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Abstract

We take a pathwise approach to classical McKean-Vlasov stochastic differential equations with additive noise, as e.g. exposed in Sznitmann [34]. Our study was prompted by some concrete problems in battery modelling [19], and also by recent progress on rough-pathwise McKean-Vlasov theory, notably Cass–Lyons [9], and then Bailleul, Catellier and Delarue [4]. Such a “pathwise McKean-Vlasov theory” can be traced back to Tanaka [36]. This paper can be seen as an attempt to advertize the ideas, power and simplicity of the pathwise approach, not so easily extracted from [4, 9, 36]. As novel applications we discuss mean field convergence without a priori independence and exchangeability assumption; common noise and reflecting boundaries. Last not least, we generalize Dawson–Gärtner large deviations to a non-Brownian noise setting.

1 Introduction

We consider the following *generalized* McKean-Vlasov stochastic differential equation (SDE) on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$,

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t))dt + dW_t \\ X_0 = \zeta. \end{cases} \quad (1)$$

The input data to the problem is the random variable

$$(\zeta, W) : \Omega \rightarrow \mathbb{R}^d \times C_T,$$

and

$$X : \Omega \rightarrow C_T := C([0, T], \mathbb{R}^d)$$

is the solution (process). We denote by $\mathcal{L}(Y)$ the law of a random variable Y . Classically, one takes W as a Brownian Motion. For us, it will be crucial to avoid any a priori specification of the noise. Indeed, we are not even asking for any filtration on the space Ω and equation (1) will be studied pathwise. For a $p \in [1, \infty)$, let $\mathcal{P}_p(\mathbb{R}^d)$ be the space of probability measures on \mathbb{R}^d with finite p -moment endowed with the p -Wasserstein metric. The drift is a function

$$b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{R}^d,$$

which is assumed uniformly Lipschitz continuous in the last two variables, cf. Assumption 1 below.

In a nutshell, McKean-Vlasov equations are SDEs which depend on the law of the solution. They have been extensively studied in the literature, for a comprehensive introduction we refer to [34]. They arise in many applications as limit of systems of interacting particles, for instance in the theory of mean field games developed by Lasry and Lions [23–25]. Other interesting

applications arise in fluid-dynamics [5, 17, 28], also with common noise features, and neuroscience [13, 27, 37] and macroeconomics [29], also involving general driving signals. Last not least, our motivation, also with regard to reflecting boundary conditions (a feature out of reach of present rough path machinery), comes from battery modelling [19].

Closely related to the McKean-Vlasov equation is the system of particles (classically) driven by independent Brownian motions W^i , with independent identically distributed (i.i.d.) initial conditions ζ^i ,

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, L^N(X_t^{(N)}))dt + dW_t^i \\ X_0^{i,N} = \zeta^i, \end{cases} \quad i = 1, \dots, N. \quad (2)$$

The particles interact with each other through the empirical measure, which is defined as follows. Given a space E (such as \mathbb{R}^d or C_T) and a vector $x^{(N)} = (x^1, \dots, x^N) \in E^N$, we call $\mathcal{P}(E)$ the space of probability measures over E and we define

$$L^N(x^{(N)}) := \frac{1}{N} \sum_{i=1}^N \delta_{x^i} \in \mathcal{P}(E).$$

Let X be a process solution to Equation (1) with inputs (ζ, W) distributed as (ζ^1, W^1) . When the number of particles, N , grows to infinity, we have the following a.s. convergence in $\mathcal{P}(C_T)$ equipped with the usual weak-* topology,

$$L^N(X^{(N)}(\omega)) \xrightarrow{*} \mathcal{L}(X), \quad \text{for } \mathbb{P} - \text{a.e. } \omega. \quad (3)$$

This result, as well as the well-posedness of equation (1) is proved in [34] when the particles are exchangeable and subjects to independent inputs. This approach can be generalized to more general diffusion coefficients [21, 22] using standard semi-martingale theory.

Cass and Lyons [9] studied McKean-Vlasov equations in the framework of rough paths. Thanks to rough-path theory they can study (rough) pathwise solutions to the McKean-Vlasov equation and this lets them go beyond the classical case when W is a semi-martingale under \mathbb{P} . They could treat the case when in equation (1) a diffusivity coefficient σ appears in front of the noise. More precisely, they treated the case when the dependence of b in the measure is linear and the diffusivity σ is independent from the law. This problem was revisited by Bailleul [3] in the case of a Lipschitz dependence of b on the measure. Finally, Bailleul et al. [4] studied the general case for both the drift and the diffusivity Lipschitz dependent on the law of the solution.

At least in the context of battery modelling with additive noise [19], no rough path machinery should be necessary, leave alone some formidable difficulties for rough differential equations to deal with reflecting boundaries [1, 15]. This was the initial motivation for our pathwise study, which soon turned out informative and rather pleasing in the generality displayed here. As our work neared completion we realized that we were not the first to go in this direction: the basic idea can be found (somewhat hidden) in a paper by Tanaka, [36, Sec.2]. (There is no shortage of citations to [36], but we are unaware of any particular work that makes use of the, for us, crucial Section 2 in that paper.) May that be as it is, advertising this aspect of Tanaka's work, as pathwise ancestor to [3, 4, 9], is another goal of this note, and in any case there is no significant overlap of our results with [36].

The main intuition in [36] and subsequent works is that equation (2) can be interpreted as equation (1) by using a transformation of the probability space and the input data. We explain this connection between the equations in Section 3.1. This approach makes it possible to reduce the study of the mean field limit to a stability result for equation (1). This implies in particular that

there is no need for asymptotical independence or exchangeability of the particles in order to obtain convergence (3). Indeed, one can show that the solution map

$$\mathcal{L}(\zeta, W) \mapsto \mathcal{L}(X)$$

that associates the law of the solution to the law of the inputs is continuous, and as soon as there is convergence for the law of the input data there is also convergence for the law of the solution. No independence, nor identical distributions (or even exchangeability) for the inputs are required, as we explain in Sections 3.1 and 3.3.

Main ideas

Given a Polish space E , we work on the space of probability measures with finite p -th moment, $\mathcal{P}_p(E)$, endowed with the Wasserstein distance \mathcal{W}_p (see Section 1.1 for the precise definition). The idea is to construct the solution map of equation (1), for a generic probability measure μ ,

$$\Phi : \mathcal{P}_p(\mathbb{R}^d \times C_T) \times \mathcal{P}_p(C_T) \rightarrow \mathcal{P}_p(C_T), \quad (\mathcal{L}(\zeta, W), \mu) \mapsto \mathcal{L}(X^\mu). \quad (4)$$

Here X^μ is the pathwise solution to equation (1) when the inputs are ζ, W and the measure in the drift is given as μ , instead of the law of X . Existence and uniqueness of the solutions of the McKean-Vlasov equation (1) follow as a fixed point argument of the parameter dependent map Φ . Indeed, one can prove that, for fixed (ζ, W) , the map $\Phi(\mathcal{L}(\zeta, W), \cdot)$ is a contraction on the space $\mathcal{P}_p(C_T)$. Hence, there is a unique fixed point $\bar{\mu} := \bar{\mu}(\mathcal{L}(\zeta, W)) = \Phi(\mathcal{L}(\zeta, W), \bar{\mu})$. This fixed point uniquely determines a pathwise solution $X^{\bar{\mu}}$ to equation (1).

Since Φ is Lipschitz continuous in all its arguments, it follows from Proposition 10 that also the map that associates the parameter to the fixed point, namely Ψ defined in (16) is Lipschitz continuous. This is the stability result that we need in order to prove convergence of the particle system.

Main results.

In this setting, we obtain the following list of results.

Theorem 1 (see Theorem 11). *Let $p \in [1, \infty)$ and assume b Lipschitz. For $i = 1, 2$, let $(\zeta^i, W^i) \in L^p(\mathbb{R}^d \times C_T, \mathbb{P}^i)$ be two sets of input data. There exist unique pathwise solutions $X^i \in L^p(C_T)$ to equation (1), driven by the respective input data. Moreover,*

$$\mathcal{W}_p(\mathcal{L}(\zeta^1, W^1, X^1), \mathcal{L}(\zeta^2, W^2, X^2)) \leq C \mathcal{W}_p(\mathcal{L}(\zeta^1, W^1), \mathcal{L}(\zeta^2, W^2)),$$

for some constant $C = C(p, T, b) > 0$.

We obtain a similar results for the case when the driver W is a random variable over the càdlàg space D_T .

Lemma 2 (see Lemmas 20 and 21). *Let*

$$\Phi : \mathcal{P}(\mathbb{R}^d \times D_T) \times \mathcal{P}(D_T) \rightarrow \mathcal{P}(D_T), \quad (\mathcal{L}(\zeta, W), \mu) \mapsto \mathcal{L}(X^\mu).$$

be defined similarly as (4). The fixed point map $\Psi : \nu \mapsto \mu = \phi(\nu, \mu)$ is continuous with respect to the weak convergence of measures.

We note that, in the case of jump processes, we have only weak continuity of the law of the solution with respect to the law of the inputs. We don't prove Lipschitz continuity with respect to the stronger Wasserstein norm \mathcal{W}_p .

As application off the main result, we have

Corollary 3 (see Theorem 22). *Consider the N -particle system (2) with (not necessarily Brownian! not necessarily independent!) random driving noise $W^{(N)} := (W^{1,N}, \dots, W^{N,N})$ and initial data $\zeta^{(N)} := (\zeta^{1,N}, \dots, \zeta^{N,N})$. Assume convergence (in p -Wasserstein sense) of the empirical measure*

$$L^N(\zeta^{(N)}(\omega), W^{(N)}(\omega)) \rightarrow \nu \in \mathcal{P}_p(\mathbb{R}^d \times C_T)$$

for a.e. ω (resp. in probability) w.r.t. \mathbb{P} . Then the empirical measure $L^N(X^{(N)})$ of the particle system converges in the same sense and the limiting law is characterized by a generalized McKean-Vlasov equation, with input data distributed like ν .

Natural non-i.i.d. situations arises in presence of common noise, cf. Section 3.3, or in the presence of heterogeneous inputs, cf. Section 3.4. In an i.i.d. setting, the required assumption is (essentially trivially) verified by the law of large number. Independent driving fractional Brownian motions, for instance, are immediately covered. Another consequence concerns the large deviations.

Definition 4. Let E be a Polish space and $(\mu^N)_{N \in \mathbb{N}}$ a sequence of Borel probability measures on E . Let $(a_N)_{N \in \mathbb{N}}$ be a sequence of positive real numbers with $\lim_{N \rightarrow \infty} a_N = \infty$. Given a lower semicontinuous function $I : E \rightarrow [0, \infty]$, the sequence μ^N is said to satisfy a *large deviations principle* with rate I if, for each Borel measurable set $A \subset E$,

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{N \rightarrow \infty} a_N^{-1} \log(\mu^N(A)) \leq \limsup_{N \rightarrow \infty} a_N^{-1} \log(\mu^N(A)) \leq -\inf_{x \in \bar{A}} I(x).$$

Here A° is the interior of A and \bar{A} its closure. Moreover, if the sublevel sets of I are compact, then I is said to be a *good* rate function.

We say that a sequence of random variables $(X^N)_{N \in \mathbb{N}}$ on E satisfies a large deviations principle, if the sequence of the distributions $(\mathcal{L}(X^N))_{N \in \mathbb{N}}$ does.

The following generalizes a classical result of Dawson–Gärtner [12], see also Deuschel et al. [14].

Corollary 5 (see Theorem 42). *In the i.i.d. case, the empirical measure $L^N(X^{(N)})$ satisfies a large deviations principle with rate function, defined on a suitable Wasserstein space over C_T ,*

$$\mu \mapsto H(\mu \mid \Phi(\mathcal{L}(\zeta, W), \mu)),$$

where H is the relative entropy and Φ is introduced below.

This result is consistent with the one obtained in [36, Theorem 5.1], for the case of drivers given as i.i.d. Brownian motions.

One can easily drop the i.i.d. assumption, and replace H by an “assumed” large deviations principle I for the convergence of the input laws. In this case the outputs satisfy a large deviations principle.

Corollary 6 (see Lemma 40). *If the empirical measure of the inputs $L^N(\zeta^{(N)}, W^{(N)})$ satisfies a large deviations principle with (good) rate function I , then the empirical measure $L^N(X^{(N)})$ satisfies a large deviations principle with (good) rate function $\mu \mapsto I(f_{\#}^{\mu}\mu)$, defined on a suitable Wasserstein space over C_T . Here f^{μ} is defined in (41).*

Think of f^{μ} as the function that reconstruct the inputs (initial condition, driving path) from the solution of an ordinary differential equation (ODE).

The method presented here can be also applied to SDE defined in a domain $D \subset \mathbb{R}^d$, assumed to be a convex polyhedron for simplicity, and with reflection at the boundary. We consider the generalized McKean-Vlasov Skorokhod problem

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t, k_t))dt + dW_t - dk_t, & X_0 = \zeta, \\ d|k|_t = 1_{X_t \in \partial D} d|k|_t, & dk_t = n(X_t) d|k|_t. \end{cases} \quad (5)$$

We have the following:

Theorem (see Theorem 33). *Let $p \in [1, \infty)$ and assume b Lipschitz. For $i = 1, 2$, let $(\zeta^i, W^i) \in L^p(\bar{D} \times C_T, \mathbb{P}^i)$ be two sets of input data. Then there exist unique pathwise solutions (X^i, k^i) to the generalized McKean-Vlasov Skorokhod problem (5), driven by the respective input data. Moreover,*

$$\mathcal{W}_p(\mathcal{L}(\zeta^1, W^1, X^1, k^1), \mathcal{L}(\zeta^2, W^2, X^2, k^2)) \leq C \mathcal{W}_p(\mathcal{L}(\zeta^1, W^1), \mathcal{L}(\zeta^2, W^2)).$$

with $C = C(p, T, b) > 0$.

Battery modelling

Our initial motivation for the heterogeneous particles case comes from modeling lithium-ion batteries. The numerical simulations of [19] indicate that the capacity of the battery and its efficiency is mainly determined by the size distribution of the lithium iron phosphate particles. It is thus important to allow for the particles to be of fixed different, predetermined sizes.

Lithium-ion batteries are the most promising storage devices to store and convert chemical energy into electrical energy and vice versa. In [19] lithium-ion batteries are studied where at least one of the two electrodes stores lithium within a many-particle ensemble, for example each particle of the electrode is made of Lithium-iron-phosphate. One of the practical achievements of [19] consists of the conclusions that the capacity of the battery and its efficiency as well is dominantly determined by the size distribution of the storage particles, ranging from 20 to 1000 nanometers. The radii r^i of the particles in the battery are distributed according to a distribution $\lambda \in \mathcal{P}([20, 1000])$. However, in the numerical simulations, it leads to better accuracy to artificially choose the radii in advance, instead of randomly sample them. For instance, assume that we want to simulate 1000 particles, whose radii can be of exactly two given sizes, r_1, r_2 , with equal probability. It is much more convenient to choose 500 particles of radius r_1 and 500 of radius r_2 , instead of sampling them from a binomial, as this could lead to imbalanced simulations and introduce an extra source of error. For this reason it is important that the theoretical results support the use of carefully chosen radii r^i of different length, such their empirical measure converges, as the number of particles grows, to a desired distribution λ . The radii so chosen, are deterministic (hence, independent), but not identically distributed.

The dynamics of the charging/discharging process is modeled in [19] by a coupled system of SDEs for the evolution of the lithium mole fractions $Y^{i,N} \in [0, 1]$ of particles $i = 1, 2, \dots, N$

of the particle ensemble. The evolution of $Y^{i,N}$ over a time interval $[0, T]$ is described by the following system of SDEs

$$\begin{cases} dY_t^{i,N} = \frac{1}{\tau_i}(\Lambda_{t,Li} - \mu_{Li}(Y_t^{i,N}))dt + dW_t^i - dk_t^{i,N} \\ d|k_t^{i,N}| = 1_{Y_t^{i,N} \in \{0,1\}} d|k_t^{i,N}|, \quad dk_t^{i,N} = n(Y_t^{i,N})d|k_t^{i,N}|, \quad i = 1, \dots, N, \\ Y_0^{i,N} = a \in [0, 1]. \end{cases} \quad (6)$$

We assume that all the particles have the same amount of lithium mole fraction $a \in [0, 1]$ at time $t = 0$. In practice, this initial condition is very close to 1, when the battery is empty and very close to 0, when the battery is charged. The particles are driven by a family of independent Brownian motions $W^{(N)} := (W^i)_{1 \leq i \leq N}$, which account for random fluctuations that can occur within the system during charging and discharging. The quantity $\tau_i \equiv \tau(r_i)$, which is related to the relaxation time, is a function of the radius r_i of the particle. As discussed earlier, the radii can only have values in a fixed range $I := [r_{min}, r_{max}] \subset (0, \infty)$. We assume that $\tau^{-1} : I \rightarrow \mathbb{R}$ is Lipschitz and bounded. The term $\mu_{Li} : \mathbb{R} \rightarrow \mathbb{R}$ is the chemical potential of the Lithium and, in this framework, it is also taken Lipschitz and bounded. The interaction between particles is encoded in the surface chemical potential

$$\Lambda_{t,Li} := q_t + \frac{1}{N} \sum_{j=1}^N \frac{\mu_{Li}(Y_t^{j,N})}{\tau_j},$$

where q_t is a given measurable and bounded function characterizing the state of charge of the battery at time $t \in [0, T]$. By the assumptions on μ_{Li} and τ , the surface chemical potential is a bounded and Lipschitz continuous function of the empirical distribution of the Lithium mole fractions and radii, $\mu^N := \frac{1}{N} \sum_{i=1}^N \delta_{(Y_t^i, r^i)}$. Moreover, we impose on the particles Skorokhod-type boundary conditions, of the same type as the ones described in Section 4. We call $n(x)$ the outer unit normal vector, which, in this case, reduces to $n(x) = (-1)^{x+1}$, for $x \in \{0, 1\}$. This will force the mole fraction of each particle to remain in $[0, 1]$.

Under the previous assumptions, the particle system (6) can be essentially treated combining the results of Theorem 33 and Corollary 28, as follows. To unify the notation to the rest of the paper, we define

$$b : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}(\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{R}, \quad (t, x, r, \nu) \mapsto q_t + \langle \nu, \frac{\mu_{Li}}{\tau} \rangle - \frac{\mu_{Li}(x)}{\tau(r)},$$

and we consider the following *generalized* McKean-Vlasov Skorokhod equation

$$\begin{cases} dX_t = b(t, X_t, R, \mathcal{L}(X_t, R))dt + dW_t - dk_t, \quad X_0 = a, \\ d|k|_t = 1_{X_t \in \partial D} d|k|_t, \quad dk_t = n(X_t) d|k|_t, \end{cases} \quad (7)$$

The input data are given by $a \in [0, 1]$, $R \in L^p(I)$ and $W \in L^p(C_T)$, for $p \in [1, \infty)$. The solution is a couple $(X, k) \in C_T([0, 1]) \times C_T$. When $W := W^{(N)} \in L^p(\Omega_N, C_T)$ and $R = R^{(N)} = (r^1, \dots, r^N) \in L^p(\Omega_N, C_T(I))$ (the radii are constant path in I), we recover the system (6). We assume that the radii r^i are sampled from a distribution $\lambda \in \mathcal{P}_p(C_T(I))$ in such a way that $L^N(R^{(N)}) \xrightarrow{*} \lambda$, this gives the limit process (X, k) solution to (7), driven by $(W, R) \in C_T \times C_T(I)$ with law $\mu_W \otimes \lambda$ (here μ_W is the Wiener measure). We summarize this in the following proposition.

Proposition 7. *Let $p \in [1, \infty)$ and let $(W^i)_{i \in \mathbb{N}}$ be a family of independent Brownian motions on \mathbb{R} . Assume $I \subset (0, \infty)$ is a closed interval and let $(r^i)_{i \in \mathbb{N}} \subset I$ be a sequence in I , such that*

$$L^N(R^{(N)}) \xrightarrow{*} \lambda \in \mathcal{P}_p(I).$$

Then, for every $N \in \mathbb{N}$, equation (6) admits a unique solution $(Y^{(N)}, k^{(N)}) := (Y^{i,N}, k^{i,N})_{i=1,\dots,N}$. Moreover,

$$L^{(N)}(Y^{(N)}, k^{(N)}) \xrightarrow{*} \mathcal{L}(X, k),$$

where (X, k) is a solution to equation (7), with input data $(W, R) \in C_T \times C_T(I)$ with law $\mu_{\mathcal{W}} \otimes \lambda$.

The proof of this proposition (which we will not give in full details) follows exactly as the proof of Corollary 28, with the difference that, instead of Theorem 11, one applies Theorem 33.

Structure of the paper.

In Section 2 we prove the well-posedness for the *generalized* McKean-Vlasov equation (1). In Section 3 we present applications to classical mean field particle approximation, heterogeneous mean field and mean field with common noise as corollaries of the main result. Finally, we adapt the result to study McKean-Vlasov equations with reflection at the boundary, see Section 4 and we show a (classical) large deviations result as a straightforward application in Section 5.

1.1 Notation

Given p in $[1, +\infty)$ and a Polish space E , with metric induced by a norm $\|\cdot\|_E$, we denote by $\mathcal{P}_p(E)$ the space of probability measures on E with finite p -moment, namely the measures μ such that

$$\int_E \|x\|_E^p d\mu(x) < +\infty.$$

For $T > 0$, we denote by $C_T(\mathbb{R}^d) := C([0, T], \mathbb{R}^d)$ (the space of continuous functions from $[0, T]$ to \mathbb{R}^d), endowed with the supremum norm $\|f\|_{\infty:T} := \sup_{t \in [0, T]} |f(t)|$, for $f \in C_T(\mathbb{R}^d)$. When there is no risk of confusion about the codomain, we denote the space of continuous functions by C_T . Moreover, when there is non risk of confusion about the time interval, we use the lighter notation $\|\cdot\|_{\infty}$. Moreover, we call $C_{T,0} = \{\gamma \in C_T \mid \gamma_0 = 0\}$, the subsets of paths that vanish at time 0.

For a domain \bar{D} in \mathbb{R}^d , we denote by $C_T(\bar{D}) := C([0, T], \bar{D})$ (continuous functions from $[0, T]$ to \bar{D}), endowed with the supremum norm $\|\cdot\|_{\infty}$.

Given $t \in [0, T]$, the projection π_t is defined as the function $\pi_t : C_T \rightarrow \mathbb{R}^d$ as $\pi_t(\gamma) := \gamma(t)$. We define the marginal at time t of $\mu \in \mathcal{P}_p(C_T)$ as $\mu_t := (\pi_t)_\# \mu \in \mathcal{P}_p(\mathbb{R}^d)$. We also denote by $\mu|_{[0,t]}$ the push forward of μ with respect to the restriction on the subinterval $[0, t]$.

Given a Polish space (E, d) , the p -Wasserstein metric on $\mathcal{P}_p(E)$ is defined as

$$\mathcal{W}_{E,p}(\mu, \nu)^p = \inf_{m \in \Gamma(\mu, \nu)} \iint_{E \times E} d(x, y)^p m(dx, dy), \quad \mu, \nu \in \mathcal{P}_p(E), \quad (8)$$

where $\Gamma(\mu, \nu)$ is the space of probability measures on $E \times E$ with first marginal equal to μ and second marginal equal to ν . We will omit the space E from the notation when there is no confusion.

We denote by $\mathcal{L}(X)$ the law of a random variable X .

We use C_p to denote constants depending only on p .

Let C_c^∞ and C^n be the set of infinitely differentiable real-valued functions of compact support defined on \mathbb{R}^d and the set of n times continuously differentiable functions on \mathbb{R}^d such that

$$\|\varphi\|_{C^n} := \sum_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} |D^\alpha \varphi| < +\infty.$$

Let Lip_1 be the space of Lipschitz continuous functions in C^0 , such that

$$\|\varphi\|_{C^0}, \sup_{x \neq y \in \mathbb{R}^d} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leq 1.$$

For $T > 0$, we denote by $D_T(\mathbb{R}^d) := D([0, T], \mathbb{R}^d)$, the space of càdlàg functions (right-continuous with left limit) from $[0, T]$ to \mathbb{R}^d . When there is no risk of confusion about the codomain, we denote the space of cadlag functions by D_T . For $\gamma \in D_T(\mathbb{R}^d)$, $\|\gamma\|_{\infty; T} := \sup_{t \in [0, T]} |f(t)|$. Moreover, when there is no risk of confusion about the time interval, we use the lighter notation $\|\cdot\|_\infty$. We endow D_T with the Skohorod metric, defined as follows

$$\sigma(\gamma, \gamma') = \inf \{ \lambda \in \Lambda \mid \|\lambda\| + \|\gamma - \gamma' \circ \lambda\|_\infty \}, \quad \gamma, \gamma' \in D_T, \quad (9)$$

where Λ is the space of strictly increasing bijections on $[0, T]$ and

$$\|\lambda\| := \sup_{s \neq t} \left| \log \left(\frac{\lambda_{s,t}}{t - s} \right) \right|, \quad \lambda \in \Lambda.$$

The space (D_T, σ) is a Polish space.

2 The main result

In this section we study the *generalized* McKean-Vlasov SDE on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$,

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t))dt + dW_t \\ X_0 = \zeta. \end{cases} \quad (10)$$

Here the drift $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ is a given Borel function, the input to the problem is the random variable

$$(\zeta, W) : \Omega \rightarrow \mathbb{R}^d \times C_T,$$

and $X : \Omega \rightarrow C_T$ is the solution. As we will see later, the law $\mathcal{L}(X)$ of the solution depends only on the law $\mathcal{L}(\zeta, W)$, for this reason we refer also to $\mathcal{L}(\zeta, W)$ as input.

Note two differences here with respect to classical SDEs: the drift depends on the solution X also through its law and W is merely a random continuous paths; in particular, it does not have to be a Brownian motion. For these differences, it is worth giving the precise definition of solution.

Definition 8. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\zeta : \Omega \rightarrow \mathbb{R}^d$, $W : \Omega \rightarrow C_T$ be random variables on it. A solution to equation (10) with input (ζ, W) is a random variable $X : \Omega \rightarrow C_T$ such that, for a.e. ω , the function $X(\omega)$ satisfies the following integral equality

$$X_t(\omega) = \zeta(\omega) + \int_0^t b(s, X_s(\omega), \mathcal{L}(X_s))ds + W_t(\omega).$$

We assume the following conditions on b :

Assumption 1. Let $p \in [1, \infty)$. The drift $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ is a measurable function and there exists a constant K_b such that,

$$|b(t, x, \mu) - b(t, x', \mu')|^p \leq K_b (|x - x'|^p + \mathcal{W}_{\mathbb{R}^d, p}(\mu, \mu')^p),$$

$$\forall t \in [0, T], x, x' \in \mathbb{R}^d, \mu, \mu' \in \mathcal{P}_p(\mathbb{R}^d).$$

Before giving the main result, we introduce some notation. For a given μ in $\mathcal{P}_p(C_T)$, we consider the SDE

$$\begin{cases} dY_t^\mu = b(t, Y_t^\mu, \mu_t)dt + dW_t \\ Y_0^\mu = \zeta. \end{cases} \quad (11)$$

We have the following well-posedness result

Lemma 9. Under Assumption 1, for every input $(\zeta, W) \in L^p(\mathbb{R}^d \times C_T)$ and $\mu \in \mathcal{P}_p(C_T)$, there exists a unique $Y^\mu \in L^p(C_T)$ which satisfies, $\forall \omega \in \Omega$,

$$Y_t^\mu(\omega) = \zeta(\omega) + \int_0^t b(s, Y_s^\mu(\omega), \mu_s)ds + W_t(\omega).$$

Moreover, denote by

$$\begin{aligned} S^\mu : \mathbb{R}^d \times C_T &\rightarrow C_T \\ (x_0, \gamma) &\mapsto S^\mu(x_0, \gamma), \end{aligned} \quad (12)$$

where $S^\mu(x_0, \gamma)$ is a solution to the ODE

$$x_t = x_0 + \int_0^t b(s, x_s, \mu_s)ds + \gamma_t. \quad (13)$$

Then, $Y^\mu = S^\mu(\zeta, W)$.

Proof. For every couple $(x_0, \gamma) \in \mathbb{R}^d \times C_T$ the ODE (13) classically admits a solution $S^\mu(x_0, \gamma)$, which is continuous with respect to the inputs (x_0, γ) . It is easy to verify that $S^\mu(\zeta, W)$ solves equation (11). We only verify that Y^μ has finite p -moments. There exists a constant $C(p, b, T)$ such that

$$\mathbb{E}\|Y^\mu\|_\infty^p \leq \mathbb{E}|\zeta|^p + C \left(1 + \int_0^T \mathbb{E} \sup_{s \in [0, t]} |Y_s^\mu|^p dt + \int_0^T \int_{\mathbb{R}^d} |x|^p d\mu_t(x) dt \right) + \mathbb{E}\|W\|_\infty^p.$$

We notice that $\int_{\mathbb{R}^d} |x|^p \mu_t(dx) \leq \int_{C_T} \|\gamma\|_\infty^p d\mu(\gamma) < +\infty$. Gronwall's inequality and the assumptions on (ζ, W) conclude the proof. \square

We call

$$\begin{aligned} \Phi : \mathcal{P}_p(\mathbb{R}^d \times C_T) \times \mathcal{P}_p(C_T) &\rightarrow \mathcal{P}_p(C_T) \\ (\mathcal{L}(\zeta, W), \mu) &\mapsto \mathcal{L}(Y^\mu) = (S^\mu)_\# \mathcal{L}(\zeta, W), \end{aligned} \quad (14)$$

the push forward of a probability measure $\mathcal{L}(\zeta, W)$ under the solution map S^μ defined in (12). We denote the parameter dependent map $\Phi(\nu, \cdot)$ by Φ^ν .

Note that X uniquely solves the McKean-Vlasov equation (10) with input (ζ, W) , if and only if $\mathcal{L}(X)$ is a fixed point of $\Phi^{\mathcal{L}(\zeta, W)}$:

- if X solves (10), then, by uniqueness for fixed $\mu = \mathcal{L}(X)$, $X = S^{\mathcal{L}(X)}(\zeta, W)$ \mathbb{P} -a.s. and so $\mathcal{L}(X)$ is a fixed point of $\Phi^{\mathcal{L}(\zeta, W)}$;
- conversely, if $\mu^{\mathcal{L}(\zeta, W)}$ is a fixed point of $\Phi^{\mathcal{L}(\zeta, W)}$, then $X = S^{\mu^{\mathcal{L}(\zeta, W)}}(\zeta, W)$ has finite p -moment and solves (10).

Hence existence and uniqueness for (10) in Theorem 11 follow from existence and uniqueness for fixed points of $\Phi^{\mathcal{L}(\zeta, W)}$, for any law $\mathcal{L}(\zeta, W)$.

For this reason, the main ingredient in the proof of Theorem 11 is the following general proposition, a version of the contraction principle with parameters. The proof is postponed to the appendix.

Proposition 10. *Let (E, d_E) and (F, d_F) be two complete metric spaces. Consider a function $\Phi : F \times E \rightarrow E$ with the following properties:*

- 1) (uniform Lipschitz continuity) there exists $L > 0$ such that

$$d_E(\Phi(Q, P), \Phi(Q', P')) \leq L [d_E(P, P') + d_F(Q, Q')].$$

- 2) (contraction) There exist a constant $0 < c < 1$ and a natural number $k \in \mathbb{N}$ such that

$$d_E((\Phi^Q)^k(P), (\Phi^Q)^k(P')) \leq cd_E(P, P') \quad \forall Q \in F, \forall P, P' \in E,$$

with $\Phi^Q(P) := \Phi(Q, P)$.

Then for every $Q \in F$ there exists a unique $P_Q \in E$ such that

$$\Phi(Q, P_Q) = P_Q.$$

Moreover,

$$\forall Q, Q' \in F, \quad d_E(P_Q, P_{Q'}) \leq \tilde{C} d_F(Q, Q'), \quad (15)$$

where $\tilde{C} := \left(\sum_{i=1}^k L^i \right) (1 - c)^{-1}$.

We give now the main result, from which most of the applications follow. It states well-posedness of the generalized McKean-Vlasov equation and Lipschitz continuity with respect to the driving signal.

Theorem 11. *Let $T > 0$ be fixed and let $p \in [1, \infty)$, assume Assumption 1.*

- (i) *For every input $(\zeta, W) \in L^p(\mathbb{R}^d \times C_T)$, the map $\Phi^{\mathcal{L}(\zeta, W)}$ has a unique fixed point, $\mu^{\mathcal{L}(\zeta, W)}$.*
- (ii) *The map that associates the law of the inputs to the fixed point, namely*

$$\begin{aligned} \Psi : \mathcal{P}_p(\mathbb{R}^d \times C_T) &\rightarrow \mathcal{P}_p(C_T) \\ \nu &\mapsto \mu^\nu \end{aligned} \quad (16)$$

is well-defined and Lipschitz continuous.

- (iii) *For every input (ζ, W) , there exists a unique solution X to the generalized McKean-Vlasov (10), given by $X = S^{\Psi(\mathcal{L}(\zeta, W))}(\zeta, W)$.*

(iv) There exists a constant $\tilde{C} = \tilde{C}(p, T, b) > 0$ such that: for every two inputs (ζ^i, W^i) , $i = 1, 2$ (defined possibly on different probability spaces) with finite p -moments, the following is satisfied

$$\mathcal{W}_{C_T, p}(\mathcal{L}(X^1), \mathcal{L}(X^2)) \leq \tilde{C} \mathcal{W}_{\mathbb{R}^d \times C_T, p}(\mathcal{L}(\zeta^1, W^1), \mathcal{L}(\zeta^2, W^2)).$$

In particular, the law of a solution X depends only on the law of (ζ, W) .

Proof. The result follows from Proposition 10, applied to the spaces $E := \mathcal{P}_p(C_T)$, $F := \mathcal{P}_p(\mathbb{R}^d \times C_T)$ and the map Φ defined in (14), provided we verify conditions 1) and 2).

Let now $\mu \in E$ be fixed, let ν^1 and ν^2 be in $\mathcal{P}_p(\mathbb{R}^d \times C_T)$ and let m be an optimal plan on $(\mathbb{R}^d \times C_T)^2$ for these two measures. We call optimal plan a measure m that satisfies the minimum in the Wasserstein distance, see (43). On the probability space $((\mathbb{R}^d \times C_T)^2, m)$, we call ζ^i, W^i the r.v. defined by the canonical projections and $Y^i = S^\mu(\zeta^i, W^i)$ the solution to equation (11) with input (ζ^i, W^i) . By definition of the Wasserstein metric, we have that

$$\mathcal{W}_{C_T, p}(\Phi(\nu^1, \mu), \Phi(\nu^2, \mu))^p = \mathcal{W}_{C_T, p}(\mathcal{L}(Y^1), \mathcal{L}(Y^2))^p \leq C_p \mathbb{E}_m \|Y^1 - Y^2\|_{\infty: T}^p.$$

The right hand side can be estimated using the equation,

$$\begin{aligned} \mathbb{E}_m \|Y^1 - Y^2\|_{\infty: T}^p &\leq C_p \mathbb{E}_m |\zeta^1 - \zeta^2|^p + C_p \mathbb{E}_m \|W^1 - W^2\|_{\infty: T}^p \\ &\quad + K_b C_p \int_0^T \mathbb{E}_m \|Y^1 - Y^2\|_{\infty: t}^p dt. \end{aligned}$$

Using Gronwall's inequality we obtain

$$\begin{aligned} \mathcal{W}_{C_T, p}(\mathcal{L}(Y^1), \mathcal{L}(Y^2))^p &\leq C_p e^{TK_b C_p} (\mathbb{E}_m |\zeta^1 - \zeta^2|^p + \mathbb{E}_m \|W^1 - W^2\|_{\infty: T}^p) \\ &= \tilde{L} \mathcal{W}_{\mathbb{R}^d \times C_T, p}(\nu^1, \nu^2)^p, \end{aligned} \quad (17)$$

where $\tilde{L} := C_p e^{TK_b C_p}$.

Let now (ζ, W) be fixed with law $\nu := \mathcal{L}(\zeta, W)$. Consider $\mu^1, \mu^2 \in E$ and call S^{μ^i} , for $i = 1, 2$, the corresponding solution map as defined in (12) (driven by the initial datum ζ and the path W). Let $t \in [0, T]$ be fixed. Using equation (11) again, we get that

$$\begin{aligned} \int_{\mathbb{R}^d \times C_T} \|S^{\mu^1}(x_0, \gamma) - S^{\mu^2}(x_0, \gamma)\|_{\infty: t}^p d\nu(x_0, \gamma) &\leq K_p C_p \int_0^t \mathcal{W}_{C_s, p}(\mu^1|_{[0, s]}, \mu^2|_{[0, s]})^p ds \\ &\quad + K_p C_p \int_0^t \int_{\mathbb{R}^d \times C_T} \|S^{\mu^1}(x_0, \gamma) - S^{\mu^2}(x_0, \gamma)\|_{\infty: s}^p d\nu(x_0, \gamma) ds. \end{aligned}$$

We deduce by the definition of Φ^ν and Wasserstein distance and applying Gronwall's lemma that

$$\begin{aligned} \mathcal{W}_{C_t, p}(\Phi^\nu(\mu^1)|_{[0, t]}, \Phi^\nu(\mu^2)|_{[0, t]})^p &\leq \int_{\mathbb{R}^d \times C_T} \|S^{\mu^1}(x_0, \gamma) - S^{\mu^2}(x_0, \gamma)\|_{\infty: t}^p d\nu(x_0, \gamma) \\ &\leq C_p K_b e^{tK_b C_p} \int_0^t \mathcal{W}_{C_s, p}(\mu^1|_{[0, s]}, \mu^2|_{[0, s]})^p ds. \end{aligned} \quad (18)$$

Taking $t = T$, we have that

$$\mathcal{W}_{C_T, p}(\Phi^\nu(\mu^1), \Phi^\nu(\mu^2))^p \leq \tilde{L} \mathcal{W}_{C_T, p}(\mu^1, \mu^2)^p. \quad (19)$$

With estimates (17) and (19) we have shown that $\bar{\Phi}$ satisfies 1).

To prove 2), we reiterate k times the application $\bar{\Phi}^\nu$ and we use (18) to obtain

$$\begin{aligned} \mathcal{W}_{C_{T,p}}((\bar{\Phi}^\nu)^k(\mu^1), (\bar{\Phi}^\nu)^k(\mu^2))^p &\leq \tilde{L}^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathcal{W}_{C_{t_1,p}}(\mu^1|_{[0,t_1]}, \mu^2|_{[0,t_1]})^p dt_1 \cdots dt_k \\ &\leq \tilde{L}^k \mathcal{W}_{C_{T,p}}(\mu^1, \mu^2)^p \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} dt_1 \cdots dt_k \\ &\leq \frac{(T\tilde{L})^k}{k!} \mathcal{W}_{C_{T,p}}(\mu^1, \mu^2)^p. \end{aligned}$$

By choosing $k > 0$ large enough, we have that $c := \frac{(T\tilde{L})^k}{k!} < 1$. This shows point 2) and concludes the proof. \square

If the driving process is progressively measurable, then so is the solution:

Proposition 12. *Let $(\mathcal{F}_t)_{t \geq 0}$ be a right-continuous, complete filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ such that ζ is \mathcal{F}_0 -measurable and W is $(\mathcal{F}_t)_{t \geq 0}$ -progressively measurable. Then the solution X to (10) is also $(\mathcal{F}_t)_{t \geq 0}$ -progressively measurable.*

Proof. The proof is classical. Fix t in $[0, T]$, then, \mathbb{P} -a.e., the restriction $X|_{[0,t]} = X|_{[0,t]}(\omega)$ on $[0, t]$ of the solution X also solves (11) on $[0, t]$ with inputs ζ and $W|_{[0,t]}$ (restriction of W on $[0, t]$) and input measure $\mu|_{[0,t]}$ (pushforward of $\mu = \mathcal{L}(X)$ by the restriction on $[0, t]$). Therefore $X|_{[0,t]}(\omega) = S_t^{\mu|_{[0,t]}}(\zeta, W|_{[0,t]})$. Since $S_t^{\mu|_{[0,t]}}$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(C_t)$ -measurable and ζ and $W|_{[0,t]}$ are \mathcal{F}_t -measurable, also $X|_{[0,t]}$ is \mathcal{F}_t -measurable, in particular $X|_{[0,t]}$ is \mathcal{F}_t -measurable. Hence X is adapted and therefore progressively measurable by continuity of its paths. \square

2.1 Weak continuity

In this note we are generally interested in proving quantitative convergence in the Wasserstein distance. However, one can show that the law of the solution of the mean field equation (10) is continuous in the weak topology of measures, with respect to the law of the inputs, in the spirit of [36].

Assumption 2. *Given a Polish space (E, d) , we endow the space $\mathcal{P}(E)$ with a metric Π_E , with the following properties*

- (i) *The metric Π_E is complete and metrizes the weak convergence of measures.*
- (ii) *For any two random variables $X, X' : \Omega \rightarrow E$, we have*

$$\Pi_E(\mathcal{L}(X), \mathcal{L}(X')) \leq \mathbb{E}d(X, X').$$

Remark 13. Let Lip_1 be the space of bounded and Lipschitz functions on E , as defined in Section 1.1. Define the Kantorovich-Rubinstein metric as

$$\Pi_E(\mu, \nu) := \sup_{\varphi \in \text{Lip}_1} \int_E \varphi d(\mu - \nu),$$

This metric satisfies Assumption 2. Note that 2 (i) follows from [7, Theorem 8.3.2 and Theorem 8.9.4]

For the drift we assume the following.

Assumption 3. The drift $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ is a measurable function and there exists a constant K such that,

■ (Lipschitz continuity)

$$|b(t, x, \mu) - b(t, x', \mu')| \leq K (|x - x'| + \Pi_{\mathbb{R}^d}(\mu, \mu')),$$

$$\forall t \in [0, T], x, x' \in \mathbb{R}^d, \mu, \mu' \in \mathcal{P}(\mathbb{R}^d).$$

■ (boundedness)

$$|b(t, x, \mu)| \leq K,$$

$$\forall t \in [0, T], x \in \mathbb{R}^d, \mu \in \mathcal{P}(\mathbb{R}^d).$$

Remark 14. Assume that there exists a function $B : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that there exists a constant $C > 0$,

$$|B(x, y)| \leq C, \quad |B(x, y) - B(x', y')| \leq C (|x - x'| + |y - y'|), \quad \forall x, x', y, y' \in \mathbb{R}^d,$$

and the drift satisfies $b(t, x, \mu) := \int_{\mathbb{R}^d} B(x, y) \mu(dy)$. Then b satisfies Assumptions 3, with $K = 3C$. This is the case treated in [36].

Lemma 15. Given $\nu \in \mathcal{P}(\mathbb{R}^d \times C_T)$, the solution map

$$\begin{aligned} S^\nu : \mathbb{R}^d \times C_T &\rightarrow C_T \\ (x_0, \gamma) &\mapsto S^\nu(x_0, \gamma), \end{aligned} \quad (20)$$

to the ODE

$$x_t = x_0 + \int_0^t b(s, x_s, (x_s)_\# \nu) ds + \gamma_t. \quad (21)$$

is well defined.

Proof. We prove the lemma by iteration. For a fixed $x_0, \gamma \in \mathbb{R}^d \times C_T$, define $x_t^0 := x_0 + \gamma_t$, and x_t^{n+1} defined implicitly as $x_t^{n+1} = x_0 + \int_0^t b(s, x_s^{n+1}, (x_s^n)_\# \nu) ds + \gamma_t$. Clearly, for every $n \in \mathbb{N}$, the function $(x_0, \gamma) \mapsto x^n$ is well defined and measurable.

We compute the following, for $t \in [0, T]$, using Assumption 3, Gronwall's Lemma and Assumption 2 (ii)

$$|x_t^n - x_t^{n+1}| \leq K e^{Kt} \int_0^t \Pi_{\mathbb{R}^d}((x_s^{n-1})_\# \nu, (x_s^n)_\# \nu) ds \leq K e^{Kt} \int_0^t \int_{\mathbb{R}^d \times C_T} |x_s^{n-1} - x_s^n| d\nu ds.$$

Iterating this inequality down to $n = 0$, we obtain that there exists a positive constant $C(T, K)$, independent of n , such that

$$|x_t^n - x_t^{n+1}| \leq \frac{C(T, K)^n}{n!}.$$

Hence, we have that, for every $x_0, \gamma \in \mathbb{R}^d \times C_T$, the sequence $(x^n(x_0, \gamma))_{n \geq 0}$ is Cauchy in $(C_T, \|\cdot\|_\infty)$. Indeed, for $\epsilon > 0$, there exists $m > 0$ big enough, such that for every $n \geq m$,

$$\|x^m - x^n\|_\infty \leq \sum_{i=m}^{n-1} \|x^i - x^{i+1}\|_\infty \leq \sum_{i=m}^{\infty} \frac{C(T, K)^i}{i!} < \epsilon.$$

We call $x(x_0, \gamma) \in C_T$ its limit as $n \rightarrow \infty$. The pointwise limit of Borel measurable functions is measurable, hence $(x_0, \gamma) \mapsto x$ is also measurable and $(x_s)_\# \nu$ is well-defined. We can thus pass to the limit in equation (21) to show that x is a solution to it. \square

Lemma 16. *The function*

$$\begin{aligned} \Psi : (\mathcal{P}(\mathbb{R}^d \times C_T), \Pi_{\mathbb{R}^d \times C_T}) &\rightarrow (\mathcal{P}(C_T), \Pi_{C_T}) \\ \nu &\mapsto (S^\nu)_\# \nu, \end{aligned} \quad (22)$$

is continuous. By Assumption 2 (i), this is equivalent to continuity with respect to the topology induced by the weak convergence of measures.

Proof. Let $(\nu^n)_{n \geq 0} \subset \mathcal{P}(\mathbb{R}^d \times C_T)$ be a sequence of probability measures that converges weakly to $\nu \in \mathcal{P}(\mathbb{R}^d \times C_T)$. From Skorohod representation theorem, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sequence $(\zeta^n, W^n) : \Omega \rightarrow \mathbb{R}^d \times C_T$ of random variables distributed as ν^n that converges almost surely to a random variable (ζ, W) distributed as ν .

Let $X^n := S^{\nu^n}(\zeta^n, W^n)$. By definition, $\mu^n := \mathcal{L}(X^n) = \Psi(\nu^n)$ and X^n solves the following SDE in the sense of Definition 8,

$$X_t^n = \zeta^n + \int_0^t b(s, X_s^n, \mathcal{L}(X_s^n)) ds + W_t^n.$$

It is easy to check that the random variables X^n are equicontinuous and equibounded and deduce that the family μ^n is tight in C_T . With an abuse of notation, assume that $(\mu^n)_{n \geq 0}$ is a subsequence that converges weakly to some $\mu \in \mathcal{P}(C_T)$, and $(X^n)_{n \geq 0}$ such that $\mathcal{L}(X^n) = \mu^n$. By using the equation, one can check that $(X^n(\omega))_{n \geq 0}$ is a Cauchy sequence in C_T for $\mathbb{P} - a.e. \omega$. Let X be the almost sure limit of X^n , as $n \rightarrow \infty$. Clearly, μ^n converges weakly to $\mathcal{L}(X)$, hence $\mathcal{L}(X) = \mu$. Passing to the limit in the equation, we can see that $\mu = \mathcal{L}(X) = \Psi(\nu)$. This concludes the proof. \square

2.2 Càdlàg drivers

In this section we follow the same reasoning as Section 2.1 to study the case when the drivers are discontinuous processes in (D_T, σ) . We first set some notation and recall some results about càdlàg functions.

Given $t \in [0, T]$, the projection π_t is defined, analogously to the continuous case, as the function $\pi_t : D_T \rightarrow \mathbb{R}^d$ as $\pi_t(\gamma) := \gamma(t)$.

Definition 17. For a function $\gamma \in D_T$, we define its càdlàg modulus as a function of $\delta \in (0, 1)$,

$$w_\gamma(\delta) = \inf_{\Pi} \max_{1 \leq i \leq n} \sup_{t_{i-1} \leq s \leq t \leq t_i} |\gamma_{s,t}|,$$

where the infimum is taken over all the partitions Π with mesh size bigger than δ .

Then we have the following lemma, from [6, equation (13.3)].

Lemma 18. *Let $(\nu^n)_{n \geq 0} \subset \mathcal{P}(D_T)$ be a sequence of probability measures converging weakly to $\nu \in \mathcal{P}(D_T)$, then there exists a set $T_\nu \subset [0, T]$ of full Lebesgue measure (actually T_ν^c is at most countable) such that ν_t^n converges weakly to ν_t , for all $t \in T_\nu$.*

Given a Polish space (E, d) , we use once again the notation Π_E to denote a distance on $\mathcal{P}(E)$ that satisfies Assumption 2.

For the drift we assume the following.

Assumption 4. The drift $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ is a measurable function and there exists a constant K such that,

■ (Lipschitz continuity)

$$|b(x, \mu) - b(x', \mu')| \leq K (|x - x'| + \Pi_{\mathbb{R}^d}(\mu, \mu')),$$

$$\forall x, x' \in \mathbb{R}^d, \mu, \mu' \in \mathcal{P}(\mathbb{R}^d).$$

■ (boundedness)

$$|b(x, \mu)| \leq K,$$

$$\forall x \in \mathbb{R}^d, \mu \in \mathcal{P}(\mathbb{R}^d).$$

Remark 19. The function b defined in Remark 14 also satisfies Assumptions 4.

2.2.1 Well-posedness and continuity

We have the following results, analogously to Section 2.1.

Lemma 20. Let b satisfy Assumptions 4. Given $\nu \in \mathcal{P}(\mathbb{R}^d \times D_T)$, the solution map

$$\begin{aligned} S^\nu : \mathbb{R}^d \times D_T &\rightarrow D_T \\ (x_0, \gamma) &\mapsto S^\nu(x_0, \gamma), \end{aligned} \quad (23)$$

to the ODE

$$x_t = x_0 + \int_0^t b(x_s, (x_s)_\# \nu) ds + \gamma_t. \quad (24)$$

is well defined.

Proof. The proof of this lemma follows exactly the proof of Lemma 15. We define the sequence $(x^n)_{n \in \mathbb{N}} \subset D_T$ and show that it is a Cauchy sequence in the uniform norm $\|\cdot\|_\infty$. By taking $\lambda(t) = t$ in the definition of σ , equation (9), one notices immediately that σ is bounded by the distance induced by $\|\cdot\|_\infty$. Hence, $(x^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in σ and the conclusion follows as in Lemma 15. \square

Lemma 21. The function

$$\begin{aligned} \Psi : (\mathcal{P}(\mathbb{R}^d \times D_T), \Pi_{\mathbb{R}^d \times D_T}) &\rightarrow (\mathcal{P}(D_T), \Pi_{D_T}) \\ \nu &\mapsto (S^\nu)_\# \nu, \end{aligned} \quad (25)$$

is continuous. By assumption, this is equivalent to continuity with respect to the topology induced by the weak convergence of measures.

Proof. Let $(\nu^n)_{n \geq 0} \subset \mathcal{P}(\mathbb{R}^d \times D_T)$ be a sequence of probability measure that converges weakly to $\nu \in \mathcal{P}(\mathbb{R}^d \times D_T)$. From Skorokhod representation theorem, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sequence $(\zeta^n, W^n) : \Omega \rightarrow \mathbb{R}^d \times D_T$, for $n \geq 0$, of random variables distributed as ν^n which converges almost surely to a random variable (ζ, W) distributed as ν .

Let $X^n := S^{\nu^n}(\zeta^n, W^n)$. By definition, $\mu^n := \mathcal{L}(X^n) = \Psi(\nu^n)$ and X^n solves the following SDE pathwise,

$$X_t^n = \zeta^n + \int_0^t b(X_s^n, \mathcal{L}(X_s^n)) ds + W_t^n.$$

By construction, the laws of W^n are tight. Equivalently, by [6, Theorem 13.2], they satisfy

$$\lim_{a \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\{\|W^n\|_\infty \geq a\} = 0, \quad (26)$$

$$\forall \epsilon > 0, \quad \lim_{\delta \rightarrow 0} \limsup_{n \in \mathbb{N}} \mathbb{P}\{w_{W^n}(\delta) \geq \epsilon\} = 0. \quad (27)$$

It follows from Assumption 4 that the random variables X^n also satisfy (26) and (27). Thus, we deduce that the family μ^n is tight in $\mathcal{P}(\mathbb{R}^d \times D_T)$.

With an abuse of notation, assume that $(\mu^n)_{n \geq 0}$ is a subsequence that converges weakly to some $\mu \in \mathcal{P}(D_T)$, and $(X^n)_{n \geq 0}$ such that $\mathcal{L}(X^n) = \mu^n$. By using the equation, we now check that $(X^n(\omega))_{n \geq 0}$ is a Cauchy sequence in (D_T, σ) for $\mathbb{P} - a.e. \omega$.

First observe that Lemma 18 and Lebesgue dominated convergence imply that $\int_0^T \Pi_{\mathbb{R}^d}(\mu_s^n, \mu_s) ds \rightarrow 0$, as $n \rightarrow \infty$. Hence $(\mu^n)_{n \in \mathbb{N}} \subset L^1([0, T], \mathcal{P}(\mathbb{R}^d))$ is a Cauchy sequence. Let now $\Omega^0 \subset \Omega$ be a set of full measure such that $(\zeta^n(\omega), W^n(\omega)) \rightarrow (\zeta(\omega), W(\omega))$, for all $\omega \in \Omega^0$, as $n \rightarrow \infty$.

Fix $\omega \in \Omega^0$, $\epsilon > 0$ there exists $N > 0$, such that for all $m, n \geq N$, we have

$$\sigma(W^n(\omega), W^m(\omega)) < \epsilon, \quad |\zeta^n(\omega) - \zeta^m(\omega)| < \epsilon, \quad \int_0^T \Pi_{\mathbb{R}^d}(\mu_s^n, \mu_s^m) ds < \epsilon.$$

We omit now the dependence of the random variables from ω . There exists $\bar{\delta} > 0$, such that for every $0 < \delta < \bar{\delta}$, $\sup_n w(X^n, \delta) < \epsilon$. We can choose $\delta = \epsilon \wedge \bar{\delta}$, and $\lambda := \lambda(\omega, \delta, m, n)$ such that

$$\|\lambda\| + \|W^n - W^m \circ \lambda\|_\infty < \delta < \epsilon, \quad (28)$$

$$\|X^m + X^m \circ \lambda\|_\infty < 2w(X^m, \delta) < 2 \sup_n w(X^n, \delta) < 2\epsilon.$$

We note that, for all $t \in [0, T]$,

$$\left| \int_{\lambda_t \vee t}^{\lambda_t \wedge t} ds \right| \leq |\lambda_t - t| \leq e^{\|\lambda\|} - 1,$$

where the last inequality follows from [6, equation (12.17)]. We can thus compute the following, for $t \in [0, T]$,

$$\begin{aligned} \left| \int_0^t b(X_s^n, \mu_s^n) ds - \int_0^{\lambda_t} b(X_s^m, \mu_s^m) ds \right| &\leq K \left| \int_{\lambda_t \vee t}^{\lambda_t \wedge t} ds \right| + \int_0^{\lambda_t \wedge t} |b(X_s^n, \mu_s^n) - b(X_s^m, \mu_s^m)| ds \\ &\leq K(e^{\|\lambda\|} - 1) + K \int_0^T \Pi_{\mathbb{R}^d}(\mu_s^n, \mu_s^m) ds + K \int_0^T |X_s^n - X_s^m| ds \\ &\leq K(e^{\|\lambda\|} - 1) + K\epsilon + K \int_0^T |X_s^n - (X^m \circ \lambda)_s| ds \\ &\quad + K \int_0^T |X_s^m - (X^m \circ \lambda)_s| ds \\ &\lesssim 4\epsilon + \int_0^T |X_s^n - (X^m \circ \lambda)_s| ds \end{aligned}$$

From which we deduce

$$\begin{aligned} |X_t^n - (X^m \circ \lambda)_t| &\leq |\zeta^n - \zeta^m| + |W_t^n - (W^m \circ \lambda)_t| + \left| \int_0^t b(X_s^n, \mu_s^n) ds - \int_0^{\lambda t} b(X_s^m, \mu_s^m) ds \right| \\ &\lesssim \epsilon + \|W^n - W^m \circ \lambda\| + \int_0^T |X_s^n - (X^m \circ \lambda)_s| ds \end{aligned}$$

We add $\|\lambda\|$ on both sides, apply Gronwall's Lemma and inequality (28) to obtain

$$\sigma(X^n(\omega), X^m(\omega)) < C(T, K)\epsilon.$$

Hence, we have that $X^n(\omega)$ is a Cauchy sequence in (D_T, σ) , for $\omega \in \Omega^0$.

Let X be the almost sure limit of X^n , as $n \rightarrow \infty$. The laws μ^n converge weakly to $\mathcal{L}(X)$, hence $\mathcal{L}(X) = \mu$. Passing to the limit in the equation, we can see that $\mu = \mathcal{L}(X) = \Psi(\nu)$. This concludes the proof. \square

3 Applications

3.1 Particle approximation

In this section we show how the results in Section 2 yield a convergence result for a particle system associated with the McKean-Vlasov equation.

Given inputs $\bar{\zeta}$ and \bar{W} (on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$), we consider the following McKean-Vlasov equation

$$\begin{cases} d\bar{X}_t = b(t, \bar{X}_t, \mathcal{L}(\bar{X}_t))dt + d\bar{W}_t \\ X_0 = \bar{\zeta}. \end{cases} \quad (29)$$

To this, given $N \in \mathbb{N}$, we associate the corresponding interacting particle system (on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$),

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}})dt + dW_t^{i,N}, & i = 1, \dots, N \\ X_0^{i,N} = \zeta^{i,N}, \end{cases} \quad (30)$$

with given input

$$\begin{aligned} (\zeta^{(N)}, W^{(N)}) : \Omega &\rightarrow (\mathbb{R}^d \times C_T)^N \\ \omega &\mapsto (\zeta^{i,N}(\omega), W^{i,N}(\omega))_{1 \leq i \leq N}. \end{aligned}$$

For a given $N \in \mathbb{N}$ and an N -dimensional vector $Y^{(N)} = (Y^1, \dots, Y^N)$ with entries in a Polish space E , we define the empirical measure associated with $Y^{(N)}$ as

$$L^N(Y^{(N)}) := \frac{1}{N} \sum_{i=1}^N \delta_{Y^i}.$$

As pointed out in the introduction, the main argument of Cass-Lyons/Tanaka approach is that the particle system (2) can be interpreted as the limiting McKean-Vlasov equation (1) by using a transformation of the probability space and the input data. The main result Theorem 11 not only implies well-posedness of both McKean-Vlasov and particle approximation, but also allows to deduce convergence of the particle system from convergence of the corresponding signals,

something which is usually easy to verify, for example, if the signals are empirical measures of independent noises.

Now we show how to interpret equations (29) and (30) as generalized McKean-Vlasov equation (10). Clearly (29) is (10) with inputs $\bar{\zeta}$ and \bar{W} . For (30), for fixed $N \in \mathbb{N}$, we consider the space $(\Omega_N, \mathcal{A}_N, \mathbb{P}_N)$, where $\Omega_N := \{1, \dots, N\}$, $\mathcal{A}_N := 2^{\Omega_N}$ and $\mathbb{P}_N := \frac{1}{N} \sum_{i=1}^N \delta_i$. On this space, we can identify any N -uple $Y^{(N)} = (Y^1, \dots, Y^N) \in E^N$, as a random variable $\Omega_N \ni i \mapsto Y^i \in E$. With this identification, the law of $Y^{(N)}$ on Ω_0 is precisely the empirical measure associated with $Y^{(N)}$, namely $L^N(Y^{(N)})$. Indeed, for each continuous and bounded function φ on E , we have

$$\mathbb{E}_{\mathbb{P}_N}[\varphi(Y^{(N)})] = \sum_{i=1}^N \frac{1}{N} \varphi(Y^i) = L^N(Y^{(N)})(\varphi).$$

We assume that $(\zeta^{(N)}(\omega), W^{(N)}(\omega))$ is valued in $(\mathbb{R}^d \times C_T)^N$ for every N and for every $\omega \in \Omega$. We fix $\omega \in \Omega$ and N and we apply the previous argument to the N -uples

$$\begin{aligned} (\zeta^{(N)}, W^{(N)})(\omega) &= ((\zeta^{1,N}, W^{1,N})(\omega), \dots, (\zeta^{N,N}, W^{N,N})(\omega)), \\ X^{(N)}(\omega) &= (X^{1,N}(\omega), \dots, X^{N,N}(\omega)). \end{aligned}$$

For fixed $\omega \in \Omega$, the law of $(\zeta^{(N)}(\omega), W^{(N)}(\omega))$ on Ω_N is the empirical measure $L^N(\zeta^{(N)}, W^{(N)})(\omega)$ and the law of $X^{(N)}(\omega)$ on Ω_N is the empirical measure $L^N(X^{(N)})(\omega)$, which appears exactly in (30), projected at time t . Hence, for fixed ω in Ω , the interacting particle system (30) is the generalized McKean-Vlasov equation (10), defined on the space $(\Omega_N, \mathcal{A}_N, \mathbb{P}_N)$ and driven by the empirical measure $L^N(\zeta^{(N)}, W^{(N)})(\omega)$.

We are ready to apply Theorem 11 to obtain the following result, which ties the convergence of the particles to the convergence of the inputs. An immediate consequence is that the empirical measure of the particle system converges if the input converges: no independence or exchangeability are required.

Theorem 22. *Let $p \in [1, \infty)$ and assume 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For a fixed $N \in \mathbb{N}$, let $(\zeta^{(N)}, W^{(N)}) = (\zeta^{i,N}, W^{i,N})_{1 \leq i \leq N} : \Omega \rightarrow (\mathbb{R}^d \times C_T)^N$ be a family of random variables. Let $\bar{\zeta} \in L^p(\Omega, \mathbb{R}^d)$ and $\bar{W} \in L^p(\Omega, C_T)$. Then,*

- i for every $\omega \in \Omega$, there exists a unique pathwise solution $X^{(N)}(\omega)$ in the sense of Definition 8 to the interacting particle system (30). Moreover, $\omega \mapsto X^{(N)}(\omega)$ is \mathcal{A} -measurable.*
- ii there exists a unique pathwise solution \bar{X} in the sense of Definition 8 to equation (29).*
- iii there exists a constant C depending on b such that for all $N \geq 1$, for \mathbb{P} -a.e. $\omega \in \Omega$,*

$$\mathcal{W}_{C_T, p}(L^N(X^{(N)}(\omega)), \mathcal{L}(\bar{X}))^p \leq C \mathcal{W}_{\mathbb{R}^d \times C_T, p}(L^N(\zeta^{(N)}(\omega), W^{(N)}(\omega)), \mathcal{L}(\bar{\zeta}, \bar{W}))^p. \quad (31)$$

Proof. Let $N \in \mathbb{N}$. Fix $\omega \in \Omega$, we apply Theorem 11 in the following setting

$$\begin{aligned} (\Omega^1, \mathcal{A}^1, \mathbb{P}^1) &:= (\Omega_N, \mathcal{A}_N, \mathbb{P}_N), & (\zeta^1, W^1)(\omega) &:= (\zeta^{(N)}(\omega), W^{(N)}(\omega)), \\ (\Omega^2, \mathcal{A}^2, \mathbb{P}^2) &:= (\Omega, \mathcal{A}, \mathbb{P}), & (\zeta^2, W^2) &:= (\bar{\zeta}, \bar{W}). \end{aligned}$$

The finite p -moment condition is satisfied by $(\bar{\zeta}, \bar{W})$ by assumption and also by $(\zeta^{(N)}(\omega), W^{(N)}(\omega))$, since

$$\begin{aligned} \|(\zeta^1, W^1)(\omega)\|_{L^p(\Omega^1)}^p &= \mathbb{E}_{\mathbb{P}_N} [|\zeta^{(N)}(\omega)|^p + \|W^{(N)}(\omega)\|_\infty^p] \\ &= \frac{1}{N} \sum_{i=1}^N |\zeta^i(\omega)|^p + \frac{1}{N} \sum_{i=1}^N \|W^i(\omega)\|_\infty^p < +\infty. \end{aligned}$$

Since the assumptions on the drift b are also satisfied, Theorem 11 establishes the existence of solutions $X^1(\omega) =: X^{(N)}(\omega)$ and $X^2 =: \bar{X}$. Moreover the map Ψ is continuous, hence $\omega \mapsto L^{(N)}(X^{(N)})(\omega)$ is \mathcal{A} -measurable, which makes $X^{(N)}(\omega) := S^{L^{(N)}(X^{(N)})(\omega)}(\zeta(\omega), W^{(N)}(\omega))$ measurable. This gives (i) and (ii). Theorem (11) also gives exactly the inequality in (iii). The proof is complete. \square

Remark 23. We stress out that, when looking at the particle system, we are applying Theorem 11 on the discrete space, for a fixed ω , and the law that appears on the drift is the empirical measure at fixed ω .

Remark 24. In the proof of point iii of Theorem 22, we can actually get the bound for every ω if we use the pathwise solution $X^{(N)}(\omega)$ (in the sense of Definition 8), as this satisfies (30) for every ω . However, the “ \mathbb{P} -a.s.” is required when dealing with a solution to the interacting particle system (30) in the usual probabilistic sense, where (30) is required to hold only \mathbb{P} -a.s..

3.2 Classical mean field limit

Now we specialize the previous result in the case of i.i.d. inputs, recovering the classical result by Sznitman [34]:

Corollary 25. *Given a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ (with the standard assumptions) and $p \in (1, \infty)$ let $(\zeta^i)_{i \geq 1} \subset L^p(\Omega, \mathbb{R}^d)$, be a family of i.i.d. random variables which are \mathcal{F}_0 -measurable and $(W^i)_{i \geq 1}$ be a family of independent adapted Brownian motions. Moreover, let $(\bar{\zeta}, \bar{W}) \in L^p(\Omega, \mathbb{R}^d \times C_T)$ be an independent copy of (ζ^1, W^1) . Then the solutions $X^{(N)}$ and \bar{X} to the interacting particles system (30) and the McKean-Vlasov SDE (29), respectively, given by Theorem 22, are progressively measurable and we have the following convergence*

$$L^N(X^{(N)}) \xrightarrow{*} \mathcal{L}(\bar{X}), \quad \mathbb{P} - a.s. \quad (32)$$

Remark 26. The classical case when b is a convolution with a regular kernel, say $b(t, x, \mu) = (K * \mu)(x)$, is treated here, as b in this case satisfies the assumption of Theorem 22.

Proof of Corollary 25. Progressive measurability for the particle system (30) follows from (11) (ii) of Theorem 11 and is a consequence of Proposition 12 for the McKean-Vlasov SDE (29).

We prove now the convergence. First recall that Theorem 22, and in particular inequality (31), applies in this case. Hence, if we can prove that the right-hand-side of (31) goes to zero, we have the desired convergence (32).

Hence, by Lemma 44, we deduce the convergence in p' -Wasserstein, for every $p' \in (1, p)$. This is the convergence of the right-hand-side of (31). The proof is complete. \square

3.3 Mean field with common noise

In this section we study a system of interacting particles with common noise. We consider the following system on the space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$,

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}}) dt + dW_t^i + dB_t \\ X_0^{i,N} = \zeta^i. \end{cases} \quad i = 1, \dots, N \quad (33)$$

Here $(\zeta^i)_{i=1, \dots, N} \subset L^p(\bar{\Omega}, \mathbb{R}^d)$ is a family of i.i.d. random variables. This system represents N interacting particles where each particle is subject to the interaction with the others as well as some randomness. There are two sources of randomness, one which acts independently on each particle and is represented by the independent family of identically distributed random variables $W^{(N)} = (W^i)_{1 \leq i \leq N} \subset L^p(\bar{\Omega}, C_T)$. The second source of randomness is the same for each particle and is represented by the random variable $B \in L^p(\bar{\Omega}, C_T)$, which is assumed to be independent from the W^i . Usually W^i and B are Brownian motions, but it is not necessary to assume it here. The Brownian motion case was considered in [11].

Our aim is to prove that the empirical measure associate to the system converges, as $N \rightarrow \infty$, to the conditional law, given B , of the solution of the following McKean-Vlasov SDE

$$\begin{cases} d\bar{X}_t = b(t, \bar{X}_t, \mathcal{L}(\bar{X}_t|B)) dt + d\bar{W}_t + dB_t \\ \bar{X}_0 = \bar{\zeta}. \end{cases} \quad (34)$$

Here $\bar{\zeta}$ is a random variable on \mathbb{R}^d and \bar{W} is random variables on C_T distributed as ζ^1 and W^1 respectively. We denote by $\mathcal{L}(X|B)$ the conditional law of X given B . Our result is the following.

Corollary 27. *Let $p \in [1, \infty)$, $p' \in (p, \infty)$, and assume 1. Let $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$ be a probability space. On this space we consider independent families $\zeta^{(N)} = (\zeta^i)_{1 \leq i \leq N} \subset L^p(\bar{\Omega}, \mathbb{R}^d)$, $W^{(N)} = (W^i)_{1 \leq i \leq N} \subset L^{p'}(\bar{\Omega}, C_T)$ of i.i.d. random variables. Let $\bar{\zeta}$ be distributed as $\zeta^{i,N}$ and let \bar{W} be distributed as $W^{i,N}$ and independent of $\bar{\zeta}$. Moreover, assume that $B \in L^p(\bar{\Omega}, C_T)$ is a random variable independent from the others. Then there exists a solution $X^{(N)} \in L^p(\bar{\Omega}, (C_T)^N)$ to equation (33) and a solution $\bar{X} \in L^p(\bar{\Omega}, C_T)$ to equation (34). Moreover, we have*

$$\mathcal{W}_{C_T, p}(L^N(X^{(N)}), \mathcal{L}(\bar{X}|B)) \rightarrow 0, \quad \bar{\mathbb{P}} - \text{a.s.} \quad \text{as } N \rightarrow \infty.$$

Proof. Since B is independent from the other variables, we can assume, without loss of generality, that our probability space is of the form $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{A} \otimes \mathcal{A}', \mathbb{P} \otimes \mathbb{P}')$, that the random variables ζ^i , $\bar{\zeta}$, W^i and \bar{W} are defined on a space $(\Omega, \mathcal{A}, \mathbb{P})$ and the random variable B is defined on the space $(\Omega', \mathcal{A}', \mathbb{P}')$.

For a fixed path $\beta \in C_T$, we consider the modified inputs, on $(\Omega, \mathcal{A}, \mathbb{P})$, $W^{i,\beta} := W^i + \beta$ and $\bar{W}^\beta := \bar{W} + \beta$. Let $X^{(N),\beta}$ (respectively X^β) be the solution to equation (30) (resp. equation (29)) with input $(\zeta^{(N)}, W^{(N),\beta})$ (resp. $(\bar{\zeta}, \bar{W}^\beta)$) given by Theorem 22. The Lipschitz bound in Theorem 22 and the independence of ζ^i and $W^{i,\beta}$, via Lemma 44, imply that, for \mathbb{P} -a.e. ω ,

$$\mathcal{W}_{C_T, p}(L^N(X^{(N),\beta}(\omega)), \mathcal{L}(X^\beta)) \rightarrow 0.$$

Now we build the solution \bar{X} and $X^{(N)}$ resp. to (34) and to (33). We claim that the maps

$$\Omega \times C_T \ni (\omega, \beta) \mapsto X^\beta(\omega) \in C_T, \quad \Omega \times C_T \ni (\omega, \beta) \mapsto X^{(N),\beta}(\omega)$$

have versions that are jointly measurable and, for such versions, we define $\bar{X}(\omega, \omega') = X^{B(\omega')}(\omega)$ and $X^{(N)}(\omega, \omega') = X^{(N), B(\omega')}(\omega)$. Note that, by the definition of X^B , for every fixed $\omega' \in \Omega'$, we have \mathbb{P} -a.s.

$$dX^{B(\omega')} = b(t, X^{B(\omega')}, \mathcal{L}_{\mathbb{P}}(X^{B(\omega')}))dt + dW_t + dB_t(\omega'),$$

where the law is taken with respect to the space $(\Omega, \mathcal{A}, \mathbb{P})$. But the independence of B from the other variables implies that, $\bar{\mathbb{P}}$ -a.s.,

$$\mathcal{L}_{\mathbb{P}}(X^B) = \mathcal{L}_{\mathbb{P} \otimes \mathbb{P}'}(X^B | B).$$

Hence \bar{X} is a solution to equation (34) on the product space $\Omega \times \Omega'$. Similarly $X^{(N)}$ is a solution to (33) on $\Omega \times \Omega'$. Therefore we have, for $\bar{\mathbb{P}}$ -a.e. (ω, ω') ,

$$\mathcal{W}_{C_T, p}(L^N(X^{(N)})(\omega, \omega'), \mathcal{L}(\bar{X} | B)(\omega')) = \mathcal{W}_{C_T, p}(L^N(X^{(N), \beta})(\omega), \mathcal{L}(X^\beta)) |_{\beta=B(\omega')} \rightarrow 0,$$

which is the desired convergence.

It remains to prove the measurability claim on X^β and $X^{(N), \beta}$. We prove it for $X^{(N), \beta}$, the proof for \bar{X} being analogous. Recall the notation in Section 2 and note that the following maps are Borel measurable

$$\begin{aligned} F_1 &: \mathcal{P}_p(C_T) \times \mathbb{R}^d \times C_T \ni (\mu, x_0, \gamma) \mapsto S^\mu(x_0, \gamma) \in C_T, \\ F_2 &: \mathcal{P}_p(\mathbb{R}^d \times C_T) \times C_T \ni (\nu, \beta) \mapsto (\cdot + (0, \beta))_{\#} \nu \in \mathcal{P}_p(C_T), \end{aligned}$$

(where $\cdot + (0, \beta)$ is the map on $\mathbb{R}^d \times C_T$ defined by $(x, \gamma) + (0, \beta) = (x, \gamma + \beta)$). Indeed, F_1 is continuous (because the solution of (11) depends continuously on the drift, the initial data and the signal), F_2 is also Lipschitz-continuous (indeed, for any (β, ν) and (β', ν') , if m is an optimal plan between ν and ν' , then $((\cdot + (0, \beta), \cdot + (0, \beta'))_{\#} m$ is an admissible plan between $F_2(\beta, \nu)$ and $F_2(\beta', \nu')$ and standard bounds give the Lipschitz property). Moreover let Ψ the map defined in (16). It is continuous, hence measurable. Now we can write, for every β in C_T , for every $i = 1, \dots, N$,

$$X^{(N), \beta, i}(\omega) = F_1(\Psi(F_2(L^N(\zeta^{(N)}(\omega), W^{(N)}(\omega)), \beta)), \zeta^i(\omega), W^i(\omega) + \beta), \quad \mathbb{P} - \text{a.s.}$$

and the right-hand side above is composition of measurable maps, hence measurable. Therefore the right-hand side is a measurable version of $X^{(N), \beta}$. The proof is complete. \square

3.4 Heterogeneous mean field

As a further application of Theorem 22 we want to consider the case of heterogeneous mean field. We will show the convergence even when the drivers are not identically distributed. This applies in particular to the results of the physical system studied in [19] as was discussed in the introduction. In that model, it is assumed that the state of each particle is influenced by its radius. Particle i has a radius r^i , which is deterministic, and it is known that the radii are distributed according to a distribution λ . We allow here for the radii to be stochastic and not necessarily identically distributed, but still independent. Moreover, we will assume the volume to change in time.

Heterogeneous mean field systems appear also in other contexts, see for example (among many others) [37], [10], which work with semimartingale inputs and use a coupling à la Sznitman [34].

On the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we consider a family $(\zeta^{(N)}, W^{(N)}) = (\zeta^i, W^i)_{i \geq 1} \subset L^p(\Omega, \mathbb{R}^d \times C_T(\mathbb{R}^d))$. This family is taken i.i.d.

In addition, for each $N \in \mathbb{N}$, we consider a family $R^{(N)} = (R^{i,N})_{1 \leq i \leq N} \subset L^p(C_T(\mathbb{R}^n)^N)$.

We construct the following interacting particle system

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, R_t^{i,N}, L^N(X_t^{(N)}, R_t^{(N)}))dt + dW_t^i \\ X_0^{i,N} = \zeta^i. \end{cases} \quad (35)$$

We call this an heterogeneous particle system because the particles are not exchangeable anymore, if the $R^{i,N}$ are not exchangeable.

We assume that the $R^{i,N}$ are independent of the ζ^i and W^i and that there exists a measure $\lambda \in \mathcal{P}_p(C_T(\mathbb{R}^n))$ such that

$$L^N(R^{(N)})(\omega) \xrightarrow{*} \lambda, \quad \mathbb{P} - \text{a.s.}$$

We also consider the following mean field equation (on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$):

$$\begin{cases} d\bar{X}_t = b(t, \bar{X}_t, \bar{R}_t, \mathcal{L}(\bar{X}_t, \bar{R}_t))dt + d\bar{W}_t \\ \bar{X}_0 = \bar{\zeta} \end{cases}$$

where $\bar{\zeta}$, \bar{W} and \bar{R} are independent random variables distributed resp. as ζ^i , W^i and λ . The following result is a corollary of Theorem 22. We also use Lemma 29 and Lemma 30 to deal with the convergence of the input data.

Corollary 28. *Let $p \in [1, \infty)$. Assume that $b : [0, T] \times \mathbb{R}^{d+n} \times \mathcal{P}_p(\mathbb{R}^{d+n}) \rightarrow \mathbb{R}^d$ is a measurable function and there exists a constant K_b such that,*

$$|b(t, x, \mu) - b(t, x', \mu')|^p \leq K_b (|x - x'|^p + \mathcal{W}_{\mathbb{R}^{d+n}, p}(\mu, \mu')^p),$$

$\forall t \in [0, T], x, x' \in \mathbb{R}^{d+n}, \mu, \mu' \in \mathcal{P}_p(\mathbb{R}^{d+n})$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. On this space we consider independent families $\zeta^{(N)} = (\zeta^i)_{i \geq 1} \subset L^p(\Omega, \mathbb{R}^d)$, $W^{(N)} = (W^i)_{i \geq 1} \in L^p(\Omega, C_T)$ of i.i.d. random variables. Let $\bar{\zeta}$ be distributed as ζ^1 and let \bar{W} be distributed as W^1 and independent of $\bar{\zeta}$. Moreover, assume that $R^{(N)} = (R^{i,N})_{1 \leq i \leq N}$ is a family of independent random variables in $L^p(\Omega, \mathbb{R}^n)$ which are independent from the others. If there is convergence of the heterogeneous part,

$$L^N(R^{(N)})(\omega) \xrightarrow{*} \lambda \quad \mathbb{P} - \text{a.s.} \quad \text{as } N \rightarrow \infty,$$

then also the solution converges,

$$L^N(X^{(N)}, R^{(N)}) \xrightarrow{*} \mathcal{L}(\bar{X}, \bar{R}), \quad \mathbb{P} - \text{a.s.} \quad \text{as } N \rightarrow \infty,$$

Proof. We start by rewriting the system (35) so that we can invoke Theorem 22. We change the state space of the system from \mathbb{R}^d to $\mathbb{R}^d \times \mathbb{R}^n$ and we define on this new space the process $Y_t^{i,N} := (X_t^{i,N}, R_t^{i,N})$. Clearly, $X^{i,N}$ is a solution to system (35) if and only if $Y^{i,N}$ solves

$$\begin{cases} dY_t^{i,N} = \begin{pmatrix} b(t, Y_t^{i,N}, L^N(Y_t^{(N)})) \\ 0 \end{pmatrix} dt + d \begin{pmatrix} W_t^i \\ R_t^{i,N} \end{pmatrix} \\ Y_0^{i,N} = \begin{pmatrix} \zeta^i \\ R_0^{i,N} \end{pmatrix}. \end{cases}$$

A similar transformation can be applied to the McKean-Vlasov equation to obtain that $\bar{Y}_t = (\bar{X}_t, \bar{R}_t)$ solves

$$\begin{cases} d\bar{Y}_t = \begin{pmatrix} b(t, \bar{Y}_t, \mathcal{L}(\bar{Y}_t)) \\ 0 \end{pmatrix} dt + d \begin{pmatrix} \bar{W}_t \\ \bar{R}_t \end{pmatrix} \\ \bar{Y}_0 = \begin{pmatrix} \bar{\zeta} \\ \bar{R}_0 \end{pmatrix}. \end{cases}$$

In this setting the inputs satisfy the assumption of Theorem 22. Hence, we obtain the following inequality. $\forall \omega \in \Omega$,

$$\begin{aligned} & \mathcal{W}_{C_T(\mathbb{R}^{d+n}), p}(L^N(X^{(N)}, R^{(N)}), \mathcal{L}(\bar{X}, \bar{R}))^p \\ & \leq C \mathcal{W}_{\mathbb{R}^d \times C_T(\mathbb{R}^{d+n}), p}(L^N(\zeta^{(N)}, R^{(N)}, W^{(N)}), \mathcal{L}(\bar{\zeta}, \bar{R}, \bar{W}))^p. \end{aligned}$$

Almost sure convergence to 0 of the right-hand side is a consequence of Lemma 30 (with $X_i := (\zeta^{i,N}, W^{i,N})$ and $Y_{i,N} := (R^{i,N})$ on the spaces $E := \mathbb{R}^d \times C_T$ and $F := \mathbb{R}^n$). The proof is complete. \square

The following variant of the strong law of large numbers will be useful to prove Lemma 30.

Lemma 29. *Let $(X_i)_{i \geq 1}$ be a sequence of i.i.d. real-valued centered random variables and let $(Y_{i,N})_{1 \leq i \leq N}$ be an independent family of real-valued independent random variables. Moreover, assume that there exists $C > 0$ such that*

$$\|X_i\|_{L^4(\mathbb{R})} \leq C, \quad \|Y_{i,N}\|_{L^4(\mathbb{R})} \leq C, \quad \forall i, N \geq 1.$$

Then,

$$S^N := \frac{1}{N} \sum_{i=1}^N X_i Y_{i,N} \rightarrow 0, \quad \mathbb{P} - \text{a.s.}$$

Proof. We first establish a bound on the fourth moment of the empirical sum S^N .

$$\mathbb{E}|S^N|^4 = \frac{1}{N^4} \sum_{i=1}^N \mathbb{E}[X_i^4] \mathbb{E}[Y_{i,N}^4] + \frac{6}{N^4} \sum_{i,j=1}^N \mathbb{E}[X_i^2] \mathbb{E}[X_j^2] \mathbb{E}[Y_{i,N}^2] \mathbb{E}[Y_{j,N}^2] \leq \frac{C}{N^2}.$$

Only those two terms in the sum do not vanish, because the X_i 's are centered. The constant C depends on the upper bounds of the random variables. Let $p < \frac{1}{4}$,

$$E_N := \left\{ |S^N| > \frac{1}{N^p} \right\}.$$

Using Chebychev inequality, we have the following

$$\sum_{N=1}^{\infty} \mathbb{P}\{E^N\} \leq \sum_{N=1}^{\infty} N^{4p} \mathbb{E}[S^N] \leq C \sum_{N=1}^{\infty} N^{4p-2}.$$

For our choice of p , we have convergence of the series. Borel Cantelli's Lemma implies that

$$\mathbb{P}\{\limsup_{N \rightarrow \infty} E^N\} = 0,$$

which in turn implies almost sure convergence of S^N . \square

Lemma 30. *Let $p \in [1, \infty)$ be fixed. Let $(X_i)_{i \geq 1}$ be a sequence of i.i.d. random variables on a space $(\Omega, \mathcal{A}, \mathbb{P})$ taking values in a Polish space E , with law $\mu \in \mathcal{P}_p(E)$. Let $(Y_{i,N})_{1 \leq i \leq N}$ be another sequence of random variables taking values on a Polish space F , which is independent from $(X_i)_{i \geq 1}$. Assume that there exists a probability measure $\lambda \in \mathcal{P}_p(F)$ such that*

$$L^N(Y^{(N)}) := \frac{1}{N} \sum_{i=1}^N \delta_{Y_{i,N}} \xrightarrow{*} \lambda, \quad \mathbb{P} - \text{a.s.} \quad (36)$$

Then,

$$L^N(X^{(N)}, Y^{(N)}) \xrightarrow{*} \mu \otimes \lambda, \quad \mathbb{P} - \text{a.s.}$$

Proof. Since $(X_i)_{i \geq 1}$ are a sequence of i.i.d. random variables, there exists a set of full measure $\Omega^x \subset \Omega$, such that $L^N(X^{(N)}(\omega)) \xrightarrow{*} \mu$, for every $\omega \in \Omega^x$. Weak convergence implies tightness of the sequence $(L^N(X^{(N)}(\omega)))$, thus, for every $\epsilon > 0$, there exists a compact set $E_\epsilon^\omega \subset E$, such that

$$L^N(X^{(N)}(\omega))((E_\epsilon^\omega)^c) < \frac{\epsilon}{2}, \quad \omega \in \Omega^x.$$

In a similar way, there exists a set of full measure $\Omega^y \subset \Omega$ such that for every $\epsilon > 0$ there exists a compact $F_\epsilon^\omega \subset F$ that satisfies $L^N(Y^{(N)}(\omega))((F_\epsilon^\omega)^c) < \frac{\epsilon}{2}$, $\omega \in \Omega^y$. For every $\omega \in \Omega^x \cap \Omega^y$, we can consider the compact $K_\epsilon^\omega = E_\epsilon^\omega \times F_\epsilon^\omega \subset E \times F$ and compute the following

$$L^N(X^{(N)}(\omega), Y^{(N)}(\omega))(K_\epsilon^\omega) \leq L^N(X^{(N)}(\omega))((E_\epsilon^\omega)^c) + L^N(Y^{(N)}(\omega))((F_\epsilon^\omega)^c) < \epsilon.$$

We have thus shown that the sequence $L^N(X^{(N)}, Y^{(N)})$ is almost surely tight. With an abuse of notation, we call L^N a converging subsequence and we take a continuous and bounded test function of the form $\varphi(x, y) := \varphi_1(x)\varphi_2(y)$ on $E \times F$. We compute the following

$$\begin{aligned} L^N(X^{(N)}, Y^{(N)})(\varphi) - (\mu \otimes \lambda)(\varphi) &= \frac{1}{N} \sum_{i=1}^N \varphi_2(Y_{i,N}) \left[\varphi_1(X_i) - \int_E \varphi_1(x) d\mu(x) \right] \\ &\quad + \frac{1}{N} \sum_{i=1}^N \int_E \varphi_1(x) d\mu(x) \left[\varphi_2(Y_{i,N}) - \int_F \varphi_2(y) d\lambda(y) \right]. \end{aligned}$$

The first term on the right hand side converges to zero thanks to Lemma 29, since the term in the brackets is a collection of bounded centered i.i.d. random variables. The second term on the right-hand side converges by assumption (36). \square

4 Reflection at the boundary

The problem of SDEs in a domain with reflection has been considered since the works by Skorokhod [31], [32]. The literature is vast and we mention the works by Tanaka [35], Lions and Sznitman [26] as two of the most important papers. The case of mean field SDEs with reflection has also been studied, see for example the works by Sznitman [33], Graham and Metivier [18], which establish well-posedness under general conditions and particle approximation for independent inputs and with Brownian motion as driving signal (possibly with a diffusion coefficient). Also other types of SDEs with mean field interactions and in domains have been studied (with different kind of reflections), see for example [20], [8].

Here we show how to adapt the main result, Theorem 11, and the argument to the case of reflecting boundary conditions. With respect to the previously cited works, we can allow general continuous paths as inputs, we do not need to assume independence nor exchangeability of particles for particle approximation.

Throughout this section, we assume that D is a bounded convex polyhedron in \mathbb{R}^d with nonempty interior (see Remark 38 below for extensions).

We are given a Borel vector field b that satisfies the following

Assumption 5. *Let $p \in [1, +\infty)$. The function $b : [0, T] \times \bar{D} \times \mathcal{P}_p(\bar{D} \times \mathbb{R}^d) \rightarrow \mathbb{R}^d$ is a measurable function and there exists a constant K_b such that,*

$$|b(t, x, \mu) - b(t, x', \mu')|^p \leq K_b (|x - x'|^p + \mathcal{W}_{\mathbb{R}^d, p}(\mu, \mu')^p),$$

$$\forall t \in [0, T], x, x' \in \mathbb{R}^d, \mu, \mu' \in \mathcal{P}_p(\bar{D} \times \mathbb{R}^d).$$

We consider the generalized McKean-Vlasov Skorokhod problem

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t, k_t))dt + dW_t - dk_t \\ X \in C_T(\bar{D}), X_0 = \zeta, \\ k \in BV_T, d|k|_t = 1_{X_t \in \partial D} d|k|_t, dk_t = n(X_t) d|k|_t. \end{cases} \quad (37)$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, the input to equation 37 is a random variable (ζ, W) with values in $\bar{D} \times C_T$, the solution is the couple (X, k) of random variables satisfying the equation above, $|k|$ denotes the total variation process of k (not the modulus of k) and $n(x)$ is the outer normal at x , for x in ∂D , see Remark 32 below for the precise meaning. A short explanation on the meaning of the k term is given later after the main result.

We give now the precise definition of solution:

Definition 31. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\zeta : \Omega \rightarrow D, W : \Omega \rightarrow C_T$ be random variables on it. A solution to the generalized McKean-Vlasov Skorokhod problem with input (ζ, W) is a couple of random variables $X : \Omega \rightarrow C_T(\bar{D})$ and $k : \Omega \rightarrow C_T$ such that, for Lebesgue-a.e. t , $\mathcal{L}(X_t, k_t)$ is in $\mathcal{P}_p(\bar{D} \times \mathbb{R}^d)$ and, for a.e. ω , equation (37) is satisfied (where $X \in C_T(\bar{D})$ means that X is $C_T(\bar{D})$ -valued, $k \in BV_T$ means that $k \in BV_T := BV([0, T]; \mathbb{R}^d)$ \mathbb{P} -a.s. and where the last line is understood in the sense of Remark 32 below).

Remark 32. Actually the last condition is only valid for smooth domains, which is not the case for D convex polyhedron (it is not smooth at the intersections of the faces of the polyhedron). For simplicity of notation, here and in what follows (also for the particle system), we keep the formulation above, with the understanding that the precise condition should be: for a.e. ω there exists a Borel function $\gamma = \gamma^\omega : [0, T] \rightarrow \mathbb{R}^d$ such that $dk_t = \gamma_t d|k|_t$ and, for $d|k|$ -a.e. t , γ_t belongs to $d(X_t)$, where

$$d(x) = \left\{ \sum_{i, x \in \partial D_i} \alpha_i n_i \mid \alpha_i \geq 0, \left| \sum_{i, x \in \partial D_i} \alpha_i n_i \right| = 1 \right\}$$

and where ∂D_i are the faces of the polyhedron with outer normals n_i .

Our main result is, as before, well-posedness of the generalized McKean-Vlasov Skorokhod problem and Lipschitz continuity with respect to law of the input.

Theorem 33. *Let $T > 0$ be fixed and let $p \in [1, \infty)$. Assume that b satisfies 5.*

- 1 For every input (ζ, W) (random variable in $L^p(\bar{D} \times C_T)$) with finite p -moment, there exists a unique solution (X, k) to the generalized McKean-Vlasov Skorokhod problem (37).
- 2 There exists a constant $\tilde{C} = \tilde{C}(p, T, b) > 0$ such that: for every two inputs (ζ^i, W^i) , $i = 1, 2$ (defined possibly on different probability spaces) with finite p -moments, the following is satisfied

$$W_{C_T(\bar{D}) \times C_T, p}(\mathcal{L}(X^1, k^1), \mathcal{L}(X^2, k^2)) \leq \tilde{C} W_{\bar{D} \times C_T, p}(\mathcal{L}(\zeta^1, W^1), \mathcal{L}(\zeta^2, W^2)).$$

In particular, the law of a solution (X, k) depends only on the law of (ζ, W) .

To prove this result, we regard the generalized McKean-Vlasov Skorokhod problem as a fixed point problem with parameter. For this, we introduce the following Skorokhod problem, for fixed μ in $\mathcal{P}_p(C_T(\bar{D}) \times C_T)$ (calling μ_t the marginal at time t):

$$\begin{cases} dY_t^\mu = b(t, Y_t^\mu, \mu_t)dt + dW_t - dh_t^\mu \\ Y^\mu \in C_T(\bar{D}), Y_0^\mu = \zeta, \\ h^\mu \in BV_T, d|h^\mu|_t = 1_{Y_t \in \partial D} d|h^\mu|_t, dh_t^\mu = n(Y_t) d|h^\mu|_t. \end{cases} \quad (38)$$

We recall the following well-posedness result for μ fixed:

Lemma 34. Fix μ in $\mathcal{P}_p(C_T(\bar{D}) \times C_T)$ and assume that b is Lipschitz and bounded as in Theorem 33. Then, for every $T > 0$, for every deterministic initial datum $\zeta \equiv x_0$ in \bar{D} and for every deterministic path $W \equiv \gamma$ in C_T , there exists a unique solution $(Y, h) = (Y^\mu, h^\mu)$ in $C_T(\bar{D}) \times C_T$ to the above equation.

This result is classical and one can see it as a consequence of well-posedness for Skorokhod problem without drift, via Lemma 35 below, in the same line of the proof of Theorem 33 (see in particular the bound (39)). We call $S^\mu : \bar{D} \times C_T \rightarrow C_T(\bar{D}) \times C_T$ the solution map to (38), that is, $S^\mu(x_0, \gamma) = (Y^\mu, h^\mu)$ where (Y^μ, h^μ) solves (38) with deterministic input $(x_0, \gamma) \in \bar{D} \times C_T$.

For a general random input (ζ, W) in $L^p(\bar{D} \times C_T)$, this result, applied to $(\zeta(\omega), W(\omega))$ for a.e. ω , gives existence and pathwise uniqueness of a solution (Y^μ, h^μ) to (38) and the representation formula $(Y^\mu, h^\mu) = S^\mu(\zeta, W)$. Moreover, again from Lemma 35 below, if the input (ζ, W) has finite p -moment, then also the solution (Y^μ, h^μ) has finite p -moment. We call

$$\begin{aligned} \Phi : \mathcal{P}_p(\bar{D} \times C_T) \times \mathcal{P}_p(C_T(\bar{D}) \times C_T) &\rightarrow \mathcal{P}_p(C_T(\bar{D}) \times C_T), \\ (\mathcal{L}(\zeta, W), \mu) &\mapsto (S^\mu)_\# \mathcal{L}(\zeta, W), \end{aligned}$$

the law of a probability measure $\mathcal{L}(\zeta, W)$, under the solution map S_T^μ of the Skorokhod problem with μ fixed. We sometimes denote $\Phi(\nu, \cdot)$ by Φ^ν .

As in the case without boundaries, note that (X, k) solves the McKean-Vlasov Skorokhod problem if and only if $\mathcal{L}(X, k)$ is a fixed point of $\Phi^{\mathcal{L}(\zeta, W)}$. Hence, Theorem 33 reduces to a fixed point problem with parameter.

A key tool in the proof of this result is the Lipschitz dependence of the boundary term k on the given path in the Skorokhod problem. The precise statement follows from [16, Theorem 2.2] (there the Skorokhod problem is formulated in the space of cadlag functions, but continuity of the solution is ensured by [35, Lemma 2.4]).

Lemma 35. Fix $T > 0$. For x_0 in \bar{D} , z in C_T . Then there exists a unique solution $(y, k) = (y^{x_0, z}, k^{x_0, z})$ in $C_T(\bar{D}) \times C_T$ to the Skorokhod problem driven by z , namely

$$\begin{cases} dy = dz - dk, \\ y \in C_T(D), \quad y_0 = x_0, \\ k \in BV_T, \quad d|k| = 1_{y \in \partial D} d|k|, \quad dk = n(y) d|k|. \end{cases}$$

Moreover there exists $C \geq 0$ (which is locally bounded in T) such that, for every x_0^1, x_0^2 in D , for every z^1, z^2 in C_T ,

$$\begin{aligned} \|y^{x_0^1, z^1} - y^{x_0^2, z^2}\|_\infty + \|k^{x_0^1, z^1} - k^{x_0^2, z^2}\|_\infty &\leq C|x_0^1 - x_0^2| + C\|z^1 - z^2\|_\infty, \\ \|y^{x_0^1, z^1} - x_0^1\|_\infty + \|k^{x_0^1, z^1}\|_\infty &\leq C\|z^1\|_\infty. \end{aligned}$$

Proof of Theorem 33. The result follows from the abstract Proposition 10, provided we verify conditions 1) and 2) on Φ .

Let $\mu \in \mathcal{P}_p(C_T(\bar{D}) \times C_T)$ be fixed, let ν^1 and ν^2 be in $\mathcal{P}_p(\bar{D} \times C_T)$ and let m be an optimal plan on $(\mathbb{R}^d \times C_T)^2$ for these two measures. On the probability space $((\bar{D} \times C_T)^2, m)$, we call $\zeta^i, W^i, i = 1, 2$, the r.v. defined by the canonical projections and $(Y^i, h^i) = S^\mu(\zeta^i, W^i)$ the solution to the Skorokhod problem (38) with input (ζ^i, W^i) . We have

$$W_p(\Phi(\nu^1, \mu), \Phi(\nu^2, \mu))^p \leq \mathcal{E}^m(\|Y^1 - Y^2\|_\infty + \|h^1 - h^2\|_\infty)^p,$$

so it is enough to bound the right-hand side. We can apply Lemma 35 to $z^i = \int_0^t b(t, Y_r^i, \mu) dr + W^i, x_0^i = \zeta^i$ and so $y^i = Y^i, k^i = h^i$: we get

$$\|Y^1 - Y^2\|_\infty + \|h^1 - h^2\|_\infty \leq C|\zeta^1 - \zeta^2| + C \int_0^T |b(t, Y_t^1, \mu) - b(t, Y_t^2, \mu)| dt + C\|W^1 - W^2\|_\infty.$$

Using the Lipschitz property of b in x (uniformly in μ), we get

$$\|Y^1 - Y^2\|_\infty + \|h^1 - h^2\|_\infty \leq C|\zeta^1 - \zeta^2| + C \int_0^T \|Y^1 - Y^2\|_\infty dt + C\|W^1 - W^2\|_\infty.$$

By Gronwall inequality

$$\|Y^1 - Y^2\|_\infty + \|h^1 - h^2\|_\infty \leq C|\zeta^1 - \zeta^2| + C\|W^1 - W^2\|_\infty.$$

We take expectation (with respect to m) of the p -power and use the optimality of m , to obtain

$$W_p(\Phi(\nu^1, \mu), \Phi(\nu^2, \mu))^p \leq C W_p(\nu^1, \nu^2)^p.$$

This ends the proof of condition 1) of Proposition 10.

Let now (ζ, W) be fixed with law $\nu := \mathcal{L}(\zeta, W)$. Consider $\mu^1, \mu^2 \in \mathcal{P}_p(C_T(\bar{D}) \times C_T)$ and call $(Y^i, h^i) = (Y^{\mu^i}, h^{\mu^i}), i = 1, 2$ the corresponding solutions to the Skorokhod problem (38) (driven by the initial datum ζ and the path W). We can apply Lemma 35 to $z^i = \int_0^t b(t, Y_r^{\mu^i}, \mu^i) dr + W, x_0^i = \zeta$ and so $y^i = Y^{\mu^i}, k^i = h^{\mu^i}$: we get

$$\|Y^1 - Y^2\|_\infty + \|h^1 - h^2\|_\infty \leq C \int_0^T |b(t, X_r^{\mu^1}, \mu^1) - b(t, X_r^{\mu^2}, \mu^2)| dr.$$

Taking the p -power and arguing as without boundaries, we get

$$\|Y^1 - Y^2\|_\infty^p + \|h^1 - h^2\|_\infty^p \leq C \int_0^T \|Y^1 - Y^2\|_\infty^p dt + C \int_0^T W_{C_t,p}(\mu^1, \mu^2)^p dt$$

and so, by Gronwall inequality,

$$\|Y^1 - Y^2\|_\infty^p + \|h^1 - h^2\|_\infty^p \leq C \int_0^T W_{C_t,p}(\mu^1, \mu^2)^p dt. \quad (39)$$

Taking expectation, we conclude

$$W_p(\Phi(\nu, \mu^1), \Phi(\nu, \mu^2))^p \leq C \int_0^T W_{C_t,p}(\mu^1, \mu^2)^p dt.$$

As for without boundaries, iterating this inequality k times for k large enough (such that $(CT)^k/k! < 1$), we get condition 2) in Proposition 10. The proof is complete. \square

As in the case without boundary, if the driving process is adapted, then so is the solution to the McKean-Vlasov Skorokhod problem. We omit the proof as it is completely analogous to the one without boundary.

Proposition 36. *Let $(\mathcal{F}_t)_t$ be a right-continuous, complete filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ such that ζ is \mathcal{F}_0 -measurable and W is $(\mathcal{F}_t)_t$ -progressively measurable. Then the solution (X, k) to (37) is also $(\mathcal{F}_t)_t$ -progressively measurable.*

Finally, following Section 3.1, we can obtain a particle approximation to the McKean-Vlasov Skorokhod problem (37), just as corollary of the main result Theorem 33. Here the corresponding particle system reads

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, L^N(X_t^{(N)}, k_t^{(N)}))dt + dW_t^{i,N} - dk_t^{i,N} \\ X^{i,N} \in C_T(\bar{D}), X_0^{i,N} = \zeta^{i,N}, \\ k^{i,N} \in BV_T, d|k^{i,N}|_t = 1_{X_t^{i,N} \in \partial D} d|k^{i,N}|_t, dk_t^{i,N} = n(X_t^{i,N})d|k^{i,N}|_t. \end{cases} \quad (40)$$

Again the solution is an N -uple of couples $(X^{i,N}, k^{i,N})_{i=1,\dots,N}$ (and again $|k^{i,N}|$ denotes the total variation process of $k^{i,N}$ and $k^{i,N} \in BV_T$ means that $k^{i,N}$ belongs to BV_T \mathbb{P} -a.s.). The following result can be proven exactly as Theorem 22 (here we use a notation analogous to that theorem).

Theorem 37. *Let $p \in [1, \infty)$ and assume b satisfies Assumption 5. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. On this space we consider, for $N \in \mathbb{N}$, a family of random variables $(\zeta^{(N)}, W^{(N)}) = (\zeta^{i,N}, W^{i,N})_{1 \leq i \leq N}$ taking values on $\bar{D} \times C_T$. Let $\bar{\zeta} \in L^p(\Omega, \bar{D})$ and $\bar{W} \in L^p(\Omega, C_T)$. Then:*

- i There exists a unique pathwise solution $(X^{(N)}, k^{(N)})$ (resp. (\bar{X}, \bar{k})) to the interacting particle system (40) (resp. equation (37)).*
- ii There exists a constant C depending on b such that for all $N \geq 1$, for a.e. $\omega \in \Omega$,*

$$\begin{aligned} & W_{C_T(\bar{D}) \times C_T,p}(L^N(X^{(N)}(\omega), k^{(N)}(\omega)), \mathcal{L}(\bar{X}, \bar{k}))^p \\ & \leq C W_{\bar{D} \times C_T,p}(L^N(\zeta^{(N)}(\omega), W^{(N)}(\omega)), \mathcal{L}(\bar{\zeta}, \bar{W}))^p. \end{aligned}$$

iii If the empirical $L^N(\zeta^{(N)}, W^{(N)})$ converges to $\mathcal{L}(\bar{\zeta}, \bar{W})$ \mathbb{P} -a.s., then also the empirical measure of the solution converges.

Remark 38. More general cases can be treated, for example oblique reflection or even more general domains D , possibly with some extra assumptions: as one can see from the proof, it is enough to have an estimate as in Lemma 35 for the boundary term. The case of oblique reflection (still with D convex polyhedron) is treated in [16] (see Assumptions 2.1 and Theorem 2.1 there). The case of more general domains is treated for example in [30, 35], though the Lipschitz constant in Lemma 35 seems in this case to depend also on z .

5 Large Deviations

In this section we assume that the driving paths W of equation (10) live on the space $C_{T,0}$ of continuous functions starting at 0. The results of Sections 2 and 3 apply also in this case.

Let $p \in [1, \infty)$. Let $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ be a drift as before and such that it satisfies 1.

As in Section 2, we define the function

$$\begin{aligned} \Phi : \mathcal{P}_p(\mathbb{R}^d \times C_{T,0}) \times \mathcal{P}_p(C_T) &\rightarrow \mathcal{P}_p(C_T) \\ (\mathcal{L}(\zeta, W), \mu) &\mapsto \mathcal{L}(X^\mu) = (S^\mu)_\# \mathcal{L}(\zeta, W), \end{aligned}$$

where S^μ is the solution map of ODE (13), as defined in (12), with $\mathbb{R}^d \times C_{T,0}$ instead of $\mathbb{R}^d \times C_T$ as a domain. Similarly, we consider the map Ψ defined as in (16), replacing C_T with $C_{T,0}$.

We introduce, for every μ in $\mathcal{P}_p(C_T)$, the map

$$f^\mu : C_T \ni \gamma \mapsto \left(\gamma_0, \gamma \cdot - \gamma_0 - \int_0^\cdot b(s, \gamma_s, \mu_s) ds \right) \in \mathbb{R}^d \times C_{T,0}. \quad (41)$$

Note that $f^\mu = (S^\mu)^{-1}$ and f^μ is continuous, in particular measurable.

Lemma 39. *Let $T > 0$ be fixed and let $p \in [1, \infty)$, assume 1. The function Ψ is a bijection, with inverse given by $\Psi^{-1}(\mu) = f^\mu_\# \mu$.*

Proof. For every ν in $\mathcal{P}_p(\mathbb{R}^d \times C_T)$ and η in $\mathcal{P}_p(C_T)$, we have

$$\Phi(\nu, \mu) = (S^\mu)_\# \nu = \eta \text{ if and only if } \nu = f^\mu_\# \eta.$$

In particular, with $\eta = \mu$, we get that $\Psi(\nu) = \mu$ if and only if $\nu = f^\mu_\# \mu$. Hence Ψ is invertible, with inverse given by $\Psi^{-1}(\mu) = f^\mu_\# \mu$ (one can also show that Ψ^{-1} is continuous). \square

For $N \in \mathbb{N}$, let $(\zeta^{(N)}, W^{(N)}) = (\zeta^{i,N}, W^{i,N})_{1 \leq i \leq N} : \Omega \rightarrow (\mathbb{R}^d \times C_{T,0})^N$ be a family of random variables. We consider the system of interacting particles on \mathbb{R}^d as defined in (30), namely

$$\begin{cases} dX^{i,N} = b(t, X^{i,N}, L^N(X^{(N)}))dt + dW_t^{i,N} \\ X_0^{i,N} = \zeta^{i,N}. \end{cases} \quad (42)$$

with solution $X^{(N)} := (X^{i,N})_{i=1, \dots, N}$. We have seen in Section 3.2 that we can define a suitable probability space $(\Omega_N, \mathcal{A}_N, \mathbb{P}_N)$, such that

$$\mathcal{L}_{\mathbb{P}_N}(\zeta^{(N)}, W^{(N)}) = L^N(\zeta^{(N)}, W^{(N)}) := \frac{1}{N} \sum_{i=1}^N \delta_{(\zeta^{i,N}, W^{i,N})},$$

and equation (10) is exactly the interacting particle system (42). Let $(\bar{\zeta}, \bar{W}) \in L^p(\mathbb{R}^d \times C_{T,0})$, we call $\bar{X} \in L^p(C_T)$ the solution to the related McKean-Vlasov equation (29).

This construction shows that Ψ is a continuous function that maps the empirical measure of the inputs into the empirical measure of the particles, namely

$$\Psi(L^N(\zeta^{(N)}, W^{(N)})) = L^N(X^{(N)}), \quad \forall N \in \mathbb{N}.$$

This suggests the following immediate application to the contraction principle for large deviations.

Lemma 40. *Let $(\zeta^{(N)}, W^{(N)}) = (\zeta^{i,N}, W^{i,N})_{1 \leq i \leq N} \subset L^p(\mathbb{R}^d \times C_{T,0})$ be a sequence of random variables and let $I : \mathcal{P}_p(\mathbb{R}^d \times C_{T,0}) \rightarrow [0, +\infty]$ be a lower semi-continuous function. Assume that $L^N(\zeta^{(N)}, W^{(N)})$ satisfies a large deviations principle with (good) rate function I , in the sense of Definition 4.*

Let $X^{(N)} = (X^{i,N})_{i=1,\dots,N}$ be the solution to the interacting particle system (42) with inputs $(\zeta^{i,N}, W^{i,N})_{i=1,\dots,N}$. Then the empirical law $L^N(X^{(N)})$ satisfies a large deviations principle with (good) rate function

$$J(\mu) := I(\Psi^{-1}(\mu)) = I(f_{\#}^{\mu}), \quad \forall \mu \in \mathcal{P}_p(C_T).$$

Proof. We know that the function Ψ is a continuous function, we can thus apply the contraction principle for large deviations which ensures that $L^N(X^{(N)})$ satisfies a large deviations principle with rate function

$$J(\mu) := \inf \{ I(\nu) \mid \forall \nu \in \mathcal{P}_p(\mathbb{R}^d \times C_{T,0}), \quad \Psi(\nu) = \mu \}, \quad \mu \in \mathcal{P}_p(C_T).$$

From the bijectivity of Ψ , given by Lemma 39, we deduce that

$$J(\mu) = I(\Psi^{-1}(\mu)) = I(f_{\#}^{\mu}), \quad \mu \in \mathcal{P}_p(C_T).$$

□

Given a Polish space E , the relative entropy between two measures $\mu, \mu' \in \mathcal{P}_p(E)$ is defined as

$$H(\mu \mid \mu') := \begin{cases} \int_E \log\left(\frac{d\mu}{d\mu'}\right) d\mu, & \mu \ll \mu', \\ +\infty, & \text{otherwise.} \end{cases}$$

We can specialize Lemma 40 to the case when the rate function of the inputs is the entropy with respect to a specific measure. In this case we obtain an even more explicit rate function for the convergence of the empirical measure of the particles.

Lemma 41. *Let $(\zeta^{(N)}, W^{(N)}) = (\zeta^{i,N}, W^{i,N})_{1 \leq i \leq N} : \Omega \rightarrow (\mathbb{R}^d \times C_{T,0})^N$ be a sequence of random variables such that: There exists $\bar{\nu} \in \mathcal{P}_p(\mathbb{R}^d \times C_{T,0})$ such that $L^N(\zeta^{(N)}, W^{(N)})$ satisfies a large deviations principle with good rate function*

$$H(\nu \mid \bar{\nu}), \quad \forall \nu \in \mathcal{P}_p(\mathbb{R}^d \times C_{T,0}).$$

Let $X^{(N)} = (X^{i,N})_{i=1,\dots,N}$ be the solution to the interacting particle system (42) with inputs $(\zeta^{i,N}, W^{i,N})_{i=1,\dots,N}$. Then the empirical law $L^N(X^{(N)})$ satisfies a large deviations principle with good rate function

$$H(\mu \mid \Phi(\bar{\nu}, \mu)), \quad \forall \mu \in \mathcal{P}_p(C_T).$$

Proof. We can apply Lemma 40 to obtain that $L^N(X^{(N)})$ satisfies a large deviations principle with rate function

$$I(\mu) := H(\Psi^{-1}(\mu) \mid \bar{\nu}), \quad \mu \in \mathcal{P}_p(C_T).$$

We show now that $H(\Psi^{-1}(\mu) \mid \bar{\nu}) = H(\mu \mid \Phi(\bar{\nu}, \mu))$. For this, note that, by Lemma 39 and by the definition of Φ ,

$$\Psi^{-1}(\mu) = f_{\#}^{\mu} \mu, \quad \bar{\nu} = f_{\#}^{\mu} \Phi(\bar{\nu}, \mu).$$

Here $f_{\#}^{\mu}$ is a push-forward via a measurable map f^{μ} with measurable inverse S^{μ} . Hence, by standard facts in measure theory, $\Psi^{-1}(\mu) \ll \bar{\nu}$ if and only if $\mu \ll \Phi(\bar{\nu}, \mu)$, in which case we have

$$\frac{d\Psi^{-1}(\mu)}{d\bar{\nu}} = \frac{d\mu}{d\Phi(\bar{\nu}, \mu)} \circ S^{\mu}.$$

Hence, in the case that $\Psi^{-1}(\mu)$ is not absolutely continuous with respect to $\bar{\nu}$, we have $H(\Psi^{-1}(\mu) \mid \bar{\nu}) = H(\mu \mid \Phi(\bar{\nu}, \mu)) = +\infty$. In the case that $\Psi^{-1}(\mu)$ is absolutely continuous with respect to $\bar{\nu}$, we have

$$H(\Psi^{-1}(\mu) \mid \bar{\nu}) = \int \frac{d\Psi^{-1}(\mu)}{d\bar{\nu}} \log \frac{d\Psi^{-1}(\mu)}{d\bar{\nu}} d\bar{\nu} = \int \frac{d\mu}{d\Phi(\bar{\nu}, \mu)} d(S_{\#}^{\mu} \bar{\nu}) = H(\mu \mid \Phi(\bar{\nu}, \mu)).$$

The proof is complete. \square

We will now apply Sanov's Theorem to i.i.d. inputs. The case when the convergence happens in the Wasserstein metric was proved in [38], and it requires an exponential integrability assumption on the law of the inputs.

Theorem 42. *Let $(\zeta^i, W^i)_{i \geq 1} \subset L^p(\mathbb{R}^d \times C_{T,0})$ be a sequence of i.i.d. random variables with law $\bar{\nu} := \mathcal{L}(\zeta^1, W^1)$. Assume that there exists $(x^0, \gamma^0) \in \mathbb{R}^d \times C_{T,0}$ such that*

$$\log \int_{\mathbb{R}^d \times C_{T,0}} \exp(\lambda(|x - x^0| + \|\gamma - \gamma^0\|_{\infty})^p) d\bar{\nu}(x, \gamma) < +\infty, \quad \forall \lambda > 0.$$

Let $X^{(N)} := (X^{i,N})_{i=1,\dots,N}$ be the solution to the interacting particle system (42) with inputs $(\zeta^{(N)}, W^{(N)}) := (\zeta^i, W^i)_{i=1,\dots,N}$. Then the empirical law $L^N(X^{(N)})$ satisfies a large deviations principle with good rate function

$$H(\mu \mid \Phi(\bar{\nu}, \mu)), \quad \forall \mu \in \mathcal{P}_p(C_T).$$

Proof. Sanov's theorem, as in [38, Theorem 1.1], gives that the empirical measure $L^N(\zeta^{(N)}, W^{(N)})$ satisfies a large deviations principle with good rate function

$$I(\nu) = H(\nu \mid \bar{\nu}), \quad \forall \nu \in \mathcal{P}_p(\mathbb{R}^d \times C_{T,0}).$$

The proof then follows from Lemma 41. \square

A Proof of Proposition 10

In this section we prove proposition 10.

First, we must show that Φ^Q has a unique fixed point. If $k = 1$, it is exactly the contraction principle, so we will assume $k > 1$. Clearly $(\Phi^Q)^k$ is a contraction, hence it has a unique fixed point P_Q . Hence,

$$d_E(\Phi^Q(P_Q), P_Q) = d_E((\Phi^Q)^{k+1}(P_Q), (\Phi^Q)^k(P_Q)) \leq c d_E(\Phi^Q(P_Q), P_Q).$$

Since $c < 1$, this implies $d_E(\Phi^Q(P_Q), P_Q) = 0$ and therefore P_Q is also a fixed point for Φ^Q . Every fixed point of Φ^Q is also a fixed point for $(\Phi^Q)^k$, hence P_Q is the only fixed point of Φ^Q .

We are left to prove (15). By induction, one can show that

$$\forall Q, Q' \in F, \forall P \in E \quad d_E((\Phi^Q)^k(P), (\Phi^{Q'})^k(P)) \leq \left(\sum_{i=1}^k L^i \right) d_F(Q, Q').$$

Using a triangular inequality as well as assumption 2) and the previous inequality we obtain

$$\begin{aligned} d_E(P_Q, P_{Q'}) &= d_E((\Phi^Q)^k(P), (\Phi^{Q'})^k(P')) \\ &\leq d_E((\Phi^Q)^k(P), (\Phi^Q)^k(P')) + d_E((\Phi^Q)^k(P'), (\Phi^{Q'})^k(P')) \\ &\leq c d_E(P_Q, P_{Q'}) + \left(\sum_{i=1}^k L^i \right) d_F(Q, Q'). \end{aligned}$$

The proof is complete.

B Wasserstein Metric

We now recall some useful information on the Wasserstein metric, which we defined in (8). For more details the reader can refer to [2]. Let $p \in [1, \infty)$.

- i The infimum in the definition of Wasserstein metric is a minimum. For each couple $\mu, \nu \in \mathcal{P}_p(E)$ there exists a measure $m \in \Gamma(\mu, \nu)$ such that

$$\mathcal{W}_{E,p}(\mu, \nu)^p = \iint_{E \times E} d(x, y)^p m(dx, dy). \quad (43)$$

- ii The Wasserstein distance of two measures on the space of paths is larger than the distance of the corresponding one-time marginals at t , for any t . Indeed, note that, for any $\mu, \nu \in \mathcal{P}_p(C_T)$, if m is in $\Gamma(\mu, \nu)$, then $m_t \in \Gamma(\mu_t, \nu_t)$, therefore we have

$$\mathcal{W}_{\mathbb{R}^d,p}(\mu_t, \nu_t)^p \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - x'|^p m_t(dx, dx') = \iint_{C_T \times C_T} |\gamma_t - \gamma'_t|^p m(d\gamma, d\gamma') \leq \mathcal{W}_{C_T,p}(\mu, \nu)^p.$$

- iii Let E be a Polish space. For any given sequence $(\mu^n)_{n \geq 1} \in \mathcal{P}_p(E)$ the following are equivalent

iii.1 (The sequence converges in Wasserstein sense) $\lim_{n \rightarrow \infty} \mathcal{W}_{E,p}(\mu_n, \mu) = 0$.

iii.2 (The sequence converges weakly and is uniformly integrable) There exists $x_0 \in E$ such that,

$$\begin{cases} \mu_n \xrightarrow{*} \mu, & \text{as } n \rightarrow \infty \\ \lim_{k \rightarrow \infty} \int_{E \setminus B_k(x_0)} d^p(x, x_0) d\mu^n(x) = 0, & \text{uniformly in } n. \end{cases}$$

Cf. [2, Proposition 7.1.5].

As a consequence of point (iii), we give a sufficient condition to pass from weak convergence of measures to convergence in the p -Wasserstein distance.

Lemma 43. *Let (E, d) be a Polish space and $\mu_n, n \in \mathbb{N}$, μ be probability measures on E , fix $q \in [1, \infty)$. If the sequence $(\mu_n)_{n \in \mathbb{N}}$ converges to μ in the weak topology on probability measures and if, for some $p \in (q, \infty)$ and some x_0 in E ,*

$$\sup_n \int_E d(x, x_0)^p \mu_n(dx) < \infty, \quad (44)$$

then μ_n converges in the q -Wasserstein metric to $\mu \in \mathcal{P}_q(E)$.

Proof. By property (iii), it is enough to show that the map $x \mapsto d(x, x_0)^q$ is uniformly integrable with respect to $(\mu_n)_n$. For this, we have, for any $R > 0$, for any n ,

$$\int_{d(x, x_0) > R} d(x, x_0)^q \mu_n(dx) \leq R^{p-q} \int_E d(x, x_0)^p \mu_n(dx).$$

By the uniform bound (44), we can choose R large enough to make the right-hand side above small for all n . This shows that $x \mapsto d(x, x_0)^q$ is uniformly integrable. \square

Lemma 44. *Given $p \in (1, \infty)$ and a separable Banach space $(E, |\cdot|)$, let $(X^i)_{i \geq 1} \in L^p(\Omega, E)$ be a family of i.i.d. random variables on this space with law μ . Then,*

$$\lim_{N \rightarrow \infty} \mathcal{W}_{E,q}(L^N(X^{(N)}), \mu) = 0, \quad q \in (1, p), \quad \mathbb{P} - a.s.$$

Proof. Since (X^i) are i.i.d., \mathbb{P} -a.s. convergence in the weak topology

$$L^N(X^{(N)}(\omega)) \xrightarrow{*} \mathcal{L}(X^1), \quad \mathbb{P} - a.s.$$

is a classical result, see for example [39] and references therein. Moreover, by the law of large numbers, we have, for a.e. ω ,

$$\int_E |x|^p dL^N(X^{(N)}(\omega))(x) = \frac{1}{N} \sum_{i=1}^N |X^i(\omega)|^p \rightarrow \mathbb{E}|X^1|^p < \infty.$$

We obtain condition (44) in Lemma 43, which concludes the proof. \square

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