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Bilinear coagulation equations

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Abstract

We consider coagulation equations of Smoluchowski or Flory type where the total merge rate has a bilinear form $\pi(y) \cdot A\pi(x)$ for a vector of conserved quantities π, generalising the multiplicative kernel. For these kernels, a gelation transition occurs at a finite time $t_{\mathrm{g}} \in (0, \infty)$, which can be given exactly in terms of an eigenvalue problem in finite dimensions. We prove a hydrodynamic limit for a stochastic coagulant, including a corresponding phase transition for the largest particle, and exploit a coupling to random graphs to extend analysis of the limiting process beyond the gelation time.

1 Introduction and Main Results

Smolouchowski [34] introduced the basic mathematical model for coagulating particles, giving an ordinary differential equation describing the distribution of particle masses, which arises from considering a microscopic particle system. The physics of the underlying system enters into the model through a choice of interaction kernel $K(x, y)$, describing the speed of the coagulation $x, y \mapsto x + y$; the particular case $K(x, y) = xy$ is known as the *multiplicative* kernel, and is particularly well-studied. It is well-known that the resulting Smolouchowski equation corresponds to the distribution of cluster sizes in the Erdős-Réyni random graphs $\mathcal{G}(N, t/N)$ in the $N \to \infty$ limit, which, together with the simplicity of the kernel, allows a fairly complete analysis of this case.

In many physical situations, the rate of coagulation will depend on more than only the mass of the particles, and so it is desirable to generalise Smoluchowski's equation. In particular, we note the works [28, 29] which allow coagulation in more than one possible way, and where the total rate of coagulation between particles of types x, y can be bounded in terms of a function φ which is conserved as particles coagulation.

There are two natural ways to frame the study of coagulation: one can either start from an interacting particle system, where existence and uniqueness is elementary, but where characterising the many particle limit may require substantial effort, or one can work directly with the mean-field Smolouchowski and Flory equations, for which existence and uniqueness require more consideration, and can fail in some cases [28]. Relatedly, there are several different ways in which one can characterise gelation. At the level of the particle system, one can study the phase transition where the size of the largest particle goes from size $\ll N$ to a size comparable to N [23]. At the level of the limiting equation, gelation refers to the point where the solution to the Smolouchowski or Flory equation $(\mu_t)_{t\geq 0}$ fails to conserve the total particle mass, which is known to be related [23, 29] to the divergence of the second moment of the particle mass.

We will study coagulation systems where, as above, coagulation can occur in several possible ways, and where the internal structure of particles can evolve, in a mass-preserving way, without coagulation. The important hypothesis is the *bilinear* structure: we ask that the total mass of the kernel $\overline{K}(x, y)$

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can be expressed in the form $\pi(y) \cdot A\pi(x)$, for a fixed matrix A and a vector π of conserved quantities. In this case, the limiting Flory equation is well-posed, globally in time, and the stochastic particle system couples exactly to a class of random graphs introduced by [5] which generalise the Erdős-Réyni graphs. We analyse the limiting equation in Theorem 1 and prove a law of large numbers for the stochastic particle systems in Theorem 2, together demonstrating that the three effects described above all occur at the same time t_{g}.

1.1 Definitions

As mentioned above, our analysis rests on the *bilinear* form of the total rate $\overline{K}$, which allows us to connect the Smoluchowski equation to random graphs in Section 4. The following definition makes this precise.

Definition 1. *A* Bilinear Coagulation System *is a 5-tuple* (S, R, π, K, J) *consisting of a complete metric space* S*, a continuous involution* R *on* S*, a finite collection of continuous maps* $\pi = (\pi_i)_{0\le i\le n+m}$*,* $n \ge 1, m \ge 0$ *from* S *to* $\mathbb{R}$*, and nonnegative kernels* K, J *on* $S\times S\times S$ *and* $S \times S$ *respectively, such that the following hold.*

i). For all $0 \le i \le n+m$ *and all* $x, y \in S$*,*

$$\pi_i = \pi_i(x) + \pi_i(y) \qquad K(x,y,\cdot) + J(x,\cdot)\text{- almost everywhere.} \tag{1}$$

ii). For $1 \le i \le n$*, the map* $\pi_i : S \to \mathbb{R}$ *takes only nonnegative values, and* π_0 *takes values in the positive integers* $\mathbb{N}$*.*

iii). The involution R *satisfies*

$$\pi_i \circ R = \begin{cases} \pi_i & 0 \le i \le n; \\ -\pi_i, & n+1 \le i \le n+m \end{cases} \tag{2}$$

and, for all $x, y \in S$*,*

$$K(Rx, Ry, \cdot) = R_\# K(x,y,\cdot); \qquad J(Rx,\cdot) = R_\# J(x,\cdot). \tag{3}$$

Here, and throughout, the subscript $_\#$ *denotes the pushforward of a measure.*

iv). There exists a constant C *such that, for all* $x \in S$*,*

$$\sum_{i=n+1}^{m} \pi_i(x)^2 \le C\varphi(x)^2 \tag{4}$$

where $\varphi(x) = \sum_{i=0}^n \pi_i(x)$*. Moreover, the sublevel sets* $S_\xi = \{x \in S : \varphi(x) \le \xi\}$ *are compact, for all* $\xi \in [0,\infty)$*.*

v). For all $x, y \in S$*, the total rate* $\overline{K}(x,y) = K(x,y,S)$ *may be expressed as*

$$\overline{K}(x,y) = \sum_{1\le i,j\le n+m} a_{ij}\pi_i(x)\pi_j(y) \tag{5}$$

for a fixed $(n+m)\times(n+m)$ symmetric real matrix $A=(a_{ij})_{1\le i,j\le n+m}$. Moreover, the matrix A is of the block-diagonal form

$$A=\begin{pmatrix} A^{+} & 0 \\ 0 & A^{\mathrm{par}} \end{pmatrix} \tag{6}$$

where A^{+}, A^{par} are $n\times n$ and $m\times m$ square matrices respectively, and all entries of A^{+} are nonnegative. Finally, for all $1\le i\le n+m$, there exists $1\le j\le n+m$ such that $a_{ij}>0$, so that no row or column of A vanishes. For J, we ask that the total rate $\overline{J}(x)=J(x,S)$ satisfies $\sup_x \frac{\overline{J}(x)}{\varphi(x)}<\infty.$

vi). For $f\in C_c(S)$, the maps

$$Kf: S\times S\to\mathbb{R}, \qquad (x,y)\mapsto \int_S f(z)K(x,y,dz); \tag{7}$$

$$Jf: S\to\mathbb{R}, \qquad x\mapsto \int_S f(z)J(x,dz) \tag{8}$$

are continuous.

Remark 1.1. *We think of $\pi_0(x)$ as counting the number of particles at time 0 which have been absorbed into x. As a result, we will ask in (A5.) below that our initial measure μ_0 is supported on $\{\pi_0\}=1$, and π_0 artificially introduces monodisperse initial conditions.*

If we are given a space S equipped only with $\pi_1,...,\pi_{n+m}$, we can replace S by $\mathbb{N}\times S$, and setting $\pi_0(a,x)=a, \pi_i(a,x)=\pi_i(x), i=1,...,n+m, (a,x)\in\mathbb{N}\times S$. In this way, and since π_0 does not enter the total rate $\overline{K}(x,y)$, the artificial requirements on π_0 above do not restrict the physics of the coagulation system.

Stochastic Particle Systems. With the setting defined above, we can introduce the interacting particle systems under consideration.

We study a system of coagulating particles $(x_j^N(t): j\le l^N(t))$, and the associated empirical measure

$$\mu_t^N=\frac{1}{N}\sum_{j=1}^{l^N(t)}\delta_{x_j^N(t)} \tag{9}$$

with the following dynamical rules.

i). The rate at which unordered pairs of particles $\{x,y\}$ in S merge to form a new particle in $A\subset S$ is $2K(x,y,A)/N$.

ii). A particle of type x evolves can a particle of type $y\in A\subset S$ with a total rate $J(x,A)$.

This is a generalisation of a Marcus–Lushnikov coagulation process [23] on S, which we will refer to as the *stochastic coagulant*. Note that a $1/N$ scaling of the pair interaction rate is used, which ensures that each molecule has a total evolution rate of order 1. Dividing jump rates by N is equivalent to accelerating time by the same factor and this alternative formulation means that the jump rates in the definition of the “stochastic coalescent” in [2] as well as of the “stochastic K-coagulant” in [29] omit the $1/N$ from the rates and rescale time when taking the $N\to\infty$ limit.

Limiting kinetic equations. We now consider various forms of the limiting Smoluchowski equation. Define a drift operator L, by specifying for all $f \in C_c(S)$,

$$\begin{aligned}\langle f, L(\mu)\rangle =& \frac{1}{2}\int_{S^3}\{f(z)-f(x)-f(y)\}K(x,y,dz)\mu(dx)\mu(dy)\\ &+\int_{S^2}\{f(y)-f(x)\}J(x,dy).\end{aligned} \tag{10}$$

The weak form of the Smolochowski equation for a process of measures $(\mu_t)_{t<T}$ on S is to ask that

$$\forall f \in C_c(S),\ t<T, \qquad \langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle f, L(\mu_s)\rangle ds. \tag{Sm}$$

The equation (Sm) captures the effects of coagulations between finite clusters. However, as discussed above, we wish to include the possibility of a macroscopic component, which we term *gel*. To include this effect, we modify the drift operator by specifying, for $f \in C_c(S)$,

$$\langle f, L_{\mathrm{g}}(\mu_t)\rangle = \langle f, L(\mu_t)\rangle - \int_S f(x)\overline{K}(x,y)\mu_t(dx)(\mu_0-\mu_t)(dy). \tag{12}$$

The weak form of the Flory equation is to ask, similarly,

$$\forall f \in C_c(S),\ t<T, \qquad \langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle f, L_{\mathrm{g}}(\mu_s)\rangle ds. \tag{Fl}$$

Here, the additional term comes into play only after μ_t ceases to conserve the quantities $\langle \pi_i, \mu_t\rangle, 1 \le i \le n+m$, and the extra term represents the interaction with the gel. This generalises the Smoluchowski coagulation equations [34] in a way analogous to Flory [38], and we use the term 'K-coagulant' for a solution to (Fl), following [29].

Precise conditions on measurability and integrability required to interpret these equations concretely are given in Appendix I.

We write

$$\begin{aligned} g_t = (M_t, E_t, P_t) &= \langle \pi, \mu_0-\mu_t\rangle \\ &= (\langle \pi_i, \mu_0-\mu_t\rangle)_{i=0}^{n+m}\end{aligned} \tag{14}$$

for the gel data, where M_t, E_t, P_t are the 0^{th}, $1^{\text{st}}-n^{\text{th}}$, and $(n+1)^{\text{th}}-(n+m)^{\text{th}}$ coordinates, respectively. Following remarks in [29], one may show that if μ_t is a solution to (Fl), then the maps $t \mapsto \langle \pi_i, \mu_t\rangle, i \le n$ are non-increasing, which guarantees that $M_t, E_t \ge 0$. We write S^{Π} for the state space of gel data, given by

$$S^{\Pi} = \mathbb{N}\times\mathbb{R}^n\times\mathbb{R}^m \tag{15}$$

and use the same notation $\pi_i, 0 \le i \le n+m$ for the projections onto the factors. When $x \in S$ and $g \in S^{\Pi}$, we use $\overline{K}(x,g)$ for the rate of absorption, given by (5) with the new meanings of $\pi_i(g)$. We will also write φ for the linear combination $\varphi = \sum_{i\le n}\pi_i$, defined on both S and S^{Π}.

Definition 2 (Conservative Solutions)**.** *Let S be a bilinear coagulation system. We say that a solution $(\mu_t)_{t<T}$ to either (Sm) or (Fl) is* conservative *if all the functions $t \mapsto \langle \pi_i, \mu_t\rangle, 0 \le i \le n+m$ are constant on $[0,T)$.*

Thus, any solution to (Sm) or (Fl) is conservative up to some time $0 \leq t_{\mathrm{g}} \leq \infty$, and non-conservative thereafter.

We will usually impose symmetry requirements (A1.) on the initial data which guarantee that $\langle \pi_i, \mu_t \rangle = 0$ for all t, for all $i = n+1, .., n+m$. As noted above, the functions $t \mapsto \langle \pi_i, \mu_t \rangle, i \leq n$ are non-increasing, whenever $(\mu_t)_{t<T}$ is a local solution to either equation. Therefore, under hypothesis (A1.), a solution $(\mu_t)_{t<T}$ to either equation is conservative if, and only if, the map $t \mapsto \langle \varphi, \mu_t \rangle$ is constant on $[0, T)$.

Let $\mathcal{M} = \mathcal{M}_{\leq 1}(S)$ be the space of measures on S with total mass at most 1. We equip $\mathcal{M}$ with the *vague* topology $\mathcal{F}(\mathcal{M}, C_c(S))$ induced by continuous, compactly supported functions on S, and fix a complete metric d compatible with this topology.

1.2 Statement of Results

We will make the following hypotheses on the initial data μ_0.

Assumption A. *We will ask that the initial data μ_0 is a sub-probability measure on a bilinear coagulation space S, satisfying the following hypotheses.*

(A1.) The measure μ_0 is even under the transformation R: $R_\# \mu_0 = \mu_0$.

(A2.) For all $i \leq n$, we have $\langle \pi_i^3, \mu_0 \rangle < \infty$.

(A3.) The set $\{\pi_i : 1 \leq i \leq n\}$ is linearly independent in the space $L^2(m)$. In particular, none of the functions $\pi_i : 1 \leq i \leq n$ are 0 μ_0-almost everywhere.

(A4.) The kernel K is μ_0-irreducible: if $A \subset S$ is such that, for all $x \in A$ and $y \in A^{\mathrm{c}}$, $\overline{K}(x, y) = 0$, then either $\mu_0(A) = 0$ or $\mu_0(A^{\mathrm{c}}) = 0$. Moreover, μ_0 is not a point mass.

(A5.) The initial data μ_0 is supported on $\{x \in S : \pi_0(x) = 1\}$.

We summarise our results on the analysis of the Flory equation (Fl) as follows.

Theorem 1. *Let S be a π_0-bilinear coagulation system, and let μ_0 be a sub-probability measure on S satisfying Assumption A. Then the equation (Fl) has a unique solution $(\mu_t)_{t \geq 0}$ starting at μ_0; we write $g_t = (M_t, E_t, P_t)$ for the gel data defined in (14). This solution has the following properties.*

1. Phase Transition. *Let t_{g} be the first time at which the solution μ_t fails to be conservative, that is:*

$$t_{\mathrm{g}} := \inf\{t \geq 0 : \langle \pi_i, \mu_t \rangle \neq \langle \pi_i, \mu_0 \rangle \text{ for some } 0 \leq i \leq n+m\} = \inf\{t \geq 0 : \langle \varphi, \mu_t \rangle < \langle \varphi, \mu_0 \rangle\}. \tag{16}$$

Then $t_{\mathrm{g}} \in (0, \infty)$, and can be given explicitly in terms of the moments of μ_0 as

$$t_{\mathrm{g}} = \mathfrak{r}(\Lambda(\mu_0))^{-1}; \qquad \Lambda(\mu_0)_{ij} = \langle (A\pi)_i \pi_j, \mu_0 \rangle, \qquad 1 \leq i, j \leq n \tag{17}$$

where $\mathfrak{r}(\cdot)$ denotes the spectral radius of a matrix.

2. Behaviour of the Second Moment. *Consider the second moments*

$$\mathcal{Q}(t) = (\langle \pi_i \pi_j, \mu_t \rangle)_{i,j=0}^n; \qquad \mathcal{E}(t) = \langle \varphi^2, \mu_t \rangle. \tag{18}$$

Then

i). $\mathcal{Q}(t)$ is finite and continuous, and so locally bounded, on $[0,\infty) \setminus \{t_{\mathrm{g}}\}$.

ii). On $[0, t_{\mathrm{g}})$, each moment $\mathcal{Q}_{ij}$ is monotonically increasing, as is $\mathcal{E}$.

iii). At the gelation time, $\mathcal{E}(t_{\mathrm{g}}) = \infty$, and $\mathcal{E}(t) \to \infty$ as $t \to t_{\mathrm{g}}$.

3. Representation of Gel Data. *For each $t \geq 0$, there exists a unique maximal n-tuple $c_t = (c_t^i)_{i=1}^n \geq 0$ such that, for all $x \in S$,*

$$\sum_{i=1}^n c_t^i \pi_i(x) = 2t \int_S \left(1 - \exp\left(-\sum_{i=1}^n c_t^i \pi_i(y)\right)\right) \overline{K}(x,y)\mu_0(dy). \tag{19}$$

c_t undergoes a phase transition at time t_{g}: if $t \leq t_{\mathrm{g}}$, then $c_t = 0$, and if $t > t_{\mathrm{g}}$ then at least one component of c_t is strictly positive. Moreover, the map $t \mapsto c_t$ is continuous.

The gel data are given in terms of c_t by

$$g_t^i = \int_S \pi_i(x) \left(1 - \exp\left(-\sum_{j=1}^n c_t^j \pi_j(x)\right)\right) \mu_0(dx). \tag{20}$$

Therefore, if $t > t_{\mathrm{g}}$ then $M_t > 0$, and $E_t > 0$ componentwise. Moreover, the map $t \mapsto g_t$ is continuous, and $g_{t_{\mathrm{g}}} = 0$.

4. Gel Dynamics. *The map $t \mapsto g_t$ is differentiable on $t \in (t_{\mathrm{g}}, \infty)$, and*

$$\frac{d}{dt} g_t^i = \sum_{j,k=1}^n \langle \pi_i \pi_j, \mu_t \rangle a_{jk} g_t^k. \tag{21}$$

5. Order of the Phase Transition, and the Size-Biasing Effect. *The map $t \mapsto c_t$ is right differentiable at t_{g}, and as a consequence, the phase transition is first order; that is, the right-derivatives of the gel data $g_t^i, i = 0, 1, ..., n$ exist and are strictly positive at t_{g}. Moreover, there exist $\theta_i \geq 0, i = 1, .., n$, such that $\sum_i \theta_i = 1$ and such that*

$$\sum_{i=1}^n \lambda_i (g'_{t_{\mathrm{g}}+})_i \geq \left(\frac{\sum_{i=1}^n \langle \lambda_i \pi_i, \mu_0 \rangle}{\langle \pi_0, \mu_0 \rangle}\right) (g'_{t_{\mathrm{g}}+})_0. \tag{22}$$

We call this a size-biasing *effect: the average of the linear combination $\sum_i \theta_i \pi_i$ over particles in the early gel is at least the average over all particles. Let us define also the total interaction rate, which will quantify the inhomogeneity of the initial data μ_0:*

$$s(x) = \int_S \overline{K}(x,y)\mu_0(dy). \tag{23}$$

If s is not constant μ_0-almost everywhere, then θ_i can be chosen so that the inequality in (22) is strict.

We also prove the following theorem, which is a law of large numbers result for the coagulating particle system $(x_j^N(t) : j \le l^N(t))$ .. Firstly, following ideas of [29], we show that the empirical measure μ_t^N converges to the limiting solution $(\mu_t)_{t\ge 0}$ in the vague topology, uniformly in time. The second part of the result is that the *stochastic gel* $g_t^N = N^{-1}\pi(x_1^N(t))$ itself satisfies a law of large numbers, converging to the true gel g_t as $N \to \infty$, where we recall x_1^N is the largest particle by π_0.

We make the following hypotheses for the law of large numbers. These are naturally satisfied when, for example, the initial particles $(x_i^N(0) : 1 \le i \le l^N(0))$ are sampled as a Poisson random measure with intensity $N\mu_0$. However, it is useful for some intermediate results to give these results in the more general form used here.

Assumption B. *Let μ_0^N be the initial data the stochastic coagulant, and let μ_0 be the initial data of the limiting Flory equation.*

(B1.) As $N \to \infty$, the initial measures $\mu_0^N = \frac{1}{N}\sum_{i\le l^N(0)} \delta_{x_i}$ converge in probability to μ_0 under the vague topology, that is:

$$d(\mu_0^N, \mu_0) \to 0 \qquad \textit{in probability.} \tag{24}$$

Moreover, μ_0^N is supported on the set $\{\pi_0 = 1\}$.

(B2.) We also have the convergence

$$\langle \pi_i, \mu_0^N\rangle \to \langle \pi_i, \mu_0\rangle \qquad \textit{in probability} \tag{25}$$

for all $0 \le i \le n+m$, and the uniform integrability

$$\sup_{N\ge 1} \mathbb{E}\langle \varphi^2, \mu_0^N\rangle < \infty; \qquad \sup_{N\ge 1} \mathbb{E}\left[\langle \varphi^2, \mu_0^N\rangle 1\left(\langle \varphi^2, \mu_0^N\rangle \ge M\right)\right] \to 0 \textit{ as } M \to \infty. \tag{26}$$

Theorem 2. *Let μ_0 be a sub-probability measure on S satisfying Assumption A, and let $(\mu_t)_{t\ge 0}, (g_t)_{t\ge 0}$ be the associated solution to (Fl) and corresponding gel. For $N \ge 1$, let μ_t^N be the stochastic coagulant with initial data satisfying Assumption B, and write $(x_j^N(t) : j \le l^N(t))$ for the particles of the stochastic system, sorted in decreasing order of π_0. Let $g_t^N = N^{-1}(\pi_i(x_1^N(t)))_{i=0}^n$ be the data of the largest particle in the stochastic system, normalised by N^{-1}. Then we have the convergence*

$$\sup_{t\ge 0} \left(d\left(\mu_t^N, \mu_t\right) + \left|g_t^N - g_t\right|\right) \to 0 \tag{27}$$

in probability. In particular, we have the following phase transition:

i). If $t \le t_{\mathrm{g}}$, then the largest particle has gel data of the order $o_{\mathrm{p}}(N)$;

ii). If $t > t_{\mathrm{g}}$, the largest particle has gel data of the order $\Theta_{\mathrm{p}}(N)$.

Moreover, if ξ_N is any sequence with $\xi_N \to \infty$ and $\frac{\xi_N}{N} \to 0$, then we may define $\widetilde{g}_t^N$ by summing the data of all particles $x_j^N(t)$ with $\pi_0(x_j^N(t)) \ge \xi_N$, and normalising by N. Then the same result holds when we replace g_t^N by $\widetilde{g}_t^N$ in (27).

Here, and throughout, we use the notation $o_{\mathrm{p}}(\cdot), \mathcal{O}_{\mathrm{p}}(\cdot), \Theta_{\mathrm{p}}(\cdot)$ for the probabilistic equivalents of $o(\cdot), \mathcal{O}(\cdot), \Theta(\cdot)$, and say that an event[1] holds *with high probability* if relevant probabilities converge to 1 as $N \to \infty$. Precise definitions can be found in [15].

[1] or, more formally, a sequence of events indexed by N

1.3 Plan of the Paper.

Our programme will be as follows.

1. In the remainder of this section, we will discuss other works on coagulating particle systems in the literature, and how they relate to our results.

2. In Section 2, we will prove that the limiting equation (Fl) has unique, globally defined solutions, based on a truncation argument from [28, 29].

3. In Section 3, we prove an initial result, Lemma 3.1, on the convergence of the stochastic coagulant, using the ideas of [29, Theorem 4.1]. This will later be used to prove later points of Theorem 1 based on probabilistic arguments for the empirical measures μ_t^N, and the random graphs G_t^N introduced in Section 4.

4. In Section 4, we show how the stochastic coagulant can be coupled to a family of inhomogenous random graphs defined in [5]. Key results for these graphs are recalled in Appendix II. The critical time t_c for these graphs may be found exactly, leading to the explicit expression in Theorem 1.

5. A weakness of the preceding sections is that, a priori, the critical time t_c for the graph processes may differ from the gelation time t_g; in Section 5, we show that this cannot happen. This is based on a preliminary version of Theorem 2, which shows convergence of (μ_t^N, g_t^N) at a single fixed time $t \geq 0$.

6. Section 6 is dedicated to a proof of item 2 of Theorem 1, concerning the second moments $\mathcal{Q}_{ij}(t) = \langle \pi_i\pi_j, \mu_t\rangle, \mathcal{E}(t) = \langle \varphi^2, \mu_t\rangle$. The statements about the subcritical and critical cases $t < t_\mathrm{g}, t = t_\mathrm{g}$ follow general ideas in [28, 29], while the statement about the supercritical case $t > t_\mathrm{g}$ uses additional ideas from the theory of random graphs.

7. Section 7 uses the ideas of previous sections to prove items 3 and 4 of Theorem 1, concerning the gel data g_t beyond the critical point.

8. Section 8 uses the analysis of the gel to extend Lemma 3.1 to show that convergence is uniform in time.

9. Section 9 proves item 5 of Theorem 1, concerning the behaviour near the critical point. This completes the proof of this theorem.

10. To finish the proof of Theorem 2, we revisit the ideas of Section 5 to prove convergence of the stochastic gel $g_t^N, \widetilde{g}_t^N$, uniformly in time. This is the focus of Section 10, and builds further on ideas of previous sections.

1.4 Literature Review

The original equation introduced by Smoluchowski considers the case of coagulating particles, whose only property is a mass belonging to $\mathbb{N}$, and where the coaguation $x, y \mapsto x+y$ has a general rate $\overline{K}(x,y)$. In this case, identifying measures $\mu \in \mathcal{M}_{\leq 1}(\mathbb{N})$ with a summable sequence, the equation analagous to (Sm) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(x) = \frac{1}{2}\sum_{y<x}\overline{K}(y, x-y)\mu_t(y)\mu_t(x-y)\mathrm{d}y - \mu_t(x)\sum_{y=1}^{\infty}\overline{K}(x,y)\mu_t(y)\mathrm{d}y. \tag{28}$$

For an extensive review the reader is referred to [2]. The case $\overline{K}(x,y) := xy$ is known as the *multiplicative* coagulation kernel and in this case with $\mu_0 = \delta_1$, the solution of (28) exhibits gelation at $t_{\mathrm{g}} = 1$.

The existence and value of the gelation time has been studied for a range of $\overline{K}$. For particles with integer masses and $\epsilon(xy)^\alpha \le \overline{K}(x,y) \le Mxy,\ M \in \mathbb{R}_+, \alpha \in (\frac{1}{2}, 1)$ Jeon [16] proved the existence of a gelation phase transition and provided an upper bound on the gelation time.

Norris [28, 29] introduced a more general form, analagous to (Sm) on a general space S, allowing particles with internal structure and where, for any pair of particles, there are multiple possible coagulation products, in the case $\overline{K}(x,y) \le C\varphi(x)\varphi(y)$ for some function φ growing no more than linearly in particle mass, a step that is important for the present work. A lower bound for the gelation time was proved in [28] and an upper bound was added under appropriate assumptions in [29]; however, these bounds do not coincide in general. Normand [26] obtained explicit results concerning the blowup of a second moment for a sexed model which gives a lower bound on the gelation time, and in a later work [27] finds explicit expressions for the gelation time for a selection of models with arms. Consequently, ours is one of the first models for which the gelation time can be found exactly; moreover, several aspects of our analysis extend what was previously known about the Smoluchowski equation, using the connection to random graphs [5].

The study of gelation as the formation of a very large connected structure by joining basic building blocks goes back at least to Flory [11] whose motivation was hydrocarbon polymerisation in the manufacture of plastics. Flory understood polymerisation as the formation of a random graph, rather than in terms of coagulation, and was aware of a sharp phase transition at the emergence of a giant connected structure, which he termed ‘gel’. A rigorous proof of the random graph phase transition was provided by Erdős and Rényi [9]. The existence of a phase transition corresponding to the formation of a giant particle, which corresponds to the phase transition in Theorem 2, was first discussed by Lushnikov [23], who uses this to explain the explosion of the second moment, corresponding to item 2 of Theorem 1, in the particular case of the multiplicative kernel. The first connection between random graph and particle approaches appears in [7], where the phase transition is proved for the particle coagulation process and an interpretation as a new proof for a phase transition in the Erdős-Rényi random graph is noted; this is also discussed in the survey article [2]. We extend this connection, and show that the bilinear form of the merger rate allows us to couple the stochastic coagulant process to *inhomogeneous* random graphs as considered by [5].

Our original motivation was to study a concept of interaction clusters introduced by Gabrielov et al. [12] in the context of the billiard model for an ideal gas. The distribution of the sizes of the interaction clusters is formally derived in [31] in terms of the solution of the Boltzmann equation. Reducing to the case of cutoff Maxwell molecules for the spatially homogeneous Boltzmann equation, the phase transition observed in [12] can be identified precisely and the cluster size distributions observed to match those arising from the Smoluchowski coagulation equation with product kernel [23, 2, 31]. Heuristically, when a collision occurs, the corresponding clusters merge, which may be represented as a coagulation event at the level of interaction clusters.

In [32] the clusters were studied for the Kac process, which is a stochastic approximation to the billiard model with elastic collisions, and the restriction to Maxwell molecules was lifted. This allowed a general collision rate including the hard sphere case and it was formally shown in a large particle number limit that the distribution of the cluster sizes converges to a version of the Smoluchowski coagulation equation with a time-dependent product kernel. In the Kac model where the rate of collision between two molecules with velocities v, w is proportional to $|v-w|^2 = |v|^2 + |w|^2 - 2v \cdot w$, a sum over particles in a cluster shows that the total merge rate depends on the mass, momentum and energy of the two clusters. Moreover, since collisions are elastic, these quantities add when two clusters merge, and are unchanged when a cluster undergoes an internal collision. This quadratic collision rate is of significant interest [22, 32, 36], although it does not have a natural physical interpretation. The explicit representation of the critical times in the present work enable us to verify the conjecture that the phase transition occurs strictly before the mean free time [32].

2 Well-Posedness of the Limiting Equation

This chapter is dedicated to a first analysis of the Smoluchowski equations (Sm, Fl), following Norris [28, 29]. Our goal in this section is to prove the following lemma on the well-posedness of (Fl).

Lemma 2.1. *For any measure $\mu_0 \in \mathcal{M}$ satisfying (A1.), the equation with gel (Fl) has a unique global solution $(\mu_t)_{t\geq 0}$ starting at μ_0. Moreover, $P_t = 0$ for all $t \geq 0$.*

Corollary 2.2. *Suppose $(\mu_t')_{t<T}$ is a conservative local solution to the equation without gel, (Sm), starting at μ_0. Then $\mu_t = \mu_t'$ for all $t < T$, and $T < t_{\mathrm{g}}$. Hence, (Sm) has a unique maximal conservative solution, given by $(\mu_t)_{t<t_{\mathrm{g}}}$.*

Our proof of Lemma 2.1 is an adaptation of the arguments in [28, Section 2] and [29, Section 2] and is based on a truncation argument. Recalling that $\varphi = \sum_{i=0}^n \pi_i$, we see that $\overline{K}(x,y) \leq \Delta\varphi(x)\varphi(y)$ for some $\Delta = \Delta(A)$. For all $\xi > 0$, we define the truncated particle space

$$S_\xi = \{x \in S : \varphi(x) \leq \xi\}. \tag{29}$$

We consider the following 'truncation at level ξ': in the empirical measure, we track only those particles inside S_ξ, and consider all other particles to belong to a 'truncated gel'. Although the particles in the truncated gel affect the dynamics in S_ξ, these contributions depend only on the total data g^ξ of the truncated gel, due to the bilinear form of the kernel. This leads to an ordinary differential equation with Lipschitz coefficients in an infinite dimensional space.

We formalise this intuition as follows. For a measure μ^ξ supported on S_ξ and $g^\xi \in S_{\mathrm{g}}$, we define a signed measure $L^\xi_{\mathrm{g}}(\mu^\xi, g^\xi)$ on S_ξ by specifying, for all $f \in C_c(S)$,

$$\begin{aligned}&\langle f, L^\xi_{\mathrm{g}}(\mu^\xi, g^\xi)\rangle \\ &\quad= \frac{1}{2}\int_{S_\xi^2}[f(x+y)1[\varphi(x+y)\leq\xi] - f(x) - f(y)]\overline{K}(x,y)\mu^\xi(dx)\mu^\xi(dy) \\ &\qquad + \int_{S_\xi}(f(y)-f(x))J(x,dy)\mu^\xi(dx) - \int_{S_\xi} f(x)\overline{K}(x,g^\xi)\mu^\xi(dx).\end{aligned} \tag{30}$$

This corresponds to the dynamics of particles inside S_ξ. The rate of change of the truncated gel data is given by

$$\begin{aligned}R^\xi_{\mathrm{g}}(\mu^\xi, g^\xi) = \frac{1}{2}\int_{S_\xi^2}&\pi(x+y)1[\varphi(x+y)>\xi]\overline{K}(x,y)\mu^\xi(dx)\mu^\xi(dy) \\ &+ \int_{S_\xi}\pi(x)\overline{K}(x,g^\xi)\mu^\xi(dx).\end{aligned} \tag{31}$$

We now seek measures μ_t^ξ supported on S_ξ and gel data $g_t^\xi=(M_t^\xi,E_t^\xi,P_t^\xi)\in S_{\mathrm{g}}$ such that, for all bounded measurable f on S_ξ,

$$\langle f,\mu_t^\xi\rangle=\langle f,\mu_0^\xi\rangle+\int_0^t\left\langle f,L_{\mathrm{g}}^\xi(\mu_s^\xi,g_s^\xi)\right\rangle ds; \tag{FI$|_\xi^1$}$$

$$g_t^\xi=g_0^\xi+\int_0^t R_{\mathrm{g}}^\xi(\mu_s^\xi,g_s^\xi)ds. \tag{FI$|_\xi^2$}$$

We will use the following existence and uniqueness result for the restricted dynamics (FI$|_\xi^1$, FI$|_\xi^2$).

Lemma 2.3. *[Existence and Uniqueness of Restricted Dynamics] Suppose μ_0^ξ is a finite measure on S_ξ which satisfies (A1.), and $g_0^\xi\in S_{\mathrm{g}}$ satisfies $\pi_i(g_0^\xi)=0$ for all $i>n$. Then there exists a unique map (μ_t^ξ,g_t^ξ) on $[0,\infty)$, which solves the restricted dynamics (FI$|_\xi^1$, FI$|_\xi^2$). Moreover, for all $t\geq 0$, μ_t^ξ is a positive, finite measure on S_ξ, $P_t^\xi=0$ for all times $t\geq 0$, and $g_t^\xi\in S_{\mathrm{g}}$.*

Sketch Proof of Lemma 2.3. This may be proved by a trivial modification of the arguments in [28, Proposition 2.2]. We define Picard iterates $((\mu_t^{(\xi,n)},g_t^{(\xi,n)}):n\geq 0,t\geq 0)$ by

$$(\mu_t^{(\xi,0)},g_t^{(\xi,0)})=(\mu_0^\xi,g_0^\xi); \tag{34}$$

$$\left(\mu_t^{(\xi,n+1)},g_t^{(\xi,n+1)}\right)=(\mu_0^\xi,g_0^\xi)+\int_0^t(L_{\mathrm{g}}^\xi,R_{\mathrm{g}}^\xi)\left(\mu_s^{(n,\xi)},g_s^{(n,\xi)}\right)ds. \tag{35}$$

One then uses bilinear continuity arguments in total variation norm $\|\cdot\|$ to show that, given a bound $\langle\varphi,\mu_0^\xi\rangle+\varphi(g_0^\xi)\leq C$, there is a positive time $T=T(\xi,C)>0$ such that the Picard iterates $(\mu_t^{(\xi,n)})_{t\leq T}$ converge uniformly in total variation on $[0,T]$, and that the limit μ_t^ξ solves (FI$|_\xi^1$, FI$|_\xi^2$), possibly allowing μ_t^ξ to be a signed measure. This argument also implies that the solution is unique on this interval. Now, we note that the quantity $\langle\varphi,\mu_t^\xi\rangle+\varphi(g_t^\xi)$ is constant in time, and therefore this construction can be repeated on $[T,2T]$, $[2T,3T]$, etc, which proves global existence and uniqueness. Finally, an integrating factor is introduced to argue that μ_t is a positive measure. In our case, it is also straightforward to see that the gel data $M_t^\xi,E_t^\xi\geq 0$, and that $P_t^\xi=0$, thanks to the symmetry (A1.). □

Proof of Lemma 2.1. We first show existence. For all $\xi<\infty$, we let (μ_t^ξ,g_t^ξ) be the solution to the dynamics (FI$|_\xi^1$, FI$|_\xi^2$) restricted to S_ξ, with initial data

$$\mu_0^\xi(dx)=1[x\in S_\xi]\,\mu_0(dx);\qquad g_0^\xi=\int_{x\notin S_\xi}\pi(x)\mu_0(dx). \tag{36}$$

Observe that, if $\xi<\xi'$, then $\widetilde{\mu}_t^\xi,\widetilde{g}_t^\xi$ given by

$$\widetilde{\mu}_t^\xi(dx)=1_{x\in S_\xi}\,\mu_t^{\xi'}(dx);\qquad \widetilde{g}_t^\xi=g_t^{\xi'}+\int_{x\in S_{\xi'}\setminus S_\xi}\pi(x)\mu_t^{\xi'}(dx) \tag{37}$$

solve the dymanics (FI$|_\xi^1$,FI$|_\xi^2$) with the same initial data μ_0^ξ,g_0^ξ. From uniqueness in Lemma 2.3, it follows that $\widetilde{\mu}_t^\xi=\mu_t^\xi;\widetilde{g}_t^\xi=g_t^\xi$. This shows that the measures μ_t^ξ are increasing in ξ, while the gel data M_t^ξ,E_t^ξ are decreasing, and P_t^ξ is identically 0, by symmetry (A1.). Therefore, the limits

$$\mu_t=\lim_{\xi\uparrow\infty}\mu_t^\xi;\qquad M_t=\lim_{\xi\to\infty}M_t^\xi;\qquad E_t=\lim_{\xi\to\infty}E_t^\xi \tag{38}$$

exist in the sense of monotone limits; one can then check that μ_t and $g_t = (M_t, E_t, 0)$ satisfy the full equation (FI), with initial values μ_0 and $g_0 = 0$.

To see uniqueness, let μ_t be the solution constructed above and write $g_t = (M_t, E_t, P_t)$ for the data of the gel. Let $\widetilde{\mu}_t$ be any solution to (FI) starting at μ_0, and let $\widetilde{g}_t = (\widetilde{M}_t, \widetilde{E}_t, \widetilde{P}_t)$ be the associated data of the gel. For all $\xi < \infty$, it is simple to verify that

$$\widetilde{\mu}_t^\xi(dx) = 1_{x\in S_\xi}\,\widetilde{\mu}_t(dx); \qquad \widetilde{g}_t^\xi = \widetilde{g}_t + \int_{S_\xi^c} \pi(x)\widetilde{\mu}_t(dx) \tag{39}$$

is a solution to the dynamics ($\mathrm{FI}|_\xi^1$, $\mathrm{FI}|_\xi^2$) on S_ξ. By uniqueness in Lemma 2.3, it follows that $\widetilde{\mu}_t^\xi = \mu_t^\xi$, and taking monotone limits, we see that $\widetilde{\mu}_t = \lim_{\xi\to\infty}\widetilde{\mu}_t^\xi = \lim_{\xi\to\infty}\mu_t^\xi = \mu_t$. The argument for $\widetilde{g}$ is identical. □

3 Convergence of the Stochastic Coagulant

We now turn to a preliminary version of Theorem 2. In this section, we will outline the proof of the convergence of the stochastic coagulant μ_t^N to a solution μ_t of (FI), locally uniformly in time. Most of the arguments are well-known for the Smoluchowski equation [28, 29], and for brevity, we will sketch the proof with an indication of the nontrivial technical details. Throughout, we fix μ_0 satisfying Assumption A, and μ_t^N with initial data μ_0^N satisfying Assumption B. Our result is as follows.

Lemma 3.1. *Suppose μ_0 satisfies Assumption A, and let $(\mu_t)_{t\geq 0}$ be the solution to (FI) starting at μ_0. Let $(\mu_t^N)_{t\geq 0}$ be stochastic coalescents with initial data μ_0^N satisfying Assumption B. Then we have the local uniform convergence*

$$\forall t_\mathrm{f} \geq 0 \qquad \sup_{t\leq t_\mathrm{f}} d(\mu_t^N, \mu_t) \to 0 \text{ in probability} \tag{40}$$

where recall that d is a complete metric inducing the vague topology.

Remark 3.2. *We will later upgrade the* local *uniform convergence to* full *uniform convergence in Lemma 8.2. We also remark that this does not immediately imply the convergence of the gel terms in Theorem 2, as the test functions involved are neither compactly supported nor even bounded. This will be dealt with in Sections 5, 10, where the proofs build on this result.*

Proof. The proof follows the well known method of proving tightness and identifying possible limit paths: Firstly, the jump rates can bounded, uniformly in time, in terms of the initial second moment $\langle\varphi^2, \mu_0^N\rangle$ and, thanks to (B2.), these are stochastically bounded: $\langle\varphi^2, \mu_0^N\rangle \in \mathcal{O}_\mathrm{p}(1)$ as $N\to\infty$. As a result, it follows that for all $t_\mathrm{f} \geq 0$, the processes $(\mu_t^N)_{0\leq t\leq t_\mathrm{f}}$ are tight in the Skorohod topology of $\mathbb{D}([0, t_\mathrm{f}], (\mathcal{M}, d))$.

Next, we wish to argue that if $\overline{\mu}$ is any subsequential limit point, then $\overline{\mu}$ coincides with the solution μ_t to (FI). For this stage, we show that for certain well-chosen $\xi > 0$, the pair

$$\mu_t^{N,\xi} = \mu_t^N 1_{S_\xi}; \qquad g_t^{N,\xi} = \langle \pi, \mu_t^N - \mu_t^{N,\xi}\rangle \tag{41}$$

converge to a pair $\overline{\mu}_t^\xi = \overline{\mu}_t 1_{S_\xi}, \overline{g}_t^\xi$ which solve the restricted evolution equations ($\mathrm{FI}|_\xi^1$, $\mathrm{FI}|_\xi^2$), started at

$$\overline{\mu}_0^\xi = \mu_0 1_{S_\xi}; \qquad \overline{g}_0^\xi = \int_{x\notin S_\xi} x\mu_0(dx). \tag{42}$$

In order to prove this convergence, we will need a pair of regularity conditions (C1-C2.) which will be displayed below. These allow us to obtain vague convergence of $\mu_t^{N,\xi}$, despite the discontinuity of the cutoff S_ξ. Moreover, one can show that these conditions are satisfied for almost all $\xi > 0$.

i). Almost surely, for almost all $t \le t_{\mathrm{f}}$,

$$\overline{\mu}_t\left(\{x\colon \varphi(x)=\xi\}\right)+\overline{\mu}_t\otimes\overline{\mu}_t\left(\{(x,y)\colon \varphi(x+y)=\xi\}\right)=0; \tag{C1.}$$

ii). This also holds for $t=0$. That is, almost surely,

$$\overline{\mu}_0\left(\{x\colon \varphi(x)=\xi\}\right)+\overline{\mu}_0\otimes\overline{\mu}_0\left(\{(x,y)\colon \varphi(x+y)=\xi\}\right)=0. \tag{C2.}$$

Thanks to the construction of solutions to the global equation (FI) in Lemma 2.1, we know that for all such ξ, $\overline{\mu}_t 1_{S_\xi}$ coincides with $\mu_t 1_{S_\xi}$. Finally, we take a limit of such $\xi\uparrow\infty$, to conclude the equality $\overline{\mu}_t=\mu_t, t\le t_{\mathrm{f}}$. Since the limit process $(\mu_t)_{0\le t\le t_{\mathrm{f}}}$ is continuous in the vague topology $(\mathcal{M},d)$, it follows that we may upgrade from Skorohod to uniform convergence:

$$\sup_{0\le t\le t_{\mathrm{f}}} d\left(\mu_t^N,\mu_t\right)\to 0 \quad \text{in probability} \tag{45}$$

as claimed. □

4 Coupling of the Stochastic Coagulant to Random Graphs

In this section, we will show that the stochastic coagulant defined in the introduction may be coupled to a *dynamic* version of the random graphs $\mathcal{G}^{\mathcal{V}}(N,tk)$ considered in [5]. This allows us to apply some results of that paper, which we summarise in Appendix II, to analyse the stochastic coagulant process and the limit equation.

Definition 3. *[Dynamic Inhomogenous Random Graphs] Fix a measure μ_0 satisfying Assumption A. Let $\mathbf{x}_N=(x_i, i=1,2,...,l^N)$ be a collection random points in S of potentially random length l^N, and sample $\tau_e\sim$ Exponential(1), independently of each other, for $e=(ij), 1\le i,j\le l^N$, and independently of $\mathbf{x}_N$. We define the kernel*

$$k(v,w)=2\overline{K}\left(x,y\right) \tag{46}$$

where the right-hand side is the total mass of the interaction kernel $\overline{K}(x,y)=K(x,y,S)$. We form the random graphs $(G_t^N)_{t\ge0}$ on $\{1,2,...,l^N\}$ by including the edge $e=(ij)$ if

$$t\ge\frac{N\tau_e}{k(x_i,x_j)}. \tag{47}$$

We write $G_t^N\sim\mathcal{G}(\mathbf{x}_N,\frac{tK}{N})$ for the distribution of G_t^N, for a single fixed $t\ge0$. We say that G_0^N satisfy Assumption B for μ_0 if the same is true of the empirical measures $\mu_0^N=N^{-1}\sum_{i\le l^N}\delta_{x_i}$. We emphasise that the x_i do not *change during the dynamics.*

This has the following immediate consequences. Firstly, the conditions in Assumption B guarantee that $\mathcal{V}=(S,\mu_0,(\mathbf{x}_N)_{N\ge1})$, is a generalised vertex space in the sense of [5], which is recalled in Definition II.1, and k is an irreducible kernel as described in Definition II.2, thanks to (A4.). Using both parts of (B2.), one can also show that the kernel k is *graphical* in the sense of Definition II.5.

For all times t, G_t^N is an instance of the inhomogeneous random graph from Definition II.3. Moreover, the process $(G_t^N)_{t\geq 0}$ is increasing, and is a Markov process, by the memoryless property of the exponential variables τ_e. We write T for the convolution operator

$$(Tf)(x) = \int_S f(y)k(x,y)\mu_0(dy) \tag{48}$$

and $\|T\|$ for the associated operator norm in $L^2(\mu_0)$.

We write also $t_\mathrm{c} = \|T\|^{-1}$. The following is the basic statement of a phase transition for G_t^N, which follows from Theorem II.8.

Lemma 4.1. *Let μ_0 satisfy Assumption A, and let G_t^N be the random graphs constructed above, such that G_0^N satify Assumption B. Write $C_1(G_t^N)$ for the size of the largest component of G_t^N. Then we have the following phase transition:*

i). If $t \leq t_\mathrm{c}$, then $N^{-1}C_1(G_t^N) \to 0$ in probability.

ii). If $t > t_\mathrm{c}$, then there exists $c = c(t)$ such that, with high probability, $C_1(G_t^N) \geq cN$.

We write $\mathcal{C}_1(G), ...\mathcal{C}_j(G), ...$ for the connected components, which we also call *clusters*, of G, in decreasing order of size, allowing $\mathcal{C}_j = \emptyset$ if G has fewer than j components and $C_j(G)$ for the number of vertices in $\mathcal{C}_j(G)$. For a cluster $\mathcal{C}$ of the graph G_t^N, we will write

$$M(\mathcal{C}) = \sum_{i\in\mathcal{C}} \pi_0(x_i), \quad E(\mathcal{C}) = \left(\sum_{i\in\mathcal{C}} \pi_j(x_i)\right)_{j=1}^{n}, \quad P(\mathcal{C}) = \left(\sum_{i\in\mathcal{C}} \pi_j(x_i)\right)_{j=n+1}^{n+m} \tag{49}$$

for the unnormalised data, and

$$\pi(\mathcal{C}) = \sum_{i\in\mathcal{C}} \pi(x_i) = (M(\mathcal{C}), E(\mathcal{C}), P(\mathcal{C})), \quad \varphi(\mathcal{C}) = \sum_{i\in\mathcal{C}}\sum_{j=0}^{n} \pi_j(x_i). \tag{50}$$

We write $\delta(\mathcal{C})$ for the point mass $\delta(\mathcal{C}) = \delta_{\pi(\mathcal{C})}$, and $\pi_\star(G_t^N)$ for the normalised empirical measure

$$\pi_\star(G_t^N) = \frac{1}{N}\sum_{\text{Clusters}} \delta(\mathcal{C}) \tag{51}$$

where the sum is over all clusters $\mathcal{C}$ of G_t^N. This is connected to the stochastic coagulants as follows:

Lemma 4.2 (Coupling of Random Graphs and Stochastic Coagulants)**.** *Fix points $\mathbf{x}_N = (x_1, ..., x_{l^N(0)})$ in S, and let $(G_t^N)_{t\geq 0}$ be the random graph process described in Definition 3 for this choice of vertex data. Consider also a stochastic coagulant $(\mu_t^N)_{t\geq 0}$ started from $\mu_0^N = \frac{1}{N}\sum_{i\leq l^N(0)} \delta_{x_i}$. Then the processes $\pi_\star(G_t^N)$ and $\pi_\# \mu_t^N$ are equal in law.*

Remark 4.3. *This is the key result which makes much of our analysis possible. Many of the remaining points of Theorem 1 above concern only the moments $\langle \pi_i, \mu_t\rangle$, $\langle \varphi^2, \mu_t\rangle$ which depend on μ_t only through the pushforward $\pi_\#\mu_t$. By applying Lemma 3.1 in the space S^Π, we can use the pushforwards $\pi_\#\mu_t^N$ as stochastic proxies to $\pi_\#\mu_t$, and thanks to Lemma 4.2, the measures $\pi_\#\mu_t^N$ can be realised as $\pi_\star(G_t^N)$ for a random graph process G_t^N. In this way, we can apply results from the theory of random graphs [5] to deduce results about solutions (μ_t) to the Smoluchowski equation (FI).*

Sketch of proof of Lemma 4.2. Let us fix $\mathbf{x}_N$. Firstly, both processes are Markov: for $\pi_\#\mu_t^N$, the follows because the total rate (5) depends only on $\pi(x), \pi(y)$, and similarly for $\pi_\star(G_t^N)$. One may also verify that the two processes undergo the same transitions at the same rates, again thanks to (5), and that the total rate is bounded in terms of $\mathbf{x}_N$. The boundedness of the total rate implies the uniquness in law for the corresponding Markov generator, which concludes the proof. □

Combining this with the approximation result Lemma 3.1 for the stochastic coagulant, we may connect the random graph process to the limit equation as follows.

Lemma 4.4 (Convergence of the Random Graphs)**.** *Let μ_0 be a measure on S satisfying Assumption A, and let $(G_t^N)_{t\ge 0}$ be the random graph processes constructed above with initial data $\mathbf{x}_N = (x_1, ...x_{l^N})$ which satisfies Assumption B. Let $(\mu_t)_{t\ge0}$ be the solution to the Smoluchowski Equation (Fl) starting at μ_0; then we have the local uniform convergence*

$$\sup_{t\le t_{\mathrm{f}}} d_\Pi(\pi_\star(G_t^N), \pi_\#\mu_t) \to 0 \tag{52}$$

in probability, for all $t_{\mathrm{f}} < \infty$, where d_Π is a metric for the vague topology $\mathcal{F}(\mathcal{M}_{\le 1}(S^\Pi), C_c(S^\Pi))$.

We can also compute the critical time associated to G_t^N explicitly:

Lemma 4.5 (Computation of critical time)**.** *Let μ_0 be a measure satisfying Assumption A, and let G_t^N be random graphs satisfying Assumption B. Then the convolution operator T constructed above is a bounded linear map on $L^2(\mu_0)$ and the inverse of the critical time for the graph phase transition, t_{c}^{-1}, is the largest eigenvalue of the $n\times n$ matrix $\Lambda(\mu_0)$ given by $\Lambda(\mu_0)_{ij} = \langle (A\pi)_i\pi_j, \mu_0\rangle$. In particular, $t_{\mathrm{c}} \in (0,\infty)$.*

Remark 4.6. *This is exactly the form claimed for t_{g} in Theorem 1. However, we have not yet established that $t_{\mathrm{c}} = t_{\mathrm{g}}$; this is the content of Lemma 5.1.*

Proof of Lemma 4.5. Firstly, by (A2.), it is easy to see that $k \in L^2(S\times S, \mu_0\times\mu_0)$, and so, by Lemma II.10, $\|T\| = t_{\mathrm{c}}^{-1}$ is the largest eigenvalue of T; its eigenspace is one-dimensional and consists of functions that are single signed, μ_0- almost everywhere. Since $0 < \|T\| < \infty$ we have $0 < t_{\mathrm{c}} < \infty$.

In order to reduce from the operator T to the matrix $\Lambda(\mu_0)$ we construct a basis $\{e_i\}_{i\ge1}$ of $L^2(\mu_0)$ such that

$$e_i(x) = \pi_i(x), \quad i = 1, 2, ..n+m \tag{53}$$

and, for $i > n+m$, e_i is orthogonal to $E = \mathrm{Span}(e_1, ..., e_{n+m})$. Note that π_0 plays no special role in the basis, because it does not appear in the rate $\overline{K}$. We also write $E_+ = \mathrm{Span}(e_1, ...e_n)$ and $E_{\mathrm{Sym}} = \mathrm{Span}(e_{n+1}, ..., e_{n+m})$. By expanding the total rate $\overline{K}(x,y)$, we see that, for all $f \in L^2(m)$,

$$(Tf)(x) = 2\sum_{i,j=1}^{n+m} a_{ij}\langle f, \pi_i\rangle_{L^2(\mu_0)}\pi_j(x) \tag{54}$$

where $\langle\cdot,\cdot\rangle_{L^2(\mu_0)}$ denotes the $L^2(\mu_0)$ inner product. Therefore, T maps into the subspace E, and is 0 on its orthogonal complement. We further note that the subspaces E_+, E_{sym} are orthogonal, and are invariant under T. Therefore, the eigenspace E^λ corresponding to $\lambda = t_{\mathrm{c}}^{-1}$ is a direct sum $E_+^\lambda \oplus E_{\mathrm{sym}}^\lambda$ of eigenspaces contained within E_+, E_{sym}.

Since E^λ is one-dimensional, one summand must be trivial, and so either $E^\lambda = E_+^\lambda \subseteq E_+$, or $E^\lambda \subseteq E_{\mathrm{sym}}$. To exclude the second possibility, we note that any $f\in E_{\mathrm{sym}}$ satisfies $f(Rx) = -f(x)$

for all x by Definition 1, while eigenfunctions of T are single-signed μ_0-almost everywhere. It therefore follows that $E^\lambda \subseteq E_+$ and that t_c^{-1} is the largest eigenvalue of $T|_{E_+}$.

The result is now immediate since (54) shows that $\Lambda(\mu_0)$ is the matrix representation of $T|_{E_+}$ respect to the basis introduced above. □

We also define ρ_t as the survival function from Lemma II.7, given by the maximal solution to

$$\rho_t(x) = 1 - \exp\left(-t(T\rho_t)(x)\right). \tag{55}$$

We note, for future use, the following properties where k is the kernel given above.

Lemma 4.7. *The survival function $\rho_t(v) = \rho(tk, x)$ takes the form*

$$\rho_t(x) = 1 - \exp\left(-\sum_{i=1}^{n} c_t^i \pi_i(x)\right) \tag{56}$$

for some $c_t^i \geq 0$. Moreover, the functions $t \mapsto c_t^i$ are continuous.

This proves the first two assertions of item 4 of Theorem 1.

Proof. Using the symmetry $k(Rx, Ry) = k(x, y)$ and Assumption (A1.), it is simple to verify that $\tilde{\rho}(x) := \rho_t(Rx)$ also satisfies the fixed point equation (55). By maximality of ρ_t, we must have $\rho_t(Rx) \leq \rho_t(x)$ for all $x \in S$, which implies that ρ_t is even under R.

Using the identification of the range of T as in Lemma 4.5, we see that there exist $c_t^i : 1 \leq i \leq n+m$ such that

$$t(T\rho_t)(x) = \sum_{i=1}^{n+m} c_t^i \pi_i(x) \tag{57}$$

and expanding k as in (54), the coefficients are given explicitly by

$$c_t^i = 2t \sum_{j=1}^{n} a_{ij} \langle \pi_j \rho_t, \mu_0 \rangle. \tag{58}$$

Since ρ_t is even, we have $c_t^i = 0$ for $i > n$, and since $\rho_t \geq 0$, $c_t^i \geq 0$ for $i = 1, ..., n$. Using (55) again, we obtain the claimed representation

$$\rho_t(x) = 1 - \exp\left(-\sum_{i=1}^{n} c_t^i \pi_i(x)\right). \tag{59}$$

The continuity follows by applying dominated convergence to (58), and using the continuity of ρ_t established in Theorem II.12. □

5 Equality of the Critical Times

In this section, we will prove that the critical time t_c for the graph process, introduced in Section 4, coincides with the gelation time for the limiting equation, defined in Section 2 as the time at which mass and energy begin to escape to infinity.

Lemma 5.1. *Let μ_0 be a measure on S satisfying Assumption A. Let $(\mu_t)_{t\geq 0}$ be the solution to (FI) starting at μ_0, with associated data M_t, E_t of the gel; recall that $t_{\rm g}$ is defined by*

$$t_{\rm g} := \inf\{t \geq 0 : \langle \varphi, \mu_t\rangle < \langle \varphi, \mu_0\rangle\} = \inf\{t\geq 0 : g_t \neq 0\}. \tag{60}$$

Let (G^N_t) be the random graph processes constructed above, and suppose that Assumption B holds for G^N_0, μ_0. Then the critical time $t_{\rm c}$ for the graph transition process coincides with the gelation time $t_{\rm g}$.

The following is a straightforward corollary.

Corollary 5.2. *Let μ_0 satisfy Assumption A, and let $(\mu_t)_{t\geq 0}$ be the solution to (FI) starting at μ_0, with gelation at $t_{\rm g}$. Then $t_{\rm g}$ is given explicitly by (17).*

Proof of Corollary 5.2. Let us form $\mathbf{x}_N$ by sampling points as a Poisson random measure with intensity $N\mu_0$. It is immediate that the resulting data $\mathbf{x}_N$ satisfies Assumption B for the measure μ_0, and the critical time $t_{\rm c}$ of the associated graphs G^N_t is given by the claimed expression (17). From the previous lemma, it now follows that the gelation time $t_{\rm g} = t_{\rm c}$, which proves the claimed result. □

The proof of Lemma 5.1 is based on the following weak version of the convergence of the gel in Theorem 2, which will be revisited in Section 10 to establish uniform convergence.

Lemma 5.3. *Let $(\mu_t)_{t\geq 0}, M_t, E_t$ be as in Lemma 5.1 and G^N_t be as in the proof of Corollary 5.2. Fix $t > 0$, and write g^N_t for the scaled mass and energy of the largest particle in G^N_t, as in Section 4:*

$$g^N_t = \frac{1}{N}\,\pi(\mathcal{C}_1(G^N_t)) = \left(\frac{1}{N}\sum_{i\in\mathcal{C}_1(G^N_t)} \pi_j(x_i)\right)_{j=0}^{n+m} = (M^N_t, E^N_t, P^N_t). \tag{61}$$

Then $M^N_t \to M_t$ and $E^N_t \to E_t$ in probability.

We first show that Lemma 5.3 implies Lemma 5.1; the remainder of this section is dedicated to the proof of Lemma 5.3.

Proof of Lemma 5.1. Let us assume, for the moment, that Lemma 5.3 holds. Throughout, let $(x_i)_{i=1}^{l^N}$ be the vertex data of the random graph process, which we recall are independent of time.

Firstly, suppose for a contradiction that $t_{\rm g} < t_{\rm c}$. Then $\varphi(g_{t_{\rm c}}) > 0$, but we bound

$$\varphi(g^N_{t_{\rm c}}) \leq \left(\frac{1}{N}C_1(G^N_{t_{\rm c}})\right)^{\frac{1}{2}} \left(\frac{1}{N}\sum_{i=1}^{l^N} \varphi(x_i)^2\right)^{\frac{1}{2}}. \tag{62}$$

The first term converges to 0 in probability, by definition of the phase transition in Theorem II.8, and the second term is bounded in L^2 by hypothesis (B2.). This implies that $\varphi(g^N_{t_{\rm c}}) \to 0$ in probability, which contradicts Lemma 5.3; we must therefore have that $t_{\rm g} \geq t_{\rm c}$.

Conversely, if $t < t_{\rm g}$, then $M_t = 0$ by definition. Now, the convergence

$$\frac{1}{N}C_1(G^N_t) = M^N_t \to 0 \tag{63}$$

in probability implies that the largest cluster is of the order $o_{\mathrm{p}}(N)$, which is only possible if $t \leq t_{\mathrm{c}}$ by Lemma 4.1. Since $t < t_{\mathrm{g}}$ was arbitrary, we must have $t_{\mathrm{g}} \leq t_{\mathrm{c}}$, and together with the previous argument, we have shown that $t_{\mathrm{g}} = t_{\mathrm{c}}$ as claimed. □

The proof of Lemma 5.3 is based on the following argument. We know, from Theorem II.11, that any 'mesoscopic' clusters contain negligable mass; thanks to the integrability assumption (A2.), the same is true for the energy. Therefore, almost all mass and energy either belongs to the 'microscopic' scale, whose convergence is quantified by Lemma 3.1, or the giant component, whose convergence is the subject of interest here. Therefore, with a suitable approximation argument, the claimed convergence will follow from the quoted results.

We begin with a preparatory lemma; throughout, we will assume the notation of Lemma 5.3. For the proof of of Lemma 5.1, and later Theorem 2, we will wish to study the convergence of integrals $\langle \varphi f, \mu_t^N\rangle$, for bounded continuous functions f with non-compact support. However, the convergence result Lemma 3.1 only gives us information when the support of f is compact. Our second preparatory lemma allows us to approximate the integrals $\langle \varphi f, \mu_t^N\rangle$ for functions f whose support is bounded in the π_0-direction.

Lemma 5.4 (A step towards uniform integrability)**.** *Let μ_0, $(\mu_t^N)_{t\geq 0}$ be as in the previous lemma. Then, for every $r > 0$,*

$$\beta(r,\eta) := \sup_{N\geq 1} \mathbb{E}\left[\sup_{t\geq 0}\left\langle \varphi 1[\varphi(x) > \eta, \pi_0(x) \leq r], \mu_t^N\right\rangle\right] \to 0 \qquad \text{as } \eta \to \infty. \tag{64}$$

Proof. We note that $\langle \varphi 1[\varphi(x) > \eta, \pi_0(x) \leq r], \mu_t^N\rangle$ depends on μ_t^N only through the pushforward $\pi_\# \mu_t^N$, since the integrand only depends on the values of π at the different particles. From Lemma 4.2, we can find random graphs G_t^N, such that $\mathbf{x}_N$ is an enumeration of the atoms of μ_0^N and $\pi_\star(G_t^N) = \pi_\# \mu_t^N$ for all times t. With this coupling, we express the integral as follows:

$$\begin{aligned}\langle \varphi 1[\varphi(x) > \eta, \pi_0(x) \leq r], \mu_t^N\rangle &= \frac{1}{N}\sum_{\text{Clusters } \mathcal{C}\subset G_t^N} \varphi(\mathcal{C}) 1[\varphi(\mathcal{C}) > \eta, \pi_0(\mathcal{C}) \leq r] \\ &= \frac{1}{N}\sum_{j=1}^{l^N(t)} \sum_{i\in \mathcal{C}_j(G_t^N)} \varphi(x_i) 1\left[\varphi(\mathcal{C}_j(G_t^N)) > \eta, \pi_0(\mathcal{C}_j(G_t^N)) \leq r\right].\end{aligned} \tag{65}$$

Using Cauchy-Schwarz, we bound

$$\begin{aligned}&\sup_{t\geq 0}\left\langle \varphi 1[\varphi(x) > \eta, \pi_0(x) \leq r], \mu_t^N\right\rangle \\ &\leq \left(\frac{1}{N}\sum_{j=1}^{l^N(0)} \varphi(x_j)^2\right)^{\frac{1}{2}} \left(\sup_{t\geq 0}\frac{1}{N}\sum_{j=1}^{l^N(t)}\sum_{i\in\mathcal{C}_j(G_t^N)} 1\left[\varphi(\mathcal{C}_j(G_t^N)) > \eta, C_j(G_t^N) \leq r\right]\right)^{\frac{1}{2}} \qquad (66) \\ &= \left(\frac{1}{N}\sum_{i=1}^{l^N(0)} \varphi(x_i)^2\right)^{\frac{1}{2}} \left(\sup_{t\geq 0}\left\langle \pi_0 1[\varphi(x) > \eta, \pi_0(x) \leq r], \mu_t^N\right\rangle\right)^{\frac{1}{2}}.\end{aligned}$$

As remarked in Definition 3, the data x_i associated with the graph nodes are constant in time, so the first factor is independent of $t \geq 0$, and is bounded in L^2 by the second assertion of (B2.). Therefore, it is sufficient to prove the claim with φ replaced by π_0.

Now we note that with probability one

$$\sup_{t\geq 0} \left\langle \pi_0 1[\varphi(x) > \eta, \pi_0(x) \leq r], \mu_t^N \right\rangle \leq r \sup_{t\geq 0} \left\langle 1[\varphi(x) > \eta], \mu_t^N \right\rangle \\ \leq \frac{r}{\eta} \sup_{t\geq 0} \left\langle \varphi, \mu_t^N \right\rangle = \frac{r}{\eta} \left\langle \varphi, \mu_0^N \right\rangle$$

and the result follows from (B2.). □

Using the preparatory lemma developed above, we now prove Lemma 5.3.

Proof of Lemma 5.3. Throughout, we let $(\mu_t^N)_{t\geq 0}$ be a stochastic coagulant coupled to a random graph process $(G_t^N)_{t\geq 0}$, as described in Section 4 with vertex data $\mathbf{x}_N = (x_i)_{i=1}^{l^N(0)}$; thanks to this construction, M_t^N is exactly the size of the largest cluster in G_t^N, and E_t^N are the sums

$$E_t^N = \left(N^{-1} \sum_{j \in \mathcal{C}_1(G_t^N)} \pi_i(x_j) \right)_{i=1}^n . \tag{67}$$

The case $t = 0$ is trivial, and can be omitted. We deal first with the 0^{th} coordinate M_t^N; the cases for the $1^{\text{st}}, \ldots, n^{\text{th}}$ coordinates E_t^N are entirely analagous.

Fix $t > 0$, and let ξ_N be a sequence, to be constructed later, such that

$$\xi_N \to \infty; \qquad \frac{\xi_N}{N} \to 0. \tag{68}$$

We now construct 'bump functions' as follows. Let $\eta_r \to \infty$ be a sequence growing sufficiently fast that, in the notation of Lemma 5.4, $\beta(r, \eta_r) \to 0$, and let

$$S_{(r)} := \{x \in S : \pi_0(x) < r, \varphi(x) \leq \eta_r\}. \tag{69}$$

Let $\widetilde{h}_r$ be the indicator $\widetilde{h}_r = 1[\pi_0(x) < r]$, and construct a continuous, compactly supported function $\widetilde{f}_r$ such that

$$0 \leq \widetilde{f}_r \leq 1; \qquad \widetilde{f}_r = 1 \text{ on } S_{(r)}; \qquad \widetilde{f}_r(x) = 0 \text{ if } \pi_0(x) \geq r. \tag{70}$$

The final condition is compatible with continuity because $\pi_0 : S \to \mathbb{N}$ is continuous and integer valued. We define $f_N = \widetilde{f}_{\xi_N}$ and $h_N = \widetilde{h}_{\xi_N}$. We now decompose the difference $M_t^N - M_t$:

$$\begin{aligned} M_t^N - M_t = \underbrace{(\langle \pi_0, \mu_t \rangle - \langle \pi_0 f_N, \mu_t \rangle)}_{:=\mathcal{T}_N^1} + \underbrace{\langle \pi_0 f_N, \mu_t - \mu_t^N \rangle}_{:=\mathcal{T}_N^2} \\ + \underbrace{\langle \pi_0 (f_N - h_N), \mu_t^N \rangle}_{:=\mathcal{T}_N^3} + \underbrace{\langle \pi_0 h_N, \mu_t^N \rangle - (\langle \pi_0, \mu_0^N \rangle - M_t^N)}_{:=\mathcal{T}_N^4} \\ + \underbrace{\langle \pi_0, \mu_0^N - \mu_0 \rangle}_{:=\mathcal{T}_N^5} . \end{aligned} \tag{71}$$

where we recall that $M_t = \langle \pi_0, \mu_0 - \mu_t \rangle$. We now estimate the errors $\mathcal{T}_N^i, i = 1, 3, 4, 5$; the remaining term $\mathcal{T}_N^2$ will be dealt with separately, and requires careful construction of the sequence ξ_N.

1. Estimate on $\mathcal{T}_N^1$. Let z_N be the lower bound $z_N = 1_{S_{(\xi_N)}}$, so that $z_N \le f_N \le 1$. As $N \to \infty$, $\pi_0 z_N \uparrow \pi_0$, and so by monotone convergence, $\langle \pi_0 z_N, \mu_t\rangle \uparrow \langle \pi_0, \mu_t\rangle$. This implies that the (nonrandom) error $\mathcal{T}_N^1 \to 0$.

2. Estimate on $\mathcal{T}_N^3$. From the definitions of f_N, h_N, we observe that

$$|\mathcal{T}_N^3(t)| = \langle \pi_0(h_N - f_N), \mu_t^N\rangle \le \langle \pi_0 1[\pi_0(x) < \xi_N, \varphi(x) > \eta_{\xi_N}], \mu_t^N\rangle. \tag{72}$$

Therefore, in the notation of Lemma 5.4, $\mathbb{E}\left[\sup_{t\ge 0}|\mathcal{T}_N^3(t)|\right] \le \beta(\xi_N, \eta_{\xi_N})$. By construction of η_r, and since $\xi_N \to \infty$, it follows that $\mathbb{E}[\sup_{t\ge 0}|\mathcal{T}_N^3(t)|] \to 0$, which implies convergence to 0 in probability.

3. Estimate on $\mathcal{T}_N^4$. Recalling that $h_N(x) = 1[\pi_0(x) < \xi_N]$ and using the coupling to random graphs, we have the equality

$$\langle \pi_0 h_N, \mu_t^N\rangle = \langle \pi_0, \mu_0^N\rangle - M_t^N 1\left[M_t^N \ge \frac{\xi_N}{N}\right] - \frac{1}{N}\sum_{j\ge 2: C_j(G_t^N)\ge \xi_N}\ \sum_{i\in C_j(G_t^N)} \pi_0(x_i) \tag{73}$$

which gives the equality

$$\mathcal{T}_N^4 = -M_t^N 1\left(M_t^N \le \frac{\xi_N}{N}\right) - \frac{1}{N}\sum_{j\ge 2: C_j(G_t^N)\ge \xi_N} \pi_0(\mathcal{C}_j(G_t^N)). \tag{74}$$

Using Cauchy-Schwarz, we bound

$$\left|\mathcal{T}_N^4(t)\right| \le \left(\frac{1}{N}\sum_{j\ge 2: C_j(G_t^N)\ge \xi_N} C_j(G_t^N)\right)^{\frac{1}{2}} \left(\frac{1}{N}\sum_{i=1}^{l^N(0)} \varphi(x_i)^2\right)^{\frac{1}{2}} + \frac{\xi_N}{N}. \tag{75}$$

The first term converges to 0 in probability by Theorem II.11 and (B2.), and the second converges to 0 since $\xi_N \ll N$. Together, these imply that $\mathcal{T}_N^4(t) \to 0$ in probability.

4. Estimate on $\mathcal{T}_N^5$. Using the first part of (B2.), we have the convergence in distribution

$$\langle \pi_0, \mu_0^N\rangle \to \langle \pi_0, \mu_0\rangle \tag{76}$$

which implies that $\mathcal{T}_N^5 \to 0$ in probability as desired.

5. Construction of ξ_N, and convergence of $\mathcal{T}_N^2$. It remains to show how a sequence ξ_N can be constructed such that $\mathcal{T}_N^2 \to 0$ in probability and such that (68) holds. Recalling the definition of $\widetilde{f}_r$ above, let $A_{r,N}^1$ be the events $A_{r,N}^1 = \{|\langle \varphi \widetilde{f}_r, \mu_t^N - \mu_t\rangle| < \frac{1}{r}\}$; as $N \to \infty$ with r fixed, both $\mathbb{P}(A_{r,N}^1) \to 1$, by Lemma 3.1. We now define N_r inductively for $r \ge 1$ by setting $N_1 = 1$, and letting N_{r+1} be the minimal $N > \max(N_r, (r+1)^2)$ such that, for all $N' \ge N$, $\mathbb{P}(A_{r+1,N'}^1) > \frac{r}{r+1}$. Now, we set $\xi_N = r$ for $N \in [N_r, N_{r+1})$. It follows that $\xi_N \to \infty$ and $\xi_N \le \sqrt{N} \ll N$, and

$$\mathbb{P}\left(C_1(G_t^N)) \ge \xi_N\right) \ge 1 - \frac{1}{\xi_N} \to 1. \tag{77}$$

Therefore, ξ_N satisfies the requirements (68) above. Moreover,

$$\mathbb{P}\left(|\mathcal{T}_N^2| < \frac{1}{\xi_N}\right) \geq \mathbb{P}\left(A^1_{\xi_N,N}\right) > 1 - \frac{1}{\xi_N} \to 1 \tag{78}$$

and so, with this choice of ξ_N, $\mathcal{T}_N^2 \to 0$ in probability. Since we have now dealt with every term appearing in the decomposition (71), it follows that $M_t^N \to M_t$ in probability, as claimed.

The arguments for the $1^{\text{st}} - n^{\text{th}}$ components E_t^N are identical to those above, using the same bound (75) on $\mathcal{T}_N^4$. □

We also note, for future use, an important corollary of this argument.

Corollary 5.5. *At the instant of gelation, the gel is negligible: $g_{t_{\mathrm{g}}} = 0$.*

Proof. For the $0^{\text{th}} - n^{\text{th}}$ components, this follows from the critical case of Theorem II.8 exactly as in (62). The remaining m components $g_t^i, i > n$ are identically 0 by the symmetry (A1.), as in Lemma 2.1. □

6 Behaviour of the Second Moments

In this section, we consider part 2 of Theorem 1, concerning the behaviour of the second moments $\mathcal{Q}(t)_{ij} = \langle \pi_i\pi_j, \mu_t\rangle, 0 \leq i, j \leq n$ and $\mathcal{E}(t) = \langle \varphi^2, \mu_t\rangle$. Following [23, 29], one might expect that the gelation time t_{g} corresponds to a divergence of $\mathcal{E}(t)$ as $t \uparrow t_{\mathrm{g}}$; by an approximation argument, we will show that this is indeed the case. We also introduce a *duality argument*, corresponding to Theorem II.13, which allows us to prove that $\mathcal{E}$ is finite on (t_{g}, ∞). The final assertion follows from the fact that $g_{t_{\mathrm{g}}} = 0$, which is the content of Corollary 5.5.

6.1 Subcritical Regime

We first deal with the subcritical regime $[0, t_{\mathrm{g}})$, to show that the second moments $\mathcal{Q}_{ij}(t), \mathcal{E}(t)$ are finite and increasing on this interval, and that t_{g} is exactly the first time at which $\mathcal{E}$ diverges.

Lemma 6.1. *Suppose μ_0 satisfies Assumption A, and let $(\mu_t)_{t\geq 0}$ be the corresponding solution to (Fl). The second moments $\mathcal{Q}(t)_{ij} = \langle \pi_i\pi_j, \mu_t\rangle, 0 \leq i, j \leq n,\ \mathcal{E}(t) = \langle \varphi^2, \mu_t\rangle$ are finite, continuous and increasing on $[0, t_{\mathrm{g}})$, and $\mathcal{E}(t) = \langle \varphi^2, \mu_t\rangle$ increases to infinity as $t \uparrow t_{\mathrm{g}}$, where t_{g} is the associated gelation time.*

The ideas of this argument follow [29], where there is a similar result for *approximately multiplicative* kernels, for which the total rate $\overline{K}(x, y)$ is bounded above *and below* by nonzero multiples of $\widetilde{\varphi}(x)\widetilde{\varphi}(y)$, where $\widetilde{\varphi}$ is a mass function playing the same rôle as our φ. Unfortunately, this cannot be applied directly, for two reasons.

i). Firstly, the total rate in (5) contains the terms $a_{ij}\pi_i(x)\pi_j(y), n \leq i, j \leq n + m$ of indefinite sign.

ii). Secondly, the remaining combination of $\pi_i, 1 \leq i \leq n$ is not *a priori* of approximately multiplicative form: particles where some π_i are small, and others large, will in general prevent such a bound from holding.

Our strategy will be as follows.

1 Firstly, we will show that if $(\mu_t)_{t\geq 0}$ solves (FI), then the pushforward measures $(\pi_\#\mu_t)_{t<t_{\mathrm{g}}(\mu_0)}$ solve a modified equation (mΠFI) on the simpler space $S^\Pi = \mathbb{N}\times[0,\infty)^n\times\mathbb{R}^m$, with a reduced kernel $K^{\Pi,\mathrm{m}}$. This allows us to eliminate the terms of indefinite sign mentioned above. This new equation has unique solutions, and so $\nu_t = \pi_\#\mu_t$ is the unique solution starting at $\nu_0 = \pi_\#\mu_0$; in particular, the second moments $\langle\varphi^2,\nu_t\rangle$, $\langle\varphi^2,\mu_t\rangle$ coincide, and gelation takes place at the same time $t_{\mathrm{g}}(\mu_0) = t_{\mathrm{g}}(\nu_0)$. Therefore, we can prove the desired result working solely at the level of (mΠFI).

2 Thanks to results of Norris [29, Theorem 2.1], if $(\nu_t)_{t\geq 0}$ is a solution to (mΠFI) with $\langle\varphi^2,\nu_0\rangle < \infty$, then there exists $t_{\mathrm{e}} = t_{\mathrm{e}}(\nu_0) > 0$ such that $\langle\varphi^2,\nu_t\rangle$ is locally integrable on $[0,t_{\mathrm{e}})$ and such that $\langle\varphi^2,\nu_t\rangle\uparrow\infty$ as $t\uparrow t_{\mathrm{e}}$.

3 We introduce a truncated state space S^Π_ϵ, which excludes particles where any $\pi_i/\pi_0, 1\leq i\leq n$ is either very large or very small, and construct new initial data ν_0^ϵ which are supported in this space. In this context, the kernel $K^{\Pi,\mathrm{m}}$ is approximately multiplicative, and so [29, Theorem 2.2] guarantees that the solutions $(\nu_t^\epsilon)_{t\geq 0}$ undergo gelation at exactly the blow-up time $t_{\mathrm{e}}(\nu_0^\epsilon)$.

4 We argue, from the characterisation of the gelation time in Section 4, that our construction gives an approximation of the gelation times: $t_{\mathrm{g}}(\nu_0^\epsilon)\to t_{\mathrm{g}}(\nu_0)$. We will argue, based on a system of ordinary differential equations for the moments $\langle\pi_i\pi_j,\nu_t\rangle = \langle\pi_i\pi_j,\mu_t\rangle$, that the blowup time is also continuous: $t_{\mathrm{e}}(\nu_0^\epsilon)\to t_{\mathrm{e}}(\nu_0)$. Together with the previous points, this proves the claimed result.

We begin by introducing the modified equation.

Lemma 6.2. *Let $K^{\Pi,\mathrm{m}}$ be the kernel on $S^\Pi = \mathbb{N}\times[0,\infty)^n\times\mathbb{R}^m$ given by*

$$K^{\Pi,\mathrm{m}}(p,q,dr) = \left(\sum_{i,j=1}^n a_{ij}p_iq_j\right)\delta_{p+q}(dr). \tag{79}$$

Consider the corresponding equation incorporating gel, for measures on S^Π, which we write as

$$\nu_t = \nu_0 + \int_0^t L^{\mathrm{m}}_{\mathrm{g}}(\nu_s)ds. \tag{mΠFI}$$

Let μ_0 be a measure on S satisfying Assumption A, and let $(\mu_t)_{t\geq 0}$ be the corresponding solution to (FI). Then the pushforward measures $\nu_t = \pi_\#\mu_t$ are the unique solution to (mΠFI) starting at $\nu_0 = \pi_\#\mu_0$.

Remark 6.3. *Under the new kernel $K^{\Pi,\mathrm{m}}$, the quantities π_i are still conserved for $0\leq i\leq n$, but not for $n+1\leq i\leq n+m$. However, since we seek to analyse $\langle\varphi^2,\mu_t\rangle, \varphi = \sum_{0\leq i\leq n}\pi_i$, we will not need any conservation properties of π_i for $i>n$ in this section.*

Sketch Proof of Lemma 6.2. Much of the proof consists of algebraic manipulations, using the definitions and hypotheses in Definition 1. In the interest of brevity, such manipulations will omitted.

Let us first consider the reflected measures $R_\#\mu_t = \mu_t\circ R^{-1}$ on S. By (A1.), $R_\#\mu_0 = \mu_0$, and using part iii) of Definition 1, one can show that for all $t\geq 0$, all finite measures μ on S and all bounded, measurable functions f on S, $\langle f\circ R, L(\mu)\rangle = \langle f, L(R_\#\mu)\rangle$. From this, and performing a similar manipulation for the gel term, it follows that $(R_\# \mu_t)_{t\ge 0}$ also solves the equation (FI) with the same initial data which implies, by uniqueness, we must have $\mu_t = R_\# \mu_t$ for all $t \ge 0$. Using this, one can now similarly prove that, for all t and f as above,

$$\begin{aligned} \langle f, L(\mu_t)\rangle &= \int_{S^3} (f(z) - f(x) - f(y)) K(Rx, y, dz)\mu_t(dx)\mu_t(dy) \\ &= \int_{S^3} (f(z) - f(x) - f(y)) K(x, Ry, dz)\mu_t(dx)\mu_t(dy). \end{aligned} \tag{81}$$

Taking a linear combination, and again performing a similar manipulation for the gel term, it follows that μ_t solves the equation analagous to (FI) for the symmetrised kernel

$$K^{\mathrm{Sym}}(x, y, dz) = \frac{1}{4}K(Rx, y, dz) + \frac{1}{2}K(x, y, dz) + \frac{1}{4}K(x, Ry, dz). \tag{82}$$

Since the coagulation rate in K^{Sym} only depends on $\pi(x), \pi(y)$, one can verify that the pushforward measures $\pi_\# \mu_t$ on S^Π solve the projected equation (mΠFI) as claimed. □

We now turn to the second point, which concerns the moment behaviour of the solutions to (mΠFI). The following result follows from ideas of [29], which we will briefly sketch.

Lemma 6.4. *Let ν_0 be a measure on S^Π satisfying Assumption A, and let $(\nu_t)_{t\ge 0}$ be the corresponding solution to (mΠFI). Then there exists $t_\mathrm{e} = t_\mathrm{e}(\nu_0) > 0$ such that $t \mapsto \langle \varphi^2, \nu_t\rangle$ is finite and increasing on $[0, t_\mathrm{e})$, and $\langle \varphi^2, \nu_t\rangle \uparrow \infty$ as $t \uparrow t_\mathrm{e}$. Moreover, $(\nu_t)_{t<t_\mathrm{e}}$ is conservative, and so $t_\mathrm{e}(\nu_0) \le t_\mathrm{g}(\nu_0)$.*

The subscript $_\mathrm{e}$ here denotes 'explosion': t_e is the first time the second moment diverges to ∞.

Sketch Proof of Lemma 6.4. This argument applies different results from [29] to our case. We say that a local solution $(\nu_t)_{t<T}$ to (mΠFI) is *strong* if the map $t \mapsto \langle \varphi^2, \nu_t\rangle$ is integrable on compact subsets of $[0, T)$. Applying the results of [29, Theorem 2.1], there exists a unique maximal strong solution $(\nu'_t)_{t<t_\mathrm{e}(\nu_0)}$ to (mΠFI), which is conservative and that $t_\mathrm{e}(\nu_0) \ge C\langle \varphi^2, \nu_0\rangle^{-1}$ for some constant C depending on A.

We next apply Corollary 2.2 to see that this solution must be an initial segment of $(\nu_t)_{t<t_\mathrm{g}(\nu_0)}$: that is, $t_\mathrm{e}(\nu_0) \le t_\mathrm{g}(\nu_0)$, and $\nu'_t = \nu_t$ for all $t \le t_\mathrm{e}(\nu_0)$. Therefore, the results of [29] will apply to our process $(\nu_t)_{t<t_\mathrm{e}(\nu_0)}$.

Since $(\nu_t)_{t<t_\mathrm{e}(\nu_0)}$ is conservative, we follow the ideas of [29, Proposition 2.7], to obtain the integral relations, for all $t < t_\mathrm{e}$ and $0 \le i, j \le n$,

$$\langle \pi_i\pi_j, \nu_t\rangle = \langle \pi_i\pi_j, \nu_0\rangle + 2\int_0^t \sum_{k,l=1}^n \langle \pi_i\pi_k, \nu_s\rangle a_{kl} \langle \pi_l\pi_j, \nu_s\rangle ds. \tag{83}$$

These immediately imply that $\langle \varphi^2, \nu_t\rangle$ is bounded on compact subsets of $[0, t_\mathrm{e})$, and in particular does not diverge before t_e. Moreover, since all terms on the right-hand side are nonnegative, these relations imply that all moments $\langle \pi_i\pi_j, \nu_t\rangle$ and $\langle \varphi^2, \nu_t\rangle$ are increasing on $[0, t_\mathrm{e})$.

Finally, we show that $\langle \varphi^2, \nu_t\rangle$ diverges near $t_\mathrm{e}(\nu_0)$. This follows from the time-of-existence estimate quoted above: for $t < t_\mathrm{e}$, the unique maximal strong solution starting at ν_t is precisely $(\nu_{s+t})_{s<t_\mathrm{e}-t}$, and so for some $C = C(A) < \infty$,

$$t_\mathrm{e} - t \ge C\langle \varphi^2, \nu_t\rangle^{-1}. \tag{84}$$

This rearranges to show that $\langle \varphi^2, \nu_t\rangle \ge C(t_\mathrm{e} - t)^{-1}$ which diverges as $t \uparrow t_\mathrm{e}$, as claimed. □

In order to obtain the full connection of the explosion and gelation times, we modify the setting to exclude the problematic particles identified above. Let

$$S^{\Pi}_\epsilon = \{p \in S^{\Pi} : \epsilon \pi_0(p) \le \pi_i(p) \le (\epsilon^{-1}+\epsilon)\pi_0(p) \text{ for all } 1 \le i \le n\}. \tag{85}$$

Note that this state space is preserved under the kernel $K^{\Pi,\mathrm{m}}$. Moreover, on the reduced state space S^{Π}_ϵ, the modified kernel $K^{\Pi,\mathrm{m}}$ is *approximately multiplicative* [29] in the sense that, for some $\delta_\epsilon > 0$ and $\Delta_\epsilon < \infty$, we have

$$\delta_\epsilon\, \varphi(p)\varphi(q) \le \overline{K^{\Pi,\mathrm{m}}}(p,q) \le \Delta_\epsilon\, \varphi(p)\varphi(q) \tag{86}$$

for all $p, q \in S^{\Pi}_\epsilon$.

We now construct approximations ν_0^ϵ to ν_0 which are supported on S^{Π}_ϵ. Let us fix μ_0 satisfying Assumption A and $\nu_0 = \pi_{\#}\mu_0$; for any $\epsilon > 0$, let ν_0^ϵ be given by specifying, for all bounded measurable functions h on S^{Π},

$$\begin{aligned}\int_{S^{\Pi}} & h(p)\nu_0^\epsilon(dp) \\ &= \int_{S^{\Pi}} h(p_0, p_1+\epsilon, \dots p_n+\epsilon, p_{n+1}, \dots p_{n+m}) 1[p_i \le \epsilon^{-1} \text{ for all } 1\le i \le n]\nu_0(dp).\end{aligned} \tag{87}$$

In this way, we shift ν_0 slightly away from the axes, while also truncating when any π_i becomes large. It follows, from existence and uniqueness, that the solution $(\nu_t^\epsilon)_{t\ge 0}$ to (mΠFl) starting at ν_0^ϵ is supported on S^{Π}_ϵ for all $t \ge 0$. We can now apply [29, Theorem 2.2] to obtain the connection between gelation and explosion for these solutions:

Lemma 6.5. *Let $(\nu_t^\epsilon)_{t\ge0}$ be the solution to (mΠFl) starting at the measure ν_0^ϵ constructed above. Let $t_\mathrm{e}(\nu_0^\epsilon)$ be the explosion time of the second moment, as above, and $t_\mathrm{g}(\nu_0^\epsilon)$ the first time that ν_t^ϵ fails to be conservative. Then $t_\mathrm{e}(\nu_0^\epsilon) = t_\mathrm{g}(\nu_0^\epsilon)$.*

This then connects the gelation phenomenon to the blowup of the second moment, as desired, but only for the special case of the truncated and shifted initial distribution. We now seek to remove this restriction to obtain the result for the original measures μ_0, ν_0. To do this, we will show that $t_\mathrm{g}(\nu_0^\epsilon) \to t_\mathrm{g}(\nu_0)$ and $t_\mathrm{e}(\nu_0^\epsilon) \to t_\mathrm{e}(\nu_0)$ as we take $\epsilon \downarrow 0$.

Lemma 6.6 (Convergence of Gelation Times)**.** *Let ν_0, ν_0^ϵ be the measures constructed above, and $t_\mathrm{g}(\nu_0), t_\mathrm{g}(\nu_0^\epsilon)$ the corresponding gelation times. Then, as $\epsilon \downarrow 0$, $t_\mathrm{g}(\nu_0^\epsilon) \to t_\mathrm{g}(\nu_0)$.*

Proof. First, we recall that $\pi_1, \dots\pi_n$ are linearly independent in $L^2(\mu_0)$, and hence in $L^2(\nu_0)$, by hypothesis. Using the convergence $\langle \pi_i\pi_j, \nu_0^\epsilon\rangle \to \langle \pi_i\pi_j, \nu_0\rangle$, it follows that for $\epsilon > 0$ small enough, and any a_i with $\sum_i |a_i| = 1$, we have $\langle (\sum_i a_i\pi_i)^2, \nu_0^\epsilon\rangle > 0$. This, in turn, guarantees that $\pi_1, \dots\pi_n$ are linearly independent in $L^2(\nu_0^\epsilon)$, for all $\epsilon > 0$ small enough.

We can now apply the explicit characterisation of t_g obtained in Lemma 4.5 for the measures ν_0^ϵ:

$$t_\mathrm{g}(\nu_0^\epsilon) = \lambda_1(\Lambda(\nu_0^\epsilon))^{-1} \tag{88}$$

where $\Lambda(\nu_0^\epsilon)$ is the matrix $\Lambda(\nu_0^\epsilon)_{ij} = \sum_{k=1}^{n+m} \langle \pi_i\pi_k, \nu_0^\epsilon\rangle a_{kj}$ and $\lambda_1(\cdot)$ denotes the largest eigenvalue of a matrix. Moreover, as $\epsilon \downarrow 0$, the coefficients of the matrices $\Lambda(\nu_0^\epsilon)$ converge to the analagous matrix $\Lambda(\nu_0)$ for the measure ν_0.

It is well-known, following for instance from [37], that as the coefficients of a matrix vary continuously, so to do the associated eigenvalues, meaning that

$$\lambda_1(\Lambda(\nu_0^\epsilon)) \to \lambda_1(\Lambda(\nu_0)) \tag{89}$$

as $\epsilon \downarrow 0$. Combining this with the characterisation of t_g above, it follows that

$$\begin{aligned} t_\mathrm{g}(\nu_0^\epsilon) = \lambda_1(\Lambda(\nu_0^\epsilon))^{-1} &\to \lambda_1(\Lambda(\nu_0))^{-1} \\ &= t_\mathrm{g}(\nu_0) \end{aligned} \tag{90}$$

as desired. □

Finally, we show the same result for the explosion times. Thanks to Lemma 6.4 and (83), the matrix of second moments $q_{ij}(t) = \langle \pi_i\pi_j, \nu_t\rangle, 1 \le i,j \le n$ satisfies a closed differential equation, with locally Lipschtiz coefficients, on $[0, t_\mathrm{e})$. We will now show that t_e is exactly the time of existence of a solution started at q_0.

Lemma 6.7. *Consider the ordinary differential equations*

$$\dot{q}_t = b(q_t); \qquad b(q) = 2qA^+q, \qquad q \in \mathrm{Mat}_n(\mathbb{R}); \tag{Q1}$$

$$\dot{z}_t = w(q_t)z_t, \qquad w : \mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_{n+1}(\mathbb{R}) \textit{ linear}; \qquad z \in \mathbb{R}^{n+1}. \tag{Q2}$$

Then, for all $(z_0, q_0) \in \mathbb{R}^{n+1} \times \mathrm{Mat}_n(\mathbb{R})$*, there exists a unique maximal solution* $\chi(t, z_0, q_0), \psi(t, q_0)$ *starting at* (z_0, q_0)*, defined until the time* $\zeta(q_0)$ *where (Q1) blows up.*

Then, for any measure ν_0 *on* S^Π*, the time of existence is exactly the explosion time:*

$$t_\mathrm{e}(\nu_0) = \zeta(q_0), \qquad (q_0)_{ij} = \langle \pi_i\pi_j, \nu_0\rangle, 1 \le i,j \le n. \tag{93}$$

Proof. Firstly, it is straightforward to verify that q_t does not depend on the initial data z_0, since (Q1) only depends on q; in particular, the blowup time ζ is a function only of q_0. It is also straightforward to verify that (Q2) cannot blow up before $\zeta(q_0)$, since on compact subsets $[0,t] \subset [0, \zeta(q_0))$, the coefficients of (Q2) are Lipschitz, uniformly in time. As a result, the time of existence for the pair (Q1, Q2) is exactly the time of existence $\zeta(q_0)$, as claimed.

To link the explosion times t_e and the time of existence $\zeta(q_0)$, the equations (83) show that the matrix $q_{ij}(t) = \langle \pi_i\pi_j, \nu_t\rangle, 1 \le i,j \le n$ and the vector $z_t = \langle \pi_0\pi_i, \nu_t\rangle, 0 \le i \le n$ solve the system (Q1, Q2) on $0 \le t < t_\mathrm{e}(\nu_0)$, which implies that $t_\mathrm{e}(\nu_0) \le \zeta(q_0)$. For the converse, for $t < t_\mathrm{e}$, we have the equality

$$\chi(z_0, q_0)_0 + \sum_{i=1}^n \psi(q_0, t)_{ii} = \langle \pi_0^2, \nu_t\rangle + \sum_{i=1}^n \langle \pi_i^2, \nu_t\rangle \tag{94}$$

where the initial data are

$$q_0 = (\langle \pi_i\pi_j, \nu_0\rangle)_{i,j=1}^n \qquad z_0 = (\langle \pi_i\pi_0, \nu_0\rangle)_{0\le i\le n}. \tag{95}$$

The left hand side is bounded on compact subsets of $[0, \zeta(q_0))$ and the right-hand side dominates $\langle \varphi^2, \nu_t\rangle$ up to a constant C, which leads to a contradiction if we assume that $t_\mathrm{e} < \zeta(q_0)$, since $\langle \varphi^2, \nu_t\rangle \uparrow \infty$ as $t \uparrow t_\mathrm{e}(\nu_0)$. We therefore have $\zeta(q_0) \le t_\mathrm{e}(\nu_0)$ which proves the equality desired. □

We will now analyse the pair of equations presented above. This will prove the desired continuity of t_{e}, and some points which will be helpful for later reference.

Lemma 6.8. *Consider the differential equations (Q1, Q2) in the previous lemma, and the sets*

$$E = \mathrm{Mat}_n([0,\infty)); \qquad E_\delta = \{q \in E : \forall i, q_{ii} > \delta\}; \qquad E^\circ = \cup_{\delta>0} E_\delta; \tag{96}$$

$$E_{\mathrm{cs}} = \{q \in E : \text{ for all } i,j \le n \text{ and } t < \zeta(q),\ \psi(q,t)_{ij}^2 \le \psi(q,t)_{ii}\psi(q,t)_{jj}\}. \tag{97}$$

Then, if $q_0 \in E_\delta$, $(\psi(q_0,t))_{t<\zeta(q_0)} \subset E_\delta$, and similarly if $q_0 \in E_{\mathrm{cs}}$, then $(\psi(q_0,t))_{t<\zeta(q_0)} \subset E_{\mathrm{cs}}$. We have the following properties:

i). Let J_ϵ be the set

$$J_\epsilon = \{q \in E : \ \zeta(q) \ge \epsilon\}. \tag{98}$$

Then for all $\epsilon, \delta > 0$, the set $J_\epsilon \cap E_\delta \cap E_{\mathrm{cs}}$ is bounded.

ii). Suppose $q_0^\epsilon \in E_{\mathrm{cs}}, \epsilon > 0$ and $q_0^\epsilon \to q_0 \in E_{\mathrm{cs}} \cap E^\circ$. Then $\zeta(q_0^\epsilon) \to \zeta(q_0)$.

iii). Suppose $I \subset \mathbb{R}_+$ is an open interval, and the map $(z_0, q_0) : I \to \mathbb{R}^{n+1} \times (E_{\mathrm{cs}} \cap E^\circ)$ is continuous, and such that $t < \zeta(q_0(t))$ for all $t \ge 0$. Then the maps $t \mapsto \psi(q_0(t), t)$ and $t \mapsto \chi(z_0(t), q_0(t), t)$ are continuous on I.

Proof. i). Let us first fix $q \in E$. First of all write $a_\star = \min\{a_{ij} : a_{ij} > 0\}$ and let i, j be such that $a_{ij} > 0$. We now estimate

$$\frac{d}{dt}\psi(t,q)_{ij} \ge 2a_{ij}\psi(t,q)_{ij}^2 \ge 2a_\star q_{ij}^2. \tag{99}$$

This differential inequality may be integrated to obtain

$$\psi(t,q)_{ij} \ge \frac{q_{ij}}{1 - 2ta_\star q_{ij}}. \tag{100}$$

In particular, this gives the upper bound $\zeta(q) \le (2a_\star q_{ij})^{-1}$, which implies the claimed boundedness of J_ϵ in the $(i,j)^{\mathrm{th}}$ coordinate whenever $a_{ij} > 0$.

We will now extend this boundedness to all n^2 coordinates when we restrict to $q \in E_\delta \cap J_\epsilon \cap E_{\mathrm{cs}}$. Let M be the maximum diagonal entry of q:

$$M = \max_{1\le i\le n} q_{ii} \tag{101}$$

and fix i where this maximum is attained; by hypothesis on A, there exists $j \le n$ such that $a_{ij} \ge a_\star > 0$. It is straightforward to see that the derivative $\frac{d}{dt}\psi(q,t)_{ij}$ is increasing along the solution, which implies the estimate

$$\psi\left(\frac{\epsilon}{2}, q\right)_{ij} \ge \frac{\epsilon}{2} b(q)_{ij} = \epsilon \sum_{k,l\le n} q_{ik}a_{kl}q_{lj} \ge \epsilon q_{ii}a_{ij}q_{jj} \ge \epsilon\delta a_\star M. \tag{102}$$

By hypothesis, $\zeta(q) \ge \epsilon$, so $\zeta(\psi(\frac{\epsilon}{2}, q)) \ge \frac{\epsilon}{2}$. Applying the bound on ζ above, we find that

$$\frac{\epsilon}{2} \le \frac{1}{2a_\star^2 \epsilon\delta M}. \tag{103}$$

Finally, since we chose $q \in E_{\mathrm{cs}}$, we have the uniform bound

$$\max_{ij} q_{ij} \le M \le (a_\star^2 \epsilon^2 \delta)^{-1}. \tag{104}$$

ii). The lower semicontinuity of explosion times is standard, and follows from the continuous dependence on the initial data. Therefore, it is sufficient to prove that $\limsup_{\epsilon\to 0}\zeta(q^\epsilon)\le\zeta(q)$.

Suppose, for a contradiction, that for some $\eta>0$, we have $\limsup_{\epsilon\to 0}\zeta(q^\epsilon)>\zeta(q)+\eta$; by passing to a subsequence, we may assume that $\zeta(q^\epsilon)>\tau+\eta$ for all ϵ, where we write $\tau=\zeta(q)$. Moreover, since $q^\epsilon_0\in E_{\mathrm{cs}}$ and $q^\epsilon\to q\in E^\circ$, we may assume that $q^\epsilon, q\in E_\delta\cap E_{\mathrm{cs}}$ for all ϵ, for some $\delta>0$, which implies that $\psi(q^\epsilon,t)\in E_\delta\cap E_{\mathrm{cs}}$ for all $t<\zeta(q^\epsilon)$ and all $\epsilon>0$.

Now, if $t\le\tau$, we have $\zeta(\psi(t,q^\epsilon))=\zeta(q^\epsilon)-t\ge\eta$, which implies the containment

$$\{\psi(t,q^\epsilon):t\le\tau,\epsilon>0\}\subset E_\delta\cap J_\eta\cap E_{\mathrm{cs}} \tag{105}$$

which we know, from item i)., to be bounded: for some $C<\infty$,

$$\{\psi(t,q^\epsilon):t\le\tau,\epsilon>0\}\subset \mathrm{Mat}_n([0,C]). \tag{106}$$

By the lemma of leaving compact sets, there exists $s<\tau$ such that, for all $t\in(s,\tau)$, $\psi_t(q)\notin \mathrm{Mat}_n([0,C])$. However, if we pick $t\in(s,\tau)$, we have $\psi_t(q^\epsilon)\to\psi_t(q)$, by the continuity of the dependence in the initial conditions, which is a contradiction. Therefore, $\limsup_{\epsilon\to 0}\zeta(q^\epsilon)\le\zeta(q)$, which proves the claimed convergence.

iii). Let us first establish the claim for ψ. Firstly, we note that by ii)., the map $t\mapsto\zeta(q_0(t))$ is continuous on I. Therefore, fixing $t\in I$, we may choose choose $\epsilon,\delta>0$ such that, if $|t-s|\le\delta$, then $s\in I$ and $s<\min\left(\zeta\left(q_0(s)\right),\zeta\left(q_0(t)\right)\right)-\epsilon$. Now, we observe that, for $s\in[t-\delta,t+\delta]$,

$$|\psi(t,q_0(t))-\psi(s,q_0(s))|\le|\psi(t,q_0(t))-\psi(t,q_0(s))|+|\psi(t,q_0(s))-\psi(s,q_0(s))|. \tag{107}$$

As $s\to t$, the first term converges to 0 by continuity of the solution $\psi(q,t)$ in the initial data q_0; it is therefore sufficient to control the second term. By the choice of δ, for all $s\in[t-\delta,t+\delta]$, we have

$$\zeta(\psi(s,q_0(s)))=\zeta(q_0(s))-s>\epsilon \tag{108}$$

so that $\psi(s,q_0(s))\in J_\epsilon$. Moreover, by compactness of $[t-\delta,t+\delta]$, there exists some $\eta>0$ such that $q_0(s)\in E_\eta$ for all $s\in[t-\delta,t+\delta]$, and since $q_0(s)\in E_{\mathrm{cs}}$ and these sets are preserved under the flow, we have $\psi(q_0(s),u)\in E_\eta\cap E_{\mathrm{cs}}$ for all $0\le u\le\zeta(q_0(s))$. However, we showed in point i). above that that the intersection of these three regions is compact and so there exists a constant $M=M(\epsilon)$: for all $s\in[t-\delta,t+\delta]$, and for all $u\le t+\delta$,

$$u<\zeta(q_0(s));\qquad |b(\psi(u,q_0(s))|\le M. \tag{109}$$

This implies the bound, for all $s\in[t-\delta,t+\delta]$,

$$|\psi(t,q_0(s))-\psi(s,q_0(s))|\le M|t-s| \tag{110}$$

which implies the claimed continuity.

The case for $\chi(z_0(t),q_0(t),t)$ is similar. Let us fix $t\in I$; following the same argument leading to (109), there exists $\delta>0, M<\infty$ such that, if $s\in[t-\delta,t+\delta]$ then $s\in I$ and for all $u\le s$, $\psi(u,q_0(u))\in\mathrm{Mat}_n([0,M])$. The equation (Q2) can now be integrated directly to obtain, for $s\in[t-\delta,t+\delta]$,

$$\chi(s,z_0(s),q_0(s))=\exp\left(\int_0^s w\left(\psi(u,q_0(u))\right)du\right)z_0(s). \tag{111}$$

In particular, it follows that $\chi(s,z_0(s),q_0(s))$ is bounded as s varies in $[t-\delta,t+\delta]$. With this, the argument for ψ can be modified to prove the same result

□

We can finally combine the previous lemmas to prove Lemma 6.1.

Proof of Lemma 6.1. Let us fix μ_0 satisfying Assumption A, and let ν_0 be its pushforward $\nu_0 = \pi_\# \mu_0$; let $(\mu_t)_{t\ge 0}$ and $(\nu_t)_{t\ge 0}$ be the solutions to (FI, mΠFI) with these starting points, respectively. By Lemma 6.2, ν_t is given by $\nu_t = \pi_\#\mu_t$ and in particular, $\mathcal{E}(t) = \langle \varphi^2, \mu_t\rangle = \langle \varphi^2, \nu_t\rangle$, $\mathcal{Q}_{ij}(t) = \langle \pi_i\pi_j, \mu_t\rangle = \langle \pi_i\pi_j, \nu_t\rangle$ and $t_{\mathrm{g}}(\nu_0) = t_{\mathrm{g}}(\mu_0)$.

From Lemma 6.4, we know that there exists $t_{\mathrm{e}} = t_{\mathrm{e}}(\nu_0) > 0$ such that $\mathcal{E}(t) = \langle \varphi^2, \nu_t\rangle$ is finite, continuous and increasing on $[0, t_{\mathrm{e}})$, and diverges to infinity as $t \uparrow t_{\mathrm{e}}$. Moreover, thanks to the differential equations (83), all components of $\mathcal{Q}(t)$ are continuous and increasing on $[0, t_{\mathrm{e}})$.

Consider next the shifted initial data ν_0^ϵ given by (87); thanks to Lemma 6.5, we know that $t_{\mathrm{g}}(\nu_0^\epsilon) = t_{\mathrm{e}}(\nu_0^\epsilon)$. By Lemma 6.6, $t_{\mathrm{g}}(\nu_0^\epsilon) \to t_{\mathrm{g}}(\nu_0)$. For the explosion times, we know from Lemma 6.7 that $t_{\mathrm{e}}(\nu_0^\epsilon) = \zeta(q_0^\epsilon)$ and $t_{\mathrm{e}}(\nu_0) = \zeta(q_0)$, where $q_0^\epsilon, q_0 \in E$ are the matrixes

$$(q_0)_{ij} = \langle \pi_i\pi_j, \nu_0\rangle; \qquad (q_0^\epsilon)_{ij} = \langle \pi_i\pi_j, \nu_0^\epsilon\rangle. \tag{112}$$

By dominated convergence, $q_0^\epsilon \to q_0$; by hypothesis (A3.), each $(q_0)_{ii} = \langle \pi_i^2, \nu_0\rangle = \langle \pi_i^2, \mu_0\rangle > 0$, so $q_0 \in E_\delta$ for some $\delta > 0$. Finally, for all $t < \zeta(q_0) = t_{\mathrm{e}}(\nu_0)$, $\psi(t, q_0)_{ij} = \langle \pi_i\pi_j, \nu_t\rangle$ which certainly satisfies the desired Cauchy-Schwarz inequality $\psi(t,q_0)_{ij}^2 \le \psi(t,q_0)_{ii}\psi(t,q_0)_{jj}$, so $q_0 \in E_{\mathrm{cs}}$. A similar argument shows that $q_0^\epsilon \in E_{\mathrm{cs}}$ for all $\epsilon > 0$, so Lemma 6.8 shows that $t_{\mathrm{e}}(\nu_0^\epsilon) = \zeta(q_0^\epsilon) \to \zeta(q_0) = t_{\mathrm{e}}(\nu_0)$. Comparing these two limits, $t_{\mathrm{g}}(\nu_0) = t_{\mathrm{e}}(\nu_0)$, concluding the proof. □

6.2 The Critical Point

Using the concepts introduced above, we next consider the behaviour at and near the critical time t_{g}.

Lemma 6.9. *In the notation of Lemma 6.1, we have*

$$\mathcal{E}(t_{\mathrm{g}}) = \infty = \lim_{t\to t_{\mathrm{g}}} \mathcal{E}(t). \tag{113}$$

Proof. We first show that $\mathcal{E}(t_{\mathrm{g}}) = \infty$. Suppose, for a contradiction, that $\mathcal{E}(t_{\mathrm{g}}) < \infty$. Then, applying [29, Proposition 2.7] as in Lemma 6.4, we see that, for some positive $\delta > 0$, there exists a strong solution $(\nu_t)_{t<\delta}$ to (Sm), starting at $\mu_{t_{\mathrm{g}}}$. This solution is conservative, so is an initial segment of the solution $(\nu_t)_{t\ge 0}$ to (FI) starting at $\mu_{t_{\mathrm{g}}}$. By uniqueness in Lemma 2.1,

$$\nu_t = \mu_{t_{\mathrm{g}}+t} \qquad \text{for all } t \ge 0. \tag{114}$$

By Corollary 5.5, $\langle \varphi, \mu_{t_{\mathrm{g}}}\rangle = \langle \varphi, \mu_0\rangle$, and by definition of t_{g},

$$\langle \varphi, \mu_{t_{\mathrm{g}}+t}\rangle < \langle \varphi, \mu_0\rangle = \langle \varphi, \mu_{t_{\mathrm{g}}}\rangle \text{ for all } t > 0. \tag{115}$$

This contradicts the fact that $(\nu_t)_{t<\delta}$ is strong, which therefore shows that $\mathcal{E}(t) = \infty$.

The second point follows, because $t \mapsto \mu_t$ is continuous, and $\mu \mapsto \langle \varphi^2, \mu\rangle$ is lower semicontinuous, when $\mathcal{M}$ is equipped with the vague topology. □

6.3 The Supercritical Regime

We finally turn to the supercritical case; our result is as follows.

Lemma 6.10. *In the notation of Lemma 6.1, the map $t \mapsto \mathcal{E}(t)$ is finite and continuous, and therefore locally bounded, on (t_{g}, ∞).*

The proof is based on a *duality argument* following Theorem II.13, which connects the measures in the supercritical regime to an auxiliary process in the subcritical case. Let $(G^N_t)_{t\ge 0}$ be the random graph processes described in Section 4 with points $\mathbf{x}_N$ sampled as a Poisson random measure of intensity $N\mu_0$; it is straightforward to see that Assumption B holds. Fix $t > t_{\mathrm{g}}$, and let $\widetilde{G}^N_t$ be the graph G^N_t with the giant component deleted.

Let $\rho_t(x) = \rho(t,x)$ be the survival function defined in Lemmas 4.7, II.7, and let $\widehat{\mu}^t_0(dx) = (1 - \rho_t(x))\mu_0(dx)$. By Lemma 2.1, there exists a unique solution $(\widehat{\mu}^t_s)_{s\ge 0}$ to the equation (FI) starting at $\widehat{\mu}^t_0$; write $\widehat{t_{\mathrm{g}}}(t)$ for its gelation time.

Let $\mathbf{y}_N = (y_i : i \le \widehat{l}^N)$ be an enumeration of the vertexes x_i not belonging to the giant component in G^N_t. By Theorem II.13, we can construct a random graph $\widehat{G}^N_t$ on $\{1, ..., \widehat{l}^N\}$,

In order to appeal to Lemmas 4.4, 5.1, we will now verify that the desired Assumptions A, B hold for the vertex space $\widehat{\mathcal{V}}$.

Lemma 6.11. *Fix $t > 0$, and let $\mu_0, G^N_t, \widehat{\mu}^t_0$ and $\widehat{\mathcal{V}}$ be as described above. Then Assumption B hold for $\mathbf{y}_N$ and $\widehat{\mu}^t_0$.*

Proof. To ease notation, we write $\widehat{\mu}_0$ for $\widehat{\mu}^t_0$, μ^N_0 for the initial empirical measure of the unmodified process corresponding to $\mathbf{x}_N$, and $\widehat{\mu}^N_0$ for the reduced empirical measure corresponding to $\mathbf{y}_N$:

$$\widehat{\mu}^N_0 = \frac{1}{N}\sum_{i=1}^{\widehat{l}^N} \delta_{y_i}. \tag{116}$$

It is straightforward to see that $\widehat{\mu}^t_0$ inherits the properties in Assumption A from μ_0, and so it is sufficient to establish Assumption B.

For (B1.), we note that part of the content of Theorem II.13 is the weak convergence

$$\widehat{\mu}^N_0 = \frac{1}{N}\sum_{i=1}^{\widehat{l}^N} \delta_{y_i} \to \widehat{\mu} \qquad \text{weakly, in probability.} \tag{117}$$

Since the vague topology is weaker than the weak topology, we immediately have the vague convergence required. Moreover, by construction, Supp$(\widehat{\mu}^N_0) \subset$ Supp(μ^N_0), so it follows from (B1.) that $\widehat{\mu}^N_0$ is supported on $\{\pi_0 = 1\}$ as required.

We will now show that (B2.) follows from the previous point, together with the moment estimates for the original initial measure μ^N_0.

Fix $\xi < \infty$, and let $\chi \in C_c(S)$ be such that $1_{S_\xi} \le \chi \le 1_{S_{\xi+1}}$. We observe that

$$\begin{aligned} \left|\langle \pi, \widehat{\mu}^N_0\rangle - \langle \pi, \widehat{\mu}_0\rangle\right| &\le \left|\langle \pi\chi, \widehat{\mu}^N_0 - \widehat{\mu}_0\rangle\right| + \langle |\pi|\, 1_{S^c_\xi}, \widehat{\mu}^N_0\rangle + \langle |\pi|\, 1_{S^c_\xi}, \widehat{\mu}_0\rangle \\ &\le \left|\langle \pi\chi, \widehat{\mu}^N_0 - \widehat{\mu}_0\rangle\right| + \frac{C}{\xi}\langle \varphi^2, \mu^N_0\rangle + \frac{C}{\xi}\langle \varphi^2, \mu_0\rangle \end{aligned} \tag{118}$$

for some constant C, thanks to the bound in part iv) of the definition (1). We now fix $\epsilon, \delta > 0$. Thanks to (A2., B2.), $\langle \varphi^2, \mu_0^N\rangle$ is bounded in L^1 and $\langle \varphi^2, \mu_0\rangle < \infty$, and so we may choose $\xi < \infty$ such that the second and third terms are at most $\epsilon/3$ with probability exceeding $1-\delta/2$, for all N. For this choice of ξ, the first term vanishes as $N\to\infty$ by vague convergence in probability, and so is at most $\frac{\epsilon}{3}$ with probability exceeding $1-\delta/2$ for all N large enough. Therefore, for all such N, we have

$$\mathbb{P}\left(|\langle \pi, \widehat{\mu}_0^N - \widehat{\mu}_0\rangle| > \epsilon\right) \le \delta \tag{119}$$

which proves the desired convergence in probability.

For the second assertion of (B2.), we note that $\langle \varphi^2, \widehat{\mu}_0^N\rangle \le \langle \varphi^2, \mu_0^N\rangle$ by the construction of $\mathbf{y}_N$, and $\langle \varphi^2, \mu_0^N\rangle$ is uniformly integrable by the hypothesis (B2.). □

We now use this preparatory result to prove Lemma 6.10.

Proof of Lemma 6.10. Let $G_t^N, \widetilde{G}_t^N, \widehat{G}_t^N$ be as above. Recalling that we consider equality of graphs to include equality of the vertex data, it follows from Theorem II.13 that

$$\mathbb{P}(\pi_\star(\widehat{G}_t^N) = \pi_\star(\widetilde{G}_t^N)) \to 1. \tag{120}$$

From Lemmas 4.4, 6.11, we obtain the following convergences in probability:

$$\pi_\star(G_t^N) \to \pi_\# \mu_t; \qquad \pi_\star(\widehat{G}_t^N) \to \pi_\# \widehat{\mu}_t^t \tag{121}$$

in the vague topology, in probability. Moreover, the difference

$$\pi_\star(G_t^N) - \pi_\star(\widetilde{G}_t^N) = \frac{1}{N}\delta(\mathcal{C}_1(G_t^N)) \tag{122}$$

converges to 0 in the vague topology in probability, since the support is eventually disjoint from any compact set, with high probability. It follows that

$$\pi_\star(\widetilde{G}_t^N) \to \pi_\# \mu_t \tag{123}$$

in the vague topology, in probability, and by uniqueness of limits, we have $\pi_\#\widehat{\mu}_t^t = \pi_\#\mu_t$. In particular, it follows that

$$\langle \varphi^2, \mu_t\rangle = \langle \varphi^2, \pi_\#\mu_t\rangle = \langle \varphi^2, \pi_\#\widehat{\mu}_t^t\rangle = \langle \varphi^2, \widehat{\mu}_t^t\rangle. \tag{124}$$

Using assumption (A2.), we can see that $tk \in L^2(S\times S, \mu_0\times\mu_0)$, and so it follows from Theorem II.13 that the graphs $\widehat{G}_t^N$ are subcritical. By Lemma 5.1, it follows that that $t < \widehat{t}_{\mathrm{g}}(t)$, and so by Lemma 6.1, we have

$$\langle \varphi^2, \mu_t\rangle = \langle \varphi^2, \widehat{\mu}_t^t\rangle < \infty. \tag{125}$$

Using Theorem II.12 and dominated convergence, the map

$$\begin{aligned} t &\mapsto q_0^t = \left(\langle \pi_i\pi_j, \widehat{\mu}_0^t\rangle\right)_{i,j=1}^n = (\langle (1-\rho_t)\pi_i\pi_j, \mu_0\rangle)_{i,j=1}^n; \\ t &\mapsto z_0^t = \left(\langle \pi_i\pi_0, \widehat{\mu}_0^t\rangle\right)_{i=0}^n = (\langle (1-\rho_t)\pi_i\pi_0, \mu_0\rangle)_{i=0}^n \end{aligned} \tag{126}$$

are continuous, and q_0^t takes values in E°. Therefore, by the general ODE considerations in Lemma 6.8 point iii)., it follows that the maps

$$t\mapsto q^t(t) = \psi(t, q_0^t) = \left(\langle \pi_i\pi_j, \widehat{\mu}_t^t\rangle\right)_{i,j=1}^n; \qquad t\mapsto z_t^t = \chi(t, z_0^t, q_0^t) = (\langle \pi_i\pi_0, \widehat{\mu}_t^t\rangle)_{i=0}^n \tag{127}$$

are finite and continuous on (t_{g}, ∞). Since $\pi_\#\widehat{\mu}_t^t = \pi_\#\mu_t$, item iii) of Lemma 6.8 shows that the maps $t\mapsto \mathcal{Q}(t)_{ij}, 0\le i,j\le n$ are finite and continuous on (t_{g}, ∞), which implies that they are bounded on compact subsets. □

Remark 6.12. *The same argument also shows that $t\mapsto \widehat{t}_{\mathrm{g}}(t)$ is continuous. This fact will be used later in the proof of Lemma 10.4.*

7 Representation and Dynamics of the Gel

7.1 Representation Formula

The duality construction used in the proof of Lemma 6.10 gives us a natural way to relate the gel data g_t to the survival function ρ_t. This is the content of the following lemma.

Lemma 7.1. *Let μ_0 be an initial data satisfying Assumption A, and let $g_t = (M_t, E_t, 0)$ be the gel data for the corresponding solution to (FI). Let $\rho_t(\cdot)$ be the corresponding survival function defined in Section 4 and Appendix II. Then we have the equality*

$$g_t = \langle \rho_t \pi, \mu_0 \rangle. \tag{128}$$

In particular, $t \mapsto g_t$ is continuous and if $t > t_{\mathrm{g}}$ then $M_t > 0$, and $E_t > 0$ componentwise.

Together with the identification of ρ_t in Lemma 4.7, this proves part 3 of Theorem 1.

Proof. We deal with the supercritical and subcritical/critical cases, $t > t_{\mathrm{g}}, t \leq t_{\mathrm{g}}$ separately.

1. Supercritical Case $t > t_{\mathrm{g}}$. Let $(\widehat{\mu}^t_s)_{s\geq 0}$ and $\widehat{t_{\mathrm{g}}}(t)$ be as in the proof of Theorem 6.10. Then, since $(\widehat{\mu}^t_s)_{s\geq 0}$ is conservative on $[0, \widehat{t_{\mathrm{g}}})$, and $t < \widehat{t_{\mathrm{g}}}(t)$, we have, for all $0 \leq i \leq n+m$,

$$\langle \pi_i, \widehat{\mu}^t_t \rangle = \langle \pi_i, \widehat{\mu}^t_0 \rangle = \int_S \pi_i(x)(1-\rho(t,x))\mu_0(dx). \tag{129}$$

As shown in Lemma 6.10, $\pi_\# \mu_t = \pi_\# \widehat{\mu}^t_t$, so we have

$$\begin{aligned} g^i_t := \langle \pi_i, \mu_0 \rangle - \langle \pi_i, \mu_t \rangle &= \langle \pi_i, \mu_0 \rangle - \langle \pi_i, \widehat{\mu}^t_t \rangle \\ &= \langle \pi_i \rho_t, \mu_0 \rangle \end{aligned} \tag{130}$$

as claimed.

2. Subcritical and Critical Cases $t \leq t_{\mathrm{g}}$. For $t < t_{\mathrm{g}}$, the result is immediate: we have g_t by definition of t_{g}, and $\rho_t = 0$ by Theorem II.7. The critical case is identical, recalling from Corollary 5.5 that $g_{t_{\mathrm{g}}} = 0$.

Continuity follows from Theorem II.12 by using dominated convergence. For the final claim, if $t > t_{\mathrm{g}}$ then $\rho_t(x) > 0$ μ_0 - almost everywhere, by Lemma II.7. By hypothesis (A3.), for all $i = 1, ..., n$, $\pi_i > 0$ on a set of positive μ_0 measure. Together, these imply that $\langle \rho_t \pi_i, \mu_0 \rangle > 0$, as claimed. □

7.2 Gel Dynamics Beyond the Critical Time

We now obtain point 4 of Theorem 1 as a consequence of the previous results. We have already proven the continuity of g_t on the whole time interval $[0, \infty)$ and the finiteness of the second moments $q_t = (\langle \pi_i \pi_j, \mu_t \rangle)^n_{i,j=1}$ in the supercritical regime. Therefore, it is sufficient to prove the following result.

Lemma 7.2. *In the notation of Theorem 1, let g_t be the data of the gel associated to $(\mu_t)_{t\geq 0}$. Then, for $t \geq t_{\mathrm{g}}$, we have*

$$g_t^i = \int_{t_{\mathrm{g}}}^t \sum_{j,k=1}^n \langle \pi_i \pi_j, \mu_t \rangle a_{jk} g_s^k ds. \tag{131}$$

Thanks to the continuity of the second moments above t_{g}, this has the differential form, holding in the classical sense,

$$\frac{d}{dt} g_t^i = \sum_{j,k=1}^n \langle \pi_i \pi_j, \mu_t \rangle a_{jk} g_t^k. \tag{132}$$

Remark 7.3. *In proving Lemma 7.2, we will split the growth of the gel into two terms $\mathcal{T}_1 + \mathcal{T}_2$, where $\mathcal{T}_1$ represents the absorption of particles into the gel, and $\mathcal{T}_2$ represents the coagulation of smaller particles. We will show that $\mathcal{T}_2 = 0$, giving the claimed result; this may be expected following the relationship between gelation and blowup of the second moment $\mathcal{E}(t)$ in Lemma 6.1, and the finiteness of $\mathcal{E}$ in the supercritical regime.*

Proof. We return to the truncated dynamics (FI$|_\xi^1$, FI$|_\xi^2$) used in the proof of Lemma 2.1. We recall that, starting at

$$\mu_0^\xi = 1_{S_\xi} \mu_0; \qquad g_0^\xi = \int_{x \notin S_\xi} x \mu_0(dx) \tag{133}$$

the solution (μ_t^ξ, g_t^ξ) to (FI$|_\xi^1$, FI$|_\xi^2$) exists and is unique, and we have

$$\mu_t^\xi = \mu_t 1_{S_\xi}; \qquad (M_t^\xi, E_t^\xi) \downarrow (M_t, E_t) \text{ as } \xi \uparrow \infty. \tag{134}$$

where $(\mu_t)_{t\geq 0}$ is the solution to (FI) starting at μ_0, and (M_t, E_t) are the nonzero components of the associated gel data.

Fix s, t such that $t_{\mathrm{g}} < s < t$. Rewriting (FI$|_\xi^2$) and using that $P_t^\xi = 0$, we have that

$$\begin{aligned} g_t^{\xi,i} - g_s^{\xi,i} = &\int_s^t \sum_{j,k=1}^n \langle \pi_i \pi_j, \mu_u^\xi \rangle a_{jk} g_u^{\xi,k} du \\ &+ \frac{1}{2} \int_s^t \int_{S_\xi^2} \pi_i(x+y) 1[\varphi(x+y) > \xi] \overline{K}(x,y) \mu_u(dx) \mu_u(dy) du. \end{aligned} \tag{135}$$

Let us write $\mathcal{T}_1(\xi), \mathcal{T}_2(\xi)$ for the two terms appearing in (135) for ease of notation.

We first show that $\mathcal{T}_1(\xi)$ converges to the expression analagous to the claimed limit in (131). By the monotonicity $\mu_u^\xi \leq \mu_u$, and local boundedness in Lemma 6.10, each $\langle \pi_i \pi_j, \mu_u^\xi \rangle$ is bounded, uniformly in $\xi < \infty$ and $u \in [s,t]$. It is also straightforward to see that the truncated gel data are bounded by $g_u^{\xi,i} \leq \langle \pi_i, \mu_0 \rangle$, so the integrand appearing in $\mathcal{T}_1(\xi)$ is bounded. Using (134) and bounded convergence, we take the limit $\xi \to \infty$ to obtain

$$\mathcal{T}_1(\xi) \to \int_s^t \sum_{j,k=1}^n \langle \pi_i \pi_j, \mu_u \rangle a_{jk} g_u^k du. \tag{136}$$

We now deal with the second term $\mathcal{T}_2(\xi)$, which we claim converges to 0. Expanding the total rate $\overline{K}$, we have

$$\mathcal{T}_2(\xi) = \int_s^t \sum_{j,k=1}^n \int_{S^2} \pi_i(x) \pi_j(x) \pi_k(y) 1\left[\varphi(x+y) > \xi\right] \mu_u(dx) \mu_u(dy). \tag{137}$$

The integrand converges to 0 pointwise as $\xi \to \infty$, and is dominated by $\pi_i(x)\pi_j(x)\pi_k(y)$. By Lemma 6.10,

$$\sup_{u\in[s,t]} \int_{S^2} \pi_i(x)\pi_j(x)\pi_k(y)\mu_u(dx)\mu_u(dy)du < \infty. \tag{138}$$

Therefore, by dominated convergence, $\mathcal{T}_2(\xi) \to 0$ as $\xi \to \infty$, as claimed. Combining this with the analysis of the first term, we have shown that

$$g_t^i - g_s^i = \int_s^t \sum_{j,k=1}^n \langle \pi_i\pi_j, \mu_u\rangle a_{jk} g_u^k du. \tag{139}$$

Taking $s \downarrow t_{\mathrm{g}}$, and using the continuity $g_s \downarrow 0$ established in Lemma 7.1, we obtain the claimed result. □

8 Uniform Convergence of the Stochastic Coagulant

We now show how previous results, describing the dynamics of g_t, imply convergence to their maximum values $\langle \pi_0, \mu_0\rangle$ as $t \to \infty$. Using this, we will be able to upgrade the previous result, Lemma 3.1, on the convergence of the stochastic coagulant to *uniform* convergence.

Lemma 8.1. *Let μ_0 be an initial measure satisfying Assumption A, and let g_t be the gel data for the associated solution $(\mu_t)_{t\geq 0})$ to (FI). As $t \uparrow \infty$, we have*

$$g_t^i \to g_\infty^i = \langle \pi_i, \mu_0\rangle \tag{140}$$

for $i = 0, ..., n$.

Proof. Let us fix $1 \leq i \leq n$, and write g_∞^i for the claimed limit $\langle \pi_i, \mu_0\rangle$; it is immediate that $g_t^i \leq g_\infty^i$ for all $t \geq 0$. Choose $t_0 > t_{\mathrm{g}}$ and $1 \leq j \leq n$ such that $a_{ij} > 0$. Thanks to Lemma 7.1, $\epsilon = a_{ij} g_{t_0}^j > 0$, and note also that g_t^j is increasing, so that this bound holds uniformly in $t \geq t_0$. Applying Lemma 7.2 and taking $t \to \infty$, we obtain the integral inequality

$$\begin{aligned} \lim_{t\to\infty} \left(g_t^i - g_{t_0}^i\right) &\geq \int_{t_0}^\infty \langle \pi_i^2, \mu_s\rangle a_{ij} g_s^j ds \geq \epsilon \int_{t_0}^\infty \langle \pi_i, \mu_s\rangle^2 ds \\ &\geq \epsilon \int_{t_0}^\infty \left(g_\infty^i - g_s^i\right)^2 ds \end{aligned} \tag{141}$$

where the limit on the left hand side exists since g_t^i is increasing. Recalling that g_t^i is bounded, the integral appearing on the right-hand side must converge, and since the integrand is decreasing in s, this is only possible if $(g_\infty^i - g_s^i)^2 \to 0$ as $s \to \infty$, as desired.

We must deal separately with π_0, since π_0 does not appear in the dynamics explicitly and the argument above does not apply. For this case, we note that the monotonicity $\rho_s \leq \rho_t$ whenever $s \leq t$ implies that ρ_t converges pointwise to a limit $\rho_\infty \leq 1$. Using Lemma 7.1 and dominated convergence, we have, for all $i = 1, ..., n$

$$\langle \pi_i\rho_\infty, \mu_0\rangle = \lim_{t\to\infty} \langle \pi_i\rho_t, \mu_0\rangle = \lim_{t\to\infty} g_t^i = \langle \pi_i, \mu_0\rangle. \tag{142}$$

This implies the containment

$$\{\rho_\infty < 1\} \subset \{\pi_i = 0\} \cup \mathcal{N}_i \tag{143}$$

for a μ_0-null set $\mathcal{N}_i$, for each $i = 1, ...n$. Taking an intersection, and since $\mu_0(\pi_i = 0$ for all $i = 1, ..n) = 0$ by irreducibility (A4.), we see that $\rho_\infty = 1$, μ_0-almost everywhere. By Lemma 7.1 and dominated convergence again,

$$M_t = g_t^0 = \langle \pi_0 \rho_t, \mu_0 \rangle \to \langle \pi_0, \mu_0 \rangle \tag{144}$$

which is the claimed limit. □

Lemma 8.2. *Fix a measure μ_0 satisfying Assumption A, and let $(\mu_t)_{t\geq 0}$ be the associated solution to (FI). Let μ_t^N be the stochastic coagulants, with initial data μ_0^N satisfying Assumption B. Then we have the* uniform *convergence*

$$\sup_{t\geq 0} d(\mu_t^N, \mu_t) \to 0 \tag{145}$$

in probability.

Proof. From the definition of the vague topology, it is sufficient to prove that, for any $f \in C_c(S)$ with $0 \leq f \leq 1$, we have the uniform convergence $\sup_{t\geq 0} \langle f, \mu_t^N - \mu_t \rangle \to 0$ in probability.

Fix $\epsilon > 0$. By Lemma 8.1, we can find $t_+ \in (t_{\mathrm{g}}, \infty)$ such that $M_{t_+} > \langle \pi_0, \mu_0 \rangle - \frac{\epsilon}{3}$. Let A_N^1 be the event

$$A_N^1 = \left\{ M_{t_+}^N > \langle \pi_0, \mu_0 \rangle - \frac{\epsilon}{3}; \quad \langle \pi_0, \mu_0^N \rangle \leq \langle \pi_0, \mu_0 \rangle + \frac{\epsilon}{3} \right\}. \tag{146}$$

By Lemma 5.3 and condition (B2.), it follows that $\mathbb{P}(A_N^1) \to 1$. On this event, we have

$$\begin{aligned} \sup_{t\geq 0} \langle f, \mu_t^N - \mu_t \rangle &\leq \sup_{0\leq t\leq t_+} \langle f, \mu_t^N - \mu_t \rangle + \sup_{t > t_+} \langle f, \mu_t^N - \mu_t \rangle \\ &\leq \sup_{0\leq t\leq t_+} \langle f, \mu_t^N - \mu_t \rangle + \left(\langle \pi_0, \mu_0^N \rangle - M_{t_+}^N \right) + \left(\langle \pi_0, \mu_0 \rangle - M_{t_+} \right) \\ &\leq \sup_{0\leq t\leq t_+} \langle f, \mu_t^N - \mu_t \rangle + \epsilon \end{aligned} \tag{147}$$

and the first term converges to 0 in probability by Lemma 3.1. □

9 Behaviour Near the Critical Point

We now prove item 5 of Theorem 1, concerning the phase transition: we will show that the gel data $g_t = (g_t^i)$ have nonnegative right-derivatives at the gelation time t_{gel}. We start from the nonlinear fixed point equation (19), which we rewrite as

$$c_t = tF(c_t); \qquad F(c)_i = 2\int_S \left(1 - \exp\left(-\sum_{k=1}^n c_k \pi_k(x)\right)\right) \sum_{j=1}^n a_{ij}\pi_j(x)\mu_0(dx). \tag{148}$$

The following proof is a modification of the arguments in [5, Theorem 3.17], which itself generalises an analagous, well-known result for the phase transition of Erdős-Réyni graphs.

Lemma 9.1. *Suppose that μ_0 satisfies Assumption A, and let c_t be as in Lemma 4.7. Then c_t is right-differentiable at t_{g}, and the right-derivative $c'_{t_{\mathrm{g}}^+} > 0$ is componentwise positive.*

Proof. Let us equip $\mathbb{R}^n$ with the inner product

$$(c, c')_{\mu_0} = \sum_{i,j=1}^{n} c_i c'_j \langle \pi_i \pi_j, \mu_0 \rangle \tag{149}$$

which is the pullback of the $L^2(\mu_0)$ inner product under $c \mapsto \sum_i c_i \pi_i$, and write $|\cdot|_{\mu_0}$ for the associated norm. Differentiating under the integral sign twice, and using (A2.), we write

$$F\left(c\right) = \Lambda c - \Sigma(c) + R\left(c\right) \tag{150}$$

where Λ is the $n \times n$ matrix found in Lemma 4.5, $\Sigma(\cdot)$ is a quadratic term, and R is a remainder:

$$\Lambda_{ij} = 2 \sum_{k=1}^{n} a_{ik} \langle \pi_k \pi_j, \mu_0 \rangle; \tag{151}$$

$$\Sigma\left(c\right)_i = \sum_{j,k,l=1}^{n} a_{ij} \langle \pi_j \pi_k \pi_l, \mu_0 \rangle c_k c_l \tag{152}$$

$$\left|R(c)\right|_{\mu_0} = o\left(|c|^2_{\mu_0}\right) \text{ as } |c| \to 0. \tag{153}$$

The signs here are chosen to guarantee that, if $c > 0$, then $\Lambda c, \Sigma(c) > 0$, and Λ is self-adjoint with respect to $(\cdot,\cdot)_{\mu_0}$. We also recall from Lemma 4.5 that the largest eigenvalue of Λ is precisely t_{g}^{-1}, and the corresponding eigenspace is 1-dimensional. Let ψ be an associated eigenvector, scaled so that $|\psi|_{\mu_0} = 1$. We note that $\sum_i \psi_i \pi_i$ is an eigenfunction of T, and in particular, the sign of ψ can be chosen so that $\sum_i \psi_i \pi_i > 0$ is strictly positive μ_0-almost everywhere; using (54) it follows that $\psi_i > 0$ for all $i = 1, ..., n$. From Lemma 4.7, Theorem II.8 and Theorem II.12, we know that $c_{t_{\mathrm{g}}} = 0$, that $c_{t_{\mathrm{g}}+\epsilon} \in [0,\infty)^n \setminus 0$ for all $\epsilon > 0$, and that $t \mapsto c_t$ is continuous at t_{g}.

Let us write $\psi^\perp$ for the orthogonal compliment of Span(ψ) with respect to $(\cdot,\cdot)_{\mu_0}$. Since Span(ψ) is exactly the eigenspace Ker$(\Lambda - t_{\mathrm{g}}^{-1}1)$, it follows from the self-adjointness of Λ that Λ maps $\psi^\perp$ into itself, and that, for $t > t_{\mathrm{g}}$ small enough, $(t\Lambda - 1)|_{\psi^\perp}$ is invertible, and that the operator norm $\|(t\Lambda - 1)|_{\psi^\perp}^{-1}\|_{\mu_0 \to \mu_0}$ is bounded as $t \downarrow t_{\mathrm{g}}$.

Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $\psi^\perp$ with respect to $(\cdot,\cdot)_{\mu_0}$, and write $c^*_t = Pc_t$ so that we have the orthogonal decomposition

$$c_t = \alpha_t \psi + c^*_t \tag{154}$$

for some $\alpha_t \in \mathbb{R}$. Noting that $\Lambda P = P\Lambda$, it follows from (148, 154) that

$$c^\star_t = P(tF(c_t)) = t\Lambda c^\star_t + tP\left(-\Sigma(c_t) + R(c_t)\right). \tag{155}$$

The function $-\Sigma(c) + R(c)$ is of quadratic growth as $|c|_{\mu_0} \to 0$, and using the invertibility of $(t\Lambda - I)|_{\psi^\perp}$ described above, it follows that there exists $\beta > 0$ such that $|c^*_t|_{\mu_0} \le \beta |c_t|^2_{\mu_0}$ whenever $|c_t|_{\mu_0} \le 1$. In turn, it follows that $|c_t|_{\mu_0} \sim \alpha_t$ as $t \downarrow t_{\mathrm{g}}$. Now, using (148) and the self-adjointness of Λ, we obtain

$$\begin{aligned} \alpha_t = (\psi, c_t)_{\mu_0} &= (t_{\mathrm{g}} \Lambda \psi, c_t)_{\mu_0} = t_{\mathrm{g}} (\psi, \Lambda c_t)_{\mu_0} \\ &= \frac{t_{\mathrm{g}}}{t} (\psi, c_t)_{\mu_0} - t_{\mathrm{g}} \left(\psi, -\Sigma(c_t) + R(c_t)\right)_{\mu_0} \\ &= \frac{t_{\mathrm{g}}}{t} \alpha_t - t_{\mathrm{g}} \left(\psi, -\Sigma(c_t) + R(c_t)\right)_{\mu_0}. \end{aligned} \tag{156}$$

We now expand to second order in α_t; for clarity, we will number the error terms $\mathcal{T}_t^i$. Since $|c_t|_{\mu_0} \sim \alpha_t$, it follows that that $|c_t^*|_{\mu_0} = \mathcal{O}(\alpha_t^2)$ and that $R(c_t) = o(\alpha_t^2)$. Expanding $\Sigma(c_t)$ using (154),

$$-\Sigma(c_t) + R(c_t) = -\alpha_t^2 \Sigma(\psi) + \mathcal{T}_t^1; \qquad |\mathcal{T}_t^1|_{\mu_0} = o(\alpha_t^2). \tag{157}$$

It therefore follows that

$$\alpha_t = t_{\mathrm{g}} \left(\frac{\alpha_t}{t} + \alpha_t^2 (\psi, \Sigma(\psi))_{\mu_0} \right) + \mathcal{T}_t^2; \qquad \mathcal{T}_t^2 = o(\alpha_t^2). \tag{158}$$

For $t > t_{\mathrm{g}}$ small enough, $\alpha_t > 0$, and we may rearrange to find

$$t - t_{\mathrm{g}} = t\, t_{\mathrm{g}}\, \alpha_t (\psi, \Sigma(\psi))_{\mu_0} + \mathcal{T}_t^3; \qquad \mathcal{T}_t^3 = o(\alpha_t) \tag{159}$$

and in particular $\alpha_t = \Theta(t - t_{\mathrm{g}})$ as $t \downarrow t_{\mathrm{g}}$, since

$$(\psi, \Sigma(\psi))_{\mu_0} = \sum_{i,j,k,l=1}^{n} a_{ij} \psi_i \psi_k \psi_l \langle \pi_j \pi_k \pi_l, \mu_0 \rangle > 0. \tag{160}$$

Finally, we obtain

$$\frac{\alpha_t}{t - t_{\mathrm{g}}} \to \frac{1}{t_{\mathrm{g}}^2 (\psi, \Sigma(\psi))_{\mu_0}} \qquad \text{as } t \downarrow t_{\mathrm{g}}. \tag{161}$$

The calculations above show that $|c_t - \alpha_t \psi| = \mathcal{O}((t - t_{\mathrm{g}})^2)$, and the claimed right-differentiability now follows. Finally, since $\psi_i > 0$ is strictly positive componentwise and $\alpha'_{t_{\mathrm{g}}+} > 0$, it follows that $c'_{t_{\mathrm{g}}+} > 0$ componentwise. □

We now show how this implies item 5 of Theorem 1. From Lemmas 4.7, 7.1, we have, for all $i = 0, 1, ..., n$

$$g_t^i = \int_S \left(1 - \exp\left(-\sum_{j=1}^{n} c_t^j \pi_j(x) \right) \right) \pi_i(x) \mu_0(dx) \tag{162}$$

Differentiating under the integral sign using hypothesis (A2.), we obtain

$$g_t^i = \sum_{j=1}^{n} c_t^j \langle \pi_i \pi_j, \mu_0 \rangle + o(c_t). \tag{163}$$

In the notation of the previous lemma, we see that for $t > t_{\mathrm{g}}$,

$$\begin{aligned} g_t^i &= (t - t_{\mathrm{g}}) \sum_{j=1}^{n} (c'_{t_{\mathrm{g}}+})_j \langle \pi_i \pi_j, \mu_0 \rangle + o(t - t_{\mathrm{g}}) \\ &= \alpha'_{t_{\mathrm{g}}+} (t - t_{\mathrm{g}}) \left\langle \sum_{j=1}^{n} \psi_j \pi_j \pi_i, \mu_0 \right\rangle + o(t - t_{\mathrm{g}}). \end{aligned} \tag{164}$$

which proves the desired right-differentiability. For the positivity, since all components of $c'_{t_{\mathrm{g}}+}$ are strictly positive, we have the lower bound for $i = 1, ..., n$

$$(g'_{t_{\mathrm{g}}+})_i \geq (c'_{t_{\mathrm{g}}+})_i \langle \pi_i^2, \mu_0 \rangle > 0. \tag{165}$$

A similar argument holds for the 0^{th} component.

Finally, we address the size-biasing effect. We wish to choose a convex combination $\theta_i : i = 1, ..., n$ such that

$$\frac{\sum_{i=1}^n \theta_i (g'_{t_g+})_i}{(g'_{t_g+})_0} \geq \frac{\sum_{i=1}^n \theta_i \langle \pi_i, \mu_0 \rangle}{\langle \pi_0, \mu_0 \rangle}. \tag{166}$$

Thanks to the calculation above, this is equivalent to proving that

$$\frac{\sum_{i,j=1}^n \theta_i \psi_j \langle \pi_i \pi_j, \mu_0 \rangle}{\sum_{k=1}^n \psi_k \langle \pi_k, \mu_0 \rangle} \geq \frac{\sum_{i=1}^n \theta_i \langle \pi_i, \mu_0 \rangle}{\langle \pi_0, \mu_0 \rangle}. \tag{167}$$

If we choose $\theta_i = \psi_i / \sum_j \psi_j$, then these follow from the Cauchy-Schwarz inequality

$$\begin{aligned} \left\langle \sum_i \psi_i \pi_i, \mu_0 \right\rangle^2 &\leq \left\langle (\sum_i \psi_i \pi_i)^2, \mu_0 \right\rangle \langle 1, \mu_0 \rangle \\ &= \left\langle (\sum_i \psi_i \pi_i)^2, \mu_0 \right\rangle \langle \pi_0, \mu_0 \rangle. \end{aligned} \tag{168}$$

We recall that the linear combination $f = \sum_i \psi_i \pi_i$ is an eigenfunction of T, and so can only be constant μ_0-almost everywhere if $s(x) = (T1)(x)$ is constant. In particular, if s is not constant μ_0-almost everywhere, the inequality (168) is strict, and hence so is (166), as desired.

10 Convergence of the Gel

We now prove the remaining part of Theorem 2, concerning the *uniform* convergence of the stochastic gel, drawing on other results we have proven. We recall that g_t^N are the data of the largest particle in the stochastic coagulant μ_t^N; to conclude the proof of Theorem 2, we must extend Lemma 5.3, to show uniform convergence in time, in probability.

Throughout this section, let μ_0 be an initial measure satisfying Assumption A, and μ_t^N be stochastic coagulants satisfying Assumption B for this choice of μ_0. We will also let G_t^N be random graphs coupled to μ_t^N as described in Section 4, so that g_t^N is the data of the largest component in G_t^N.

This subsection is structured as follows. We recall that, in the proof of Lemma 5.3, we used the result on mesoscopic clusters from [5]: if $\xi_N \to \infty$ and $\frac{\xi_N}{N} \to 0$, then for all $t \geq 0$,

$$\frac{1}{N} \sum_{j \geq 2: C_j(G_t^N) \geq \xi_N} C_j(G_t^N) \to 0 \tag{169}$$

in probability. We will first state a lemma which extends this convergence to be uniform in time. Equipped with this lemma, and previous results, we will show how the proof of Lemma 5.3 can be modified to establish uniform convergence, and prove the analagous result when we sum over all clusters exceeding a deterministic size $\xi_N \ll N$. Finally, we return to prove the preliminary lemma.

The key lemma which we will require is the following, which generalises the result of Bollobás et al. recalled in Lemma II.11.

Lemma 10.1. *Let G_t^N be as above, and let ξ_N be any sequence such that $\xi_N \to \infty$, $\frac{\xi_N}{N} \to 0$. Then we have the uniform estimate*

$$\sup_{t \geq 0} \left[\frac{1}{N} \sum_{j \geq 2: C_j(G_t^N) \geq \xi_N} C_j(G_t^N) \right] \to 0 \qquad \textit{in probability.} \tag{170}$$

The proof of this lemma will be deferred until Subsection 10.2

10.1 Proof of Theorem 2

It remains to prove that the convergence of the stochastic approximations g^N_t, $\widetilde{g}^N_t$ to the gel, given by the gel data of the largest cluster, and of all clusters exceeding a certain scale ξ_N respectively. This is the content of the following two lemmas.

Lemma 10.2. *In the notation above, we have the uniform convergence*

$$\sup_{t\geq 0}\left|g^N_t - g_t\right| \to 0 \qquad \textit{in probability.} \tag{171}$$

Lemma 10.3. *Fix a sequence ξ_N such that $\xi_N \to \infty$ and $\frac{\xi_N}{N} \to 0$, and let $\widetilde{g}^N_t$ be given by*

$$\widetilde{g}^N_t = \frac{1}{N}\sum_{j\geq 1: \mathcal{C}_j(G^N_t)\geq \xi_N} \pi(\mathcal{C}_j(G^N_t)) = \left(\langle \pi_i 1[\pi_0\geq \xi_N], \mu^N_t\rangle\right)_{i=0}^{n+m}. \tag{172}$$

Then

$$\sup_{t\geq 0}\left|\widetilde{g}^N_t - g^N_t\right| \to 0 \qquad \textit{in probability.} \tag{173}$$

We now prove these two lemmas, looking primarily at the 0^{th} coordinate. The other coordinates follow, with minor modifications which will be discussed later.

Proof of Lemma 10.2. This is an extension of the proof of Lemma 5.3, from where much of the notation is taken. We deal first with the 0^{th} coordinate $M^N_t - M_t$. Let η_r be a fast-growing sequence such that $\beta(r,\eta_r)\to 0$ in the notation of Lemma 5.4, and let $S_{(r)}, \widetilde{f}_r, \widetilde{h}_r$ be as in Lemma 5.3. Let also ξ_N be a sequence, to be constructed later, such that

$$\xi_N \to \infty; \qquad \frac{\xi_N}{N}\to 0 \tag{174}$$

and write $f_N = \widetilde{f}_{\xi_N}, h_N = \widetilde{h}_{\xi_N}$. We recall also the decomposition (71)

$$M^N_t - M_t = \sum_{i=1}^{5}\mathcal{T}^i_N(t) \tag{175}$$

where the definitions of the error terms are given in (71). The bounds obtained on $\mathcal{T}^3_N(t), \mathcal{T}^5_N(t)$ in the proof of Lemma 5.3 are already uniform in time; we will now show how the previous proof can be modified to estimate the other terms uniformly in time.

1. Estimate on $T^1_N(t)$ $\mathcal{T}^1_N(t)$ is the nonrandom error $\langle \pi_0,\mu_t\rangle - \langle \pi_0 f_N, \mu_t\rangle$. The estimate in Lemma 5.3 shows that $\langle \pi_0 f_N,\mu_t\rangle \uparrow \langle \pi_0,\mu_t\rangle$ for each fixed $t\geq 0$. The maps $t\mapsto \langle \pi_0 f_N,\mu_t\rangle$, $t\mapsto\langle \pi_0,\mu_t\rangle$ are both continuous on $[0,\infty)$, by the definition of the Flory dynamics (Fl) and Lemma 7.1 respectively. Let us extend both of these maps to $[0,\infty]$ by defining both to be 0 at $t=\infty$; the extensions are continuous, by Lemma 8.1. Therefore, by Dini's theorem, it follows that $\langle \pi_0 f_N,\mu_t\rangle \to \langle \pi_0,\mu_t\rangle$ uniformly, which implies that $\sup_{t\geq 0}|T^1_N(t)|\to 0$ as desired.

2. Estimate on $\mathcal{T}^4_N$. As in (74), we have the equality, for all $t \geq 0$

$$\mathcal{T}^4_N(t) = -M^N_t 1\left(M^N_t \leq \frac{\xi_N}{N}\right) - \frac{1}{N}\sum_{j\geq 2: C_j(G^N_t)\geq \xi_N} \pi_0(\mathcal{C}_j(G^N_t)) \tag{176}$$

where we have used the coupling of the random graphs $(G^N_t)_{t\geq 0}$ to the stochastic coagulant. Therefore, we have the uniform bound

$$\sup_{t\geq 0}\left|\mathcal{T}^4_N(t)\right| = \langle \varphi^2, \mu^N_0\rangle^{\frac{1}{2}}\left(\sup_{t\geq 0}\frac{1}{N}\sum_{j\geq 2: C_j(G^N_t)\geq \xi_N} C_j(G^N_t)\right)^{\frac{1}{2}} + \frac{\xi_N}{N} \tag{177}$$

which converges to 0, by Lemma 10.1, (B2.), and because $\xi_N \ll N$.

3. Construction of ξ_N, and convergence of $\mathcal{T}^2_N$. To conclude the proof of the supercritical case, it remains to show how a sequence ξ_N can be constructed such that $\mathcal{T}^2_N \to 0$ uniformly, in probability. Recalling the definitions of $\widetilde{f}_r$ above, let $A_{r,N}$ be the events

$$A_{r,N} = \left\{\sup_{t\geq 0}|\langle \pi_0 \widetilde{f}_r, \mu^N_t - \mu_t\rangle| < \frac{1}{r}\right\}. \tag{178}$$

Then, as $N\to\infty$ with r fixed, $\mathbb{P}(A^1_{r,N}) \to 1$ by Lemma 8.2. We now define N_r inductively for $r\geq 1$ inductively, as in Lemma 5.3, by setting $N_1 = 1$ and letting N_{r+1} be the minimal $N > N_r$ such that, for all $N' \geq N$,

$$N \geq (r+1)^2; \qquad \mathbb{P}(A_{r+1,N'}) > \frac{r}{r+1}. \tag{179}$$

Now, we set $\xi_N = r$ for $N \in [N_r, N_{r+1})$. It follows that ξ_N satisfies the requirements above, and

$$\mathbb{P}\left(\sup_{t\geq 0}|\mathcal{T}^2_N| < \frac{1}{\xi_N}\right) \geq \mathbb{P}\left(A^1_{\xi_N,N}\right) > 1 - \frac{1}{\xi_N} \to 1 \tag{180}$$

Therefore, with this choice of ξ_N, $\mathcal{T}^2_N \to 0$ uniformly in probability on $t \geq 0$.

This concludes the proof for the 0^{th} coordinate M^N_t; the 1^{st} - n^{th} coordinates are identical. For the remaining m coordinates, we replace f_N by $\frac{1}{2}(f_N(x) + f_N(Rx))$, which makes $\mathcal{T}^1_N$ identically 0 by symmetry, and use the bound $\pi_i(x)^2 \leq c\varphi(x)^2$ in estimating $\mathcal{T}^4_N$. □

Proof of Lemma 10.3. We now turn to the case where, instead of considering the largest cluster, we sum over the (possibly empty) set of clusters of size at least ξ_N, for a deterministic sequence ξ_N. In this way, we have

$$\widetilde{g}^N_t = \langle \pi 1[\pi_0 \geq \xi_N], \mu^N_t\rangle. \tag{181}$$

Let us write $h_N(x) = 1[\pi_0(x) < \xi_N]$, so that $\widetilde{g}^N_t = \langle \pi, \mu^N_0\rangle - \langle \pi h_N, \mu^N_t\rangle$. With this notation,

$$g^N_t - \widetilde{g}^N_t = \langle \pi h_N, \mu^N_t\rangle - \left(\langle \pi, \mu^N_0\rangle - g^N_t\right) \tag{182}$$

is exactly the term $\mathcal{T}^4_N$ estimated in the proofs of Lemma 5.3, 10.2, for the new choice of ξ_N. The estimate (177) therefore applies to bound $\sup_{t\geq 0}\left|g^N_t - \widetilde{g}^N_t\right|$, and the hypotheses on ξ_N are sufficient to guarantee that the right-hand side converges to 0 in probability. □

10.2 Proof of Lemma 10.1

We now turn to the proof of Lemma 10.1; our strategy is as follows. First, we prove uniform convergence on compact subsets $I \subset (t_\mathrm{g}, \infty)$ in Lemma 10.4. We will then show how this may be extended to the whole interval $[0, \infty)$, by arguing separately for an initial interval $[0, t_-]$ and for large times $[t_+, \infty)$.

Lemma 10.4. *Let G^N_t and ξ_N be as above. Fix a compact subset $I \subset (t_\mathrm{g}, \infty)$. Then we have the convergence*

$$\sup_{t\in I}\left[\frac{1}{N}\sum_{j\ge 2: C_j(G^N_t)\ge \xi_N} C_j(G^N_t)\right] \to 0 \qquad \textit{in probability.} \tag{183}$$

Proof of Lemma 10.4. It is sufficient to show that for every $t > t_\mathrm{g}$ the claim holds for some I of the form $I = (t_-, t_+) \subset (t_\mathrm{g}, \infty)$ containing t. As in Theorem 6.10, let $\widehat{\mu}^t_0$ be the measure on S given by $\widehat{\mu}^t_0(dx) = (1-\rho_t(x))\mu_0(dx)$. We also write $\widehat{t_\mathrm{g}}(t)$ for the gelation time of the solution $(\widehat{\mu}^t_s)_{s\ge 0}$ to (Fl) starting at $\widehat{\mu}^t_0$. We showed in the proof of Theorem 6.10 that, for all $t > t_\mathrm{g}$, $\widehat{t_\mathrm{g}}(t) > t$, and the map $t \mapsto \widehat{t_\mathrm{g}}(t)$ is continuous. Therefore, for any $t > t_\mathrm{g}$, we can choose $t_\pm$ such that

$$t_\mathrm{g} < t_- < t < t_+ < \widehat{t_\mathrm{g}}(t_-). \tag{184}$$

We form $\widetilde{G}^N_{t_-}$ from $G^N_{t_-}$ by deleting all vertexes of the giant component of $C_1(G^N_{t_-})$. We now form a new graph, $\widetilde{G}^N_{t_-,t_+}$ by including all edges between vertexes of $\widetilde{G}^N_{t_-}$ which are present in the graph $G^N_{t_+}$.

From Theorem II.13 and Lemma 6.11, we can construct a sequence $\mathbf{y}_N, N \ge 1$ satisfying Assumption B for $\widehat{\mu}^{t_-}_0$ and random graphs $\widehat{G}^N_{t_-} \sim \mathcal{G}(\mathbf{y}_N, t_-K/N)$, such that

$$\mathbb{P}\left(\widehat{G}^N_{t_-} = \widetilde{G}^N_{t_-}\right) \to 1. \tag{185}$$

We now form $\widehat{G}^N_{t_-,t_+}$ from $\widehat{G}^N_{t_-}$ by adding those edges present in $G^N_{t_+}$. By the Markov property of the graph process $(G^N_s)_{t\ge 0}$, these edges are independent of the construction of $\widehat{G}^N_{t_-}$, and so $\widehat{G}^N_{t_-,t_+} \sim \mathcal{G}(\mathbf{y}_N, t_+K/N)$.

Since Assumption B applies to $\mathbf{y}_N$ and $\widehat{\mu}^{t_-}_0$, Lemma 5.1 shows that the critical time for $\mathcal{G}(\mathbf{y}_N, tK/N)$ is exactly the gelation time of $(\widehat{\mu}^{t_-}_s)_{s\ge 0}$, which we have written as $\widehat{t_\mathrm{g}}(t_-)$. By the choices of $t_\pm$, $t_+ < \widehat{t_\mathrm{g}}(t_-)$, and in particular, $\widehat{G}^N_{t_-,t_+}$ is still subcritical. By construction,

$$\mathbb{P}\left(\widehat{G}^N_{t_-,t_+} = \widetilde{G}^N_{t_-,t_+}\right) \to 1. \tag{186}$$

For $s \in [t_-, t_+]$, let $\mathcal{C}_1'(G^N_s)$ be the connected component of G^N_s which contains $\mathcal{C}_1(G^N_{t_-})$, and let $C_1'(G^N_s)$ be its size. By definition, $C_1'(G^N_s) \le C_1(G^N_s)$ and so

$$\sum_{j\ge 2} C_j(G^N_s) 1\left[C_j(G^N_s) \ge \xi_N\right] \quad \le \quad \sum_{j\ge 1} C_j(G^N_s) 1\left[C_j(G^N_s) \ge \xi_N, \mathcal{C}_j(G^N_s) \ne \mathcal{C}_1'(G^N_s)\right]. \tag{187}$$

Moreover, the right-hand side is increasing as s runs over $[t_-, t_+]$, since it can be rewritten as

$$\ldots\ldots = \sum_{i=1}^{l^N} 1\left[\exists j : i \in \mathcal{C}_j(G^N_s),\ C_j(G^N_s) \ge \xi_N,\ i \notin \mathcal{C}_1(G^N_{t_-})\right] \tag{188}$$

and each summand can only increase in s as the clusters grow. Evaluating at the endpoint t_+, the construction of $\widetilde{G}^N_{t_-,t_+}$ gives

$$\sum_{j\ge 1} C_j(G^N_{t_+})1\left[C_j(G^N_{t_+} \ge \xi_N, \mathcal{C}_j(G^N_{t_+}) \neq \mathcal{C}'_1(G^N_{t_+})\right] = \sum_{j\ge 1} C_j(\widetilde{G}^N_{t_-,t_+})1\left[C_j(\widetilde{G}^N_{t_-,t_+}) \ge \xi_N\right]. \tag{189}$$

Combining (186, 187, 189) we see that, with high probability,

$$\sup_{s\in[t_-,t_+]}\left[\frac{1}{N}\sum_{j\ge 2: C_j(G^N_s)\ge \xi_N} C_j(G^N_s)\right] \le \frac{1}{N}C_1(\widehat{G}^N_{t_-,t_+}) + \frac{1}{N}\sum_{j\ge 2: C_j(\widehat{G}^N_{t_-,t_+})\ge\xi_N} C_j(\widehat{G}^N_{t_-,t_+}). \tag{190}$$

The first term of the right-hand side converges to 0 in probability because $\widehat{G}^N_{t_-,t_+}$ is subcritical, and the second term converges to 0 in probability by Theorem II.11. □

Proof of Lemma 10.1. For μ_0 as in the hypothesis, let M_t be the mass of the gel associated to the solution $(\mu_t)_{t\ge 0}$ to (FI). Fix $\epsilon > 0$; without loss of generality, assume that $\epsilon < 1$. By continuity from Lemma 7.1 and Lemma 8.1, we can choose $t_\pm \in (t_{\mathrm{g}}, \infty)$ such that

$$M_{t_-} < \frac{\epsilon}{3}; \qquad M_{t_+} > \mu_0(S) - \frac{\epsilon}{3}.. \tag{191}$$

Consider now the events

$$A^1_N = \left\{\frac{1}{N}C_1(G^N_{t_-}) < \frac{2\epsilon}{3}; \quad \frac{1}{N}C_1(G^N_{t_+}) > \mu_0(S) - \frac{\epsilon}{2}; \quad \langle 1, \mu^N_0\rangle < \mu_0(S) + \frac{\epsilon}{2}\right\}; \tag{192}$$

$$A^2_N = \left\{\frac{1}{N}\sum_{j\ge 2: C_j(G^N_{t_-})\ge\xi_N} C_j(G^N_{t_-}) < \frac{\epsilon}{3}\right\}. \tag{193}$$

Thanks to the coupling described in Section 4, Lemma 5.3 implies that $\mathbb{P}(A^1_N) \to 1$, and $\mathbb{P}(A^2_N) \to 1$ from Theorem II.11. On the event $A^1_N \cap A^2_N$, we bound as follows.

i). For the initial interval $[0, t_-]$, an argument similar to that of Lemma 10.4 shows that, on this event,

$$\begin{aligned}\sup_{t\in[0,t_-]}\frac{1}{N}\sum_{\substack{j\ge 2:\\ C_j(G^N_t)\ge\xi_N}} C_j(G^N_t) &\le \frac{1}{N}\sum_{\substack{j\ge 1:\\ C_j(G^N_{t_-})\ge\xi_N}} C_j(G^N_{t_-}) \\ &= \frac{1}{N}C_1(G^N_{t_-}) + \frac{1}{N}\sum_{\substack{j\ge 2:\\ C_j(G^N_{t_-})\ge\xi_N}} C_j(G^N_{t_-}) < \epsilon.\end{aligned} \tag{194}$$

ii). For late times $t \in [t_+, \infty)$, the largest cluster $\mathcal{C}_1(G^N_t)$ is at least the size of the cluster containing $\mathcal{C}_1(G^N_{t_+})$. Therefore,

$$\inf_{t\ge t_+}\frac{1}{N}C_1(G^N_t) \ge \frac{1}{N}C_1(G^N_{t_+}) > \mu_0(S) - \frac{\epsilon}{2} \tag{195}$$

and so

$$\sup_{t\geq t_+}\left[\frac{1}{N}\sum_{j\geq 2: C_j(G^N_t)\geq \xi_N} C_j(G^N_t)\right] \leq \sup_{t\geq t_+}\left[\frac{1}{N}\sum_{j\geq 2} C_j(G^N_t)\right] \leq \langle 1, \mu^N_0\rangle - \frac{1}{N}C_1(G^N_{t_+}) < \epsilon. \tag{196}$$

Now, consider the events

$$A^3_N = \left\{\sup_{t\in[t_-,t_+]}\left[\frac{1}{N}\sum_{j\geq 2: C_j(G^N_t)\geq \xi_N} C_j(G^N_t)\right] < \epsilon\right\}; \tag{197}$$

$$A_N = A^1_N \cap A^2_N \cap A^3_N. \tag{198}$$

By Lemma 10.4, $\mathbb{P}(A^3_N)\to 1$, and so $\mathbb{P}(A_N)\to 1$. On the event A_N, we have

$$\sup_{t\geq 0}\left[\frac{1}{N}\sum_{j\geq 2: C_j(G^N_t)\geq \xi_N} C_j(G^N_t)\right] < \epsilon \tag{199}$$

which proves the claimed convergence in probability. □

Appendix I Weak Formulation of Smoluchowski and Flory Equations

Throughout, we work with the weak formulation of the Smoluchowski and Flory equations described in the introduction. In order to make sense of every term for a putative solution $(\mu_t)_{t<T}$, we ask for the following conditions to hold.

i). For all Borel sets $A\subset S$, the map $t\mapsto \mu_t(A)$ is measurable;

ii). For all bounded, measurable functions $f: S\to\mathbb{R}_+$ of compact support, $\langle f, \mu_0\rangle < \infty$;

iii). For all compact subsets $S'\subset S$ and all $t<T$,

$$\int_0^t ds \int_{S'\times S} \overline{K}(x,y)\mu_s(dx)\mu_s(dy) < \infty; \tag{200}$$

If these hold, then we say can make sense of the following weak form of the Smoluchowski equation (Sm).

iv). For all $f\in C_c(S)$ and $t<T$,

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle f, L(\mu_s)\rangle ds. \tag{201}$$

Appendix II Introduction to Inhomogenous Random Graphs

As discussed in the introduction, the connection between gelation and random graphs is well understood, and the multiplicative kernel corresponds to the well-known Erdős-Réyni random graphs [11, 9, 2]. However, for our purposes, not all particles are equal: particles with large values of $\pi_i(x)$ will undergo more collisions and exhibit quantitatively different behaviour, and so we will need a more sophisticated model of random graphs to accommodate this inhomogeneity. In this section, we will review the theory of *inhomogenous random graphs* developed in [5], which will play the same rôle for our model that the Erdős-Réyni model does for the multiplicative kernel. We now summarise the key definitions and results from [5] which we use in our work.

Definition II.1. *A* generalised vertex space *is a triple* $\mathcal{V} = (\mathcal{S}, m, (\mathbf{x}_N)_{N\geq 1})$*, consisting of*

- *A separable metric space* $\mathcal{S}$*, equipped with its Borel* σ*-algebra;*
- *A measure* m *on* $\mathcal{S}$*, with* $m(\mathcal{S}) \in (0, \infty)$*;*
- *A family of random variables* $\mathbf{x}_N = (x_1^{(N)}, ..., x_{l^N}^{(N)})$ *taking values in* $\mathcal{S}$*, and of potentially random length* l^N*, such that the empirical measures*

$$m_N = \frac{1}{N} \sum_{k=1}^{l^N} \delta_{x_k^{(N)}} \tag{202}$$

converge to m *in the weak topology* $\mathcal{F}(C_b(\mathcal{S}))$*, in probability.*

In the special case where $m(\mathcal{S}) = 1$ *and* $l^N = N$*, we say that* $(\mathcal{S}, m, (\mathbf{x}_N)_{N\geq 1})$ *is a* vertex space.

Definition II.2. *A* kernel *is a symmetric, measurable map* $k : \mathcal{S} \times \mathcal{S} \to [0, \infty)$*. We say that* k *is* irreducible *if, whenever* $A \subset S$ *is such that* $k(x, y) = 0$ *for all* $x \in A$ *and* $y \in A^c$*, then either* $m(A) = 0$ *or* $m(A^c) = 0$*.*

Definition II.3 (Inhomogenous random graphs)**.** *Given a kernel* k *and a generalised vertex space* $\mathcal{V}$*, we let* G^N *be a random graph on* $\{1, 2, .., l^N\}$ *given as follows. Conditional on the values of* $\mathbf{x}_N$*, the edge* $e = (ij)$ *is included with probability*

$$p_{ij} = 1 - \exp\left(-\frac{k(x_i^{(N)}, x_j^{(N)})}{N}\right) \tag{203}$$

and such that the presence of different edges is (conditionally) independent. We write $G^N \sim \mathcal{G}^{\mathcal{V}}(N, k)$*. We also consider the* vertex data $\mathbf{x}_N = (x_i^{(N)})_{i=1}^{l^N}$ *to be part of the data of* G_t^N*, so that an equality of random graphs* $G = G'$ *includes the equality of the vertex data.*

Remark II.4. *This differs slightly from the main definition in [5], but is rather one of the alternatives considered in [5][Remark 2.4]*

To treat a general class of kernels k, additional regularity is required, to prevent pathologies. This is the content of the following defintion:

Definition II.5 (Graphical Kernel)**.** *We say that a kernel* k *on a vertex space* $\mathcal{V} = (\mathcal{S}, m, (\mathbf{v}_N)_{N\geq 1})$ *is* graphical *if the following hold.*

i). k is almost everywhere continuous on $\mathcal{S} \times \mathcal{S}$;

ii). $k \in L^1(\mathcal{S} \times \mathcal{S}, m \otimes m)$;

iii). If $G^N \sim \mathcal{G}^{\mathcal{V}}(N, k)$, then

$$\frac{1}{N}\mathbb{E}\left[e\left(G^N\right)\right] \to \frac{1}{2}\int_{\mathcal{S}\times\mathcal{S}} k(v,w)m(dv)m(dw) \tag{204}$$

where $e(\cdot)$ denotes the number of edges of the graph.

Definition II.6. *Given a graph G, we write $\mathcal{C}_j(G) : j = 1, 2...$ for the connected components of G, in decreasing order of their sizes $\#\mathcal{C}_j(G) = C_j(G)$. If there are fewer than j connected components, then $\mathcal{C}_j(G) = \emptyset$ and $C_j(G) = 0$.*

The phase transition is given in terms of the convolution operator

$$(Tf)(v) = \int_{\mathbb{R}^d} k(v,w)f(w)m(dw) \tag{205}$$

for functions f such that the right-hand side is defined (i.e., finite or $+\infty$) for m-almost all v; for instance, if $f \geq 0$ then Tf is well-defined, possibly taking the value ∞. We define

$$\|T\| = \sup\{\|Tf\|_{L^2(m)} : \|f\|_{L^2(m)} \leq 1, f \geq 0\}. \tag{206}$$

If T defines a bounded linear map from $L^2(m)$ to itself, then $\|T\|$ is precisely its operator norm in this setting; otherwise, $\|T\| = \infty$. It is straightforward to show that if $k \in L^2(S \times S, m \otimes m)$ then $T : L^2(m) \to L^2(m)$ is a Hilbert-Schmidt operator, and that $\|T\|_{\text{HS}} = \|k\|_{L^2(m)} < \infty$. In this case, $\|T\|$ is certainly finite, and is the operator norm of $T : L^2(m) \to L^2(m)$. The example of interest to us will fall into this case.

The analysis of the random graphs uses a branching process, similar to that used in the standard analysis of Erdős-Rényi graphs. Many quantities of the graph can be expressed in terms of the 'survival probability' $\rho(k, v)$ when the data v of the first vertex is given. To avoid the unnecessary complication of making this into a precise definition, we use the following characterisation, which is equivalent by [5, Theorem 6.2].

Theorem II.7. *Let k be an irreducible kernel on a generalised vertex space $\mathcal{V}$, such that $k \in L^1(\mathcal{S} \times \mathcal{S}, m \times m)$, and such that, for all x,*

$$\int_S k(x,y)m(dy) < \infty. \tag{207}$$

Consider the nonlinear fixed-point equation

$$\forall x \in S, \qquad \rho(x) = 1 - e^{-(T\rho)(x)} \tag{208}$$

where T is the convolution operator (48). Then (208) has a maximal solution $\rho_k(x) = \rho(k; x)$; that is, for any other solution $\tilde{\rho}$,

$$\forall x \in S, \qquad \tilde{\rho}(x) \leq \rho(k, x). \tag{209}$$

It therefore follows that $0 \leq \rho_k(x) \leq 1$ for all x. The maximal solution is necessarily unique, and so this uniquely defines ρ_k. Moreover, we have the following dichotomy:

i). If $\|T\| \leq 1$, then $\rho(k, x) = 0$ for all x;

ii). If $\|T\| > 1$, then $\rho(k, x) > 0$ for all m-almost all x.

This can be stated dynamically as follows. Consider the survival function 'at time t', given by $\rho(tk, x)$, which we will write throughout as $\rho_t(x)$. Then

- *If $t \leq \|T\|^{-1}$, then $\rho_t(x) = 0$ for all x;*
- *If $t > \|T\|^{-1}$, then $\rho_t(x) > 0$ for all x.*

We can now state the main results on the phase transition, given by [5, Theorem 3.1 and Corollary 3.2].

Theorem II.8 (Phase Transition)**.** *Let k be a graphical and irreducible kernel for a vertex space $\mathcal{V}$, with $0 < \|T\| < \infty$. Let $G^N \sim \mathcal{G}^{\mathcal{V}}(N, k)$ be random graphs on a common probability space. Then we have the convergence*

$$\frac{1}{N} C_1(G_t^N) \to \int_{\mathcal{S}} \rho(tk, v) m(dv) \qquad \text{in probability.} \tag{210}$$

Therefore, if $(G_t^N)_{t \geq 0}$ is a dynamic family of random graphs $G_t^N \sim \mathcal{G}^{\mathcal{V}}(N, tk)$, then we have the following dichotomy:

i). If $t \leq t_{\mathrm{c}} = \|T\|^{-1}$, then there is no giant component, in particular

$$\frac{C_1(G_t^N)}{N} \to 0 \tag{211}$$

in probability.

ii). If $t > t_{\mathrm{c}} = \|T\|^{-1}$, then there is a giant component: there exists $c = c(t) > 0$ such that

$$\mathbb{P}(C_1(G_t^N) > cN) \to 1. \tag{212}$$

Remark II.9. *Following [5], based on this dichotomy, we say that*

i). G^N is subcritical *if $\|T\| < 1$;*

ii). G^N is critical *if $\|T\| = 1$;*

iii). G^N is supercritical *if $\|T\| > 1$.*

The next result characterises t_{g} in terms of the point spectrum $\sigma_{\mathrm{p}}(T)$ as an operator on $L^2(m)$, and appears as [5, Lemma 5.15]

Theorem II.10 (Spectrum of T)**.** *Let $\mathcal{V}$ be a generalised vertex space and k be a graphical, irreducible kernel on $\mathcal{V}$ such that $k \in L^2(S \times S, m \times m)$. Then the operator T defined in (48) has an eigenvalue $t_{\mathrm{c}}^{-1} = \|T\|$ in $L^2(m)$, and the corresponding eigenspace is 1-dimensional. Moreover, there exists an eigenfunction f such that $f > 0$ m-almost everywhere.*

The third result we will recall is [5, Theorem 3.6], which considers clusters of a scale $\xi_N \ll N$, excluding the largest cluster. We term these *mesoscopic* clusters.

Theorem II.11. *Let $G^N \sim \mathcal{G}^{\mathcal{V}}(N,k)$, for a (generalised) vertex space $\mathcal{V}$ and an irreducible graphical kernel k. Let ξ_N be a sequence with*

$$\xi_N \to \infty; \qquad \frac{\xi_N}{N} \to 0. \tag{213}$$

Then

$$\frac{1}{N} \sum_{j\geq 2: C_j(G^N)\geq \xi_N} C_j(G^N) \to 0 \tag{214}$$

in probability.

We will also make use of the following monotonicity and continuity properties, from [5, Theorem 6.4].

Theorem II.12. *Let k be a kernel on a vertex space $\mathcal{V}$, and let $\rho_t(\cdot) = \rho(tk,\cdot)$ be the survival function defined above. Then the map $t \mapsto \rho_t(\cdot)$ is monotonically increasing, in the sense that for all $0 \leq s \leq t$ and for all x, $\rho_s(x) \leq \rho_t(x)$. We also have the following continuity property. Let $t_n \to t$ be a monotone sequence, either increasing or decreasing. Then*

$$\rho_{t_n}(x) \to \rho_t(x) \qquad \text{for } m\text{- almost all } x, \text{ and} \tag{215}$$

$$\int_{\mathcal{S}} \rho_{t_n}(x)m(dx) \to \int_{\mathcal{S}} \rho_t(x)m(dx). \tag{216}$$

The final result which we will need is a 'duality' result, connecting the supercritical and subcritical behaviours. This is given by [5, Theorem 12.1].

Theorem II.13. *Let k be an irreducible graphical kernel on a generalised vertex space $\mathcal{V}$, such that $\|T\| > 1$. Let $G^N \sim \mathcal{G}^{\mathcal{V}}(N,k)$, and form $\widetilde{G}^N$ by deleting all vertexes in the largest component $\mathcal{C}_1(G^N)$. Then, defined on the same underlying probability space, there is a generalised vertex space $\widehat{\mathcal{V}} = (\mathcal{S}, \widehat{m}, (\mathbf{y}_N)_{N\geq 1})$ with*

$$\widehat{m}(dx) = (1-\rho(k;x))m(dx) \tag{217}$$

and such that $\mathbf{y}_N$ is an enumeration of those x_i not belonging to the component $\mathcal{C}_1(G^N)$, and a random graph $\widehat{G}^N \sim \mathcal{G}^{\widehat{\mathcal{V}}}(N,k)$ such that

$$\mathbb{P}(\widetilde{G}^N = \widehat{G}^N) \to 1. \tag{218}$$

Furthermore, if $k \in L^2(\mathcal{S}\times\mathcal{S}, m\otimes m)$, then $\widehat{G}^N$ is subcritical.

We emphasise here that we have defined the equality $\widetilde{G}^N = \widehat{G}^N$ to include equality of the values x_i associated to each vertex; this follows from the construction in [5], since the values $\mathbf{y}_N$ associated to $\widehat{G}^N$ are exactly those x_i not belonging to the giant component. This generalises the standard 'duality result' of Bollobás [3] for Erdős-Rényi graphs.

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