subset N_{q_1,\tau}$, $M_{q_2,\tau} \subset M_{q_1,\tau}$, and we have $N_{q,\tau_1} \subset N_{q,\tau_2}$, $M_{q,\tau_1} \subset M_{q,\tau_2}$ for $\tau_1 < \tau_2$. We define the operator $\Psi_{q,\tau} \colon \prod_{j=1}^k L^\infty(\Omega_j)^2 \times N_{q,\tau} \times \mathcal{H}_* \to X_{q'}^*$,

$$\begin{aligned}
&\langle \Psi_{q,\tau}(\zeta_{\mathrm{imp}}, v, g, f_0), \bar V\rangle_{X_{q'}} \\
&= \int_{\Omega_0} \Big\{ \sum_{i,j=1}^4 b_{ij}(\cdot, v)\nabla v_j \cdot \nabla \bar V_i + \widetilde r_0(\cdot, v)(\mathrm{e}^{\frac{v_3+v_4}{v_2}} - 1)(\bar V_3 + \bar V_4)\Big\}\,\mathrm{d}x \\
&- \sum_{j=1}^k \int_{\Omega_j} \Big\{\widetilde r_{j1}(\cdot, v)(\mathrm{e}^{\frac{\zeta_{2j-1}}{v_2}} - \mathrm{e}^{\frac{\zeta_{2j}+v_3}{v_2}})\bar V_3 + \widetilde r_{j2}(\cdot, v)(\mathrm{e}^{\frac{\zeta_{2j}}{v_2}} - \mathrm{e}^{\frac{\zeta_{2j-1}+v_4}{v_2}})\bar V_4\Big\}\,\mathrm{d}x \\
&- \int_{\Omega_0} \Big(f_0 - F_n\big(x, v_2, \frac{v_3+v_1}{v_2}\big) + F_p\big(x, v_2, \frac{v_4 - v_1}{v_2}\big)\Big)\,\bar V_1\,\mathrm{d}x \\
&- \sum_{j=1}^k \int_{\Omega_j} \sum_{l=-1}^0 q_{2j+l}\, F_{2j+l}\big(\cdot, v_2, \frac{\zeta_{2j+l} - q_{2j+l} v_1}{v_2}\big)\,\bar V_1\,\mathrm{d}x \\
&- \int_{\Gamma_N} \sum_{i=1}^4 g_i \bar V_i\,\mathrm{d}\Gamma, \quad \bar V \in X_{q'}.
\end{aligned} \tag{16}$$

Here $q' = q/(q-1)$ denotes the dual exponent of q. Now we introduce the operator $\widetilde{\mathcal{F}}_{q,\tau} \colon \prod_{j=1}^k L^\infty(\Omega_j)^2 \times M_{q,\tau} \times \mathcal{H}_* \to X_{q'}^*$,

$$\widetilde{\mathcal{F}}_{q,\tau}(\zeta_{\mathrm{imp}}, V, v_D, g, f_0) = \Psi_{q,\tau}(\zeta_{\mathrm{imp}}, V + Lv_D, g, f_0).$$

Finally, let $\mathcal{R}_j$, $\mathcal{I}_j \colon L^\infty(\Omega_j)^2 \times M_{q,\tau} \to L^\infty(\Omega)$ be the operators (see (7))

$$\begin{aligned}
\mathcal{R}_j(\zeta_{2j-1}, \zeta_{2j}, V, v_D) &= \Big(\widetilde r_{j1}(\cdot, V + Lv_D) + \widetilde r_{j2}(\cdot, V + Lv_D)\,\mathrm{e}^{v_4/v_2}\Big)\mathrm{e}^{\zeta_{2j-1}/v_2} \\
&\quad - \Big(\widetilde r_{j1}(\cdot, V + Lv_D)\,\mathrm{e}^{v_3/v_2} + \widetilde r_{j2}(\cdot, V + Lv_D)\Big)\mathrm{e}^{\zeta_{2j}/v_2}, \\
\mathcal{I}_j(\zeta_{2j-1}, \zeta_{2j}, V, v_D) &= \sum_{l=-1,0} F_{2j+l}\big(\cdot, v_2, \tfrac{\zeta_{2j+l} - q_{2j+l} v_1}{v_2}\big), \quad j = 1, \ldots, k.
\end{aligned}$$

Let us remember that $\varphi = v_1 = V_1 + Lv_{D1}$, $T = v_2 = V_2 + Lv_{D2}$, $\zeta_n = v_3 = V_3 + Lv_{D3}$ and $\zeta_p = v_4 = V_4 + Lv_{D4}$.

A weak formulation of the system (6) – (13) is

Problem $(\widetilde{\mathbf{P}})$:

Find $(q, \tau, \zeta_{\mathrm{imp}}, V, v_D, g, f)$ such that $q \in (2,p]$, $\tau > 1$, $\zeta_{\mathrm{imp}} \in \prod_{j=1}^k L^\infty(\Omega_j)^2$,

$$(V, v_D) \in M_{q,\tau},\ (g, f) \in \mathcal{H},\ \widetilde{\mathcal{F}}_{q,\tau}(\zeta_{\mathrm{imp}}, V, v_D, g, f_0) = 0,$$
$$\mathcal{R}_j(\zeta_{2j-1}, \zeta_{2j}, V, v_D) = 0, \quad \mathcal{I}_j(\zeta_{2j-1}, \zeta_{2j}, V, v_D) = f_j, \quad j = 1, \ldots, k.$$

We call a solution $(q, \tau, \zeta_{\mathrm{imp}}, V, v_D, g, f)$ of $(\widetilde{\mathrm{P}})$ a thermodynamic equilibrium, if

$$v_i = V_i + L v_{Di} = \text{const}, \quad i = 2, 3, 4, \quad v_3 + v_4 = 0,$$
$$\zeta_{2j-1} = \zeta_{2j} + v_3, \quad \zeta_{2j} = \zeta_{2j-1} + v_4, \quad j = 1, \dots, k.$$

Especially, in thermodynamic equilibrium all reactions are in simultaneous equilibrium. Note, that the last condition, $\zeta_{2j} = \zeta_{2j-1} + v_4$, is a direct consequence of the two relations $v_3 + v_4 = 0$, $\zeta_{2j-1} = \zeta_{2j} + v_3$. Moreover, let us remark, that the equilibrium values of ζ_{imp} needn't be constant and can be functions of the space variable.

Let us give a short outlook on the methods used in the paper. In a first step (see Section 4) we globally eliminate the quantities ζ_{imp} by evaluating the constraints

$$\mathcal{R}_j(\zeta_{2j-1}, \zeta_{2j}, V, v_D) = 0, \quad \mathcal{I}_j(\zeta_{2j-1}, \zeta_{2j}, V, v_D) = f_j, \quad j = 1, \dots, k.$$

Thus we deduce from Problem $(\widetilde{\mathrm{P}})$ a reduced Problem (P) which is equivalent to $(\widetilde{\mathrm{P}})$.

In the second step (see Section 5) we establish a local existence and uniqueness result for (P) near a thermodynamic equilibrium. For this purpose first we will ensure that for boundary data v_{Di}, g_i, $i = 1, \dots, 4$, which are compatible with thermodynamic equilibrium, and for given densities $f_0, \dots, f_k$ there exists a thermodynamic equilibrium. Then we will use the Implicit Function Theorem to prove the existence of a unique stationary solution to (P) in a neighbourhood of this thermodynamic equilibrium.

We apply a weak formulation in $W^{1,p}$-function spaces such that the requirements of the Implicit Function Theorem can be validated. To obtain the necessary differentiability properties we use properties of Nemyzki operators established in [12]. Additionally, we take advantage of regularity results for strongly coupled linear elliptic systems with mixed boundary conditions in [10]. Let us mention, that the methods used here can be applied only for two-dimensional domains Ω_0.

4. Elimination of the constraints. Weak formulation (P).

The first step consists in a discussion of the constraints (7) for fixed $j \in \{1, \dots, k\}$,

$$R_{j1} - R_{j2} = 0, \qquad u_{2j-1} + u_{2j} = f_j \quad \text{on } \Omega_j.$$

We use the state equations (10), the rate formulas (12) and obtain on Ω_j two equations for the quantities ζ_{2j-1}, ζ_{2j},

$$\begin{aligned} &\Big(\widetilde{r}_{j1} + \widetilde{r}_{j2}\, \mathrm{e}^{v_4/v_2}\Big)\, \mathrm{e}^{\zeta_{2j-1}/v_2} - \Big(\widetilde{r}_{j1}\, \mathrm{e}^{v_3/v_2} + \widetilde{r}_{j2}\Big)\, \mathrm{e}^{\zeta_{2j}/v_2} = 0, \\ &F_{2j-1}(\cdot, v_2, \frac{\zeta_{2j-1} - q_{2j-1} v_1}{v_2}) + F_{2j}(\cdot, v_2, \frac{\zeta_{2j} - q_{2j} v_1}{v_2}) = f_j. \end{aligned} \tag{17}$$

The first equation in (17) yields

$$\zeta_{2j-1} = \zeta_{2j} + v_2 \ln \frac{\widetilde{r}_{j1}(\cdot, v)\, \mathrm{e}^{v_3/v_2} + \widetilde{r}_{j2}(\cdot, v)}{\widetilde{r}_{j1}(\cdot, v) + \widetilde{r}_{j2}(\cdot, v)\, \mathrm{e}^{v_4/v_2}} = \zeta_{2j} + Q_j(\cdot, v), \tag{18}$$

where the function $Q_j \colon \Omega_j \times \widetilde{\mathcal{V}}_* \to \mathbb{R}$ is of the class $D(\Omega_j, \Sigma, \widetilde{\mathcal{V}}_*)$. For arguments v with $v_3 = -v_4$ we find that

$$\frac{\partial Q_j}{\partial v_1}(x, v_1, v_2, v_3, -v_3) = 0 \quad \forall (x, v_1, v_2, v_3) \in (\Omega_j \setminus \Sigma) \times \mathbb{R} \times (0, \infty) \times \mathbb{R}. \tag{19}$$

Inserting relation (18) into the second equation of (17) leads to

$$F_{2j-1}(\cdot, v_2, \frac{\zeta_{2j} + Q_j(\cdot, v) - q_{2j-1}v_1}{v_2}) + F_{2j}(\cdot, v_2, \frac{\zeta_{2j} - q_{2j}v_1}{v_2}) = P_j(\cdot, \zeta_{2j}, v) = f_j,$$

where the function $P_j \colon \Omega_j \times \mathbb{R} \times \widetilde{\mathcal{V}}_* \to \mathbb{R}_+$ is of the class $D(\Omega_j, \Sigma, \mathbb{R} \times \widetilde{\mathcal{V}}_*)$.

LEMMA 1. *There exists a unique function $S_j = S_j(x, v, f_j)$ such that*

$$P_j(\cdot, \zeta_{2j}, v) = f_j \tag{20}$$

if and only if $\zeta_{2j} = S_j(\cdot, v, f_j)$. The function $S_j \colon \Omega_j \times \mathcal{V}_j \to \mathbb{R}$ is of the class $D(\Omega_j, \Sigma, \mathcal{V}_j)$.

Proof. 1. The assumptions on F_{2j-1}, F_{2j} formulated in (A4) ensure that for all $K > 1$ there is a $c_K > 0$ such that

$$\frac{\partial P_j}{\partial \zeta_{2j}}(x, \zeta_{2j}, v) \geq c_K \quad \forall (x, \zeta_{2j}, v) \in (\Omega_j \setminus \Sigma) \times [-K, K]^2 \times [\tfrac{1}{K}, K] \times [-K, K]^2, \tag{21}$$

$$\lim_{\zeta_{2j} \to -\infty} P_j(x, \zeta_{2j}, v) = 0, \qquad \lim_{\zeta_{2j} \to \infty} P_j(x, \zeta_{2j}, v) = \infty \quad \forall (x, v) \in (\Omega_j \setminus \Sigma) \times \widetilde{\mathcal{V}}_*.$$

2. First, let $x \in \Omega_j \setminus \Sigma$ be fixed. By the intermediate value theorem we obtain for arbitrarily given $v \in \widetilde{\mathcal{V}}_*$ a unique solution $\zeta_{2j} = S_j(x, v, f_j)$ of (20). Moreover, if $v \in [-K, K] \times [1/K, K] \times [-K, K]^2$ then $|S_j(x, v, f_j)| \leq c_K$. Multiplying the relation

$$P_j(x, S_j(x, v, f_j), v) - P_j(x, S_j(x, \bar{v}, \bar{f}_j), \bar{v}) = f_j - \bar{f}_j$$

by $S_j(x, v, f_j) - S_j(x, \bar{v}, \bar{f}_j)$, using the locally strong monotonicity property induced by (21) and dividing by $|S_j(x, v, f_j) - S_j(x, \bar{v}, \bar{f}_j)|$ we find

$$c_{\widetilde{K}} |S_j(x, v, f_j) - S_j(x, \bar{v}, \bar{f}_j)| \leq |f_j - \bar{f}_j| + |P_j(x, S_j(x, \bar{v}, \bar{f}_j), v) - P_j(x, S_j(x, \bar{v}, \bar{f}_j), \bar{v})|.$$

Using the continuity properties of P_j we thus obtain the continuity property of S_j for fixed $x \in \Omega_j \setminus \Sigma$ as required for functions of the class $D(\Omega_j, \Sigma, \mathcal{V}_j)$. Differentiating the relation $P_j(x, S_j(x, v, f_j), v) = f_j$ by v and f_j, respectively we obtain

$$\frac{\partial S_j}{\partial v}(x, v, f_j) = -\Big(\frac{\partial P_j}{\partial \zeta_{2j}}(x, S_j(x, v, f_j), v)\Big)^{-1} \frac{\partial P_j}{\partial v}(x, S_j(x, v, f_j), v),$$

$$\frac{\partial S_j}{\partial f_j}(x, v, f_j) = \Big(\frac{\partial P_j}{\partial \zeta_{2j}}(x, S_j(x, v, f_j), v)\Big)^{-1}.$$

Having in mind that P_j is of the class $D(\Omega_j, \Sigma, \mathbb{R} \times \widetilde{\mathcal{V}}_*)$ and the property (21) we can derive the local boundedness and continuity properties of the derivatives of S_j with respect to z and f_j on $\Omega_j \setminus \Sigma$ which are required for a function in the class $D(\Omega_j, \Sigma, \mathcal{V}_j)$.

3. For $x \in \Sigma$ we set $S_j(x, v, f_j) = 0$.

4. It remains to show the measurability properties of the function S_j postulated for functions of the class $D(\Omega_j, \Sigma, \mathcal{V}_j)$. Since the function P_j is in $Car(\Omega_j \setminus \Sigma, \mathbb{R} \times \widetilde{\mathcal{V}}_*)$, Theorem 3 guarantees that for all $\epsilon > 0$ there exists a closed set $A_\epsilon \subset (\Omega_j \setminus \Sigma)$ such that $\mathrm{mes}((\Omega_j \setminus \Sigma) \setminus A_\epsilon) < \epsilon$ and $P_j|_{A_\epsilon \times \mathbb{R} \times \widetilde{\mathcal{V}}_*}$ is continuous. For arbitrarily fixed $\epsilon > 0$, let $x, \bar{x} \in A_\epsilon$. Let (v, f_j), $(\bar{v}, \bar{f}_j) \in \mathcal{V}_j$ such that (v, f_j), $(\bar{v}, \bar{f}_j) \in [-K, K] \times [1/K, K] \times [-K, K]^2 \times [1/K, K]$ and $|S_j(x, v, f_j)|$, $|S_j(\bar{x}, \bar{v}, \bar{f}_j)| \leq K$ for suitable $K > 1$. Multiplying the relation

$$P_j(x, S_j(x, v, f_j), v) - P_j(\bar{x}, S_j(\bar{x}, \bar{v}, \bar{f}_j), \bar{v}) = f_j - \bar{f}_j$$

by $S_j(x,v,f_j) - S_j(\bar{x},\bar{v},\bar{f}_j)$, using the locally strong monotonicity property induced by (21) and dividing by $|S_j(x,v,f_j) - S_j(\bar{x},\bar{v},\bar{f}_j)|$ we find

$$c_K|S_j(x,v,f_j) - S_j(\bar{x},\bar{v},\bar{f}_j)| \le |f_j - \bar{f}_j| + |P_j(x,S_j(\bar{x},\bar{v},\bar{f}_j),v) - P_j(\bar{x},S_j(\bar{x},\bar{v},\bar{f}_j),\bar{v},\bar{f})|.$$

Since $P_j|_{A_\epsilon\times\mathbb{R}\times\widetilde{\mathcal{V}}_*}$ is continuous this estimate ensures, that $S_j|_{A_\epsilon\times\mathcal{V}_j}$ is continuous, too. Therefore, again by Theorem 3, $S_j\colon (\Omega_j\setminus\Sigma)\times\mathcal{V}_j\to\mathbb{R}$ as well as $S_j\colon \Omega_j\times\mathcal{V}_j\to\mathbb{R}$ are Caratheodory functions. ∎

Using the relation $\zeta_{2j} = S_j(\cdot,v,f_j)$ and (18) we rewrite the reaction rates of the ionization reactions of acceptors and donors in (1), (2) in the form

$$r_{j1}\Big(\mathrm{e}^{\frac{\zeta_{2j-1}}{v_2}} - \mathrm{e}^{\frac{\zeta_{2j}+v_3}{v_2}}\Big) = r_{j2}\Big(\mathrm{e}^{\frac{\zeta_{2j}}{v_2}} - \mathrm{e}^{\frac{\zeta_{2j-1}+v_4}{v_2}}\Big) = \frac{\widetilde{r}_{j1}\widetilde{r}_{j2}\,\mathrm{e}^{S_j(\cdot,v,f_j)/v_2}}{\widetilde{r}_{j1}+\widetilde{r}_{j2}\,\mathrm{e}^{v_4/v_2}}\Big(1-\mathrm{e}^{\frac{v_3+v_4}{v_2}}\Big) \quad \text{on } \Omega_j.$$

In other words these reaction rates take the form of a Shockley Read Hall generation-recombination term with a kinetic coefficient depending on v and f_j. To obtain a uniform notation we define for $(v,f_j)\in\mathcal{V}_j$ the functions

$$\begin{aligned} &\widehat{r}_0(\cdot,v,f_0) = \widetilde{r}_0(\cdot,v) \quad \text{on } \Omega_0,\\ &\widehat{r}_j(\cdot,v,f_j) = \frac{\widetilde{r}_{j1}\widetilde{r}_{j2}\,\mathrm{e}^{S_j(\cdot,v,f_j)/v_2}}{\widetilde{r}_{j1}+\widetilde{r}_{j2}\,\mathrm{e}^{v_4/v_2}} \quad \text{on } \Omega_j,\quad j=1,\dots,k. \end{aligned} \tag{22}$$

Remark 3. Remark 1 and Lemma 1 guarantee that $\widehat{r}_j\colon \Omega_j\times\mathcal{V}_j\to\mathbb{R}_+$ are of the class $D(\Omega_j,\Sigma,\mathcal{V}_j)$, $j=0,\dots,k$.

We introduce for $(v,f_j)\in\mathcal{V}_j$ the functions

$$\begin{aligned} H_0(\cdot,v,f_0) &= f_0 - F_n(\cdot,v_2,\frac{v_3+v_1}{v_2}) + F_p(\cdot,v_2,\frac{v_4-v_1}{v_2}) \quad \text{on } \Omega_0,\\ H_j(\cdot,v,f_j) &= q_{2j-1}F_{2j-1}(\cdot,v_2,\frac{S_j(\cdot,v,f_j)+Q_j(\cdot,v)-q_{2j-1}v_1}{v_2})\\ &\quad + q_{2j}F_{2j}(\cdot,v_2,\frac{S_j(\cdot,v,f_j)-q_{2j}v_1}{v_2}) \quad \text{on } \Omega_j,\quad j=1,\dots,k, \end{aligned} \tag{23}$$

where Q_j is given in (18).

Remark 4. Due to (A4) $H_0\colon \Omega_0\times\mathcal{V}_0\to\mathbb{R}$ is of the class $D(\Omega_0,\Sigma,\mathcal{V}_0)$. The function $-H_0(x,\cdot,v_2,v_3,v_4,f_0)\colon \mathbb{R}\to\mathbb{R}$ is monotonic increasing for $(x,v_2,v_3,v_4,f_0)\in(\Omega_0\setminus\Sigma)\times(0,\infty)\times\mathbb{R}^3$. For all $(v_2,v_3,v_4,f_0)\in(0,\infty)\times\mathbb{R}^3$ there exists a constant $c=c(v_2,v_3,v_4,f_0)>1$ such that $|H_0(x,v,f_0)|\le c(1+\mathrm{e}^{c|v_1|})$ for $x\in\Omega_0\setminus\Sigma$, $v_1\in\mathbb{R}$.

Lemma 2. *The functions $H_j\colon \Omega_j\times\mathcal{V}_j\to\mathbb{R}$ are of the class $D(\Omega_j,\Sigma,\mathcal{V}_j)$. For all $(v,f_j)\in\mathcal{V}_j$ there exists a constant $c=c(f_j)>1$ such that $H_j(x,v,f_j)\le c$ for all $x\in\Omega_j\setminus\Sigma$. The function $-H_j(x,\cdot,v_2,v_3,-v_3,f_j)\colon \mathbb{R}\to\mathbb{R}$ is monotonic increasing for all $(x,v_2,v_3,f_j)\in(\Omega_j\setminus\Sigma)\times(0,\infty)\times\mathbb{R}\times(0,\infty)$, $j=1,\dots,k$.*

Proof. According to (A4) and the properties of the functions S_j and Q_j we obtain that the functions $H_j\colon \Omega_j\times\mathcal{V}_j\to\mathbb{R}$ are of the class $D(\Omega_j,\Sigma,\mathcal{V}_j)$. Due to the definition of H_j and equation (20) we find $H_j(x,v,f_j)\le(|q_{2j-1}|+|q_{2j}|)\,f_j$ for all $(x,v,f_j)\in(\Omega_j\setminus\Sigma)\times\mathcal{V}_j$.

For the proof of the last assertion we differentiate (20) with respect to v_1,

$$\frac{\partial F_{2j-1}}{\partial y}\Big\{\frac{\partial S_j}{\partial v_1}+\frac{\partial Q_j}{\partial v_1}-q_{2j-1}\Big\}\frac{1}{v_2}+\frac{\partial F_{2j}}{\partial y}\Big\{\frac{\partial S_j}{\partial v_1}-q_{2j}\Big\}\frac{1}{v_2}=0.$$

According to (19) we have $\frac{\partial Q_j}{\partial v_1} = 0$ in arguments $(x, v_1, v_2, v_3, -v_3)$. Therefore the last equation leads to

$$\frac{\partial S_j}{\partial v_1} = \Big(q_{2j-1}\frac{\partial F_{2j-1}}{\partial y} + q_{2j}\frac{\partial F_{2j}}{\partial y}\Big)\Big(\frac{\partial F_{2j-1}}{\partial y} + \frac{\partial F_{2j}}{\partial y}\Big)^{-1}. \tag{24}$$

Next, we differentiate H_j with respect to v_1 and obtain

$$\frac{\partial H_j}{\partial v_1} = q_{2j-1}\frac{\partial F_{2j-1}}{\partial y}\Big\{\frac{\partial S_j}{\partial v_1} + \frac{\partial Q_j}{\partial v_1} - q_{2j-1}\Big\}\frac{1}{v_2} + q_{2j}\frac{\partial F_{2j}}{\partial y}\Big\{\frac{\partial S_j}{\partial v_1} - q_{2j}\Big\}\frac{1}{v_2}.$$

Using that $\frac{\partial Q_j}{\partial v_1} = 0$ in arguments $(x, v_1, v_2, v_3, -v_3)$ and inserting (24) we find

$$\frac{\partial H_j}{\partial v_1} = -\frac{1}{v_2}\,\frac{\partial F_{2j-1}}{\partial y}\,\frac{\partial F_{2j}}{\partial y}\Big(\frac{\partial F_{2j-1}}{\partial y} + \frac{\partial F_{2j}}{\partial y}\Big)^{-1}$$

for arguments $(x, v_1, v_2, v_3, -v_3, f_j)$. (A4) guarantees that $\frac{\partial F_{2j+l}}{\partial y}$ is nonnegative, $l = -1, 0$. Since $\frac{\partial H_j}{\partial v_1}(x, v_1, v_2, v_3, -v_3, f_j)$ is nonpositive for $(x, v_1, v_2, v_3, f_j) \in (\Omega_j \setminus \Sigma) \times \mathbb{R} \times (0,\infty) \times \mathbb{R} \times (0,\infty)$, and v_2 is positive, we obtain the desired result. ■

Now, for $q \in (2, p]$, $\tau > 1$, we introduce the operator

$$G_{q,\tau}\colon N_{q,\tau} \times L^\infty(\Omega_0) \times \prod_{j=1}^k \{f_j \in L^\infty(\Omega_j)\colon \operatorname{ess\,inf} f_j > 0\} \to \prod_{j=1}^k L^\infty(\Omega_j)^2,$$

$$G_{q,\tau}(v, f) = \Big(G^1_{q,\tau}(v, f_1), \dots, G^{2k}_{q,\tau}(v, f_k)\Big),$$

$$G^{2j-1}_{q,\tau},\ G^{2j}_{q,\tau}\colon N_{q,\tau} \times \{f_j \in L^\infty(\Omega_j)\colon \operatorname{ess\,inf} f_j > 0\} \to L^\infty(\Omega_j),$$

which are defined pointwise a.e. on Ω_j by

$$G^{2j-1}_{q,\tau}(v, f_j)(x) = S_j(x, v(x), f_j(x)) + Q_j(x, v(x)), \quad G^{2j}_{q,\tau}(v, f_j)(x) = S_j(x, v(x), f_j(x)),$$

$j = 1, \dots, k$, (see (18) and Lemma 1). Next, we use the notation

$$w = (v_D, g, f), \quad v = V + Lv_D$$

and define the operator $\mathcal{F}_{q,\tau}\colon M_{q,\tau} \times \mathcal{H} \to X^*_{q'}$ by

$$\mathcal{F}_{q,\tau}(V, w) = \Psi_{q,\tau}(G_{q,\tau}(V + Lv_D, f), V + Lv_D, g, f_0). \tag{25}$$

In other words (see (16), (22) and (23)) we have

$$\begin{aligned}\langle \mathcal{F}_{q,\tau}(V, w), \psi\rangle_{X_{q'}} &= \int_{\Omega_0} \sum_{i,j=1}^4 b_{ij}(\cdot, v)\nabla v_j \cdot \nabla\psi_i \,\mathrm{d}x - \int_{\Gamma_N} \sum_{i=1}^4 g_i\psi_i \,\mathrm{d}\Gamma \\ &\quad + \sum_{j=0}^k \int_{\Omega_j} \Big\{\widehat{r}_j(\cdot, v, f_j)(\mathrm{e}^{(v_3+v_4)/v_2} - 1)(\psi_3 + \psi_4) - H_j(\cdot, v, f_j)\psi_1\Big\}\,\mathrm{d}x,\end{aligned}$$

$\psi \in X_{q'}$. In this notation another weak formulation of the system (6) – (13) is

Problem (P):

Find (q, τ, V, w) such that $q \in (2, p]$, $\tau > 1$, $(V, w) \in X_q \times \mathcal{H}_p \times \mathcal{H}$,
$(V, v_D) \in M_{q,\tau}$, $\mathcal{F}_{q,\tau}(V, w) = 0$.

If (q,τ,V,w) is a solution to (P) then $(\widetilde{q},\widetilde{\tau},V,w)$ with $\widetilde{q}\in(2,q]$ and $\widetilde{\tau}\geq\tau$ is a solution to (P), too.

We call a solution (q,τ,V,w) of (P) a thermodynamic equilibrium, if

$$v_i = V_i + Lv_{Di} = \text{const},\quad i=2,3,4,\quad v_3+v_4=0.$$

REMARK 5 (Relation between the Problems (P) and $(\widetilde{\mathrm{P}})$). There exists the following relation between the Problems (P) and $(\widetilde{\mathrm{P}})$: $(q,\tau,\zeta_{\mathrm{imp}},V,v_D,g,f)$ is a solution to Problem $(\widetilde{\mathrm{P}})$ if and only if (q,τ,V,v_D,g,f) is a solution to Problem (P) and $\zeta_{\mathrm{imp}} = G_{q,\tau}(V+Lv_D,f)$. Especially, $(q,\tau,\zeta_{\mathrm{imp}},V,v_D,g,f)$ is a thermodynamic equilibrium of Problem $(\widetilde{\mathrm{P}})$ if and only if (q,τ,V,v_D,g,f) is a thermodynamic equilibrium of (P) and $\zeta_{\mathrm{imp}} = G_{q,\tau}(V+Lv_D,f)$. Therefore we can consider both problems to be equivalent and to represent weak formulations of the system (6) – (13). In particular, our main results (formulated for Problem (P) in Theorem 1, Theorem 2 and Corollary 1) carry over to the Problem $(\widetilde{\mathrm{P}})$.

5. Results for (P).

LEMMA 3 (Differentiability). *We assume* (A1) – (A5). *The operator* $\mathcal{F}_{q,\tau}\colon M_{q,\tau}\times\mathcal{H}\to X^*_{q'}$ *is continuously differentiable for all exponents* $q\in(2,p]$ *and all* $\tau>1$.

Proof. Let $q\in(2,p]$ and $\tau>1$ be arbitrarily fixed. We write $v=V+v^D$, where $v^D = Lv_D\in Y_p$. Remember that $L\colon\mathcal{H}_p\to Y_p$ is a continuous linear operator. Moreover, for $(V,v_D)\in M_{q,\tau}$ the pair (V,v^D) belongs to

$$\widehat{M}_{q,\tau} = \Big\{(V,v^D)\in X_q\times Y_p\colon V+v^D\in N_{q,\tau}\Big\}.$$

We prove that the operator $\widehat{\mathcal{F}}_{q,\tau}\colon\widehat{M}_{q,\tau}\times\mathcal{H}\to X^*_{q'}$,

$$\widehat{\mathcal{F}}_{q,\tau}(V,v^D,g,f) = \Psi_{q,\tau}(G_{q,\tau}(V+v^D,f),V+v^D,g,f_0)$$

is continuously differentiable. Then the desired result follows by the chain rule. We split up the operator $\widehat{\mathcal{F}}_{q,\tau} = A^0+A^1-B$, where $A^0\colon\widehat{M}_{q,\tau}\times L^\infty(\Omega_0)\times\prod_{j=1}^k\big\{y\in L^\infty(\Omega_j)\colon\operatorname{ess\,inf} y>0\big\}\to X^*_{q'}$, $A^1\colon\widehat{M}_{q,\tau}\to X^*_{q'}$, $B\colon L^\infty(\Gamma_N)^4\to X^*_{q'}$,

$$\langle A^0(V,v^D,f),\psi\rangle_{X_{q'}} = \int_{\Omega_0}\sum_{i,j=1}^4 b_{ij}(\cdot,v)\nabla V_j\cdot\nabla\psi_i\,\mathrm{d}x + \sum_{j=0}^k\int_{\Omega_j}\Big\{\widehat{r}_j(\cdot,v,f_j)(\mathrm{e}^{(v_3+v_4)/v_2}-1)(\psi_3+\psi_4) - H_j(\cdot,v,f_j)\psi_1\Big\}\,\mathrm{d}x$$

$$\langle A^1(V,v^D),\psi\rangle_{X_{q'}} = \int_{\Omega_0}\sum_{i,j=1}^4 b_{ij}(\cdot,v)\nabla v^D_j\cdot\nabla\psi_i\,\mathrm{d}x,$$

$$\langle Bg,\psi\rangle_{X_{q'}} = \int_{\Gamma_N}\sum_{i=1}^4 g_i\psi_i\,\mathrm{d}\Gamma,\qquad v=V+v^D,\quad\forall\psi\in X_{q'}.$$

For the proof for $A^0\colon\widehat{M}_{q,\tau}\times L^\infty(\Omega_0)\times\prod_{j=1}^k\big\{y\in L^\infty(\Omega_j)\colon\operatorname{ess\,inf} y>0\big\}\to X^*_{q'}$ we refer to [12, p. 1465, Lemma 2.2]. Again using [12, Lemma 2.2] we find that $A^1\colon\widehat{M}_{q,\tau}\to X^*_{p'}$

is continuously differentiable, and the continuous embedding $W^{1,p} \hookrightarrow W^{1,q}$ then ensures also the differentiability of $A^1\colon \widehat{M}_{q,\tau} \to X^*_{q'}$. Note that our assumptions guarantee the validity of (H2.1), (H2.2), (H2.3) in [12]. Assertions concerning the operator B are trivial. Especially, for the linearization of $\mathcal{F}_{q,\tau}$ with respect to V we have

$$\begin{aligned} \langle \partial_V \mathcal{F}_{q,\tau}(V,w)\overline{V},\psi\rangle_{X_{q'}} &= \int_{\Omega_0} \sum_{i,j=1}^{4} \big(b_{ij}(\cdot,v)\nabla\overline{V}_j + \partial_v b_{ij}(\cdot,v)\cdot\overline{V}\,\nabla v_j\big)\cdot\nabla\psi_i\,\mathrm{d}x \\ &\quad - \sum_{j=0}^{k}\int_{\Omega_j} \partial_v H_j(\cdot,v,f_j)\cdot\overline{V}\,\psi_1\,\mathrm{d}x \\ &\quad + \sum_{j=0}^{k}\int_{\Omega_j} \partial_v\big[\widehat{r}_j(\cdot,v,f_j)(\mathrm{e}^{(v_3+v_4)/v_2}-1)\big]\cdot\overline{V}\,(\psi_3+\psi_4)\,\mathrm{d}x \end{aligned} \tag{26}$$

for all $\overline{V}\in X_q$ and $\psi\in X_{q'}$. ■

Next, we describe necessary conditions for the data such that a thermodynamic equilibrium can exist. Let

$$\Lambda = \Big\{w=(v_D,g,f)\in\mathcal{H}_p\times\mathcal{H}\colon v_{Di}=\text{const},\ g_i=0,\ i=2,3,4,\ v_{D2}>0,\ v_{D3}+v_{D4}=0\Big\}.$$

THEOREM 1 (Thermodynamic equilibrium). *We assume* (A1) – (A5). *Let* $w^*=(v^*_D,g^*,f^*)\in\Lambda$ *be given.*
i) Then there exist an exponent $q_0\in(2,p]$, *a constant* $\tau>1$, *and a function* $V^*_1\in W^{1,q_0}_0(\Omega_0\cup\Gamma_N)$ *such that the pair* $(V^*,v^*_D)=((V^*_1,0,0,0),v^*_D)\in M_{q_0,\tau}$ *and the equation* $\mathcal{F}_{q_0,\tau}(V^*,w^*)=0$ *holds. In other words,* (q_0,τ,V^*,w^*) *is a solution to* (P).
ii) (q_0,τ,V^*,w^*) *is a thermodynamic equilibrium of* (P).

Proof. 1. For the given $w^*=(v^*_D,g^*,f^*)$ we define the functions $h_j\colon\Omega_j\times\mathbb{R}\to\mathbb{R}$ by

$$h_j(x,\phi) = -H_j(x,(\phi,0,0,0)+Lv^*_D,f^*_j)$$

and consider the operator $\mathcal{E}\colon H^1_0(\Omega_0\cup\Gamma_N)\to H^{-1}(\Omega_0\cup\Gamma_N)$,

$$\begin{aligned} \langle\mathcal{E}(\phi),\overline{\phi}\rangle_{H^1_0(\Omega_0\cup\Gamma_N)} &= \int_{\Omega_0}\varepsilon\nabla(\phi+Lv^*_{D1})\cdot\nabla\overline{\phi}\,\mathrm{d}x - \int_{\Gamma_N} g^*_1\overline{\phi}\,\mathrm{d}\Gamma \\ &\quad + \sum_{j=0}^{k}\int_{\Omega_j} h_j(\cdot,\phi)\,\overline{\phi}\,\mathrm{d}x, \quad \overline{\phi}\in H^1_0(\Omega\cup\Gamma_N). \end{aligned} \tag{27}$$

The properties of Γ_D, ε and H_j stated in (A1), (A3) and Remark 4, Lemma 2 supply the strong monotonicity of the operator $\mathcal{E}$. Next we prove the hemicontinuity of $\mathcal{E}$. We show that the mapping $t\mapsto\langle\mathcal{E}(\phi+t\hat{\phi}),\overline{\phi}\rangle_{H^1_0(\Omega_0\cup\Gamma_N)}$ for arbitrarily given ϕ, $\hat{\phi}$, $\overline{\phi}\in H^1_0(\Omega_0\cup\Gamma_N)$ is continuous on $[0,1]$. Let $t_0\in[0,1]$, $t_n\to t_0$, $t_n\in[0,1]$. Then

$$\begin{aligned} &\langle\mathcal{E}(\phi+t_n\hat{\phi})-\mathcal{E}(\phi+t_0\hat{\phi}),\overline{\phi}\rangle_{H^1_0(\Omega_0\cup\Gamma_N)} \\ &\le c|t_n-t_0|\|\hat{\phi}\|_{H^1}\|\overline{\phi}\|_{H^1} + \sum_{j=0}^{k}\Big|\int_{\Omega_j}\Big[h_j(\cdot,\phi+t_n\hat{\phi})-h_j(\cdot,\phi+t_0\hat{\phi})\Big]\overline{\phi}\,\mathrm{d}x\Big|. \end{aligned} \tag{28}$$

According to Remark 4, Lemma 2 we have $h_j(x, \phi + t_n\hat{\phi}) \to h_j(x, \phi + t_0\hat{\phi})$ and

$$|h_j(x, \phi + t_n\hat{\phi})| \le \widetilde{c}(1 + \mathrm{e}^{\widetilde{c}\,(|\phi|+|\hat{\phi}|)}) \quad \text{f.a.a. } x \in \Omega_j,\ j = 0, \dots, k.$$

Now we use the embedding result of Trudinger [16] for two-dimensional Lipschitzian domains which tells us that $\|\mathrm{e}^{|v|}\|_{L^2} \le d(\|v\|_{H^1})$ for all $v \in H^1(\Omega_0)$, where $d\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, monotonic increasing function with $\lim_{y\to\infty} d(y) = \infty$. Since $\overline{\phi} \in L^2(\Omega_0)$ we get a integrable upper bound for the integrands in the last term in (28) and Lebesgue's Dominated Convergence Theorem leads to the hemicontinuity of $\mathcal{E}$. Since $\mathcal{E}$ is strongly monotone and hemicontinuous there exists a unique solution $\phi \in H_0^1(\Omega_0 \cup \Gamma_N)$ of $\mathcal{E}(\phi) = 0$. Especially we have $\|\phi\|_{H^1} \le \widehat{c}$, where $\widehat{c}$ depends only on the data w^*.

2. Now we prove that this solution possesses more regularity. We define

$$\begin{aligned}\langle \mathcal{G}, \overline{\phi}\rangle_{H_0^1(\Omega_0\cup\Gamma_N)} &= \int_{\Omega_0} \Big\{ -\varepsilon \nabla v_1^{D*} \cdot \nabla\overline{\phi} + \phi\overline{\phi}\Big\}\,\mathrm{d}x + \int_{\Gamma_N} g_1^*\,\overline{\phi}\,\mathrm{d}\Gamma \\ &\quad - \sum_{j=0}^{k} \int_{\Omega_j} h_j(\cdot, \phi)\overline{\phi}\,\mathrm{d}x, \\ \langle \mathcal{E}_0(\phi), \overline{\phi}\rangle_{H_0^1(\Omega_0\cup\Gamma_N)} &= \int_{\Omega_0} \Big\{\varepsilon\nabla\phi\cdot\nabla\overline{\phi} + \phi\,\overline{\phi}\Big\}\mathrm{d}x, \quad \overline{\phi} \in H_0^1(\Omega_0 \cup \Gamma_N).\end{aligned}$$

Since $v_1^{D*} = Lv_{D1}^* \in W^{1,p}(\Omega_0)$ is a fixed element there is a $\overline{c} > 0$ such that $|v_1^{D*}| \le \overline{c}$. From the properties of H_j in Remark 4, Lemma 2 we find $|h_j(x, \phi)| \le c(v^{D*})\,(1 + \mathrm{e}^{c|v_1^{D*}+\phi|}) \le \tilde{c}(v^{D*})(1 + \mathrm{e}^{c\,\overline{c}|\phi|})$ f.a.a. $x \in \Omega_j$, $j = 0, \dots, k$. And therefore the embedding result of Trudinger mentioned in the first step of this proof yields

$$\|h_j(\cdot, \phi)\|_{L^2(\Omega_j)} \le \tilde{c}(z^{D*})\,(1 + d(\|\phi\|_{H^1})) \le \hat{c}, \quad j = 0, \dots, k.$$

Furthermore, using that $w^* \in \Lambda$ is fixed it results that $\mathcal{G} \in W^{-1,p}(\Omega_0 \cup \Gamma_N)$. Thus taking benefit from Grögers regularity result for elliptic equations with mixed boundary conditions [10] applied to the equation $\mathcal{E}_0(\phi) = \mathcal{G}$ we obtain a $q_0 \in (2, p]$ such that $\phi \in W^{1,q_0}(\Omega_0 \cup \Gamma_N)$ and $\|\phi\|_{W^{1,q_0}} \le c_{q_0}\|\mathcal{G}\|_{W^{-1,p}(\Omega_0\cup\Gamma_N)}$. (According to (A1) it is guaranteed that $\Omega_0 \cup \Gamma_N$ is regular in the sense of Gröger.)

3. The continuous embedding $W^{1,q_0}(\Omega_0) \hookrightarrow C(\bar{\Omega}_0)$ and the properties of L ensure that $\|\phi + v_1^{D*}\|_{C(\bar{\Omega}_0)} \le c(q_0, w^*)$. We set $V_1^* = \phi$, $V_i^* = 0$, $i = 2, 3, 4$, and use that $w^* \in \Lambda$. Thus we find a constant $\tau > 1$ such that $(V^*, v_D^*) = ((V_1^*, 0, 0, 0), v_D^*) \in M_{q_0,\tau}$ and $\mathcal{F}_{q_0,\tau}(V^*, w^*) = 0$ which means (q_0, τ, V^*, w^*) is a solution to Problem (P). Moreover, (q_0, τ, V^*, w^*) is a thermodynamic equilibrium of (P). ∎

We denote by $\mathcal{LIS}(X, Y)$ the set of linear isomorphisms between two Banach spaces X and Y.

Lemma 4 (Isomorphism property of the linearization). *We assume* (A1) – (A5). *Let* $w^* = (v_D^*, g^*, f^*) \in \Lambda$ *be given. Let* (q_0, τ, V^*, w^*) *be the equilibrium solution from Theorem 1. Then there exists some* $q_1 \in (2, q_0]$ *such that the operator* $\partial_V\mathcal{F}_{q_1,\tau}(V^*, w^*)$ *belongs to* $\mathcal{LIS}(X_{q_1}, X_{q_1'}^*)$.

Proof. 1. Let $q \in (2, q_0]$ and $\overline{V} \in X_q$. The linearization is given in (26) and must be calculated in the point (V^*, w^*). Let $v^* = V^* + Lv_D^*$. Since $\nabla v_i^* = 0$, $i = 2, 3, 4$,

$v_3^* + v_4^* = 0$ and

$$\partial_v \Big[\widehat{r}_j(\cdot, v^*, f_j^*)\Big(\mathrm{e}^{(v_3^*+v_4^*)/v_2^*} - 1\Big)\Big] \cdot \overline{V} = \partial_v \widehat{r}_j(\cdot, v^*, f_j^*) \cdot \overline{V} \Big(\mathrm{e}^{(v_3^*+v_4^*)/v_2^*} - 1\Big)$$
$$+\widehat{r}_j(\cdot, v^*, f_j^*)\,\mathrm{e}^{(v_3^*+v_4^*)/v_2^*}\Big(\frac{1}{v_2^*}(\overline{V}_3 + \overline{V}_4) - \frac{v_3^* + v_4^*}{v_2^{*2}}\overline{V}_2\Big),$$

we obtain according to (26) that

$$\begin{aligned}\langle \partial_V \mathcal{F}_{q,\tau}(V^*, w^*)\overline{V}, \psi\rangle_{X_{q'}} = &\int_{\Omega_0} \sum_{i,j=1}^{4} b_{ij}(\cdot, v^*)\nabla \overline{V}_j \cdot \nabla\psi_i \,\mathrm{d}x \\ &- \sum_{j=0}^{k} \int_{\Omega_j} \partial_v H_j(\cdot, v^*, f_j^*) \cdot \overline{V}\,\psi_1 \,\mathrm{d}x \\ &+ \sum_{j=0}^{k} \int_{\Omega_j} \widehat{r}_j(\cdot, v^*, f_j^*)\,\frac{\overline{V}_3 + \overline{V}_4}{v_2^*}\,(\psi_3 + \psi_4)\,\mathrm{d}x.\end{aligned} \tag{29}$$

2. For v^* we introduce the linear mapping $D(v^*)\colon X_q \to X_q$, which is pointwise defined by

$$\overline{V}(x) = D(v^*)\overline{Z}(x), \quad D(v^*) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v_2^{*2} & 0 & 0 \\ 0 & v_2^* v_3^* & v_2^* & 0 \\ 0 & v_2^* v_4^* & 0 & v_2^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & T^{*2} & 0 & 0 \\ 0 & T^*\zeta_n^* & T^* & 0 \\ 0 & T^*\zeta_p^* & 0 & T^* \end{pmatrix}.$$

Obviously $D(v^*)$ belongs to the set $\mathcal{LIS}(X_q, X_q)$. Next we define the operator $A_q = \partial_V \mathcal{F}_{q,\tau}(V^*, w^*) \circ D(v^*) \in \mathcal{L}(X_q, X_{q'}^*)$. Our aim is to prove that there exists a $q_1 \in (2, q_0]$ such that $A_{q_1} \in \mathcal{LIS}(X_{q_1}, X_{q_1'}^*)$. Using (29) and the relation $v_3^* + v_4^* = 0$ we obtain

$$\begin{aligned}\langle A_q \overline{Z}, \psi\rangle_{X_{q'}} = &\int_{\Omega_0} \sum_{i,j=1}^{4} a_{ij} \nabla \overline{Z}_j \cdot \nabla\psi_i \,\mathrm{d}x \\ &- \sum_{j=0}^{k} \int_{\Omega_j} \partial_v H_j(\cdot, v^*, f_j^*) \cdot D(v^*)\overline{Z}\,\psi_1 \,\mathrm{d}x \\ &+ \sum_{j=0}^{k} \int_{\Omega_j} \widehat{r}_j(\cdot, v^*, f_j^*)\,(\overline{Z}_3 + \overline{Z}_4)\,(\psi_3 + \psi_4)\,\mathrm{d}x,\end{aligned} \tag{30}$$

where the matrix a with $a_{ij} = \sum_{k=1}^4 b_{ik}(\cdot, v^*) D(v^*)_{kj}$, $i, j = 1, \ldots, 4$, has the form (see also (15))

$$a = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & v_2^{*2}\widetilde{\kappa} + v_2^*\omega_0 & v_2^*\omega_1 & v_2^*\omega_2 \\ 0 & v_2^*\omega_1 & v_2^*(\widetilde{\sigma}_n + \widetilde{\sigma}_{np}) & v_2^*\widetilde{\sigma}_{np} \\ 0 & v_2^*\omega_2 & v_2^*\widetilde{\sigma}_{np} & v_2^*(\widetilde{\sigma}_p + \widetilde{\sigma}_{np}) \end{pmatrix},$$

$$\omega_0 = \omega_1(v_3^* + \widetilde{P}_n v_2^*) + \omega_2(v_4^* + \widetilde{P}_p v_2^*),$$

where $\widetilde{\kappa}$, $\widetilde{\sigma}_n$, $\widetilde{\sigma}_p$, $\widetilde{\sigma}_{np}$, $\widetilde{P}_n$ and $\widetilde{P}_p$ are taken in the argument (x, v^*). Since the matrix (a_{ik}) is symmetric and positive definite (see also Remark 2), there exists a constant $a^* > 0$

such that

$$\sum_{i,j=1}^{4} a_{ij}(x) y_j y_i \geq a^* ||y||^2_{\mathbb{R}^4} \quad \forall y \in \mathbb{R}^4,\ \forall x \in \Omega_0 \setminus \Sigma. \tag{31}$$

3. Now we follow ideas in the proof of [12, Theorem 4.1]. We write the operator A_q in form of a sum $A_q = E_q + K_q$ with operators $E_q, K_q \colon X_q \to X^*_{q'}$, where

$$\begin{aligned}
\langle E_q \overline{Z}, \psi \rangle_{X_{q'}} &= \int_{\Omega_0} \Big\{ \sum_{i,j=1}^{4} a_{ij} \nabla \overline{Z}_j \cdot \nabla \psi_i + \sum_{i=1}^{4} \overline{Z}_i\, \psi_i \Big\}\, \mathrm{d}x, \\
\langle K_q \overline{Z}, \psi \rangle_{X_{q'}} &= -\sum_{i=1}^{4} \int_{\Omega_0} \overline{Z}_i\, \psi_i\, \mathrm{d}x - \sum_{j=0}^{k} \int_{\Omega_j} \partial_v H_j(\cdot, v^*, f_j^*) \cdot D(v^*) \overline{Z}\, \psi_1\, \mathrm{d}x \\
&\quad + \sum_{j=0}^{k} \int_{\Omega_j} \widehat{r}_j(\cdot, v^*, f_j^*)\, (\overline{Z}_3 + \overline{Z}_4)(\psi_3 + \psi_4)\, \mathrm{d}x.
\end{aligned}$$

Thanks to the compact embedding of $W^{1,q}(\Omega_0)$ into $L^\infty(\Omega_0)$ the operator K_q is compact. The operator E_q is injective. The regularity result of Gröger [10, Theorem 1, Remark 14] guarantees that there exists a $q_1 \in (2, q_0]$ such that E_{q_1} is surjective. Then by Banach's Open Mapping Theorem and Nikolsky's criterion for Fredholm operators the operator A_{q_1} turns out to be a Fredholm operator of index zero.

4. Next, we prove that A_{q_1} is injective. A_{q_1} has the form (30). Let $A_{q_1} \bar{Z} = 0$, $\bar{Z} \in X_{q_1}$. Using the test function $\psi = (0, \overline{Z}_2, \overline{Z}_3, \overline{Z}_4)$ and exploiting the strong ellipticity condition for (a_{ij}) from (31), the fact that $\Gamma_D \neq \emptyset$ and the property that $\widehat{r}_j(\cdot, v^*, f^*) \geq 0$, $j = 0, \ldots, k$, we get that $\overline{Z}_i = 0$, $i = 2, 3, 4$. Now we use the test function $\psi = (\overline{Z}_1, 0, 0, 0)$ for the equation $A_{q_1} \bar{Z} = 0$ and arrive at

$$\int_{\Omega_0} \varepsilon |\nabla \overline{Z}_1|^2\, \mathrm{d}x - \sum_{j=0}^{k} \int_{\Omega_j} \frac{\partial}{\partial v_1} H_j(\cdot, v^*, f_j^*)\, \overline{Z}_1^2\, \mathrm{d}x = 0.$$

Since the functions H_j are continuously differentiable and $-H_j(x, \cdot, v_2, v_3, -v_3, f_j)$ is monotonic increasing (see Remark 4, Lemma 2) we have $-\frac{\partial}{\partial v_1} H_j(x, v^*, f_j^*) \geq 0$ on $\Omega_j \setminus \Sigma$, $j = 0, \ldots, k$, which together with (A3) and (A1) leads to $\overline{Z}_1 = 0$. Thus the injectivity of $A_{q_1} \colon\ X_{q_1} \to X^*_{q_1'}$ follows. Consequentely, $A_{q_1} \in \mathcal{L}(X_{q_1}, X^*_{q_1'})$ is bijective, and by Banach's theorem we have $A_{q_1} \in \mathcal{LIS}(X_{q_1}, X^*_{q_1'})$.

5. In summary, since $D(v^*) \in \mathcal{LIS}(X_{q_1}, X_{q_1})$ we obtain the desired result that $\partial_V \mathcal{F}_{q_1,\tau}(V^*, w^*) \in \mathcal{LIS}(X_{q_1}, X^*_{q_1'})$. ■

Now we are able to formulate the main result for Problem (P).

Theorem 2 (Local existence and uniqueness of steady states). *We assume* (A1) – (A5). *Let* $w^* = (v_D^*, g^*, f^*) \in \Lambda$ *be given, and let* (q_0, τ, V^*, w^*) *be the equilibrium solution to Problem* (P), $v^* = V^* + L v_D^*$ *(see Theorem 1).*

Then there exist a $q_1 \in (2, q_0]$ *such that the following assertion holds: There exist neighbourhoods* $U \subset X_{q_1}$ *of* V^* *and* $W \subset \mathcal{H}_p \times \mathcal{H}$ *of* $w^* = (v_D^*, g^*, f^*)$ *and a* C^1*-map* $\Phi \colon W \to U$ *such that* $V = \Phi(w)$ *iff*

$$\mathcal{F}_{q_1,\tau}(V, w) = 0, \quad (V, v_D) \in M_{q_1,\tau}, \quad V \in U, \quad w = (v_D, g, f) \in W.$$

Proof. According to Lemma 4 there is an exponent $q_1 > 2$ such that $\partial_V \mathcal{F}_{q_1,\tau}(V^*, w^*) \in \mathcal{LIS}(X_{q_1}, X^*_{q_1'})$. Therefore the assertion of the theorem is a direct consequence of the Implicit Function Theorem. ∎

Finally, let us discuss some special choice of the Dirichlet boundary data. We assume that Γ_D consists of $m \geq 2$ relatively open connected components Γ_D^l with $\operatorname{mes} \Gamma_D^l > 0$, $l = 1, \dots, m$, the closures of which are pairwise disjoint. We prescribe the boundary data $v_D = (\varphi_D, T_D, \zeta_{nD}, \zeta_{pD})$ as follows:

$$\begin{aligned} &\varphi_D = \psi^l(T^l) + U^l, \quad T_D = T^l = \text{const} > 0, \quad \zeta_{nD} = -U^l, \\ &\zeta_{pD} = U^l = \text{const} \quad \text{on } \Gamma_D^l, \quad l = 1, \dots, m. \end{aligned} \tag{32}$$

The functions $\psi^l : (0, +\infty) \to \mathbb{R}$ are related to the built-in potentials on the Ohmic contacts Γ_D^l (see [14]). We assume that these functions are locally Lipschitz continuous and, for the sake of simplicity, that they do not depend explicitly on x. Such boundary data fulfil the first assumption in (D) (see Section 2).

Next we define the set

$$\Lambda_1 = \Big\{ w = (v_D, g, f) \in \mathcal{H}_p \times \mathcal{H} \colon v_D \text{ fulfils (32)}, \ g_i = 0, \ i = 2, 3, 4 \Big\}.$$

Corollary 1. *We assume* (A1) – (A5). *Let* $w = (v_D, f, g) \in \Lambda_1$ *be given. Then there are constants* $q \in (2, p]$, $\tau > 1$, $\epsilon > 0$ *such that the following assertions hold true: If*

$$|T^l - T^1| + |U^l - U^1| < \epsilon, \quad l = 2, \dots, m, \tag{33}$$

then there exists a $V \in X_q$ *such that* (q, τ, V, w) *is a solution to Problem* (P). *This solution lies in a neighbourhood of an equilibrium solution* (q, τ, V^*, w^*), *and in this neighbourhood there are no solutions* $(q, \tau, \widetilde{V}, w)$ *to* (P) *with* $\widetilde{V} \neq V$.

Proof. Let $w = (v_D, f, g) \in \Lambda_1$ be given. We define $v_D^* = (\varphi_D^*, T_D^*, \zeta_{nD}^*, \zeta_{pD}^*)$ as

$$\varphi_D^* = \psi^l(T^1) + U^1, \quad T_D^* = T^1, \quad \zeta_{nD}^* = -U^1, \quad \zeta_{pD}^* = U^1 \quad \text{on } \Gamma_D^l, \quad l = 1, \dots, m,$$

and we get that $w^* = (v_D^*, f, g) \in \Lambda$. Let (q_0, τ, V^*, w^*) be the equilibrium solution to Problem (P). Note that $1/\tau < T^1 < \tau$. According to Theorem 2 there exists constants $q \in (2, q_0]$, $\epsilon' > 0$ such that the equation $\mathcal{F}_{q,\tau}(V, w) = 0$ has a locally unique solution $V \in X_q$ if

$$\|w - w^*\|_{\mathcal{H}_p \times \mathcal{H}} = \|v_D - v_D^*\|_{\mathcal{H}_p} < \epsilon'. \tag{34}$$

Using (35) and the local Lipschitz continuity of the functions ψ^l we find a constant $c(p, \tau) > 0$ such that

$$\|v_D - v_D^*\|_{\mathcal{H}_p} \leq c(p, \tau) \sum_{l=2}^m \Big(|T^l - T^1| + |U^l - U^1| \Big)$$

if $1/\tau < T^l < \tau$, $l = 2, \dots, m$. Choosing ϵ in (33) sufficiently small the inequality (34) can be fulfilled. ∎

6. Remarks.

REMARK 6. There are various papers using the Implicit Function Theorem to study stationary problems from semiconductor modelling (see e.g. [1, 6, 7]).

Alabau [1] considered a symmetric one-dimensional diode without generation-recombination of electrons and holes. There the Implicit Function Theorem was used to show that the solutions of the stationary isothermal problem are locally unique for arbitrary reversed bias voltage.

In [6] we studied a multi species version of a stationary energy model with n different species. There we assumed that all species are mobile in contrast to the impurities contained in the model equations of the present paper and that more general reaction as considered here are involved. We used the scale of $W^{1,p}$-spaces and obtained results as in Theorem 1 and Theorem 2 of the present paper.

We investigated here the stationary energy model only for two-dimensional domains, but we allowed that the submatrix b_{ij}, $i, j = 2, 3, 4$, in (15) is dense. Griepentrog [7] considered a stationary energy model (without additional impurities) under the assumption that $\sigma_{np} = P_n = P_p = 0$ in (15). Then the matrix b becomes triangular. But he replaced the conservation law for the total energy in (6) by the heat flow equation

$$-\nabla \cdot (\kappa \nabla T) = \sigma_n |\nabla \zeta_n|^2 + \sigma_p |\nabla \zeta_p|^2 - R_0(\zeta_n + \zeta_p).$$

Using the Implicit Function Theorem in the scale of Sobolev-Campanato spaces he obtained a local existence and uniqueness result for three-dimensional domains, too.

REMARK 7. Gröger studied in [9] an isothermal instationary problem of the kind (3) – (5). He obtained results concerning existence, uniqueness as well as the asymptotic behaviour of solutions.

7. Appendix.

We assumed that the boundary values on Γ_D belong to the space $W^{1-1/p,p}(\Gamma_D)$ for some $p > 2$. Let this space be equipped with the norm (see [8])

$$\|h\|^p_{W^{1-1/p,p}(\Gamma_D)} = \int_{\Gamma_D} |h|^p \, \mathrm{d}\Gamma + \int_{\Gamma_D} \int_{\Gamma_D} \frac{|h(x) - h(y)|^p}{|x - y|^p} \, \mathrm{d}\Gamma(x) \, \mathrm{d}\Gamma(y), \tag{35}$$

$h \in W^{1-1/p,p}(\Gamma_D)$. We define a continuation operator $L : W^{1-1/p,p}(\Gamma_D) \to W^{1,p}(\Omega_0)$ as follows.

LEMMA 5. *There exists a $p_0 > 2$ such that for all $p \in [2, p_0]$ the following assertions hold. For all $v_D \in W^{1-1/p,p}(\Gamma_D)$ there exists a unique solution $v^D \in W^{1,p}(\Omega_0)$ of the Laplace equation*

$$\Delta v^D = 0 \text{ in } \Omega_0, \quad v^D = v_D \text{ on } \Gamma_D, \quad \frac{\partial v^D}{\partial \nu} = 0 \text{ on } \Gamma_N. \tag{36}$$

This solution is given by $v^D = L v_D$ where L belongs to $\mathcal{L}(W^{1-1/p,p}(\Gamma_D), W^{1,p}(\Omega_0))$.

Proof. We give only the main ideas of the proof (for some of the details see [8, 10]). Since $\Omega_0 \cup \Gamma_N$ is regular in the sense of Gröger, there exists a $p_0 > 2$ such that for any

$p \in [2, p_0]$ the mapping $I_p\colon W_0^{1,p}(\Omega_0 \cup \Gamma_N) \to W^{-1,p}(\Omega_0 \cup \Gamma_N)$,

$$\langle I_p v, w\rangle_{W_0^{1,p'}(\Omega_0\cup\Gamma_N)} = \int_{\Omega_0} \nabla v \cdot \nabla w \,\mathrm{d}x, \quad v \in W_0^{1,p}(\Omega_0 \cup \Gamma_N), \quad w \in W_0^{1,p'}(\Omega_0 \cup \Gamma_N)$$

is an isomorphism (see [10]). Let $p \in [2, p_0]$ be fixed and let $v_D \in W^{1-1/p,p}(\Gamma_D)$. We apply to v_D the linear, continuous continuation operator $C_p\colon W^{1-1/p,p}(\Gamma_D) \to W^{1-1/p,p}(\Gamma)$,

$$\widetilde{v}_D = C_p(v_D), \quad \|\widetilde{v}_D\|_{W^{1-1/p,p}(\Gamma)} \le c\,\|v_D\|_{W^{1-1/p,p}(\Gamma_D)}.$$

Now we use the right inverse of the trace operator $\gamma_p^{-1}\colon W^{1-1/p,p}(\Gamma) \to W^{1,p}(\Omega_0)$, which is linear and continuous, and obtain

$$\widetilde{v}^D = \gamma_p^{-1}(\widetilde{v}_D), \quad \|\widetilde{v}^D\|_{W^{1,p}(\Omega_0)} \le c\,\|\widetilde{v}_D\|_{W^{1-1/p,p}(\Gamma)}.$$

We write v^D in the form $v^D = \widetilde{v}^D + h$. Then, according to (36), h has to fulfil the equation

$$-\Delta h = \Delta \widetilde{v}^D \text{ in } \Omega_0, \quad h = 0 \text{ on } \Gamma_D, \quad \frac{\partial h}{\partial \nu} = -\frac{\partial \widetilde{v}^D}{\partial \nu} \text{ on } \Gamma_N,$$

or

$$I_p h = r, \quad \langle r, w\rangle_{W_0^{1,p'}(\Omega_0\cup\Gamma_N)} = -\int_{\Omega_0} \nabla \widetilde{v}^D \cdot \nabla w \,\mathrm{d}x, \quad w \in W_0^{1,p'}(\Omega_0 \cup \Gamma_N).$$

The right hand side r belongs to $W^{-1,p}(\Omega_0 \cup \Gamma_N)$ and $\|r\|_{W^{-1,p}(\Omega_0\cup\Gamma_N)} \le c\,\|\widetilde{v}^D\|_{W^{1,p}(\Omega_0)}$. Since the operator I_p is an isomorphism we find that $h = I_p^{-1} r$ and $\|h\|_{W_0^{1,p}(\Omega_0\cup\Gamma_N)} \le c\,\|r\|_{W^{-1,p}(\Omega_0\cup\Gamma_N)}$. Using that $v^D = \widetilde{v}^D + h$ and the previous estimates we end up with

$$\|v^D\|_{W^{1,p}(\Omega_0\cup\Gamma_N)} \le c\,\|v_D\|_{W^{1-1/p,p}(\Gamma_D)}.$$

Note that for Dirichlet boundary data $v_D \ge K > 0$ a.e. on Γ_D the test of the Laplace equation (36) with $-(v^D - K)^-$ supplies that $v^D \ge K$ a.e. in Ω_0, too. ∎

Finally, we need the Theorem of Scorza-Dragoni (see [13]) in a form which can easily be derived from a version of this theorem given in [3, Chap. VIII].

Theorem 3. *Let $M \subset \mathbb{R}^n$ be a bounded measurable set, $B \subset \mathbb{R}^l$ a Borel set. A function $h\colon M \times B \to \mathbb{R}$ belongs to $Car(M, B)$ if and only if for all $\epsilon > 0$ there exists a closed subset $A_\epsilon \subset M$ such that $mes(M \setminus A_\epsilon) \le \epsilon$ and $h|_{A_\epsilon \times B}$ is continuous.*

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