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Asymptotic analysis of a tumor growth model with fractional operators

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Abstract

In this paper, we study a system of three evolutionary operator equations involving fractional powers of selfadjoint, monotone, unbounded, linear operators having compact resolvents. This system constitutes a generalized and relaxed version of a phase field system of Cahn–Hilliard type modelling tumor growth that has originally been proposed in Hawkins-Daarud et al. (*Int. J. Numer. Math. Biomed. Eng.* **28** *(2012), 3–24*). The original phase field system and certain relaxed versions thereof have been studied in recent papers co-authored by the present authors and E. Rocca. The model consists of a Cahn–Hilliard equation for the tumor cell fraction φ, coupled to a reaction-diffusion equation for a function S representing the nutrient-rich extracellular water volume fraction. Effects due to fluid motion are neglected. Motivated by the possibility that the diffusional regimes governing the evolution of the different constituents of the model may be of different (e.g., fractional) type, the present authors studied in a recent note a generalization of the systems investigated in the abovementioned works. Under rather general assumptions, well-posedness and regularity results have been shown. In particular, by writing the equation governing the evolution of the chemical potential in the form of a general variational inequality, also singular or nonsmooth contributions of logarithmic or of double obstacle type to the energy density could be admitted. In this note, we perform an asymptotic analysis of the governing system as two (small) relaxation parameters approach zero separately and simultaneously. Corresponding well-posedness and regularity results are established for the respective cases; in particular, we give a detailed discussion which assumptions on the admissible nonlinearities have to be postulated in each of the occurring cases.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded, and connected set with smooth boundary Γ and unit outward normal $\mathbf{n}$; let $T > 0$ be given. Setting $Q_t := \Omega \times (0,t)$ for $t \in (0,T)$ and $Q := \Omega \times (0,T)$, as well as $\Sigma := \Gamma \times (0,T)$, we investigate in this paper the evolutionary system

$$\alpha\,\partial_t\mu + \partial_t\varphi + A^{2\rho}\mu = P(\varphi)(S-\mu) \qquad \text{in } Q, \tag{1.1}$$

$$\mu = \beta\,\partial_t\varphi + B^{2\sigma}\varphi + F'(\varphi) \qquad \text{in } Q, \tag{1.2}$$

$$\partial_t S + C^{2\tau} S = -P(\varphi)(S-\mu) \qquad \text{in } Q, \tag{1.3}$$

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \quad S(0) = S_0, \qquad \text{in } \Omega. \tag{1.4}$$

In the above system, $A^{2\rho}$, $B^{2\sigma}$, $C^{2\tau}$, with $\rho > 0$, $\sigma > 0$, $\tau > 0$, denote fractional powers of the selfadjoint, monotone, and unbounded linear operators A, B, and C, respectively, which are supposed to be densely defined in $H := L^2(\Omega)$ and to have compact resolvents. Moreover, F' denotes

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the derivative of a double-well potential F. Typical and physically significant examples of F are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$F_{reg}(r) := \frac{1}{4}\,(r^2-1)^2\,, \quad r \in \mathbb{R}, \tag{1.5}$$

$$F_{log}(r) := \begin{cases} (1+r)\ln(1+r) + (1-r)\ln(1-r) - c_1 r^2\,, & r \in (-1,1) \\ 2\log(2) - c_1\,, & r \in \{-1,1\} \\ +\infty\,, & r \notin [-1,1] \end{cases}\,, \tag{1.6}$$

$$F_{2obs}(r) := c_2(1-r^2) \quad \text{if } |r| \le 1 \quad \text{and} \quad F_{2obs}(r) := +\infty \quad \text{if } |r| > 1. \tag{1.7}$$

Here, the constants c_i in (1.6) and (1.7) satisfy $c_1 > 1$ and $c_2 > 0$, so that the corresponding functions are nonconvex. In cases like (1.7), one has to split F into a nondifferentiable convex part F_1 (the indicator function of $[-1,1]$, in the present example) and a smooth perturbation F_2. Accordingly, in the term $F'(\varphi)$ appearing in (1.2), one has to replace the derivative F_1' of the convex part F_1 by the subdifferential ∂F_1 and interpret (1.2) as a differential inclusion or as a variational inequality involving F_1 rather than F_1'. Furthermore, the function P occurring in (1.1) and (1.3) is nonnegative and smooth, and the terms on the right-hand sides in (1.4) are prescribed initial data.

The above system is a generalization of a system of PDEs that constitutes a relaxed version of a model for tumor growth originally introduced in [47] that was investigated in the papers [12–14] co-authored by the authors of this note and E. Rocca. In these works, we studied the special situation when $A^{2\rho} = B^{2\sigma} = C^{2\tau} = -\Delta$ with zero Neumann boundary conditions, and established general results concerning well-posedness, regularity, and optimal control. In particular, in [12, 13] a thorough asymptotic analysis, coupled with rigorous error estimates, was performed for the situation when the relaxation parameters $\alpha > 0$ and $\beta > 0$ approach zero, either separately or simultaneously. Notice also that in the case $P \equiv 0$ the equation (1.3) decouples from the other two equations (1.1), (1.2); the latter system of equations has for the case $\alpha = 0$ recently been the subject of a series of investigations by the present authors (cf. the papers [15–18]).

In this paper, we intend to perform a corresponding asymptotic analysis for the general system (1.1)–(1.4), where we take advantage of the well-posedness and regularity results that were established in our recent paper [19]. It will be demonstrated that for each of the three limit processes

$$\alpha \searrow 0\,,\, \beta > 0, \qquad \alpha > 0\,,\, \beta \searrow 0, \qquad \alpha \searrow 0\,, \beta \searrow 0,$$

meaningful limit problems occur for which the existence of solutions can be shown. In this analysis, it will turn out that each of the three limit processes needs specific assumptions for the fractional operators and the admissible nonlinearities. We will also address questions of uniqueness and continuous dependence, where, again, specific assumptions are necessary for the three cases.

Modeling the dynamics of tumor growth has recently become an important issue in applied mathematics (see, e.g., [23, 68]), and some different models have been introduced and discussed, numerical simulations have been provided and a comparison with the behavior of other special materials has been in order; for all that we just refer to, e.g., the works [2, 22, 23, 30, 33, 34, 46, 56, 69]. In particular, about diffuse interface models, we point out that these models mostly follow the Cahn–Hilliard framework (see [5]) that originated from the theory of phase transitions and is extensively employed in materials science and multiphase fluid flow. Among these models, two main classes can be categorized: the first one looks at the tumor and healthy cells as inertialess fluids and takes the effects generated by the fluid flow development into account by postulating a Darcy or a Brinkman law; in this direction, we refer to [24,27,36,37,40,41,43,49,53,63,67] (cf. [3,21,25,26,32,44,65,66] as well, where local or nonlocal Cahn–Hilliard systems with Darcy or Brinkman law are dealt with). Moreover, further mechanisms such as chemotaxis and active transport can be considered in the phenomenology. On the other hand, the second class of models, including the one from which the system (1.1)–(1.4) originates, actually neglects the velocity and admits as variables concentrations and chemical potential. A variety of contributions inside this class is provided by the works [6, 8, 11, 31, 38, 39, 42, 55, 57–60].

To our knowledge, except for the recent papers [19, 20], fractional operators have not been studied in either of these two groups of models, although one may also wonder about nonlocal operators. We point out that in recent years fractional operators provide a challenging subject for mathematicians: they have been successfully utilized in many different situations, and a wide literature already exists about equations and systems with fractional terms. For an overview of recent contributions, we refer to the papers [15, 16] and [9], which offer to the interested reader a number of suggestions to deepen the knowledge of the field. In our approach here, we adopt the setting of [19] and consequently work on fractional operators defined via spectral theory. This framework includes, in particular, powers of a second-order elliptic operator with either Dirichlet or Neumann or Robin homogeneous boundary conditions, and other operators like, e.g., fourth-order ones or systems involving the Stokes operator. A precise definition for our fractional operators $A^{2\rho}$, $B^{2\sigma}$, $C^{2\tau}$ along with their properties will be given in the first part of Section 2 below. As far as a biological background for the system (1.1)–(1.3) is concerned, we claim that in our approach the three fractional operators, which may be considerably different the one from the other, are employed for the dynamics of tumor growth and diffusion processes. The three operators $A^{2\rho}$, $B^{2\sigma}$, $C^{2\tau}$ are allowed to be a variation of fractional Laplacians, but also other elliptic operators, and may show different orders. Indeed, some components in tumor development, such as immune cells, exhibit an anomalous diffusion dynamics (as it observed in experiments [28]), but other components, like chemical potential and nutrient concentration are possibly governed by different fractional or non-fractional flows. However, taking all this into account, it is the case of pointing out that fractional operators are becoming more and more implemented in the field of biological applications: to this concern, a selection of notable and meaningful references is given by [1, 7, 28, 29, 45, 48, 50, 51, 54, 62, 64, 70].

The paper is organized as follows: in the next section, we list our assumptions and notations, and we state some results for the system (1.1)–(1.4) that are valid if both of the relaxation parameters are positive. The following sections then bring the asymptotic analysis as the parameters $\alpha > 0$ and $\beta > 0$ approach zero, where each of the relevant cases will be treated in a separate section. Throughout this paper, we make use of the elementary Young inequality

$$ab \le \gamma a^2 + \frac{1}{4\gamma}\, b^2 \quad \text{for every } a, b \in \mathbb{R} \text{ and } \gamma > 0. \tag{1.8}$$

Moreover, given a Banach space X, we denote by $\|\cdot\|_X$ its norm and by X^* its dual. The dual pairing between X^* and X is denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_X$. The only exception from this rule is the space $H := L^2(\Omega)$, for which $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$ denote the standard norm and inner product, respectively.

2 General assumptions and known results

In this section, we give precise assumptions and notations and state some results for the relaxed system where $\alpha > 0$ and $\beta > 0$. Now, we start introducing our assumptions. As for the operators, we first postulate that

(A1) $A : D(A) \subset H \to H$, $B : D(B) \subset H \to H$, and $C : D(C) \subset H \to H$, are unbounded, monotone, selfadjoint, linear operators with compact resolvents.

Therefore, there are sequences $\{\lambda_j\}$, $\{\lambda'_j\}$, $\{\lambda''_j\}$, and $\{e_j\}$, $\{e'_j\}$, $\{e''_j\}$, of eigenvalues and corresponding eigenfunctions such that

$$Ae_j = \lambda_j e_j, \quad Be'_j = \lambda'_j e'_j, \quad Ce''_j = \lambda''_j e''_j, \quad \text{with} \quad (e_i, e_j) = (e'_i, e'_j) = (e''_i, e''_j) = \delta_{ij}, \quad \text{for all } i, j \in \mathbb{N}, \tag{2.1}$$

$$0 \le \lambda_1 \le \lambda_2 \le \dots, \quad 0 \le \lambda'_1 \le \lambda'_2 \le \dots, \quad \text{and} \quad 0 \le \lambda''_1 \le \lambda''_2 \le \dots, \quad \text{where} \quad \lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \lambda'_j = \lim_{j\to\infty} \lambda''_j = +\infty, \tag{2.2}$$

$$\{e_j\}, \{e'_j\}, \text{ and } \{e''_j\}, \text{ are complete systems in } H. \tag{2.3}$$

As a consequence, we can define the powers of these operators with arbitrary positive real exponents as done below. As far as the first operator is concerned, we have for $\rho > 0$

$$V_A^\rho := D(A^\rho) = \Big\{ v \in H : \sum_{j=1}^\infty |\lambda_j^\rho (v, e_j)|^2 < +\infty \Big\} \quad \text{and} \tag{2.4}$$

$$A^\rho v = \sum_{j=1}^\infty \lambda_j^\rho (v, e_j) e_j \quad \text{for } v \in V_A^\rho, \tag{2.5}$$

the series being convergent in the strong topology of H, due to the properties (2.4) of the coefficients. We endow V_A^ρ with the graph norm, i.e., we set

$$(v, w)_{V_A^\rho} := (v, w) + (A^\rho v, A^\rho w) \quad \text{and} \quad \|v\|_{V_A^\rho} := (v, v)_{V_A^\rho}^{1/2} \quad \text{for } v, w \in V_A^\rho, \tag{2.6}$$

and obtain a Hilbert space. In the same way, we can define the powers B^σ and C^τ for every $\sigma > 0$ and $\tau > 0$, starting from (2.1)–(2.3) for B and C. We therefore set $V_B^\sigma := D(B^\sigma)$ and $V_C^\tau := D(C^\tau)$, endowed with the norms $\|\cdot\|_{V_B^\sigma}$ and $\|\cdot\|_{V_C^\tau}$ induced by the inner products

$$(v, w)_{V_B^\sigma} := (v, w) + (B^\sigma v, B^\sigma w) \quad \text{and} \quad (v, w)_{V_C^\tau} := (v, w) + (C^\tau v, C^\tau w), \quad \text{for } v, w \in V_B^\sigma \text{ and } v, w \in V_C^\tau, \text{ respectively.} \tag{2.7}$$

Since $\lambda_j \ge 0$ for every j, one immediately deduces from the definition of A^ρ that

$$\begin{aligned} &A^\rho : V_A^\rho \subset H \to H \quad \text{is maximal monotone, and} \\ &\varepsilon I + A^\rho : V_A^\rho \to H \quad \text{is for every } \varepsilon > 0 \text{ a topological isomorphism with the inverse} \\ &(\varepsilon I + A^\rho)^{-1} v = \sum_{j=0}^\infty \big(\varepsilon + \lambda_j^r\big)^{-1} (v, e_j) e_j \quad \text{for } v \in H, \end{aligned} \tag{2.8}$$

where $I : H \to H$ is the identity operator. Similar results hold for B^σ and C^τ. It is clear that, for every $\rho_1,\, \rho_2 > 0$, we have the Green type formula

$$(A^{\rho_1+\rho_2} v, w) = (A^{\rho_1} v, A^{\rho_2} w) \quad \text{for every } v \in V_A^{\rho_1+\rho_2} \text{ and } w \in V_A^{\rho_2}, \tag{2.9}$$

and that similar relations holds for the other two types of fractional operators. Due to these properties, we can define proper extensions of the operators that allow values in dual spaces. In particular, we can write variational formulations of the equation (1.1)–(1.3). It is convenient to use the notations

$$V_A^{-\rho} := (V_A^\rho)^*, \quad V_B^{-\sigma} := (V_B^\sigma)^*, \quad \text{and} \quad V_C^{-\tau} := (V_C^\tau)^*, \quad \text{for } \rho > 0,\, \sigma > 0,\, \tau > 0. \tag{2.10}$$

Thus, we have that

$$A^{2\rho} \in \mathcal{L}(V_A^{\rho}, V_A^{-\rho}), \quad B^{2\sigma} \in \mathcal{L}(V_B^{\sigma}, V_B^{-\sigma}), \quad \text{and} \quad C^{2\tau} \in \mathcal{L}(V_C^{\tau}, V_C^{-\tau}), \tag{2.11}$$

as well as

$$A^{\rho} \in \mathcal{L}(H, V_A^{-\rho}), \quad B^{\sigma} \in \mathcal{L}(H, V_B^{-\sigma}), \quad \text{and} \quad C^{\tau} \in \mathcal{L}(H, V_C^{-\tau}). \tag{2.12}$$

The symbols $\langle \,\cdot\,,\,\cdot\,\rangle_{V_A^\rho}$ and $\langle \,\cdot\,,\,\cdot\,\rangle_{V_C^\tau}$ will be used for the duality pairings between $V_A^{-\rho}$ and V_A^{ρ} and between $V_C^{-\tau}$ and V_C^{τ}, respectively. Moreover, we identify H with a subspace of $V_A^{-\rho}$ in the usual way, i.e., such that

$$\langle v, w\rangle_{V_A^\rho} = (v,w) \quad \text{for every } v \in H \text{ and } w \in V_A^{\rho}. \tag{2.13}$$

Analogously, we have that $H \subset V_B^{-\sigma}$ and $H \subset V_C^{-\rho}$ and use similar notations. Notice (see, e.g., [15, Sect. 3]) that all of the embeddings

$$V_A^{r_2} \subset V_A^{r_1} \subset H, \quad \text{for } 0 < r_1 < r_2, \tag{2.14}$$

$$H \subset V_A^{-r_1} \subset V_A^{-r_2}, \quad \text{for } 0 < r_1 < r_2, \tag{2.15}$$

$$V_B^{\sigma_2} \subset V_B^{\sigma_1} \subset H, \quad \text{for } 0 < \sigma_1 < \sigma_2, \tag{2.16}$$

$$V_C^{\tau_2} \subset V_C^{\tau_1} \subset H, \quad \text{for } 0 < \tau_1 < \tau_2, \tag{2.17}$$

are dense and compact.

From now on, we assume:

(A2) ρ, σ and τ are fixed positive real numbers.

For the nonlinear functions entering the equations (1.1)–(1.3) of our system, we postulate the properties listed below:

(A3) $F = F_1 + F_2$, where:

$$F_1 : \mathbb{R} \to [0, +\infty] \quad \text{is convex and lower semicontinuous with} \quad F_1(0) = 0. \tag{2.18}$$

$$F_2 \in C^1(\mathbb{R}), \quad \text{and} \quad F_2' \quad \text{is Lipschitz continuous with Lipschitz constant } L > 0. \tag{2.19}$$

$$F(s) \geq \overline{c}_1 s^2 - \overline{c}_2 \quad \text{for some positive constants } \overline{c}_1 \text{ and } \overline{c}_2 \text{ and every } s \in \mathbb{R}. \tag{2.20}$$

$$P : \mathbb{R} \to [0, +\infty) \quad \text{is bounded and Lipschitz continuous.} \tag{2.21}$$

We set, for convenience,

$$f_1 := \partial F_1 \quad \text{and} \quad f_2 := F_2'\,, \tag{2.22}$$

and denote by $D(F_1)$ and $D(f_1)$ the effective domain of F_1 and f_1, respectively. We notice that f_1 is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ and use the same symbol f_1 for the maximal monotone operators induced in L^2 spaces. For every $s \in D(f_1)$, we denote by $f_1^\circ(s)$ the element of minimal modulus in $f_1(s)$. Moreover, if the subdifferential $\partial F_1(s)$ is a singleton for every $s \in D(f_1)$ (which is, e.g., the case if $F_1 \in C^1(\mathbb{R})$), then we identify the singleton $\{f_1(s)\}$ with the real number $f_1(s)$ and treat the mapping $s \mapsto f_1(s)$ as a real-valued function without further comment.

Using (2.9) and its analogues for B and C, we can give a weak formulation of the equations (1.1)–(1.3). Moreover, we present (1.2) as a variational inequality. For the data, we make the following assumptions:

(A4) $\mu_0 \in H, \quad \varphi_0 \in V_B^\sigma$ with $F_1(\varphi_0) \in L^1(\Omega)$, and $S_0 \in H$.

By assuming $\alpha \geq 0$ and $\beta \geq 0$, we then look for a triple (μ, φ, S) satisfying

$$\mu \in L^2(0,T;V_A^\rho), \tag{2.23}$$

$$\varphi \in L^\infty(0,T;V_B^\sigma), \quad \beta\,\partial_t\varphi \in L^2(0,T;H), \tag{2.24}$$

$$\partial_t(\alpha\mu + \varphi) \in L^2(0,T;V_A^{-\rho}), \tag{2.25}$$

$$S \in H^1(0,T;V_C^{-\tau}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_C^\tau), \tag{2.26}$$

$$F_1(\varphi) \in L^1(Q), \tag{2.27}$$

and solving the system

$$\langle\, \partial_t(\alpha\,\mu(t) + \varphi(t)), v\,\rangle_{V_A^\rho} + (A^\rho\mu(t), A^\rho v) = \bigl(P(\varphi(t))(S(t) - \mu(t)), v\bigr)$$
$$\text{for every } v \in V_A^\rho \text{ and for a.e. } t \in (0,T), \tag{2.28}$$

$$\bigl(\beta\,\partial_t\varphi(t), \varphi(t) - v\bigr) + \bigl(B^\sigma\varphi(t), B^\sigma(\varphi(t) - v)\bigr)$$
$$+ \int_\Omega F_1(\varphi(t)) + \bigl(f_2(\varphi(t)), \varphi(t) - v\bigr) \leq \bigl(\mu(t), \varphi(t) - v\bigr) + \int_\Omega F_1(v)$$
$$\text{for every } v \in V_B^\sigma \text{ and for a.e. } t \in (0,T), \tag{2.29}$$

$$\langle\, \partial_t S(t), v\,\rangle_{V_C^\tau} + (C^\tau S(t), C^\tau v) = -\bigl(P(\varphi(t))(S(t) - \mu(t)), v\bigr)$$
$$\text{for every } v \in V_C^\tau \text{ and for a.e. } t \in (0,T), \tag{2.30}$$

$$(\alpha\mu + \varphi)(0) = \alpha\mu_0 + \varphi_0\,, \quad (\beta\varphi)(0) = \beta\varphi_0\,, \quad \text{and} \quad S(0) = S_0\,. \tag{2.31}$$

Here, it is understood that $\int_\Omega F_1(v) = +\infty$ whenever $F_1(v) \notin L^1(\Omega)$.

Remark 2.1. The above formulation is meaningful for nonnegative coefficients α and β. This holds, in particular, for (2.31). However, depending on whether these coefficients are positive or zero, the initial conditions can be reformulated in a more explicit way, namely,

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \quad \text{and} \quad S(0) = S_0, \qquad \text{if } \alpha > 0 \text{ and } \beta > 0, \tag{2.32}$$

$$(\alpha\mu + \varphi)(0) = \alpha\mu_0 + \varphi_0, \quad \text{and} \quad S(0) = S_0, \qquad \text{if } \alpha > 0 \text{ and } \beta = 0, \tag{2.33}$$

$$\varphi(0) = \varphi_0, \quad \text{and} \quad S(0) = S_0, \qquad \text{if } \alpha = 0 \text{ and } \beta \geq 0. \tag{2.34}$$

Observe that (2.28)–(2.30) are equivalent to their time-integrated variants, in particular for (2.29) we have

$$\int_0^T \bigl(\beta\,\partial_t\varphi(t), \varphi(t) - v(t)\bigr)\,dt + \int_0^T \bigl(B^\sigma\varphi(t), B^\sigma(\varphi(t) - v(t))\bigr)\,dt$$
$$+ \int_Q F_1(\varphi) + \int_0^T \bigl(f_2(\varphi(t)), \varphi(t) - v(t)\bigr)\,dt$$
$$\leq \int_0^T \bigl(\mu(t), \varphi(t) - v(t)\bigr)\,dt + \int_Q F_1(v) \quad \text{for every } v \in L^2(0,T;V_B^\sigma), \tag{2.35}$$

where we put $\int_Q F_1(v) = +\infty$ whenever $F_1(v) \notin L^1(Q)$.

The following result was proved in [19, Thms. 2.3 and 2.5]:

Theorem 2.2. *Let the assumptions* **(A1)–(A4)** *be fulfilled, and assume that $\alpha > 0$ and $\beta > 0$. Then there exists a triple (μ, φ, S) with the regularity* (2.23)–(2.27) *that solves the problem* (2.28)–(2.30) *and the initial conditions* (2.32)*. Moreover, this solution satisfies the estimate*

$$\begin{aligned}
&\|\partial_t(\alpha\,\mu+\varphi)\|_{L^2(0,T;V_A^{-\rho})} + \alpha^{1/2}\|\mu\|_{L^\infty(0,T;H)} + \|A^\rho\mu\|_{L^2(0,T;H)} \\
&+ \beta^{1/2}\,\|\partial_t\varphi\|_{L^2(0,T;H)} + \|\varphi\|_{L^\infty(0,T;V_B^\sigma)} + \|F(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} \\
&+ \|S\|_{H^1(0,T;V_C^{-\tau})\cap C^0([0,T];H)\cap L^2(0,T;V_C^\tau)} + \|P^{1/2}(\varphi)(S-\mu)\|_{L^2(0,T;H)} \\
&\le \widehat{K}_1\left(\alpha^{1/2}\|\mu_0\| + \|B^\sigma\varphi_0\| + \|F(\varphi_0)\|_{L^1(\Omega)} + \|S_0\| + 1\right),
\end{aligned} \tag{2.36}$$

with a constant $\widehat{K}_1 > 0$ that depends only on Ω, the constants $\overline{c}_1$ and $\overline{c}_2$ from (2.20)*, and P. If, in addition, the condition*

$$\mu_0 \in V_A^\rho, \quad \varphi_0 \in V_B^{2\sigma} \ \text{ with } \ f_1^\circ(\varphi_0) \in H, \quad S_0 \in V_C^\tau, \tag{2.37}$$

is fulfilled, then the above solution enjoys the further regularity

$$\mu \in H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \cap L^2(0,T;V_A^{2\rho}), \tag{2.38}$$
$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma), \tag{2.39}$$
$$S \in H^1(0,T;H) \cap L^\infty(0,T;V_C^\tau) \cap L^2(0,T;V_C^{2\tau}). \tag{2.40}$$

Moreover, if the embedding conditions

$$V_A^\rho \subset L^4(\Omega) \quad \text{and} \quad V_C^\tau \in L^4(\Omega) \tag{2.41}$$

are fulfilled, then the above solution is uniquely determined.

Remark 2.3. The first embedding in (2.41) is, for instance, satisfied if $A^{2\rho} = A : = -\Delta$ with the domain $H^2(\Omega)\cap H_0^1(\Omega)$ (thus, with zero Dirichlet conditions, but similarly for zero Neumann boundary conditions with domain $\{v \in H^2(\Omega) :\ \partial_{\mathbf{n}} v = 0 \text{ on } \Gamma\}$). Indeed, we have $V_A^\rho = H_0^1(\Omega)$ in this case. Clearly, the same embedding holds true if ρ is sufficiently close to $1/2$.

Remark 2.4. More generally, we could add known forcing terms u_μ, u_φ and u_S to the right-hand sides of equations (1.1), (1.2) and (1.3), respectively, and accordingly modify the definition of solution. If we assume that

$$u_\mu,\ u_\varphi\,, u_S \in L^2(0,T;H)\,, \tag{2.42}$$

then we have a similar well-posedness result. In estimate (2.36), one has to modify the right-hand side by adding the norms corresponding to (2.42) (possibly multiplied by negative powers of α and β). This remark is useful for performing a control theory of the above system with distributed controls.

Remark 2.5. We cannot repeat the proof given in [19], here. We only note for later use that the result is achieved by approximation using the Moreau–Yosida regularizations F_1^λ and f_1^λ of F_1 of f_1 at the level $\lambda > 0$ introduced in, e.g., [4, p. 28 and p. 39]. We set, for convenience,

$$F^\lambda := F_1^\lambda + F_2 \quad \text{and} \quad f^\lambda := f_1^\lambda + f_2\,. \tag{2.43}$$

Denoting by $J^\lambda := (\mathrm{id} + \lambda\, f_1)^{-1}$ (where id : $\mathbb{R} \to \mathbb{R}$ is the identity mapping) the resolvent mapping associated with the maximal monotone graph f_1 for $\lambda > 0$, we recall some well-known properties of this regularization, namely,

$$F_1^\lambda(s) = \int_0^s f_1^\lambda(s')\, ds' , \quad 0 \le F_1(J^\lambda(s)) \le F_1^\lambda(s) \le F_1(s) \quad \text{for every } s \in \mathbb{R}, \tag{2.44}$$

$$f_1^\lambda(s) \in f_1(J^\lambda(s)) \quad \text{for every } s \in \mathbb{R}\,, \tag{2.45}$$

and it follows from (2.20) that there are constants $\widehat{c}_1 > 0$, $\widehat{c}_2 > 0$, and $\Lambda > 0$, such that, for all $\lambda \in (0, \Lambda)$, we have

$$F^\lambda(s) \ge \widehat{c}_1\, s^2 \,-\, \widehat{c}_2 \quad \text{for every } s \in \mathbb{R}\,. \tag{2.46}$$

In the following, we always tacitly assume that $0 < \lambda < \Lambda$ when working with Moreau–Yosida approximations.

Now, we replace F_1 in (2.29) by F_1^λ to obtain the system

$$\alpha \, \langle \partial_t \mu^\lambda(t), v \rangle_{V_A^\rho} + \big(\partial_t \varphi^\lambda(t), v\big) + (A^\rho \mu^\lambda(t), A^\rho v) = \big(P(\varphi^\lambda(t))(S^\lambda(t) - \mu^\lambda(t)), v\big)$$
$$\text{for every } v \in V_A^\rho \text{ and for a.e. } t \in (0,T), \tag{2.47}$$

$$\beta \,\big(\partial_t \varphi^\lambda(t), \varphi^\lambda(t) - v\big) + \big(B^\sigma \varphi^\lambda(t), B^\sigma(\varphi^\lambda(t) - v)\big)$$
$$+ \int_\Omega F_1^\lambda(\varphi^\lambda(t)) + \big(f_2(\varphi^\lambda(t)), \varphi^\lambda(t) - v\big)$$
$$\le \big(\mu^\lambda(t), \varphi^\lambda(t) - v\big) + \int_\Omega F_1^\lambda(v) \quad \text{for every } v \in V_B^\sigma \text{ and for a.e. } t \in (0,T), \tag{2.48}$$

$$\langle \partial_t S^\lambda(t), v \rangle_{V_C^\tau} + (C^\tau S^\lambda(t), C^\tau v) = -\big(P(\varphi^\lambda(t))(S^\lambda(t) - \mu^\lambda(t)), v\big)$$
$$\text{for every } v \in V_C^\tau \text{ and for a.e. } t \in (0,T), \tag{2.49}$$

$$\mu^\lambda(0) = \mu_0\,, \quad \varphi^\lambda(0) = \varphi_0\,, \quad \text{and} \quad S^\lambda(0) = S_0\,. \tag{2.50}$$

Observe that (2.48) is equivalent to both its time-integrated analogue and the pointwise variational equation (since F_1^λ is differentiable and f_1^λ is its globally Lipschitz continuous derivative)

$$\beta\big(\partial_t \varphi^\lambda(t), v\big) + \big(B^\sigma \varphi^\lambda(t), B^\sigma v\big) + \big(f^\lambda(\varphi^\lambda(t)), v\big) = \big(\mu^\lambda(t), v\big)$$
$$\text{for every } v \in V_B^\sigma \text{ and for a.e. } t \in (0,T). \tag{2.51}$$

In the proof of [19, Thm. 2.3], it was shown under slightly weaker assumptions on F that the system (2.47), (2.49)–(2.51) has for every $\lambda \in (0,\Lambda)$ a unique solution triple $(\mu^\lambda, \varphi^\lambda, S^\lambda)$ satisfying (2.23)–(2.26) and the estimate

$$\big\|\partial_t(\alpha\, \mu^\lambda + \varphi^\lambda)\big\|_{L^2(0,T;V_A^{-\rho})} \,+\, \alpha^{1/2}\,\big\|\mu^\lambda\big\|_{L^\infty(0,T;H)} \,+\, \big\|A^\rho \mu^\lambda\big\|_{L^2(0,T;H)}$$
$$+\, \beta^{1/2}\,\big\|\partial_t \varphi^\lambda\big\|_{L^2(0,T;H)} \,+\, \big\|B^\sigma(\varphi^\lambda)\big\|_{L^\infty(0,T;H)} \,+\, \big\|F^\lambda(\varphi^\lambda) + C_0\big\|_{L^\infty(0,T;L^1(\Omega))}$$
$$+\, \big\|S^\lambda\big\|_{H^1(0,T;V_C^{-\tau})\cap L^\infty(0,T;H)\cap L^2(0,T;V_C^\tau)} \,+\, \big\|P^{1/2}(\varphi^\lambda)(S^\lambda - \mu^\lambda)\big\|_{L^2(0,T;H)} \,\le\, \widehat{C}_1\,, \tag{2.52}$$

where the constant $\widehat{C}_1 > 0$ is independent of $\alpha\,, \beta\,, \lambda$ and has the same structure as the right-hand side of (2.36), and where $C_0 > 0$ is a constant such that $F^\lambda(s) + C_0 \ge 0$ for all $s \in \mathbb{R}$. Owing to (2.46), we may take $C_0 = \widehat{c}_2$ in our case. A fortiori, (2.46) and (2.52) imply that, by choosing a possibly larger $\widehat{C}_1$, we may assume that

$$\|\varphi^\lambda\|_{L^\infty(0,T;V_B^\sigma)} \le \widehat{C}_1 . \tag{2.53}$$

Since, by the global Lipschitz continuity of f_2, the nonlinearity F_2 grows at most quadratically, we then can also infer the bounds

$$\|F_1^\lambda(\varphi^\lambda)\|_{L^\infty(0,T;L^1(\Omega))} + \|F_2(\varphi^\lambda)\|_{L^\infty(0,T;L^1(\Omega))} \le \widehat{C}_1 . \tag{2.54}$$

The existence result and the global bound (2.36) then follow from a passage to the limit as $\lambda \searrow 0$ in the system (2.47)–(2.50) and in (2.52).

In the general case, the equation for φ is just the variational inequality (2.29), and we cannot write anything that is similar to (1.2), since no estimate for $f_1(\varphi)$ is available. However, if one reinforces the assumptions on the structure, then one can recover (1.2) at least as a differential inclusion. The crucial condition is the following:

$$\psi(v) \in H \quad \text{and} \quad \big(B^{2\sigma}v, \psi(v)\big) \ge 0, \quad \text{for every } v \in V_B^{2\sigma} \text{ and every monotone}$$
$$\text{and Lipschitz continuous function } \psi : \mathbb{R} \to \mathbb{R} \text{ vanishing at the origin.} \tag{2.55}$$

We notice that this assumption is fulfilled if $B^{2\sigma} = -\Delta$ with zero Neumann boundary conditions. Indeed, in this case it results that $V_B^{2\sigma} = \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0\}$ and, for every ψ as in (2.55) and $v \in V_B^{2\sigma}$, we have that $\psi(v) \in H^1(\Omega)$ (since $v \in H^1(\Omega)$) and

$$\big(B^{2\sigma}v, \psi(v)\big) = \int_\Omega (-\Delta v)\, \psi(v) = \int_\Omega \nabla v \cdot \nabla \psi(v) = \int_\Omega \psi'(v)|\nabla v|^2 \ge 0.$$

More generally, in place of the Laplace operator, we can take the principal part of an elliptic operator in divergence form with Lipschitz continuous coefficients, provided that the normal derivative is replaced by the conormal derivative. In any case, we can take the Dirichlet boundary conditions instead of the Neumann boundary conditions, since the functions ψ for which (2.55) is required satisfy $\psi(0) = 0$.

The following result has been proved in [19, Thm. 2.6].

Theorem 2.6. *Let the assumptions* **(A1)**–**(A4)** *be fulfilled, and assume that* $\alpha > 0$ *and* $\beta > 0$. *If, in addition,* (2.55) *is satisfied, then there exist a solution* (μ, φ, S) *to the problem* (2.28)–(2.31) *and some* ξ *such that*

$$\varphi \in L^2(0,T;V_B^{2\sigma}) \quad \textit{and} \quad \xi \in L^2(0,T;H), \tag{2.56}$$

$$\beta\, \partial_t\varphi + B^{2\sigma}\varphi + \xi + f_2(\varphi) = \mu \quad \textit{and} \quad \xi \in f_1(\varphi) \quad \textit{a.e. in } Q. \tag{2.57}$$

Moreover, also ξ *is unique if* (2.41) *holds true, and if we also assume that the condition* (2.37) *is valid, then the unique solution* (μ, φ, S) *and the associated* ξ *satisfy* (2.38)–(2.40) *as well as*

$$\varphi \in L^\infty(0,T;V_B^{2\sigma}) \quad \textit{and} \quad \xi \in L^\infty(0,T;H). \tag{2.58}$$

We conclude our preparations with a technical lemma that relates to each other the solutions to (2.28)–(2.31) for different pairs (α_i, β_i), $i = 1, 2$.

Lemma 2.7. *Suppose that* **(A1)–(A4)** *are fulfilled, and let* $(\mu_{\alpha_i,\beta_i}, \varphi_{\alpha_i,\beta_i}, S_{\alpha_i,\beta_i})$ *be solutions to* (2.28)–(2.31) *in the sense of Theorem 2.2 for the parameters* $(\alpha_i, \beta_i) \in (0,1]$, $i = 1,2$. *Then there is some* $\widehat{M} > 0$, *which only depends on the global constant*

$$\widehat{K}_1 \left(\alpha^{1/2}\|\mu_0\| \,+\, \|B^\sigma\varphi_0\| \,+\, \|F(\varphi_0)\|_{L^1(\Omega)} \,+\, \|S_0\| + 1\right)$$

in the right-hand side of (2.36), *such that, for every* $t \in (0,T)$ *and every* $\delta > 0$, *we have*

$$\begin{aligned}(1-\alpha_1 L-\delta)\int_{Q_t}\big|\varphi_{\alpha_1,\beta_1}-\varphi_{\alpha_2,\beta_2}\big|^2 &\le \frac{1}{4\delta}\int_{Q_t}\big|(\alpha_1\mu_{\alpha_1,\beta_1}+\varphi_{\alpha_1,\beta_1})-(\alpha_2\mu_{\alpha_2,\beta_2}+\varphi_{\alpha_2,\beta_2})\big|^2\\ &\quad+\widehat{M}\big|\alpha_1-\alpha_2\big| \,+\, \alpha_1\big|\beta_1-\beta_2\big|\int_0^t\|\partial_t\varphi_{\alpha_2,\beta_2}(s)\|\,\|\varphi_{\alpha_1,\beta_1}(s)-\varphi_{\alpha_2,\beta_2}(s)\|\,ds\,. \qquad (2.59)\end{aligned}$$

Proof. For convenience, we set $\varphi_i := \varphi_{\alpha_i,\beta_i}$, $\mu_i := \mu_{\alpha_i,\beta_i}$, for $i = 1,2$. Then we multiply (2.29), written for $\beta_1, \mu_1, \varphi_1$, by α_1, insert $v = \varphi_2(t)$, and add the term $(\varphi_1(t), \varphi_1(t)-\varphi_2(t))$ to both sides of the resulting inequality. We then obtain, almost everywhere in $(0,T)$, the inequality

$$\begin{aligned}&(\alpha_1\beta_1\,\partial_t\varphi_1,\varphi_1-\varphi_2) \,+\, \alpha_1(B^\sigma\varphi_1,B^\sigma(\varphi_1-\varphi_2)) \,+\, (\varphi_1,\varphi_1-\varphi_2)\\ &\le\, (\alpha_1\,\mu_1+\varphi_1,\varphi_1-\varphi_2) \,-\, (\alpha_1\,f_2(\varphi_1),\varphi_1-\varphi_2) \,+\, \alpha_1\int_\Omega(F_1(\varphi_2)-F_1(\varphi_1)).\end{aligned}$$

Similarly, arguing on the inequality for $\beta_2, \mu_2, \varphi_2$, we get

$$\begin{aligned}&(\alpha_2\beta_2\,\partial_t\varphi_2,\varphi_2-\varphi_1) \,+\, \alpha_2(B^\sigma\varphi_2,B^\sigma(\varphi_2-\varphi_1)) \,+\, (\varphi_2,\varphi_2-\varphi_1)\\ &\le\, (\alpha_2\,\mu_2+\varphi_2,\varphi_2-\varphi_1) \,-\, (\alpha_2\,f_2(\varphi_2),\varphi_2-\varphi_1) \,+\, \alpha_2\int_\Omega(F_1(\varphi_1)-F_1(\varphi_2)).\end{aligned}$$

Adding the two inequalities, and rearranging terms, we find that almost everywhere in $(0,T)$ it holds the inequality

$$\begin{aligned}&\big(\alpha_1\beta_1\,\partial_t\varphi_1-\alpha_2\beta_2\,\partial_t\varphi_2,\varphi_1-\varphi_2\big) \,+\, \|\varphi_1-\varphi_2\|^2 \,+\, \alpha_1\big\|B^\sigma(\varphi_1-\varphi_2)\big\|^2\\ &\le\, \big((\alpha_1\mu_1+\varphi_1)-(\alpha_2\mu_2+\varphi_2),\varphi_1-\varphi_2\big) \,-\, (\alpha_1-\alpha_2)(B^\sigma\varphi_2,B^\sigma(\varphi_1-\varphi_2))\\ &\quad-\, \alpha_1(f_2(\varphi_1)-f_2(\varphi_2),\varphi_1-\varphi_2) \,-\, (\alpha_1-\alpha_2)(f_2(\varphi_2),\varphi_1-\varphi_2)\\ &\quad-\, (\alpha_1-\alpha_2)\int_\Omega(F_1(\varphi_1)-F_1(\varphi_2))\,. \qquad (2.60)\end{aligned}$$

Now, recalling (2.19), we see that

$$-\,\alpha_1(f_2(\varphi_1)-f_2(\varphi_2),\varphi_1-\varphi_2) \,\le\, \alpha_1 L\|\varphi_1-\varphi_2\|^2\,. \qquad (2.61)$$

Moreover, we have the identity

$$\begin{aligned}&\big(\alpha_1\beta_1\,\partial_t\varphi_1-\alpha_2\beta_2\,\partial_t\varphi_2,\varphi_1-\varphi_2\big)\\ &=\, \frac{\alpha_1\beta_1}{2}\,\frac{d}{dt}\,\|\varphi_1-\varphi_2\|^2 \,+\, \big((\alpha_1-\alpha_2)\beta_2 \,+\, \alpha_1(\beta_1-\beta_2)\big)(\partial_t\varphi_2,\varphi_1-\varphi_2)\,. \qquad (2.62)\end{aligned}$$

At this point, we integrate the inequality (2.60) over $(0,t)$. Omitting two nonnegative terms on the left-hand side, invoking (2.61) and (2.62), and applying the Cauchy–Schwarz and Young inequalities, we find that

$$
\begin{aligned}
&(1-\alpha_1 L)\int_{Q_t} |\varphi_1-\varphi_2|^2 \,\le\, \delta\int_{Q_t}|\varphi_1-\varphi_2|^2 \,+\, \frac{1}{4\delta}\int_{Q_t}|(\alpha_1\mu_1+\varphi_1)-(\alpha_2\mu_2+\varphi_2)|^2\\
&+\,|\alpha_1-\alpha_2|\Big(\int_0^t \|B^\sigma\varphi_2(s)\|\,\|B^\sigma(\varphi_1(s)-\varphi_2(s))\|\,ds \,+\, \int_{Q_t}\big(F_1(\varphi_1)+F_1(\varphi_2)\big)\\
&\qquad\qquad +\,\int_0^t \big(\|f_2(\varphi_2(s))\| \,+\, \beta_2\|\partial_t\varphi_2(s)\|\big)\,\|\varphi_1(s)-\varphi_2(s)\|\,ds\Big)\\
&+\,\alpha_1\,|\beta_1-\beta_2|\int_0^t \|\partial_t\varphi_2(s)\|\,\|\varphi_1(s)-\varphi_2(s)\|\,ds\,.
\end{aligned}
\tag{2.63}
$$

Finally, observe that the expression in the bracket multiplying $|\alpha_1-\alpha_2|$ is, owing to (2.36), bounded in terms of the constant $\widehat{K}_1$. From this, the assertion follows. □

3 The case $\alpha \searrow 0,\ \beta > 0$.

In order to indicate their dependence on the parameters $\alpha,\ \beta$, we denote in the following solution triples of the problem (2.28)–(2.31) by $(\mu_{\alpha,\beta},\varphi_{\alpha,\beta},S_{\alpha,\beta})$, for $\alpha,\beta\in[0,1)$. In this section, we investigate their asymptotic behavior as $\alpha\searrow 0$ and $\beta>0$. Obviously, the main difficulty in the limit processes is to pass through the limit in the nonlinearities, which requires a strong convergence of the arguments $\varphi_{\alpha,\beta}$, in particular. Denoting in the following by $\mathbf{1}$ both the functions that are identically equal to unity on Ω or Q, we assume, in addition to the general assumptions **(A1)**–**(A4)**:

(A5) At least one of the following three conditions is satisfied:

(i) λ_1 is positive.

(ii) $P(s)\ge P_0$ for all $s\in\mathbb{R}$ and some fixed $P_0>0$.

(iii) $\lambda_1=0$ is a simple eigenvalue of A and $\mathbf{1}$ is an eigenfunction belonging to V_B^σ; moreover, $D(F_1)=\mathbb{R}$, and there are constants $\widehat{c}_3>0$ and $\widehat{c}_4\ge 0$ such that

$$
|s'| \,\le\, \widehat{c}_3\,F_1(s)+\widehat{c}_4 \quad \text{whenever } s\in\mathbb{R} \text{ and } s'\in f_1(s). \tag{3.1}
$$

Remark 3.1. The condition **(A5)**,(i) is satisfied by the standard second-order elliptic operators with zero Dirichlet boundary conditions (however, also zero mixed and Robin boundary conditions can be considered, with proper definitions of the domains of the operators). The case **(A5)**,(ii) is, unfortunately, not too realistic in the practical application to tumor growth models, in which, usually, P should also attain the value zero. Finally, we comment on **(A5)**,(iii). The condition $\lambda_1=0$ is satisfied, e.g., if A is the Laplace operator $-\Delta$ with zero Neumann boundary conditions. Furthermore, in this case, the eigenvalue $\lambda_1=0$ is simple, and the corresponding eigenfunctions are constants, since Ω is supposed to be connected. Furthermore, we have $\mathbf{1}\in V_B^\sigma$ for many standard elliptic operators with zero Neumann boundary conditions (and even with zero Dirichlet boundary conditions if σ is small, for instance, if $B=-\Delta$ with $D(B)=H^2(\Omega)\cap H_0^1(\Omega)$ and $\sigma<1/4$). Moreover, the condition (3.1) excludes the logarithmic and double obstacle potentials, but it still allows f_1 to be multi-valued, since it does not require that F_1 is differentiable; it is, however, satisfied for a wide class of smooth potentials of polynomial (and even first-order exponential) type such as F_{reg}.

Remark 3.2. Clearly, we have that

$$\|A^\rho v\|^2 = \sum_{j=1}^\infty |\lambda_j^\rho (v,e_j)|^2 \ge \lambda_1^{2\rho} \sum_{j=1}^\infty |(v,e_j)|^2 = \lambda_1^{2\rho}\,\|v\|^2 \quad \text{for every } v \in V_A^\rho .$$

Hence, in the case **(A5)**(i) in which $\lambda_1 > 0$, the function $v \mapsto \|A^\rho v\|$ is a norm on V_A^r that is equivalent to (2.6). On the contrary, in the case **(A5)**(iii), we have $\lambda_1 = 0$ and the above function is just a seminorm on V_A^ρ. However, the assumptions that $\lambda_1 = 0$ is a simple eigenvalue and that the eigenfunctions are constants imply the Poincaré type inequality (cf. [15, Eq. (3.5)])

$$\|v\| \le \widehat{c}\,\|A^\rho v\| \quad \text{for some } \widehat{c} > 0, \text{ for every } v \in V_A^\rho \text{ with } \mathrm{mean}(v) = 0, \tag{3.2}$$

where $\mathrm{mean}(v)$ denotes the mean value of v. Then, a standard argument based on (3.2) and the compactness of the embedding $V_A^\rho \subset H$ yield that the mapping

$$v \mapsto |\mathrm{mean}(v)| + \|A^\rho v\| \quad \text{for } v \in V_A^\rho, \tag{3.3}$$

defines a norm on V_A^ρ which is equivalent to $\|\cdot\|_{V_A^\rho}$.

In the following, we denote by C_i, $i \in \mathbb{N}$, positive constants that may depend on the data of the system but not on the parameters α, β, λ. We suppose that $\beta > 0$ is fixed and $\{\alpha_n\}$ is any sequence satisfying $\alpha_n \searrow 0$. In view of the global bounds (2.36), we may without loss of generality assume that there are functions $\zeta, \xi, \mu_{0,\beta}, \varphi_{0,\beta}, S_{0,\beta}$ such that, as $n \to \infty$,

$$\alpha_n\,\mu_{\alpha_n,\beta} \to 0 \quad \text{strongly in } L^\infty(0,T;H), \tag{3.4}$$

$$\partial_t\big(\alpha_n\,\mu_{\alpha_n,\beta} + \varphi_{\alpha_n,\beta}\big) \to \zeta \quad \text{weakly in } L^2(0,T,V_A^{-\rho}), \tag{3.5}$$

$$A^\rho \mu_{\alpha_n,\beta} \to \xi \quad \text{weakly in } L^2(0,T;H), \tag{3.6}$$

$$\varphi_{\alpha_n,\beta} \to \varphi_{0,\beta} \quad \text{weakly-star in } H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma), \tag{3.7}$$

$$S_{\alpha_n,\beta} \to S_{0,\beta} \quad \text{weakly-star in } H^1(0,T;V_C^{-\tau}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_C^\tau). \tag{3.8}$$

Obviously, (3.4), (3.5) and (3.7) imply that $\zeta = \partial_t \varphi_{0,\beta}$. We now claim that the condition **(A5)** implies that, at least for a subsequence,

$$\mu_{\alpha_n,\beta} \to \mu_{0,\beta} \quad \text{weakly in } L^2(0,T;V_A^\rho), \tag{3.9}$$

which entails, in particular, that $\xi = A^\rho \mu_{0,\beta}$.

This follows directly if $\lambda_1 > 0$: indeed, as observed in Remark 3.2, the mapping $v \mapsto \|A^\rho v\|$ defines a norm on V_A^ρ which is equivalent to $\|\cdot\|_{V_A^\rho}$ in this case, and thus the boundedness of $\{\|A^\rho \mu_{\alpha_n,\beta}\|_{L^2(0,T;H)}\}_{n\in\mathbb{N}}$ entails that (3.9) holds true at least for a subsequence.

Suppose next that $\lambda_1 = 0$ and that **(A5)**,(ii) is fulfilled. Then we can test the equation (2.47) in the Moreau–Yosida approximation, written at the time s, by $v = \mu^\lambda_{\alpha_n,\beta}(s)$ and integrate over $(0,t)$ where $t \in (0,T]$. We then obtain the inequality

$$\begin{aligned} &\frac{\alpha_n}{2}\,\big\|\mu^\lambda_{\alpha_n,\beta}(t)\big\|^2 + \int_0^t \big(\|A^\rho \mu^\lambda_{\alpha_n,\beta}(s)\|^2 + P_0\,\|\mu^\lambda_{\alpha_n,\beta}(s)\|^2\big)ds \\ &\le \frac{\alpha_n}{2}\,\|\mu_0\|^2 + \int_0^t\!\!\int_\Omega \big(P(\varphi^\lambda_{\alpha_n,\beta})\,S^\lambda_{\alpha_n,\beta} - \partial_t \varphi^\lambda_{\alpha_n,\beta}\big)\,\mu^\lambda_{\alpha_n,\beta}\,. \end{aligned} \tag{3.10}$$

Invoking the global bounds (2.52) and Young's inequality, we readily see that the right-hand side is bounded by an expression of the form

$$\frac{P_0}{2} \int_0^t \|\mu^\lambda_{\alpha_n,\beta}(s)\|^2 \, ds \, + \, C_1 \, .$$

Therefore, it turns out that $\left\|\mu^\lambda_{\alpha_n,\beta}\right\|_{L^2(0,T;V_A^\rho)} \le C_2$. Letting $\lambda \searrow 0$, and invoking the semicontinuity of norms, we then conclude that

$$\left\|\mu_{\alpha_n,\beta}\right\|_{L^2(0,T;V_A^\rho)} \le C_2 \quad \forall \, n \in \mathbb{N}, \tag{3.11}$$

which yields the validity of (3.9) on a subsequence also in this case.

It remains to show (3.9) if $\lambda_1 = 0$ and **(A5)**,(iii) is satisfied. We recall that (2.45) yields $f_1^\lambda(s) \in f_1(J^\lambda(s))$ for all $s \in \mathbb{R}$ so that we can apply (3.1) with s replaced by $J^\lambda(s)$ and $s' = f_1^\lambda(s)$. Hence, by also using (2.44), we find for every $\lambda \in \mathbb{R}$ and $s \in \mathbb{R}$ the chain of inequalities

$$|f_1^\lambda(s)| \le \widehat{c}_3 \, F_1(J^\lambda(s)) + \widehat{c}_4 \le \widehat{c}_3 \, F_1^\lambda(s) + \widehat{c}_4 \, . \tag{3.12}$$

Now recall that f_2 is globally Lipschitz continuous on $\mathbb{R}$, whence it follows that f_2 grows at most linearly and F_2 grows at most quadratically. Hence, invoking also (2.46), we can infer that, for every $s \in \mathbb{R}$,

$$\begin{aligned} |f^\lambda(s)| &\le |f_1^\lambda(s)| + |f_2(s)| \le \widehat{c}_3 \, F_1^\lambda(s) + \widehat{c}_4 + |f_2(s)| \\ &\le \widehat{c}_3 \, F^\lambda(s) + \widehat{c}_3 \, |F_2(s)| + \widehat{c}_4 + |f_2(s)| \le \widehat{c}_3 \, F^\lambda(s) + C_3(1+s^2) \\ &\le \left(\widehat{c}_3 + C_3 \, \widehat{c}_1^{\,-1}\right) F^\lambda(s) + C_4 \, . \end{aligned} \tag{3.13}$$

Therefore, we can conclude from (2.52) and (2.46) the bounds

$$\left\|f^\lambda(\varphi^\lambda_{\alpha_n,\beta})\right\|_{L^\infty(0,T;L^1(\Omega))} + \left\|\varphi^\lambda_{\alpha_n,\beta}\right\|_{L^\infty(0,T;H)} \le C_5 \, . \tag{3.14}$$

At this point, we insert $v = \pm\mathbf{1} \in V_B^\sigma$ in (2.51) to find the estimate

$$\begin{aligned} \pm \int_\Omega \mu^\lambda_{\alpha_n,\beta}(t) &\le C_6 \Big(\beta \, \|\partial_t \varphi^\lambda_{\alpha_n,\beta}(t)\| + \|B^\sigma \varphi^\lambda_{\alpha_n,\beta}(t)\| \, \|B^\sigma \mathbf{1}\| + \left\|f^\lambda(\varphi^\lambda_{\alpha_n,\beta})(t)\right\|_{L^1(\Omega)}\Big) \\ &\le C_6 \Big(\beta \, \|\partial_t \varphi^\lambda_{\alpha_n,\beta}(t)\| + \|\varphi^\lambda_{\alpha_n,\beta}(t)\|_{V_B^\sigma} \, \|B^\sigma \mathbf{1}\| + C_5\Big) \, , \end{aligned}$$

which, owing to (2.52) and (3.14), then shows that

$$\left\|\text{mean}(\mu^\lambda_{\alpha_n,\beta})\right\|_{L^2(0,T)} \le C_7 \, (1+\beta) \, .$$

Combining this with (2.52), and recalling the equivalence of the norms (3.3) and $\|\cdot\|_{V_A^\rho}$ given in Remark 3.2, we have finally shown that the sequence $\{\|\mu^\lambda_{\alpha_n,\beta}\|_{L^2(0,T;V_A^\rho)}\}_{n\in\mathbb{N}}$ is bounded. Passage to the limit as $\lambda \searrow 0$, and the semicontinuity of norms, then yield that also $\{\|\mu_{\alpha_n,\beta}\|_{L^2(0,T;V_A^\rho)}\}_{n\in\mathbb{N}}$ is bounded. With this, we can conclude the validity of (3.9) on a subsequence also in this case.

With (3.9) shown for all of the cases considered in **(A5)**, we can continue our analysis. At first, thanks to (3.7), (3.8), and known compactness results (see, e.g., [61, Sect. 8, Cor. 4]), we may without loss of generality assume that

$$\varphi_{\alpha_n,\beta} \to \varphi_{0,\beta} \quad \text{strongly in } C^0([0,T];H), \tag{3.15}$$

$$S_{\alpha_n,\beta} \to S_{0,\beta} \quad \text{strongly in } L^2(0,T;H). \tag{3.16}$$

Then, by the Lipschitz continuity of both P and f_2,

$$P(\varphi_{\alpha_n,\beta}) \to P(\varphi_{0,\beta}) \quad \text{strongly in } C^0([0,T];H), \tag{3.17}$$

$$f_2(\varphi_{\alpha_n,\beta}) \to f_2(\varphi_{0,\beta}) \quad \text{strongly in } C^0([0,T];H). \tag{3.18}$$

Next, we observe that the convergence properties (3.9), (3.16), and (3.17) imply that

$$P(\varphi_{\alpha_n,\beta})\big(S_{\alpha_n,\beta}-\mu_{\alpha_n,\beta}\big) \to P(\varphi_{0,\beta})\big(S_{0,\beta}-\mu_{0,\beta}\big) \quad \text{weakly in } L^1(Q)\,.$$

On the other hand, $P(\varphi_{\alpha_n,\beta})\big(S_{\alpha_n,\beta}-\mu_{\alpha_n,\beta}\big)$ is bounded in $L^2(0,T;H)$ due to (2.52), since P is bounded. Hence, we deduce that

$$P(\varphi_{\alpha_n,\beta})\big(S_{\alpha_n,\beta}-\mu_{\alpha_n,\beta}\big) \to P(\varphi_{0,\beta})\big(S_{0,\beta}-\mu_{0,\beta}\big) \quad \text{weakly in } L^2(0,T;H)\,. \tag{3.19}$$

Now we are in a position to take the limit as $n \to \infty$ in the time-integrated versions of (2.28) and (2.30), respectively, written with time-dependent test functions v. We then obtain that the triple $(\mu,\varphi,S) := (\mu_{0,\beta},\varphi_{0,\beta},S_{0,\beta})$ satisfies (2.28) for $\alpha=0$, (2.30), and the initial conditions (2.34).

It remains to show the validity of (2.29) or of its time-integrated version (2.35). To this end, notice that the convex functional $v \mapsto \int_\Omega F_1(v)$, extended with value $+\infty$ whenever $F_1(v) \notin L^1(\Omega)$, is proper, convex and lower semicontinuous in H. Hence, the convergence (3.15) and the bound (2.36) imply that

$$0 \le \int_\Omega F_1(\varphi_{0,\beta}(t)) \le \liminf_{n\to\infty} \int_\Omega F_1(\varphi_{\alpha_n,\beta}(t)) \le C_8 \quad \text{for every } t \in [0,T],$$

for some uniform constant C_8. It therefore follows that $F_1(\varphi_{0,\beta}) \in L^\infty(0,T;L^1(\Omega))$, and Fatou's lemma allows us to infer that

$$0 \le \int_Q F_1(\varphi_{0,\beta}) \le \liminf_{n\to\infty} \int_Q F_1(\varphi_{\alpha_n,\beta}) < +\infty. \tag{3.20}$$

Moreover, the quadratic form $v \mapsto \int_0^T (B^\sigma v(t), B^\sigma v(t))\,dt$ is weakly sequentially lower semicontinuous on $L^2(0,T;V_B^\sigma)$, which entails that

$$\int_0^T \big(B^\sigma\varphi_{0,\beta}(t), B^\sigma\varphi_{0,\beta}(t)\big)\,dt \le \liminf_{n\to\infty} \int_0^T \big(B^\sigma\varphi_{\alpha_n,\beta}(t), B^\sigma\varphi_{\alpha_n,\beta}(t)\big)\,dt\,. \tag{3.21}$$

Using all of the above convergence results, we can therefore conclude that, for every $v \in L^2(0,T;V_B^\sigma)$,

$$\begin{aligned}
&\int_Q F_1(\varphi_{0,\beta}) + \int_0^T \big(B^\sigma\varphi_{0,\beta}(t), B^\sigma(\varphi_{0,\beta}(t)-v(t)\big)\,dt \\
&\le \liminf_{n\to\infty}\left(\int_Q F_1(\varphi_{\alpha_n,\beta}) + \int_0^T \big(B^\sigma\varphi_{\alpha_n,\beta}(t), B^\sigma(\varphi_{\alpha_n,\beta}(t)-v(t))\big)\,dt\right) \\
&\le \liminf_{n\to\infty}\left(\int_0^T \big(\mu_{\alpha_n,\beta}(t) - f_2(\varphi_{\alpha_n,\beta}(t)) - \beta\,\partial_t\varphi_{\alpha_n,\beta}(t), \varphi_{\alpha_n,\beta}(t)-v(t)\big)\,dt + \int_Q F_1(v)\right) \\
&= \int_0^T \big(\mu_{0,\beta}(t) - f_2(\varphi_{0,\beta}(t)) - \beta\,\partial_t\varphi_{0,\beta}(t), \varphi_{0,\beta}(t)-v(t)\big)\,dt + \int_Q F_1(v),
\end{aligned} \tag{3.22}$$

which shows the validity of (2.35) for $(\mu,\varphi,S) = (\mu_{0,\beta},\varphi_{0,\beta},S_{0,\beta})$. From the above analysis, we can conclude the following existence and convergence result.

Theorem 3.3. *Suppose that the conditions* **(A1)–(A5)** *are fulfilled, let* $\beta > 0$ *be fixed and* $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1]$ *be a sequence such that* $\alpha_n \searrow 0$*. Then there are a subsequence* $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ *and functions* $(\mu_{\alpha_{n_k},\beta}, \varphi_{\alpha_{n_k},\beta}, S_{\alpha_{n_k},\beta})$*, which solve the system* (2.28)–(2.31) *for* $\alpha = \alpha_{n_k}$ *in the sense of Theorem 2.2, such that there is a triple* $(\mu_{0,\beta}, \varphi_{0,\beta}, S_{0,\beta})$ *with the following properties:*

$$\partial_t\big(\alpha_{n_k}\,\mu_{\alpha_{n_k},\beta} + \varphi_{\alpha_{n_k},\beta}\big) \to \partial_t\varphi_{0,\beta} \quad \textit{weakly in } L^2(0,T;V_A^{-\rho}), \tag{3.23}$$

$$\mu_{\alpha_{n_k},\beta} \to \mu_{0,\beta} \quad \textit{weakly-star in } L^2(0,T;V_A^{\rho}), \tag{3.24}$$

$$\varphi_{\alpha_{n_k},\beta} \to \varphi_{0,\beta} \quad \textit{weakly-star in } H^1(0,T;H)\cap L^\infty(0,T;V_B^{\sigma}), \tag{3.25}$$

$$S_{\alpha_{n_k},\beta} \to S_{0,\beta} \quad \textit{weakly-star in } H^1(0,T;V_C^{-\tau})\cap L^\infty(0,T;H)\cap L^2(0,T;V_C^{\tau}). \tag{3.26}$$

In addition, $F_1(\varphi_{0,\beta}) \in L^\infty(0,T;L^1(\Omega))$*, and* $(\mu_{0,\beta}, \varphi_{0,\beta}, S_{0,\beta})$ *solves the system* (2.28)–(2.30) *for* $\alpha = 0$ *and satisfies the initial conditions* (2.34)*. Finally, it holds the additional regularity*

$$\mu \in L^2(0,T;V_A^{2\rho}). \tag{3.27}$$

Proof. Except for (3.27), everything was already proved above. The validity of (3.27) follows directly from comparison in (2.28), since, owing to the boundedness of P, we have $P(\varphi_{0,\beta})\big(S_{0,\beta} - \mu_{0,\beta}\big) - \partial_t\varphi_{0,\beta} \in L^2(0,T;H)$. □

Next, we give a regularity result that resembles the corresponding results (2.38)–(2.40) in Theorem 2.2 for the case when both $\alpha > 0$ and $\beta > 0$. Note that we cannot expect the same regularity here, since a vanishing α entails a loss of coercivity with respect to the solution component μ.

Theorem 3.4. *Suppose that* **(A1)–(A4)**, (2.37), (2.41)*, and at least one of the two conditions*

$$\text{(i) } \lambda_1 > 0, \quad \textit{and} \quad \text{(ii) } P(s) \ge P_0 > 0 \ \textit{ for all } s \in \mathbb{R}, \tag{3.28}$$

are fulfilled. Then the solution $(\mu_{0,\beta}, \varphi_{0,\beta}, S_{0,\beta})$ *established in Theorem 3.3 enjoys the additional regularity*

$$\mu_{0,\beta} \in L^\infty(0,T;V_A^{2\rho}), \tag{3.29}$$

$$\varphi_{0,\beta} \in W^{1,\infty}(0,T;H)\cap H^1(0,T;V_B^{\sigma}), \tag{3.30}$$

$$S_{0,\beta} \in H^1(0,T;H)\cap L^\infty(0,T;V_C^{\tau})\cap L^2(0,T;V_C^{2\tau}). \tag{3.31}$$

Proof. Let, for convenience, $(\mu,\varphi,S) := (\mu_{0,\beta}, \varphi_{0,\beta}, S_{0,\beta})$. We only give a formal proof of the assertion based on the Moreau–Yosida approximation, which is for $\lambda > 0$ given by the system (2.47) with $\alpha = 0$, (2.51) (in place of (2.48)), (2.49), together with the initial condition (2.34). For a rigorous proof, one would have to carry out the following arguments on the level of the time-discretized version introduced in [19]. Since this requires a considerable writing effort without bringing new insights in comparison with the calculations in [19], we prefer to argue formally, here. To this end, we differentiate (2.51) with respect to t and take $v = \partial_t\varphi^\lambda(t)$ in the resulting equation. In addition, we insert $v = \partial_t\mu^\lambda(t)$ in (2.47), add the two resulting equations, and integrate their sum over $(0,t)$ where $t \in (0,T]$. Noting that the two terms involving $\partial_t\varphi^\lambda\,\partial_t\mu^\lambda$ cancel each other, we arrive at the identity

$$\begin{aligned}
&\frac{\beta}{2}\,\|\partial_t\varphi^\lambda(t)\|^2 + \frac{1}{2}\,\|A^\rho\mu^\lambda(t)\|^2 + \int_{Q_t}|B^\sigma(\partial_t\varphi^\lambda)|^2 + \int_{Q_t}(f_1^\lambda)'(\varphi^\lambda)\,|\partial_t\varphi^\lambda|^2 \\
&= \int_{Q_t}P(\varphi^\lambda)\,(S^\lambda - \mu^\lambda)\,\partial_t\mu^\lambda + \frac{\beta}{2}\,\|\partial_t\varphi^\lambda(0)\|^2 + \frac{1}{2}\,\|A^\rho\mu_0\|^2 - \int_{Q_t}f_2'(\varphi^\lambda)\,|\partial_t\varphi^\lambda|^2\,,
\end{aligned} \tag{3.32}$$

where, due to the general assumptions, all of the terms on the left-hand side are nonnegative and the last term on the right-hand side is from (2.52) already known to be bounded independently of λ. Now observe that, by formal insertion of $\partial_t\varphi^\lambda(0)$ in (2.51) for $t=0$, it follows from (2.37) that

$$\beta\,\|\partial_t\varphi^\lambda(0)\|^2 \,=\, \bigl(-B^{2\sigma}\varphi_0 - f^\lambda(\varphi_0) + \mu_0, \partial_t\varphi^\lambda(0)\bigr) \,\le\, \frac{\beta}{2}\,\|\partial_t\varphi^\lambda(0)\|^2 \,+\, C_1, \tag{3.33}$$

where, here and in the remainder of this proof, we denote by C_i, $i\in\mathbb{N}$, positive constants that do not depend on λ. Next, an integration by parts yields that

$$\begin{aligned}
&-\int_{Q_t} P(\varphi^\lambda)\,\mu^\lambda\,\partial_t\mu^\lambda \,=\, \frac12\int_\Omega\bigl(P(\varphi_0)\,|\mu_0|^2 - P(\varphi^\lambda(t))\,|\mu^\lambda(t)|^2\bigr) \,+\, \frac12\int_{Q_t} P'(\varphi^\lambda)\partial_t\varphi^\lambda\,|\mu^\lambda|^2\\
&\le\, C_2 - \frac12\int_\Omega P(\varphi^\lambda(t))\,|\mu^\lambda(t)|^2 \,+\, C_3\int_0^t \|\mu^\lambda(s)\|_{L^4(\Omega)}^2\,\|\partial_t\varphi^\lambda(s)\|\,ds\\
&\le\, C_2 - \frac12\int_\Omega P(\varphi^\lambda(t))\,|\mu^\lambda(t)|^2 \,+\, C_4\int_0^t \|\mu^\lambda(s)\|_{V_A^\rho}^2\,\|\partial_t\varphi^\lambda(s)\|^2\,ds\,,
\end{aligned}\tag{3.34}$$

where we used Hölder's inequality and (2.41). Note that, by virtue of (2.21), the second term on the right-hand side of (3.34) is nonpositive so that it can be moved with the right sign on the left-hand side of (3.32). Moreover, we notice that $\|\mu^\lambda\|_{L^2(0,T;V_A^\rho)}$ is uniformly bounded with respect to λ as shown in the proof of Theorem 3.3, and we will account for this information in applying Gronwall's lemma. Moreover, integrating by parts and using also the already known bounds (2.52), the inequality $P\le P^{1/2}(\sup P^{1/2})$ and the Young inequality, we infer that

$$\begin{aligned}
&\int_{Q_t} P(\varphi^\lambda)\,S^\lambda\,\partial_t\mu^\lambda\\
&=\int_\Omega\bigl(P(\varphi^\lambda(t)\,S^\lambda(t)\,\mu^\lambda(t) - P(\varphi_0)\,S_0\,\mu_0\bigr) \,-\, \int_{Q_t}\bigl(P'(\varphi^\lambda)\,\partial_t\varphi^\lambda\,S^\lambda + P(\varphi^\lambda)\,\partial_t S^\lambda\bigr)\mu^\lambda\\
&\le\, C_5 + \frac14\int_\Omega P(\varphi^\lambda(t))\,|\mu^\lambda(t)|^2 \,+\, C_6\int_0^t \|\partial_t\varphi^\lambda(s)\|\,\|S^\lambda(s)\|_{V_C^\tau}\,\|\mu^\lambda(s)\|_{V_A^\rho}\,ds\\
&\quad+\,\frac12\int_{Q_t}|\partial_t S^\lambda|^2 + C_7\int_{Q_t} P(\varphi^\lambda)|\mu^\lambda|^2\,.
\end{aligned}\tag{3.35}$$

Note that the third term on the right-hand side can be treated for instance as

$$\begin{aligned}
&C_6\int_0^t \|\partial_t\varphi^\lambda(s)\|\,\|S^\lambda(s)\|_{V_C^\tau}\,\|\mu^\lambda(s)\|_{V_A^\rho}\,ds\\
&\le\, C_6\int_0^t \|S^\lambda(s)\|_{V_C^\tau}\,\|\mu^\lambda(s)\|_{V_A^\rho}\,\bigl(1+\|\partial_t\varphi^\lambda(s)\|^2\bigr)\,ds,
\end{aligned}\tag{3.36}$$

and both $\|S^\lambda\|_{L^2(0,T;V_C^\tau)}$ and $\|\mu^\lambda\|_{L^2(0,T;V_A^\rho)}$ are uniformly bounded with respect to λ (cf. (2.36)). Finally, we test (2.49) by $\partial_t S^\lambda(t)$ and integrate over $(0,t)$. Then we obtain

$$\begin{aligned}
&\int_{Q_t}|\partial_t S^\lambda|^2 \,+\, \frac12\,\|C^\tau S^\lambda(t)\|^2 \,\le\, \frac12\,\|C^\tau S_0\|^2 \,+\, \int_0^t\bigl(z^\lambda(s),\partial_t S^\lambda(s)\bigr)\,ds\\
&\le\, \frac12\,\|C^\tau S_0\|^2 \,+\, \frac14\int_{Q_t}|\partial_t S^\lambda|^2 \,+\, \int_{Q_t}|z^\lambda|^2,
\end{aligned}\tag{3.37}$$

where $z^\lambda := P(\varphi^\lambda)(\mu^\lambda - S^\lambda)$ is already known to be bounded in $L^2(0,T;H)$, independently of λ, by (2.52) and the boundedness of P. Combining (3.32)–(3.37), and invoking Gronwall's lemma, we have therefore shown the estimate

$$\|\partial_t\varphi^\lambda\|^2_{L^\infty(0,T;H)\cap L^2(0,T;V_B^\sigma)} + \|S^\lambda\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V_C^\tau)} + \sup_{t\in(0,T)}\Big(\|A^\rho\mu^\lambda(t)\|^2 + \int_\Omega P(\varphi^\lambda(t))|\mu^\lambda(t)|^2\Big) \le C_8. \tag{3.38}$$

In particular, it follows from (3.28) (see Remark 3.2) that

$$\|\mu^\lambda\|_{L^\infty(0,T;V_A^\rho)} \le C_9. \tag{3.39}$$

It remains to show the boundedness of S^λ in $L^2(0,T;V_C^{2\tau})$ and of μ^λ in $L^\infty(0,T;V_A^{2\rho})$. But this follows immediately from (2.49) and (2.47), respectively, by comparison. At this point, we take the limit as $\lambda \searrow 0$ and invoke the semicontinuity of norms to infer that the derived bounds are valid also in the limit. This concludes the proof of the assertion. □

Remark 3.5. It is also possible to prove a uniqueness result for the case $\alpha = 0$, $\beta > 0$, under restrictive additional assumptions. Since the related analysis requires a major detour in the line of argumentation and is carried out in detail in the recent paper [20], we do not present it here. Note also that in the case $P \equiv 0$ the system (2.28), (2.29) coincides for $\alpha = 0$ and $\beta \ge 0$ with the system that has recently been studied by the present authors in a series of papers (see [15–17]); for precise results in this much simpler case, in which (2.28), (2.29) decouple from (2.30), we refer to these works.

4 The case $\alpha > 0$, $\beta \searrow 0$.

In this section, we investigate the asymptotic behavior of the solutions $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, S_{\alpha,\beta})$ as $\alpha > 0$ and $\beta \searrow 0$. In this case, an additional coercivity condition for μ like (3.28) is not necessary. Instead, the main difficulty is to establish a strong convergence for the phase variable φ. Indeed, we have to make the following additional assumption:

(A6) It holds $\alpha L < 1$.

Now, let $\{\beta_n\}_{n\in\mathbb{N}} \subset (0,1]$ be any sequence such that $\beta_n \searrow 0$. Then, according to the global bound (2.36), we may without loss of generality assume the existence of functions $\zeta, \mu_{\alpha,0}, \varphi_{\alpha,0}, S_{\alpha,0}$ such that, at least for a subsequence as $n \to \infty$,

$$\beta_n\,\partial_t\varphi_{\alpha,\beta_n} \to 0 \quad \text{strongly in } L^2(0,T;H), \tag{4.1}$$

$$\partial_t\big(\alpha\,\mu_{\alpha,\beta_n} + \varphi_{\alpha,\beta_n}\big) \to \zeta \quad \text{weakly in } L^2(0,T,V_A^{-\rho}), \tag{4.2}$$

$$\mu_{\alpha,\beta_n} \to \mu_{\alpha,0} \quad \text{weakly-star in } L^\infty(0,T;H) \cap L^2(0,T;V_A^\rho), \tag{4.3}$$

$$\varphi_{\alpha,\beta_n} \to \varphi_{\alpha,0} \quad \text{weakly-star in } L^\infty(0,T;V_B^\sigma), \tag{4.4}$$

$$S_{\alpha,\beta_n} \to S_{\alpha,0} \quad \text{weakly-star in } H^1(0,T;V_C^{-\tau}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_C^\tau). \tag{4.5}$$

Now, combining (4.2)–(4.4), we see that $\zeta = \partial_t(\alpha\,\mu_{\alpha,0} + \varphi_{\alpha,0})$, and we infer from (4.3) and (4.4) that $\{\alpha\,\mu_{\alpha,\beta_n} + \varphi_{\alpha,\beta_n}\}_{n\in\mathbb{N}}$ is also bounded in the Banach space $L^2(0,T;V_A^\rho + V_B^\sigma)$, where

$$V_A^\rho + V_B^\sigma := \{v + w : \ v \in V_A^\rho \ \text{ and } \ w \in V_B^\sigma\}, \quad \text{and} \tag{4.6}$$

$$\|y\|_{V_A^\rho + V_B^\sigma} := \inf\{\|v\|_{V_A^\rho} + \|w\|_{V_B^\sigma} \ : \ v \in V_A^\rho, \ \ w \in V_B^\sigma, \ \text{ and } \ y = v + w\}. \tag{4.7}$$

Since both V_A^ρ and V_B^σ are compactly embedded in H, so is $V_A^\rho + V_B^\sigma$, and we can infer from the Aubin–Lions compactness lemma (see, e.g., [52, Thm. 5.1, p. 58]) that

$$\alpha\,\mu_{\alpha,\beta_n} + \varphi_{\alpha,\beta_n} \ \to \ \alpha\,\mu_{\alpha,0} + \varphi_{\alpha,0} \quad \text{strongly in } \ L^2(0,T;H). \tag{4.8}$$

Next, we aim at showing that $\{\varphi_{\alpha,\beta_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(0,T;H)$, which would imply that, possibly taking another subsequence,

$$\begin{aligned} &\varphi_{\alpha,\beta_n} \ \to \ \varphi_{\alpha,0} \quad \text{strongly in } \ L^2(0,T;H) \ \text{ and} \\ &\quad \varphi_{\alpha,\beta_n}(t) \ \to \ \varphi_{\alpha,0}(t) \quad \text{strongly in } \ H, \text{ for a.e. } t \in (0,T). \end{aligned} \tag{4.9}$$

To prove the claim, we employ Lemma 2.7 with the special choice (α,β_n) and (α,β_m), where $n > m$ so that $0 < \beta_n < \beta_m$. Thanks to (2.59), we have, for every $\delta > 0$,

$$\begin{aligned} (1 - \alpha L - \delta) \int_Q \bigl|\varphi_{\alpha,\beta_n} - \varphi_{\alpha,\beta_m}\bigr|^2 \ \le \ \frac{1}{4\delta} \int_Q \bigl|(\alpha\,\mu_{\alpha,\beta_n} + \varphi_{\alpha,\beta_n}) - (\alpha\,\mu_{\alpha,\beta_m} + \varphi_{\alpha,\beta_m})\bigr|^2 \\ + \ \alpha\bigl|\beta_n - \beta_m\bigr| \int_0^T \|\partial_t\varphi_{\alpha,\beta_m}(s)\|\,\|\varphi_{\alpha,\beta_n}(s) - \varphi_{\alpha,\beta_m}(s)\|\,ds\,. \end{aligned} \tag{4.10}$$

Now observe that $|\beta_n - \beta_m| \ = \ (\beta_m - \beta_n) \ \le \ \beta_m$. Moreover, $\{\varphi_{\alpha,\beta_n} - \varphi_{\alpha,\beta_m}\}$ is bounded in $L^2(0,T;H)$. Hence, by virtue of (4.1) and (4.8), the right-hand side of (4.10) converges to zero as $n > m$ and $m \to \infty$. Therefore, choosing $\delta \in (0, 1 - \alpha L)$, we conclude from (4.10) that the above claim is valid. We thus may assume that (4.9) holds true. But this implies that also

$$\mu_{\alpha,\beta_n} \ \to \ \mu_{\alpha,0} \quad \text{strongly in } \ L^2(0,T;H), \tag{4.11}$$

$$\begin{aligned} &P(\varphi_{\alpha,\beta_n}) \ \to \ P(\varphi_{\alpha,0}) \quad \text{and} \quad f_2(\varphi_{\alpha,\beta_n}) \ \to \ f_2(\varphi_{\alpha,0}) \\ &\qquad \text{both strongly in } \ L^2(0,T;H), \end{aligned} \tag{4.12}$$

using (4.8) and the Lipschitz continuity of P and f_2. Moreover, the Aubin–Lions lemma yields that also

$$S_{\alpha,\beta_n} \ \to \ S_{\alpha,0} \quad \text{strongly in } \ L^2(0,T;H), \tag{4.13}$$

and, as in (3.19), it is readily verified that

$$P(\varphi_{\alpha,\beta_n})(S_{\alpha,\beta_n} - \mu_{\alpha,\beta_n}) \to P(\varphi_{\alpha,0})(S_{\alpha,0} - \mu_{\alpha,0}) \quad \text{weakly in } L^2(0,T;H). \tag{4.14}$$

Now we are in a position to take the limit as $n \ \to \ \infty$ in the time-integrated versions of (2.28) and (2.30), respectively, written with time-dependent test functions. We then obtain that the triple $(\mu,\varphi,S) := (\mu_{\alpha,0},\varphi_{\alpha,0},S_{\alpha,0})$ satisfies (2.28) and (2.30), and (4.2) entails that $\alpha\,\mu_{\alpha,\beta_n} + \varphi_{\alpha,\beta_n} \ \to \ \alpha\,\mu_{\alpha,0} + \varphi_{\alpha,0}$ weakly in $C^0([0,T];V_A^{-\rho})$, which shows, in particular, that $(\alpha\,\mu_{\alpha,0} + \varphi_{\alpha,0})(0) = \alpha\,\mu_0 + \varphi_0$, i.e., the first of (2.33). At the same time, we conclude from (4.5) the weak convergence $S_{\alpha,\beta_n} \ \to \ S_{\alpha,0}$ in $C^0([0,T];V_C^{-\tau})$; therefore, we also have the second of the initial conditions (2.33).

It remains to show the validity of (2.29) or its time-integrated version (2.35), for $\beta = 0$. To this end, notice that (2.36), (4.9) and the lower semicontinuity of the functional $v \mapsto \int_\Omega F_1(v)$ in H imply that

$$0 \le \int_\Omega F_1(\varphi_{\alpha,0}(t)) \le \liminf_{n\to\infty} \int_\Omega F_1(\varphi_{\alpha,\beta_n}(t)) \le C \quad \text{for a.e. } t \in (0,T), \tag{4.15}$$

for some constant C independent of β_n. Thus, it follows that $F_1(\varphi_{\alpha,0}) \in L^\infty(0,T;L^1(\Omega))$ and, by Fatou's lemma,

$$0 \le \int_Q F_1(\varphi_{\alpha,0}) \le \liminf_{n\to\infty} \int_Q F_1(\varphi_{\alpha,\beta_n}) < +\infty. \tag{4.16}$$

Moreover, the quadratic form $v \mapsto \int_0^T (B^\sigma v(t), B^\sigma v(t))\,dt$ is weakly sequentially lower semicontinuous on $L^2(0,T;V_B^\sigma)$. Therefore, a similar calculation (which needs no repetition here) as in (3.20) yields the validity of (2.35).

In conclusion, we have the following result.

Theorem 4.1. *Suppose that the conditions* **(A1)–(A4)** *and* **(A6)** *are fulfilled. Moreover, let $\alpha > 0$ and $\{\beta_n\} \subset (0,1)$ be a sequence with $\beta_n \searrow 0$ as $n \to \infty$. Then there are a subsequence $\{\beta_{n_k}\}_{k\in\mathbb{N}}$ and functions $(\mu_{\alpha,\beta_{n_k}}, \varphi_{\alpha,\beta_{n_k}}, S_{\alpha,\beta_{n_k}})$, which solve the system* (2.28)–(2.31) *for $\beta = \beta_{n_k}$ in the sense of Theorem 2.2, and a triple $(\mu_{\alpha,0}, \varphi_{\alpha,0}, S_{\alpha,0})$ with the following properties:*

$$\beta_{n_k}\,\partial_t\varphi_{\alpha,\beta_{n_k}} \to 0 \quad \textit{strongly in } L^2(0,T;H), \tag{4.17}$$

$$\partial_t\bigl(\alpha\,\mu_{\alpha,\beta_{n_k}} + \varphi_{\alpha,\beta_{n_k}}\bigr) \to \partial_t\bigl(\alpha\,\mu_{\alpha,0} + \varphi_{\alpha,0}\bigr) \quad \textit{weakly in } L^2(0,T,V_A^{-\rho}), \tag{4.18}$$

$$\mu_{\alpha,\beta_{n_k}} \to \mu_{\alpha,0} \quad \textit{weakly-star in } L^\infty(0,T;H) \cap L^2(0,T;V_A^\rho) \tag{4.19}$$

$$\textit{and strongly in } L^2(0,T;H), \tag{4.20}$$

$$\varphi_{\alpha,\beta_{n_k}} \to \varphi_{\alpha,0} \quad \textit{weakly-star in } L^\infty(0,T;V_B^\sigma) \textit{ and strongly in } L^2(0,T;H), \tag{4.21}$$

$$S_{\alpha,\beta_{n_k}} \to S_{\alpha,0} \quad \textit{weakly-star in } H^1(0,T;V_C^{-\tau}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_C^\tau). \tag{4.22}$$

In addition, $F_1(\varphi_{\alpha,0}) \in L^\infty(0,T;L^1(\Omega))$, and $(\mu_{\alpha,0}, \varphi_{\alpha,0}, S_{\alpha,0})$ solves the system (2.28)–(2.30) *for $\beta = 0$ and satisfies the initial conditions* (2.33).

It seems to be difficult to derive additional regularity results for $\alpha > 0$ and $\beta = 0$, and we give a comment on this in the forthcoming Remark 5.4. However, we can show a more important uniqueness result. To this end, we need to make a compatibility assumption that strongly relates the operators A^ρ and B^σ to each other. We have the following result.

Theorem 4.2. *Assume, in addition to* **(A1)–(A4)** *and* **(A6)**, *that the following embeddings are continuous:*

$$V_B^\sigma \subset V_A^\rho, \quad V_A^\rho \subset L^4(\Omega), \quad V_B^\sigma \subset L^4(\Omega) \quad \textit{and} \quad V_C^\tau \subset L^4(\Omega). \tag{4.23}$$

Then the solution to the system (2.28)–(2.30), (2.33) *for $\alpha > 0$ and $\beta = 0$ established in Theorem 4.1 is uniquely determined.*

Proof. We point out that the third condition in (4.23) is a straightforward consequence of the first and second ones. The continuity of the embedding $V_B^\sigma \subset V_A^\rho$ implies the existence of a constant κ (which we will refer to) such that

$$\|A^\rho v\|^2 \le \kappa\bigl(\|B^\sigma v\|^2 + \|v\|^2\bigr) \quad \text{for all } v \in V_B^\sigma. \tag{4.24}$$

Let (μ_i, φ_i, S_i), $i = 1, 2$, be two solution triples. We denote $w_i := \alpha\mu_i + \varphi_i$, for $i = 1, 2$, and set $\mu := \mu_1 - \mu_2$, $\varphi := \varphi_1 - \varphi_2$, $S := S_1 - S_2$, and $w := w_1 - w_2$. Then we have a.e. in $(0, t)$, and for $i = 1, 2$, that

$$\langle \partial_t w_i, v\rangle_{V_A^\rho} + (A^\rho \mu_i, v) \;=\; (P(\varphi_i)(S_i - \mu_i), v) \quad \forall\, v \in V_A^\rho, \tag{4.25}$$

$$\begin{aligned} &\alpha(B^\sigma\varphi_i, B^\sigma(\varphi_i - v)) + \alpha \int_\Omega F_1(\varphi_i) + \alpha(f_2(\varphi_i), \varphi_i - v) + (\varphi_i, \varphi_i - v) \\ &\le\; (w_i, \varphi_i - v) + \alpha \int_\Omega F_1(v) \quad \forall\, v \in V_B^\sigma, \end{aligned} \tag{4.26}$$

$$\langle \partial_t S_i, v\rangle_{V_C^\tau} + (C^\tau S_i, C^\tau v) \;=\; (P(\varphi_i)(\mu_i - S_i), v) \quad \forall\, v \in V_C^\tau. \tag{4.27}$$

Next, we insert $v = \varphi_2$ in the inequality (4.26) for $i = 1$, $v = \varphi_1$ in the inequality for $i = 2$, add the resulting inequalities and multiply the result by a positive constant M which is yet to be specified. Then we integrate over $(0, t)$, where $t \in (0, T)$. Note that all of the terms involving F_1 cancel. Hence, also using (2.19), we obtain the inequality

$$M\alpha \int_{Q_t} |B^\sigma\varphi|^2 \;+\; M(1 - \alpha L) \int_{Q_t} |\varphi|^2 \;\le\; M \int_{Q_t} w\,\varphi\,,$$

and Young's inequality yields that for every $\delta > 0$ (which is yet to be chosen) it holds that

$$M\alpha \int_{Q_t} |B^\sigma\varphi|^2 \;+\; (M(1 - \alpha L) - \delta) \int_{Q_t} |\varphi|^2 \;\le\; \frac{M^2}{4\delta} \int_{Q_t} |w|^2\,. \tag{4.28}$$

Now we subtract the equations (4.25) for $i = 1, 2$ from each other and insert $v = w$ in the resulting equation. Similarly, we subtract the equations (4.27) for $i = 1, 2$ from each other and insert $v = S$ in the resulting equation. Finally, we add the two results. Integration over $(0, t)$ then yields the identity

$$\begin{aligned} &\frac{1}{2}\,\|w(t)\|^2 \;+\; \alpha \int_{Q_t} |A^\rho\mu|^2 \;+\; \int_{Q_t} A^\rho\mu\, A^\rho\varphi \;+\; \frac{1}{2}\,\|S(t)\|^2 \;+\; \int_{Q_t} |C^\tau S|^2 \\ &=\; \int_{Q_t} \bigl(P(\varphi_1)(S_1 - \mu_1) - P(\varphi_2)(S_2 - \mu_2)\bigr)\,(w - S)\,. \end{aligned} \tag{4.29}$$

Now observe that Young's inequality and (4.24) yield that

$$\begin{aligned} &-\int_{Q_t} A^\rho\mu\, A^\rho\varphi \;\le\; \frac{\alpha}{2} \int_{Q_t} |A^\rho\mu|^2 \;+\; \frac{2}{\alpha} \int_{Q_t} |A^\rho\varphi|^2 \\ &\le\; \frac{\alpha}{2} \int_{Q_t} |A^\rho\mu|^2 \;+\; \frac{2\kappa}{\alpha} \int_{Q_t} (|B^\sigma\varphi|^2 + |\varphi|^2)\,. \end{aligned} \tag{4.30}$$

It remains to estimate the right-hand side of (4.29) which we denote by Z. We have

$$\begin{aligned} Z \;&=\; \int_{Q_t} \bigl(P(\varphi_1) - P(\varphi_2)\bigr)(S_1 - \mu_1)(w - S) \;+\; \int_{Q_t} P(\varphi_2)(S - \mu)\,(w - S) \\ &=:\; Z_1 + Z_2, \end{aligned} \tag{4.31}$$

with obvious notation. Using the Hölder and Young inequalities, and invoking (4.23), we see that

$$\begin{aligned} |Z_1| \;&\le\; C_1 \int_0^t \|\varphi(s)\|_{L^4(\Omega)} \bigl(\|S_1(s)\|_{L^4(\Omega)} + \|\mu_1(s)\|_{L^4(\Omega)}\bigr) \bigl(\|w(s)\| + \|S(s)\|\bigr)\,ds \\ &\le\; \delta \int_0^t \|\varphi(s)\|_{V_B^\sigma}^2\,ds \;+\; \frac{C_2}{\delta} \int_0^t \Phi(s)\,\bigl(\|w(s)\|^2 + \|S(s)\|^2\bigr)\,ds\,, \end{aligned} \tag{4.32}$$

where the function $\Phi(s) := \|S_1(s)\|^2_{V_C^\tau} + \|\mu_1(s)\|^2_{V_A^\rho}$ is known to belong to $L^1(0,T)$. Here, and in the remainder of the proof, C_i, $i \in \mathbb{N}$, denote positive constants that depend only on the global data of the system.

Finally, we estimate Z_2. Omitting an obvious nonpositive term, we have, by virtue of Young's inequality, and since $\alpha\mu = w - \varphi$,

$$\begin{aligned}
|Z_2| \,&\le\, C_3 \int_0^t \bigl(\|S(s)\|\,\|w(s)\| \,+\, \|S(s)\|\,\|\mu(s)\| \,+\, \|\mu(s)\|\,\|w(s)\|\bigr)\,ds \\
&\le\, \delta\,\alpha^2 \int_{Q_t} |\mu|^2 \,+\, C_4\Bigl(1 + \frac{1}{\delta\alpha^2}\Bigr) \int_{Q_t} \bigl(|S|^2 + |w|^2\bigr) \\
&\le\, 2\delta \int_0^t \|\varphi(s)\|^2_{V_B^\sigma}\,ds \,+\, C_5\Bigl(1 + \delta + \frac{1}{\delta\alpha^2}\Bigr) \int_{Q_t} \bigl(|S|^2 + |w|^2\bigr)\,.
\end{aligned} \tag{4.33}$$

Combining (4.28)–(4.33), we have thus shown the estimate

$$\begin{aligned}
&\Bigl(M\alpha - \frac{2\kappa}{\alpha} - 3\delta\Bigr) \int_{Q_t} |B^\sigma\varphi|^2 \,+\, \Bigl(M(1-\alpha L) - \frac{2\kappa}{\alpha} - 4\delta\Bigr) \int_{Q_t} |\varphi|^2 \\
&+\, \frac{1}{2}\bigl(\|w(t)\|^2 + \|S(t)\|^2\bigr) \,+\, \frac{\alpha}{2}\int_{Q_t} |A^\rho\mu|^2 \,+\, \int_{Q_t} |C^\tau S|^2 \\
&\le \int_0^t \Bigl[\frac{C_2}{\delta}\,\Phi(s) \,+\, C_6\,\bigl(M^2+1\bigr)\Bigl(1 + \delta + \frac{1}{\delta} \,+\, \frac{1}{\delta\alpha^2}\Bigr)\Bigr]\,\bigl(\|w(s)\|^2 + \|S(s)\|^2\bigr)\,ds\,.
\end{aligned} \tag{4.34}$$

At this point, we make the choices

$$M > M_0 := \max\left\{\frac{2\kappa}{\alpha^2}\,,\,\frac{2\kappa}{\alpha(1-\alpha L)}\right\} \quad \text{and} \quad 0 < \delta < \frac{1}{4}(M - M_0)\,\min\{\alpha, 1-\alpha L\}\,.$$

Then the brackets in the first two terms on the left-hand side become positive, and we may apply Gronwall's lemma to conclude that $w = S = \varphi = 0$, whence also $\mu = 0$. $\square$

Remark 4.3. It ought to be clear from the above arguments that in the case that controls u_μ, u_φ, u_S in $L^2(0,T;H)$ are added to the right-hand sides of (2.28)–(2.30), we have an existence result resembling Theorem 3.3, and, under the assumptions of Theorem 4.2, we obtain a corresponding continuous dependence result in the norms appearing on the left-hand side of (4.34).

5 The case $\alpha \searrow 0$, $\beta \searrow 0$.

In this section, we investigate the asymptotic behavior of the solutions $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, S_{\alpha,\beta})$ as $\alpha \searrow 0$ and $\beta \searrow 0$. Quite unexpectedly, in this case the additional assumption **(A6)** is not needed. In a sense, this means that the presence of a strong perturbation $\alpha\partial_t\mu$ as in the previous section does not just produce an approximation but really changes the character of the unperturbed system if α is too large. On the other hand, we have to assume:

(A7) The eigenvalue λ_1 is positive.

Recall that then the mapping $v \mapsto \|A^\rho v\|$ defines a norm on V_A^ρ which is equivalent to the graph norm $\|\cdot\|_{V_A^\rho}$ (see Remark 3.2).

To begin with, let $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1]$ and $\{\beta_n\}_{n\in\mathbb{N}} \subset (0,1]$ be sequences such that $\alpha_n \searrow 0$ and $\beta_n \searrow 0$, and let $(\mu_{\alpha_n,\beta_n}, \varphi_{\alpha_n,\beta_n}, S_{\alpha_n,\beta_n})$ denote solutions to (2.28)–(2.31) in the sense of Theorem 2.2 associated with $(\alpha,\beta) = (\alpha_n,\beta_n)$, for $n \in \mathbb{N}$. According to (2.36), and invoking **(A7)**, we may without loss of generality assume that there are limits $\zeta, \mu_{0,0}, \varphi_{0,0}, S_{0,0}$ such that, as $n \to \infty$,

$$\alpha_n\,\mu_{\alpha_n,\beta_n} \to 0 \quad \text{strongly in } L^\infty(0,T;H), \tag{5.1}$$

$$\beta_n\,\partial_t\varphi_{\alpha_n,\beta_n} \to 0 \quad \text{strongly in } L^2(0,T;H), \tag{5.2}$$

$$\partial_t(\alpha_n\,\mu_{\alpha_n,\beta_n} + \varphi_{\alpha_n,\beta_n}) \to \zeta \quad \text{weakly in } L^2(0,T;V_A^{-\rho}), \tag{5.3}$$

$$\mu_{\alpha_n,\beta_n} \to \mu_{0,0} \quad \text{weakly in } L^2(0,T;V_A^{\rho}), \tag{5.4}$$

$$\varphi_{\alpha_n,\beta_n} \to \varphi_{0,0} \quad \text{weakly-star in } L^\infty(0,T;V_B^{\sigma}), \tag{5.5}$$

$$S_{\alpha_n,\beta_n} \to S_{0,0} \quad \text{weakly-star in } H^1(0,T;V_C^{-\tau}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_C^{\tau}). \tag{5.6}$$

From (5.1), (5.3) and (5.5) it follows that $\zeta = \partial_t\varphi_{0,0}$ and, in addition,

$$\alpha_n\,\mu_{\alpha_n,\beta_n} + \varphi_{\alpha_n,\beta_n} \to \varphi_{0,0} \quad \text{weakly in } H^1(0,T;V_A^{-\rho}), \tag{5.7}$$

whence also

$$(\alpha_n\,\mu_{\alpha_n,\beta_n} + \varphi_{\alpha_n,\beta_n})(0) \to \varphi_{0,0}(0) \quad \text{weakly in } V_A^{-\rho}. \tag{5.8}$$

Then, in view of (5.6)–(5.8), it turns out that both the initial conditions in (2.34) are fulfilled.

Next, we observe that we can argue exactly as we did in the previous section to obtain (4.8). Hence, we infer that the sequence $\{\alpha_n\,\mu_{\alpha_n,\beta_n} + \varphi_{\alpha_n,\beta_n}\}_{n\in\mathbb{N}}$ converges strongly in $L^2(0,T;H)$. We thus find from (5.1) that

$$\begin{aligned} &\varphi_{\alpha_n,\beta_n} \to \varphi_{0,0} \quad \text{strongly in } L^2(0,T;H) \text{ and} \\ &\quad \varphi_{\alpha_n,\beta_n}(t) \to \varphi_{0,0}(t) \quad \text{strongly in } H, \text{ for a.e. } t \in (0,T), \end{aligned} \tag{5.9}$$

the latter without loss of generality. Consequently, by Lipschitz continuity, we have that

$$f_2(\varphi_{\alpha_n,\beta_n}) \to f_2(\varphi_{0,0}) \quad \text{and} \quad P(\varphi_{\alpha_n,\beta_n}) \to P(\varphi_{0,0}) \quad \text{strongly in } L^2(0,T;H). \tag{5.10}$$

From this point, we may follow the lines of the previous sections to conclude the following result.

Theorem 5.1. *Assume that* **(A1)**–**(A4)** *and* **(A7)** *are fulfilled and let the sequences* $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1]$ *and* $\{\beta_n\}_{n\in\mathbb{N}} \subset (0,1]$ *satisfy* $\alpha_n \searrow 0$ *and* $\beta_n \searrow 0$*. Moreover, let* $(\mu_{\alpha_n,\beta_n}, \varphi_{\alpha_n,\beta_n}, S_{\alpha_n,\beta_n})$ *be solutions to the system* (2.28)–(2.31) *in the sense of Theorem 2.2 for* $(\alpha,\beta) = (\alpha_n,\beta_n)$ *for* $n \in \mathbb{N}$*. Then, there are a subsequence* $\{n_k\}_{k\in\mathbb{N}}$ *of* $\mathbb{N}$ *and a triple* $(\mu_{0,0}, \varphi_{0,0}, S_{0,0})$ *such that the following holds true:*

$$\alpha_{n_k}\,\mu_{\alpha_{n_k},\beta_{n_k}} \to 0 \quad \textit{strongly in } L^\infty(0,T;H), \tag{5.11}$$

$$\beta_{n_k}\,\partial_t\varphi_{\alpha_{n_k},\beta_{n_k}} \to 0 \quad \textit{strongly in } L^2(0,T;H), \tag{5.12}$$

$$\partial_t(\alpha_{n_k}\,\mu_{\alpha_{n_k},\beta_{n_k}} + \varphi_{\alpha_{n_k},\beta_{n_k}}) \to \partial_t\varphi_{0,0} \quad \textit{weakly in } L^2(0,T;V_A^{-\rho}), \tag{5.13}$$

$$\mu_{\alpha_{n_k},\beta_{n_k}} \to \mu_{0,0} \quad \textit{weakly in } L^2(0,T;V_A^{\rho}), \tag{5.14}$$

$$\varphi_{\alpha_{n_k},\beta_{n_k}} \to \varphi_{0,0} \quad \textit{weakly-star in } L^\infty(0,T;V_B^{\sigma}) \textit{ and strongly in } L^2(0,T;H), \tag{5.15}$$

$$S_{\alpha_{n_k},\beta_{n_k}} \to S_{0,0} \quad \textit{weakly-star in } H^1(0,T;V_C^{-\tau}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_C^{\tau}). \tag{5.16}$$

Moreover, $F_1(\varphi_{0,0}) \in L^\infty(0,T;L^1(\Omega))$*, and* $(\mu_{0,0}, \varphi_{0,0}, S_{0,0})$ *is a solution to* (2.28)–(2.30) *for* $\alpha = \beta = 0$ *that satisfies the initial conditions* (2.34)*.*

It seems difficult to prove a uniqueness result for the solution to the limiting problem under rather general assumptions. The arguments developed in [10] and [35] strongly use the fact that $A^{2\rho}$, $B^{2\sigma}$ and $C^{2\tau}$ are the same operator (namely, the Laplace operator with zero Neumann boundary conditions), and this kind of assumption is quite unpleasant in the context of the present paper. Thus, we prefer to keep the operators A, B and C and the exponents ρ, σ and τ independent from each other. To do this, we have to make restrictive assumptions and compatibility conditions on the structure of the system. We recall that λ'_j and L are the eigenvalues of B and the Lipschitz constant of f_2, respectively, and make the following requirement:

(A8) The following conditions are satisfied:

$F_1 \in C^1(\mathbb{R}), \quad |f_1(s)| \leq \widehat{c}_5(|s|^3 + 1)$ and $(f_1(s) - f_1(s'))(s - s') \geq \gamma\, |s - s'|^2$
for some constants $\widehat{c}_5$, $\gamma > 0$ and every $s, s' \in \mathbb{R}$. (5.17)

We have $(\lambda'_1)^{2\sigma} + \gamma > L$. (5.18)

It holds the continuous embedding $V_B^\sigma \subset L^4(\Omega)$. (5.19)

P is a positive constant P_0. (5.20)

Remark 5.2. The first condition excludes singular potentials and holds for the regular potential F_{reg} given by (1.5). As for the strong monotonicity condition in (5.17), one can split F into $F_1 + F_2$ according to (2.18)–(2.20) by setting

$$F_1(s) := \frac{1}{4}\, s^4 + \frac{1}{2}\, s^2 \quad \text{and} \quad F_2(s) := -s^2 + \frac{1}{4}\,.$$

Then, $f'_1(s) = F''_1(s) = 3s^2 + 1 \geq 1$ for every $s \in \mathbb{R}$, so that we can take $\gamma = 1$ in (5.17). The compatibility condition (5.18) is not satisfied by double-well potentials if $\lambda'_1 = 0$, since it reduces to $\gamma > L$ in this case and is satisfied only if F is convex. On the contrary, if we consider the above splitting of the regular potential F_{reg}, then we see that (5.18) holds if λ'_1 is large enough, namely if $\lambda'_1 > 1$, since $\gamma = 1$ and $L = 2$. If B is, e.g., the Laplace operator with zero Dirichlet boundary conditions, then this kind of assumption on λ'_1 is satisfied provided that Ω is small enough within a class of domains having the same shape. Finally, embeddings similar to (5.19) have already been commented in Remark 2.3, and (5.20) is not realistic in the framework of the tumor model, unfortunately.

Theorem 5.3. *Besides* **(A1)–(A4)***, assume that* **(A8)** *is satisfied as well. Then the problem* (2.28)–(2.31) *with* $\alpha = \beta = 0$ *has at most one solution satisfying* (2.23)–(2.27).

Proof. First of all, we prove that any solution (μ, φ, S) satisfies an equation like (1.2). The inequality (2.29) with $\beta = 0$ becomes

$$\big(B^\sigma\varphi(t), B^\sigma(\varphi(t) - w)\big) + \int_\Omega F_1(\varphi(t)) \leq \big(\mu(t) - f_2(\varphi(t)), \varphi(t) - w\big) + \int_\Omega F_1(w)$$

for a.e. $t \in (0, T)$ and every $w \in V_B^\sigma$. We can take, in particular, $w = \varphi(t) - \varepsilon v$ with any $v \in V_B^\sigma$ and $\varepsilon \in (0, 1)$. We then obtain that

$$\big(B^\sigma\varphi(t), B^\sigma v\big) + \int_\Omega \frac{F_1(\varphi(t)) - F_1(\varphi(t) - \varepsilon v)}{\varepsilon} \leq \big(\mu(t) - f_2(\varphi(t)), v\big).$$

Now, for fixed t, we apply the mean value theorem, the growth condition in (5.17) and the Young inequality. Using the integral remainder, we have a.e. in Ω that

$$\begin{aligned} &|F_1(\varphi(t)) - F_1(\varphi(t)-\varepsilon v)| = \left| \int_0^1 f_1(\varphi(t)-\theta\varepsilon v)\varepsilon v\, d\theta \right| \\ &\leq \widehat{c}_5\big((|\varphi(t)|+\varepsilon|v|)^3+1\big)\varepsilon\,|v| \leq \varepsilon\, C(|\varphi(t)|^4+|v|^4+1)\,, \end{aligned}$$

with some constant C proportional to $\widehat{c}_5$. Since $|\varphi(t)|^4 + |w|^4 \in L^1(\Omega)$ thanks to the embedding (5.19), we can apply the Lebesgue dominated convergence theorem and let ε tend to zero. We deduce an inequality holding for every $v \in V_B^\sigma$. Since v is arbitrary, we obtain the equality

$$\big(B^\sigma\varphi(t), B^\sigma v\big) + \int_\Omega f_1(\varphi(t))\, v = \big(\mu(t) - f_2(\varphi(t)), v\big), \tag{5.21}$$

which is valid for a.e. $t \in (0,T)$ and every $v \in V_B^\sigma$.

We notice that the integral in (5.21) cannot be written as $(f_1(\varphi(t)), v)$, since we do not know that $f_1(\varphi(t))$ belongs to H. However, it belongs to $L^{4/3}(\Omega)$, since $\varphi(t) \in V_B^\sigma \subset L^4(\Omega)$ and the growth condition in (5.17)) is in force. For this reason, if we also assume that V_B^σ is dense in $L^4(\Omega)$, then we can write (1.2) in the sense of $V_B^{-\sigma}$ by accounting for the consequent embedding $L^{4/3}(\Omega) \subset V_B^{-\sigma}$. However, we will use just (5.21) as it is.

We are ready to prove uniqueness. We pick two solutions (μ_i, φ_i, S_i), $i = 1, 2$, and set for convenience $\mu := \mu_1 - \mu_2$, $\varphi_1 - \varphi_2$, and $S := S_1 - S_2$. We write (2.28) and (2.30) for both solutions and take the differences. By recalling (5.20), we arrive at the identities

$$\begin{aligned} &\langle\, \partial_t\varphi(t), v\,\rangle_{V_A^\rho} + \big(A^\rho\mu(t), A^\rho v\big) \;=\; P_0\big(S(t)-\mu(t), v\big), \\ &\langle\, \partial_t S(t), v\,\rangle_{V_C^\tau} \;+\; \big(C^\tau S(t), C^\tau v\big) \;=\; -P_0\big(S(t)-\mu(t), v\big), \end{aligned}$$

which hold for every $v \in V_A^\rho$ and $v \in V_C^\tau$, respectively. Now, we integrate these equations with respect to time and, for X Banach space and $w \in L^1(0,T;X)$, we use the notation

$$(1 * w)(t) := \int_0^t w(s)\, ds \quad \text{for every } t \in [0,T].$$

Hence, we obtain

$$\begin{aligned} &\big(\varphi(t), v\big) + \big(A^\rho(1*\mu)(t), A^\rho v\big) \;=\; P_0\big((1*(S-\mu))(t), v\big), \\ &\big(S(t), v\big) \;+\; \big(C^\tau(1*S)(t), C^\tau v\big) \;=\; -P_0\big((1*(S-\mu))(t), v\big), \end{aligned}$$

for the same test functions v as before. At this point, we choose $v = \mu(t)$ and $v = S(t)$ in these identities, respectively. Then, we integrate with respect to time, sum up and rearrange. We deduce, for every $t \in [0,T]$, that

$$\begin{aligned} &\frac12\,\|A^\rho(1*\mu)(t)\|^2 + \int_0^t \|S(s)\|^2\, ds \\ &+ \frac12\,\|C^\tau(1*S)(t)\|^2 + \frac{P_0}{2}\,\|1*(S-\mu))(t)\|^2 = -\int_0^t \big(\varphi(s), \mu(s)\big)\, ds. \end{aligned} \tag{5.22}$$

Next, we write equation (5.21) for both solutions and choose $v = \varphi(t)$ in the difference. By integrating with respect to time, we infer that

$$\int_0^t \|B^\sigma \varphi(s)\|^2\,ds + \int_{Q_t} \big(f_1(\varphi_1) - f_1(\varphi_2)\big)\varphi = \int_0^t \big(\mu(s), \varphi(s)\big)\,ds - \int_{Q_t} \big(f_2(\varphi_1) - f_2(\varphi_2)\big)\varphi.$$

Now, we account for the obvious inequality $\|B^\sigma v\|^2 \geq (\lambda_1')^{2\sigma}\|v\|^2$, the assumption (5.17), and the Lipschitz continuity of f_2 (cf. (2.19) and (2.22)), in order to deduce that

$$(\lambda_1')^{2\sigma} \int_0^t \|\varphi(s)\|^2\,ds + \gamma \int_{Q_t} |\varphi|^2 \leq \int_0^t \big(\mu(s), \varphi(s)\big)\,ds + L \int_{Q_t} |\varphi|^2 \tag{5.23}$$

for every $t \in [0, T]$. Now, we add (5.22) and (5.23), note that there is a cancellation and finally apply (5.18). Hence, we conclude in particular that $\varphi = 0$, $S = 0$, $1 * (S - \mu) = 0$. The latter implies that $S - \mu = 0$, whence $\mu = 0$ as well. We have thus proved that $\varphi_1 = \varphi_2$, $S_1 = S_2$ and $\mu_1 = \mu_2$. □

Remark 5.4. The same assumption **(A8)** (possibly reinforced by also supposing that f_1 and f_2 are C^1 functions) can be used to prove a regularity result in the case $\alpha > 0$ and $\beta = 0$. Here we sketch a formal proof under suitable assumptions on the initial data, by observing that **(A8)** ensures the validity of (5.21) also in this case. Indeed, the argument used in the proof of Theorem 5.3 to derive (5.21) only regards the variational inequality satisfied by φ for $\beta = 0$ and thus still holds if $\alpha > 0$. We differentiate (2.28) and (2.30) with respect to time and test the equalities we get by $\partial_t \mu$ and $\partial_t S$, respectively. At the same time, we test the time derivative of (5.21) by $\partial_t \varphi$. Then, we sum up and integrate over $(0, t)$. The terms involving the product $(\partial_t \mu, \partial_t \varphi)$ cancel each other, and we obtain (by omitting the integration variable s to shorten the lines) the identity

$$\begin{aligned} &\alpha \int_0^t \|\partial_t \mu\|^2\,ds + \frac{1}{2}\,\|A^\rho \mu(t)\|^2 + \int_0^t \|B^\sigma \partial_t \varphi\|^2\,ds + \int_{Q_t} f_1'(\varphi)\,|\partial_t \varphi|^2 \\ &\quad + \int_0^t \|\partial_t S\|^2\,ds + \frac{1}{2}\,\|C^\tau S(t)\|^2 + \frac{P_0}{2}\,\|S(t) - \mu(t)\|^2 \\ &= \frac{1}{2}\,\|A^\rho \mu_0\|^2 + \frac{1}{2}\,\|C^\tau S_0\|^2 + \frac{P_0}{2}\,\|S_0 - \mu_0\|^2 - \int_{Q_t} f_2'(\varphi)\,|\partial_t \varphi|^2 . \end{aligned}$$

Now, from one side, we have that $\|B^\sigma \partial_t \varphi\|^2 \geq (\lambda_1')^{2\sigma}\|\partial_t \varphi\|^2$. On the other hand, (5.17) and (2.19) imply that $f_1'(\varphi) \geq \gamma$ and $|f_2'(\varphi)| \leq L$ a.e. in Q. Therefore, we derive, for every $\delta > 0$, that

$$\begin{aligned} &\int_0^t \|B^\sigma \partial_t \varphi\|^2\,ds + \int_{Q_t} f_1'(\varphi)\,|\partial_t \varphi|^2 + \int_{Q_t} f_2'(\varphi)\,|\partial_t \varphi|^2 \\ &\geq \delta \int_0^t \|B^\sigma \partial_t \varphi\|^2\,ds + \big((1-\delta)(\lambda_1')^{2\sigma} + \gamma - L\big) \int_0^t \|\partial_t \varphi\|^2\,ds\,. \end{aligned}$$

By choosing $\delta > 0$ such that $(1-\delta)(\lambda_1')^{2\sigma} + \gamma > L$ on account of (5.18), we conclude that

$$\begin{aligned} &\|\partial_t \mu\|_{L^2(0,T;H)} + \|A^\rho \mu\|_{L^\infty(0,T;V_A^\rho)} + \|\partial_t \varphi\|_{L^2(0,T;V_B^\sigma)} \\ &\quad + \|\partial_t S\|_{L^2(0,T;H)} + \|C^\tau S\|_{L^\infty(0,T;H)} + \|S - \mu\|_{L^\infty(0,T;H)} \leq C\,, \end{aligned}$$

where C depends only on the structural assumptions and the norms of the initial data involved in the calculation.

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