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On decomposition of embedded prismatoids in $\mathbb{R}^3$ without additional points

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On decomposition of embedded prismatoids in $\mathbb{R}^3$ without additional points

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ABSTRACT. This paper considers three-dimensional *prismatoids* which can be embedded in $\mathbb{R}^3$. A subclass of this family are *twisted prisms*, which includes the family of non-triangulable Schönhardt polyhedra [12, 10]. We call a prismatoid *decomposable* if it can be cut into two smaller prismatoids (which have smaller volumes) without using additional points. Otherwise it is *indecomposable*. The indecomposable property implies the non-triangulable property of a prismatoid but not vice versa.

In this paper we prove two basic facts about the decomposability of embedded prismatoid in $\mathbb{R}^3$ with convex bases. Let P be such a prismatoid, call an edge *interior edge* of P if its both endpoints are vertices of P and its interior lies inside P. Our first result is a condition to characterise indecomposable twisted prisms. It states that a twisted prism is indecomposable without additional points if and only if it allows no interior edge. Our second result shows that any embedded prismatoid in $\mathbb{R}^3$ with convex base polygons can be decomposed into the union of two sets (one of them may be empty): a set of tetrahedra and a set of indecomposable twisted prisms, such that all elements in these two sets have disjoint interiors.

1. INTRODUCTION

Decomposing a geometric object into simpler parts is one of the most fundamental problems in computational geometry.

In 2d, this problem is well-solved. Given a polygonal region, whose boundary is a planar straight line graph $G = (V, E)$, there are many efficient algorithms to create a *constrained triangulation* of G whose vertex set is V and it contains all edges of E. Moreover, no additional vertices is needed. Lee and Lin [8] and Chew [4] independently proved that there exists a triangulation of G, called the *constrained Delaunay triangulation*, such that it is as close as to the Delaunay triangulation of V, while it preserves all edges of E. Moreover, Chew showed that this triangulation can be constructed in optimal $O(n \log n)$ time [4].

The problem of triangulating 3d polyhedra is very difficult, even we restricted ourself to only consider simple polyhedra (without holes). It is known that not all simple polyhedra can be triangulated without adding new vertices, so-called *Steiner points*. The famous example of Schönhardt [12] (known as the Schönhardt polyhedron) shows that a twisted non-convex triangular prism cannot be triangulated without adding new vertices, see Figure 1 Left. Other examples of non-triangulable polyhedra are constructed, see e.g. [1, 6, 3, 10, 2, 13].

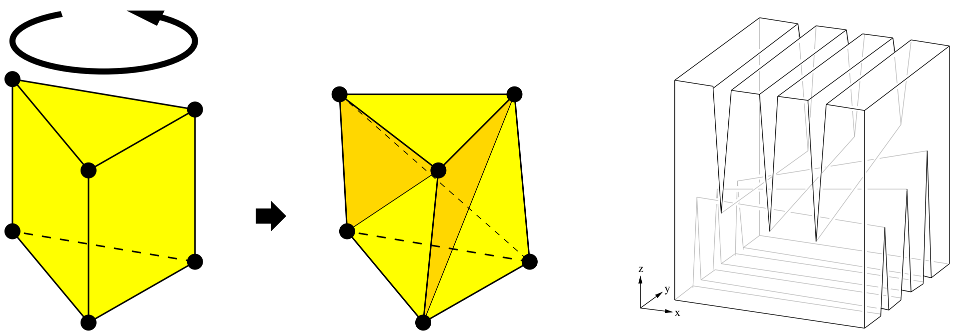

FIGURE 1. Left: The Schönhardt polyhedron. Right: A Chazelle polyhedron.

The existence of non-triangulable polyhedra is a major difficulty in many 3d problems. Ruppert and Seidel [11] proved that the problem to determine whether a simple non-convex polyhedron can be

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triangulated without Steiner points is NP-complete. It is necessary to use additional points, so-called *Steiner points*, to triangulate 3d polyhedra. Chazelle [3] constructs a family of polyhedra and proved that they require a large number of Steiner points to be triangulated, see Figure 1 Right.

Despite the fact that such polyhedra exist, there is not much study about the geometry and topology of such polyhedra. Rambau [10] first showed that any *non-convex twisted prisms* over an n-gon ($n \geq 3$) cannot be triangulated without Steiner points. Furthermore, he showed that the non-triangulability of such polyhedra does not depend on how much it is twisted. This generalised Schönhardt polyhedron into a family of polyhedra with such property. We call polyhedra of this family *Rambau polyhedra*. The Schönhardt polyhedron is the simplest case of a Rambau polyhedron.

The geometry of a Rambau polyhedron is a special prism such that its top and base polygons are (i) planar, (ii) congruent, and (iii) parallel to each other. In general, a twisted prism is not necessarily a Rambau polyhedron. Indeed, a slightly perturbed Rambau polyhedron whose base polygon has more than 3 vertices might become triangulable. On the other hand, if a prism (not necessarily a Rambau polyhedron) is twisted sufficiently large, the result prism will not be triangulable without Steiner points. The proof of this fact is rather simple. A basic fact (proved in Section 3) is that for a prism whose base polygon has more than 5 vertices, it needs interior edges to be decomposed. When a prism is twisted sufficiently large, it will reach a state that no interior edge can be inserted. Hence it must be non-triangulable. Note that a Rambau polyhedron might allow interior edges to be inserted. Motivated by this phenomenon, we want to find the critical conditions between the existence and non-existence of a tetrahedralisation for this kind of polyhedra.

This paper considers three-dimensional *prismatoids* which can be embedded in $\mathbb{R}^3$. A subclass of this family are *twisted prisms*, which includes the family of non-triangulable Schönhardt polyhedra [12, 10].

We call a prismatoid *decomposable* if it can be cut into two smaller prismatoids (which have smaller volumes) without using additional points. Otherwise it is *indecomposable*. The indecomposable property implies the non-triangulable property of a prismatoid but not vice versa.

In this paper we prove two basic facts about the decomposability of embedded prismatoid in $\mathbb{R}^3$ with convex bases. Let P be such a prismatoid, call an edge *interior edge* of P if its both endpoints are vertices of P and its interior lies inside P. Our first result is a condition to characterise indecomposable twisted prisms. It states that a twisted prism is indecomposable without additional points if and only if it allows no interior edge. Our second result shows that any embedded prismatoid in $\mathbb{R}^3$ with convex base polygons can be decomposed into the union of two sets (one of them may be empty): a set of tetrahedra and a set of indecomposable twisted prisms, such that all elements in these two sets have disjoint interiors.

Outline. The rest of this paper is organised as follows: Section 2 gives the necessary definitions of the family of prismatoids and twisted prismatoids studied in this paper. Section 3 presents the new results of this paper.

2. Preliminaries

2.1. **Prisms, Antiprisms, Prismatoids.** In geometry, a *prism* is a solid that has two polygonal faces that are parallel and congruent [7, 14]. In other words, it is a 3d polyhedron comprising an n-sided polygonal base (possibly non convex), a second base which is a translated copy (rigidly moved without rotation) of the first, and n other faces (necessarily all parallelograms) joining corresponding sides of the two bases. All cross-sections parallel to the bases are translations of the bases. Antiprisms are

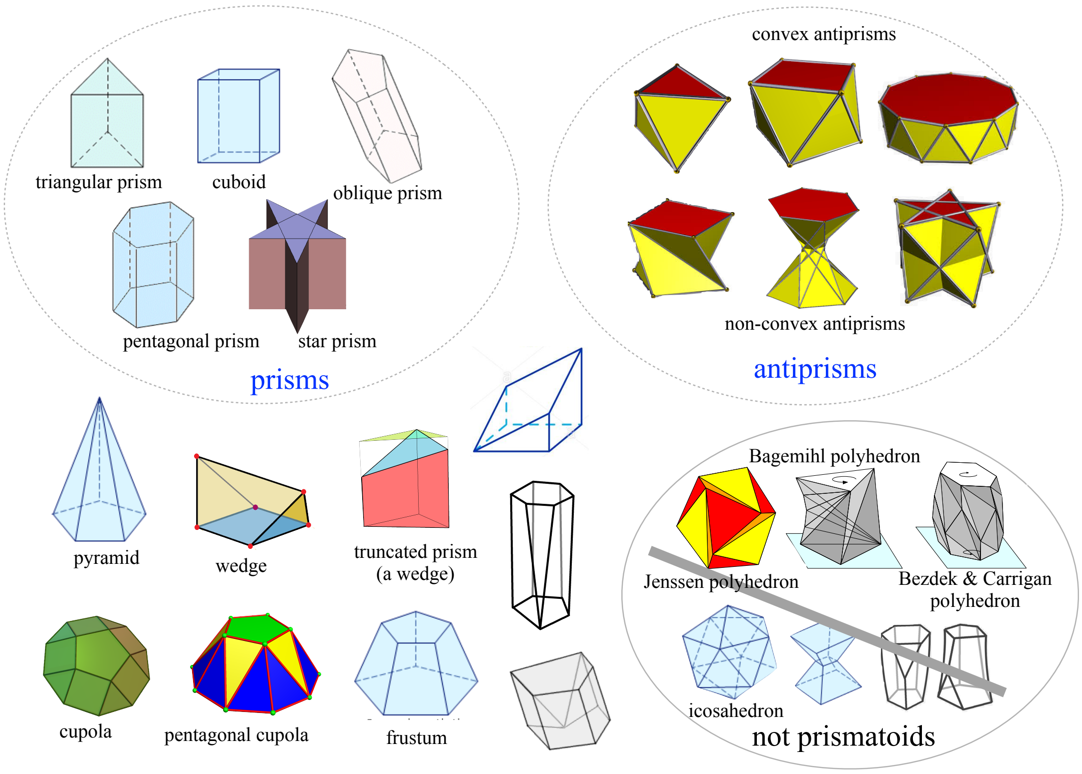

FIGURE 2. The family of prismatoids. Right-bottom shows some common figures which are not prismatoids. Figures are from WWW internet.

similar to prisms except the bases are twisted relative to each other, and that the side faces are triangles, rather than quadrilaterals. Formally, an n-sided *antiprism* is a 3d polyhedron composed of two parallel copies of an n-sided polygonal base (possibly non convex), connected by an alternating band of triangles. Both prisms and antiprisms are subclasses of prismatoids. In geometry, a *prismatoid* is a polyhedron whose vertices all lie in two parallel planes. Its lateral faces can be trapezoids or triangles [7, 14]. The family of prismatoids includes many common geometric shapes, such as pyramids, wedges, prisms, antiprisms, frusta (truncated pyramids), etc. Figure 2 shows various examples in the family of prismatoids as well as some common solids which are not prismatoids by its definition.

2.2. $S_{n,m}$**-Prismatoids.** This section defines a family of prismatoids considered in this paper. Simply saying, these prismatoids have convex bases which are connected by a band of triangular facets. Additionally they can be embedded in $\mathbb{R}^3$ without self-intersections. The precise definition is given below.

Without loss of generality, we will place a prismatoid in such a way such that the two base facets are parallel to the horizontal plane $H_0 := \{(x, y, 0)\ ;\ x, y \in \mathbb{R}\}$. Moreover, one of its facets, called *bottom facet*, lies in H_0, and the other facet, called *top facet*, lies in the plane $H_h := \{(x, y, h)\ ;\ x, y, h \in \mathbb{R}, h > 0\}$.

Let n, m be two integers satisfying (1) $1 \leq n, 1 \leq m$ and (2) $n + m \geq 4$. We define an $S_{n,m}$-*prismatoid* (or shortly $S_{n,m}$) P, as a 3d solid such that:

(i) its top facet is a convex n-gon in H_h, its bottom facet is a convex m-gon in H_0;
(ii) the side facets of P between its bottom and top facets are all triangles; and
(iii) P is topologically a 3-ball embedded in $\mathbb{R}^3$.

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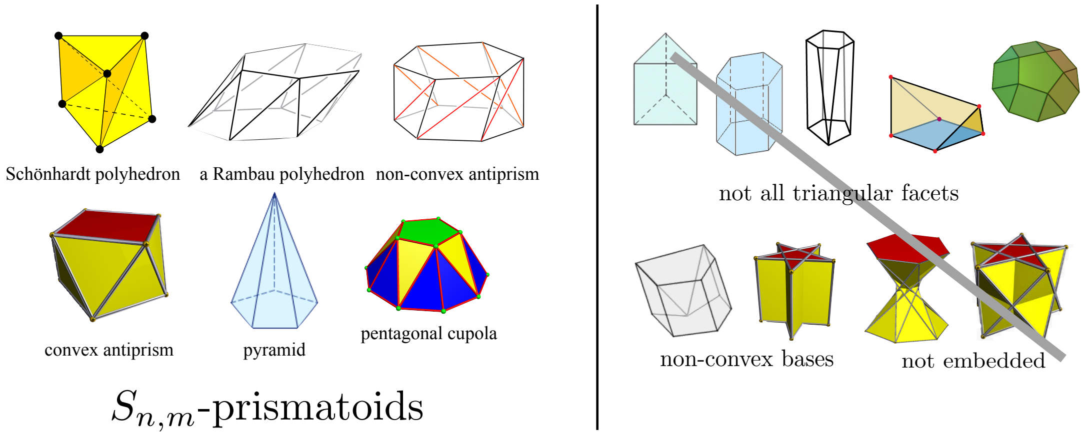

FIGURE 3. Prismatoids on the left are $S_{n,m}$-prismatoids, while those on the right are not. Figures are from WWW internet.

Many prismatoids are $S_{n,m}$-prismatoids. For examples, $S_{1,m}$'s ($m \geq 3$) are pyramids. $S_{2,m}$'s ($m \geq 2$) are wedges with triangular facets. In particular, both $S_{1,3}$'s and $S_{2,2}$ are tetrahedra. Antiprisms which can be embedded in $\mathbb{R}^3$ with no self-intersected facets are $S_{n,n}$-prismatoids, see Figure 3 Left. However, all prisms as well as many of other prismatoids are not $S_{n,m}$-prismatoids, see Figure 3 Right. Obviously, if a non $S_{n,m}$-prismatoid satisfies (iii), i.e., it can be embedded in $\mathbb{R}^3$ without self-intersection, it will become an $S_{n,m}$ by a slight perturbation in its vertex set.

Given an $S_{n,m}$-prismatoid P, there are $n + m$ triangles in its band. There is a bijection between the band of triangles and a binary string of $n + m$ $0/1$ bits. This transformation is first constructed in [5].

We first construct a *flattened band* D of triangles in the plane. It is done by cutting the band of P along one of its edges and then flatten it into the plane. There are $n + 1$ vertices and n edges on the top of D, and $m + 1$ vertices and m edges on the bottom of D. These edges are one-to-one correspond to the boundary edges of the top and bottom facets of P. The two vertical boundary edges of D are identified as the same edge which we cut open. The triangles of the band are bijectively mapped into the triangles of D, i.e., the images of the triangles of the band triangulates D. We label each triangle in D as '0' if it has an edge on the top and a vertex on the bottom, as '1' if it has an edge on the bottom and a vertex on the top. Now the set of triangles from left to right corresponds to a string like "$0100101\ldots$". An example of this transformation is shown in Figure 4.

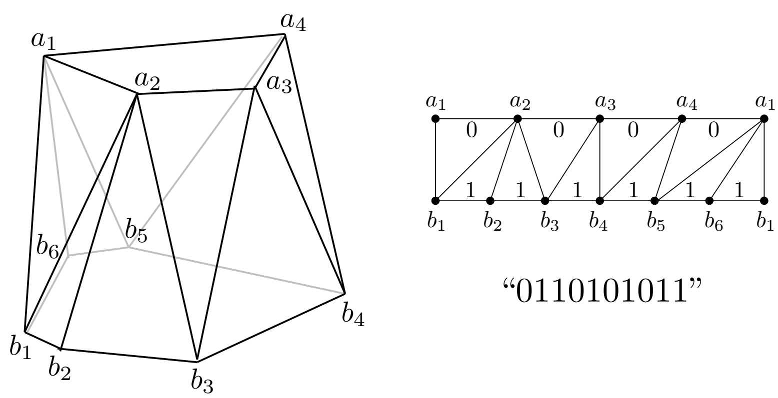

FIGURE 4. An $S_{4,6}$-prismatoid is shown in the left and the binary string corresponds to its band of triangles is shown in the right.

With this transformation, the combinatorial structure of an $S_{n,m}$ can be characterised by a binary string. However, this string does not recognise the geometry of the prismatoid. For example, a convex and a non-convex $S_{n,m}$'s may have exactly the same binary string.

2.3. **Twisted Prisms.** We use the above transformation to define a special class of $S_{n,n}$-prismatoids. Recall that an antiprism can be obtained by twist a prism. There are two directions, clockwise or counterclockwise, in the plane. Depending on which direction it is twisted. We will get two non-convex antiprisms which are similar but with combinatorially different boundary facets, see Figure 5. We call an $S_{n,n}$-prismatoid a *twisted prism* if the band of its triangles corresponds to a binary string which contains no two consecutive 0's or $1's$, i.e., a string like "$01010101\ldots$", or "$10101010\ldots$".

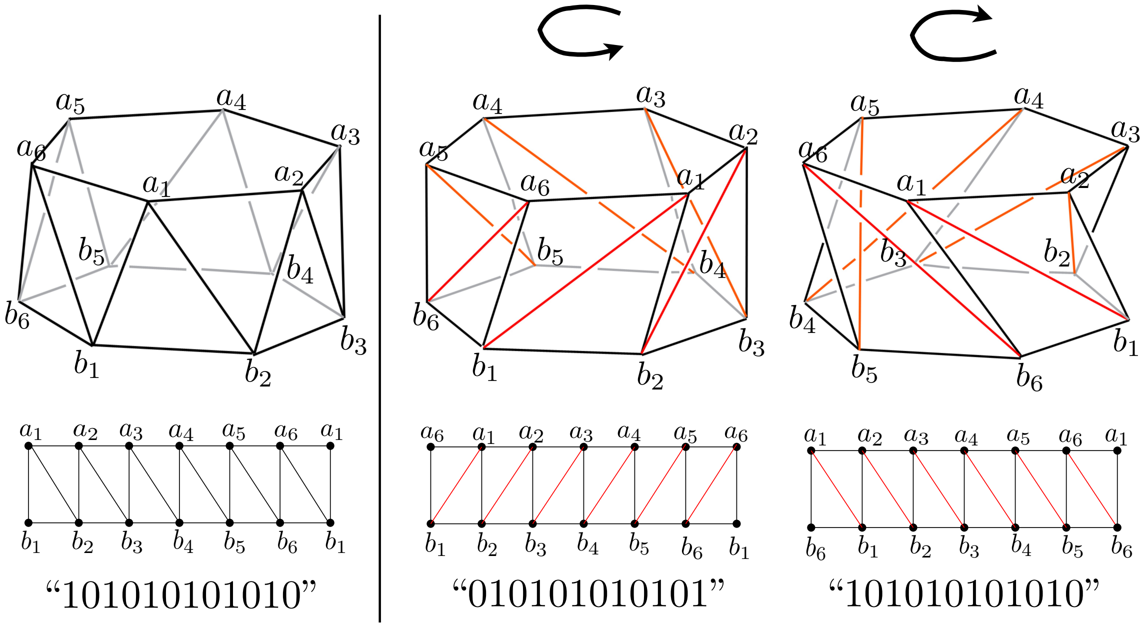

FIGURE 5. Twisted prisms (top), bands (middle), and strings (bottom). Left is a convex hexagonal antiprism. Right are two non-convex antiprisms resulted by twisting the top facet of left counterclockwise or clockwise, respectively.

The *degree* of a vertex of an $S_{n,m}$-prismatoid is the number of edges shared at this vertex. An equivalent definition of a twisted prism is: a twisted prism is an $S_{n,n}$-prismatoid whose vertices all have degree 4.

Note that a twisted prism might be convex or non-convex. We are interesting a special type of non-convex twisted prisms. Let P be a non-convex twisted prism whose base is an n-gon. We call P a *pure* non-convex twisted prism if there are exactly n non-convex edges in its boundary. For examples, the two non-convex prisms in Figure 5 are pure. In particular, all Rambau's non-convex twisted prisms are pure.

We comment that our definition of twisted prisms is slightly more general than Rambau's definition [10] in such a way that it does not require that the top and bottom facets are strictly congruent. They may be two different convex n-gons.

2.4. **Decompositions of $S_{n,m}$-Prismatoids.** Let P be a prismatoid. a *triangulation* $\mathcal{T}$ of P is a 3d geometric simplicial complex such that the union of all simplices of $\mathcal{T}$ is P, i.e., the underlying space of $\mathcal{T}$ is P. A triangulation of P may contain additional vertices, which are not vertices of P. These vertices are called *Steiner points* of P. In this paper, we are only interested in those triangulations of P which contain no Steiner points. We say a prismatoid is *triangulable* if it admits a triangulation without Steiner points. Otherwise, it is *non-triangulable*. It is well-known that some prismatoids are non-triangulable, such as the Schönhardt polyhedron as well as Rambau polyhedra.

Let P_1 and P_2 be two $S_{n,m}$-prismatoids, respectively. Let $\mathrm{Vert}(P_1)$ and $\mathrm{Vert}(P_2)$ be the vertex sets of P_1 and P_2, respectively, and let $\mathrm{Vol}(P_1)$ and $\mathrm{Vol}(P_2)$ be the volumes of P_1 and P_2, respectively We say that P_1 is *smaller than* P_2 if

(1) $\mathrm{Vert}(P_1) \subseteq \mathrm{Vert}(P_2)$; and
(2) $\mathrm{Vol}(P_1) < \mathrm{Vol}(P_2)$.

(1) means that P_1 and P_2 share the same vertex set of P_2, while (2) means that the volume of P_1 is strictly less than that of P_2. Note that (2) must hold if the number of vertices of P_1 is strictly less than that of P_2, i.e., $\mathrm{Vert}(P_1) \subsetneq \mathrm{Vert}(P_2)$ Note that if P_1 and P_2 have different vertices, than they are not comparable.

We say that an $S_{n,m}$-prismatoid is *decomposable* if it is either a single tetrahedron (i.e, an $S_{1,3}$ or $S_{2,2}$) or there exists a partition of it into two smaller $S_{n,m}$-prismatoids without using Steiner point such that the two prismatoids share no interior points, i.e., they only share at their common boundary facets. Otherwise, it is *indecomposable*.

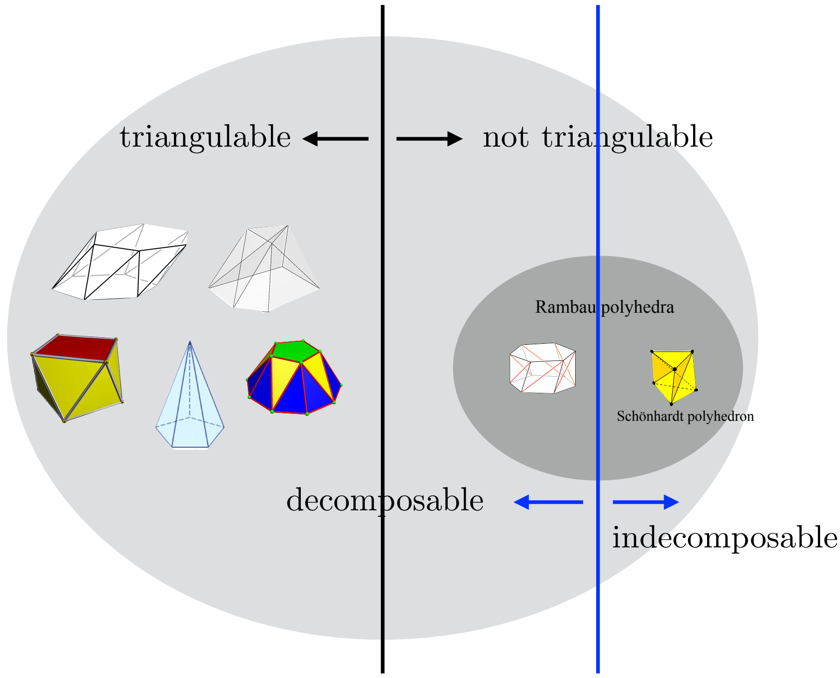

FIGURE 6. An illustration of the difference of the definitions of being triangulable and decomposable of a prismatoid.

The difference of being triangulable and being decomposable for a given prismatoid is following (see Figure 6): a triangulable prismatoid is also decomposable but not vice versa. A non-triangulable prismatoid might still be decomposable. While an indecomposable prismatoid must be non-triangulable.

3. New Results on Decomposition of $S_{n,m}$'s

In this section, we will prove two theorems about the decomposability of $S_{n,m}$'s.

Theorem 1. *A twisted prism is indecomposable if and only if it does not contain interior edges.*

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Theorem 2. *An $S_{n,m}$-prismatoid P can be decomposed with no Steiner points into the union of two sets $\mathcal{T}$ and $\mathcal{P}$, where $\mathcal{T}$ is a set of tetrahedra and $\mathcal{P}$ is a set of indecomposable twisted prisms. All elements in $\mathcal{T}$ and $\mathcal{P}$ have disjoint interiors.*

3.1. **Outline of the proof.** Recall an *ear* of a two-dimensional polygon is defined as a vertex v of this polygon such that the line segment between the two neighbors of v lies entirely in the interior of the polygon The two-ears-theorem [9] states that every simple polygon with more than three vertices has at least two ears, vertices that can be removed from the polygon without introducing any crossings. This theorem can be used to show that every two-dimensional simple polygon can be triangulated.

The analogue of "an earöf a 3d polyhedron is a degree 3 vertex, which has exactly 3 boundary edges of this polyhedron connecting to it, see Figure 7 Left. The following lemma shows that if an $S_{n,m}$ contains a degree 3 vertex then it can be reduced to a smaller prismatoid which does not contain that vertex. In other words, a degree 3 vertex can be removed from it, see Figure 7 Right.

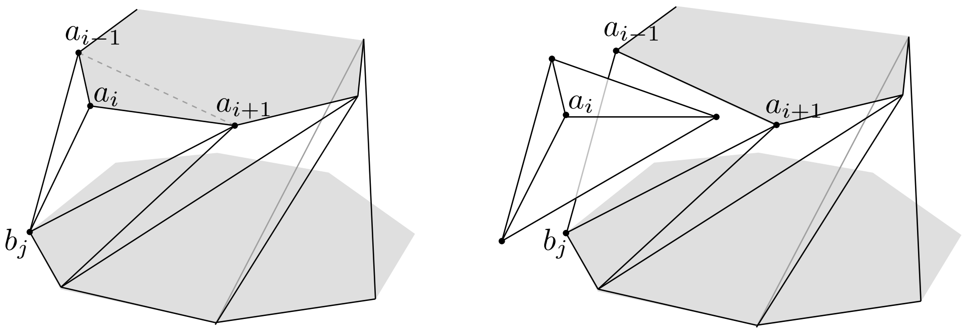

FIGURE 7. Left: an $S_{n,m}$-prismatoid contains a degree 3 vertex a_i. Right: this prismatoid is separated by the tetrahedron $\{a_{i-1}, a_i, a_{i+1}, b_j\}$ and a $(n-1, m)$-prismatoid.

Lemma 3. *If an $S_{n,m}$-prismatoid with more than 5 vertices contains a degree 3 vertex, then it can be dissected into a tetrahedron and a smaller $S_{n,m}$-prismatoid without Steiner point.*

Proof. Let P be an $S_{n,m}$. A triangular face is an *interior face* of P if its three vertices are vertices of P and it is not a boundary facet of P. We prove this lemma in the following two steps:

(1) a degree 3 vertex defines an interior face of P; and
(2) P can be separated by cutting along this interior face.

Without loss of generality, we assume P contains a degree 3 vertex a_i in its top facet, and the three boundary edges of P containing a_i are $\{a_i, a_{i-1}\}$, $\{a_i, a_{i+1}\}$, $\{a_i, b_j\}$. Then the face $\{a_{i-1}, a_{i+1}, b_j\}$ is an interior face, see Figure 7 Left.

Our proof of (2) is based on the following observation. Let our eye be at a_i, and we're looking into the interior of P. Our viewing volume is restricted by a cone with apex a_i and three boundary faces $f_1 := \{a_i, a_{i-1}, b_j\}$, $f_2 := \{a_i, a_{i+1}, b_j\}$, and $f_3 := \{a_i, a_{i-1}, a_{i+1}\}$. Note that f_1, f_2 are original boundary facets of P. Since the edge $\{a_{i-1}, a_{i+1}\}$ lies in the interior of the top facet, the triangle f_3 is an ear in top facet. By the property (iii) of P, which requires that P contains no self-intersected boundary facets.

The above facts together imply the fact that all interior points of the tetrahedron $\{a_{i-1}, a_i, a_{i+1}, b_j\}$ are interior points of P. Furthermore, the visibility to the four corners from any interior point of $\{a_{i-1}, a_i, a_{i+1}, b_j\}$ is not block by any boundary facet of P.

Therefore, the tetrahedron $\{a_{i-1}, a_i, a_{i+1}, b_j\}$ can be separated from P which results an $S_{n-1,m}$-prismatoid P' with $\{a_{i-1}, a_{i+1}, b_j\}$ as its boundary facet. □

By the above lemma, as long as an $S_{n,m}$-prismatoid contains a degree 3 vertex, it is decomposable. Since a wedge can not be a twisted prism. The above lemma immediately implies the following fact.

Corollary 4. *All $S_{2,m}$-prismatoids, $m \geq 2$, can be triangulated without Steiner points.*

If a twisted prism which is not pure can still be decompose by removing a tetrahedron. Therefore we can easily get the following corollary.

Corollary 5. *All $S_{n,m}$-prismatoids except pure non-convex twisted prisms are decomposable.*

Since not all pure non-convex twisted prisms are indecomposable, we still need to find the condition to characterise whether a pure non-convex twisted prism is indecomposable or not. Consider a twisted prism P. Call an edge $\{a_i, b_j\}$ an *interior edge* of P if both a_i and b_j are vertices of P and $\{a_i, b_j\}$ is not a boundary edge of P, and the interior of $\{a_i, b_j\}$ lies in P, see Figure 14 Left. The following lemma is crucial to reach this condition. It shows as long as there exists an interior edge of P, then P is decomposable. An example of this lemma is shown in Figure 14 Right. The proof of this lemma is given in Section 3.2.

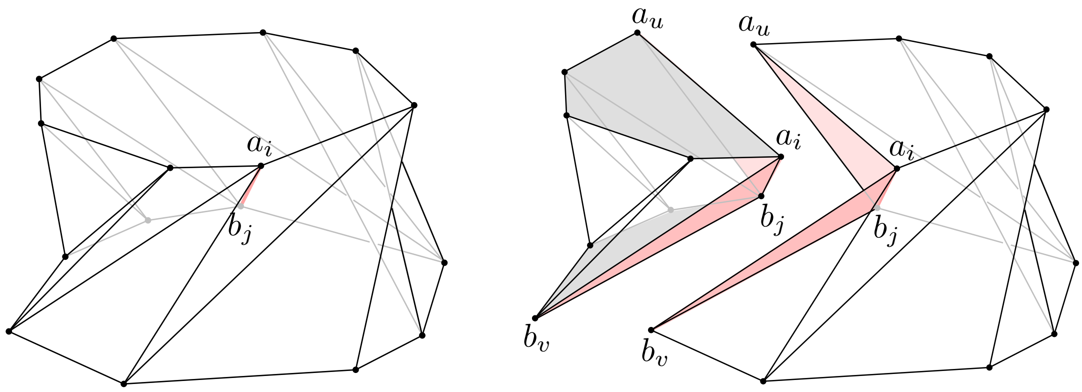

FIGURE 8. A twisted prism (left) (an $S_{8,8}$) contains an interior edge $\{a_i, b_j\}$ (shown in pink). It is decomposed into two prismatoids, an $S_{4,5}$ and an $S_{5,6}$ (right) at this interior edge $\{a_i, b_j\}$ and two chosen interior faces $\{a_i, a_u, b_j\}$ and $\{a_i, b_v, b_j\}$ of the prism.

Lemma 6. *If a pure non-convex twisted prism contains an interior edge, then it can be decomposed into two smaller prismatoids without Steiner point.*

Theorem 1 is then proved by the reserve of Lemma 6.

Proof of Theorem 2. This theorem can be proven by combining Lemma 3 and Lemma 6.

Proof. Given an $S_{n,m}$, as long as it is not a twisted prism, it can be dissected into a set of tetrahedra and a twisted prism. If the twisted prism admits at least one interior edge, it can be dissected into two smaller simplicial prismatoids. The above process can be repeated until either no twisted prism is remaining or the remaining twisted prisms are indecomposable. ☐

3.2. **Proof of Lemma 6.** We will prove this lemma by showing: if an interior edge exists, then there must exist four interior faces which share at this edge, and the original twisted prism can be separated into two smaller prismatoids by these faces on their boundary.

Let P be a pure non-convex twisted prism whose base is a convex n-gon, $n > 3$. Without loss of generality, we assume that the top facet of P is twisted counterclockwise against its bottom facet, see

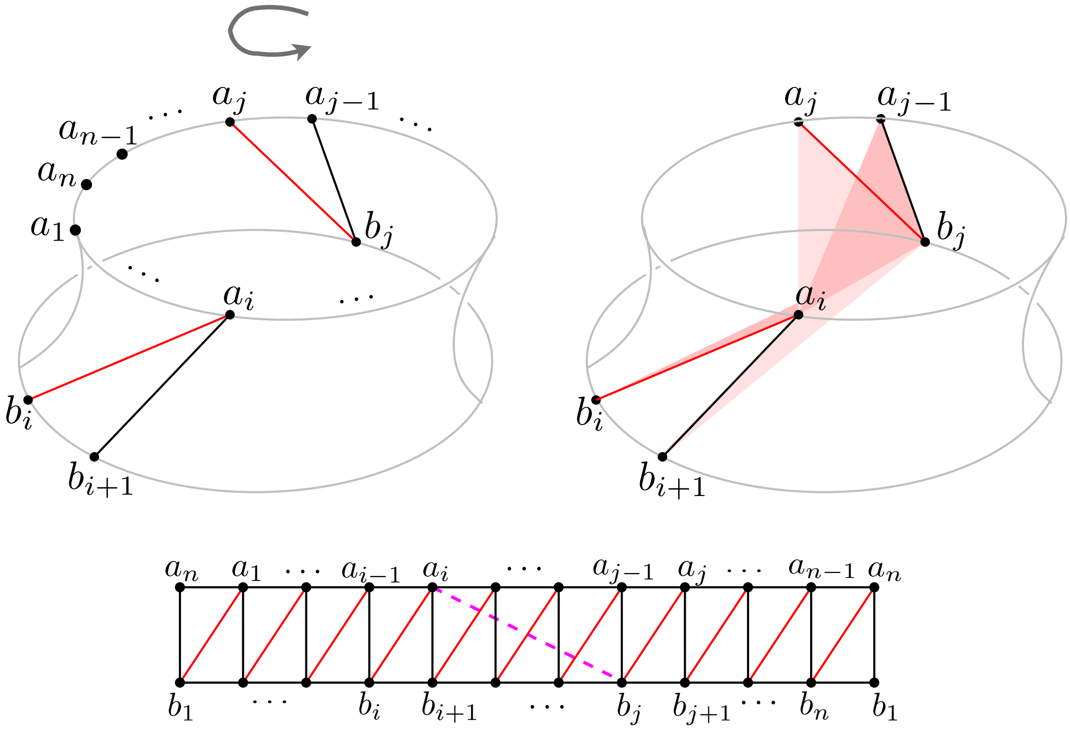

FIGURE 9. The labelling of vertices of a pure non-convex twisted prism (top) and its transformed band in the plane (bottom). Red edges are locally non-convex edges of this prism. The edge $\{a_i, b_j\}$ is an interior edge. Left: the four edges in the band of the prism. Right: the four faces at the interior edge $\{a_i, b_j\}$.

Figure 9. Also, we label the vertices of the top and bottom facets of P in a way such that the edge $\{a_i, b_i\}$ are locally non-convex edge of P, $i = 1, \cdots, n$, see Figure 9.

Let $\{a_i, b_j\}$ be an interior edge of P. The indices i and j are within the cyclic sequence $\{1, \cdots, n\}$. By our specific labelling of the vertices, i.e., $\{a_i, b_i\}$ refers to a non-convex edge, i and j must satisfy the following condition (additions and subtractions of indices are all modulo n):

$$j \notin \{i, i+1, i+2\} \text{ (equivalently } i \notin \{j, j-1, j-2\}). \tag{1}$$

Consider the edges connecting at vertices a_i and b_j in the band, which are:

$$\begin{array}{c|cc} a_i & \{a_i, b_i\} & \{a_i, b_{i+1}\} \\ \hline b_j & \{b_j, a_{j-1}\} & \{b_j, a_j\} \end{array} \tag{2}$$

Each of these boundary edges forms a face which share at the edge $\{a_i, b_j\}$. There are four faces, which can be sorted into two groups, F_{a_i} which are faces containing two vertices in the top facet, and F_{b_j} which are faces containing two vertices in the bottom facet (see Figure 9), i.e.,

$$\begin{array}{c|cc} F_{a_i} & \{a_i, a_{j-1}, b_j\} & \{a_i, a_j, b_j\} \\ \hline F_{b_j} & \{a_i, b_i, b_j\} & \{a_i, b_{i+1}, b_j\} \end{array} \tag{3}$$

Given a pair of distinct indices $i, j \in \{1, \cdots, n\}$ satisfying (1), the four faces in Table (3) exist and they are distinct.

We prove that these four faces in Table (3) are interior faces of P. It is sufficient to show that all these faces satisfy the following two facts:

(i) they do not intersect any boundary facet of P in its interior, and

(ii) all interior vertices of these faces are interior vertices of P.

Our proof of these two facts is based on observation, which suggests an intuitive geometric proof.

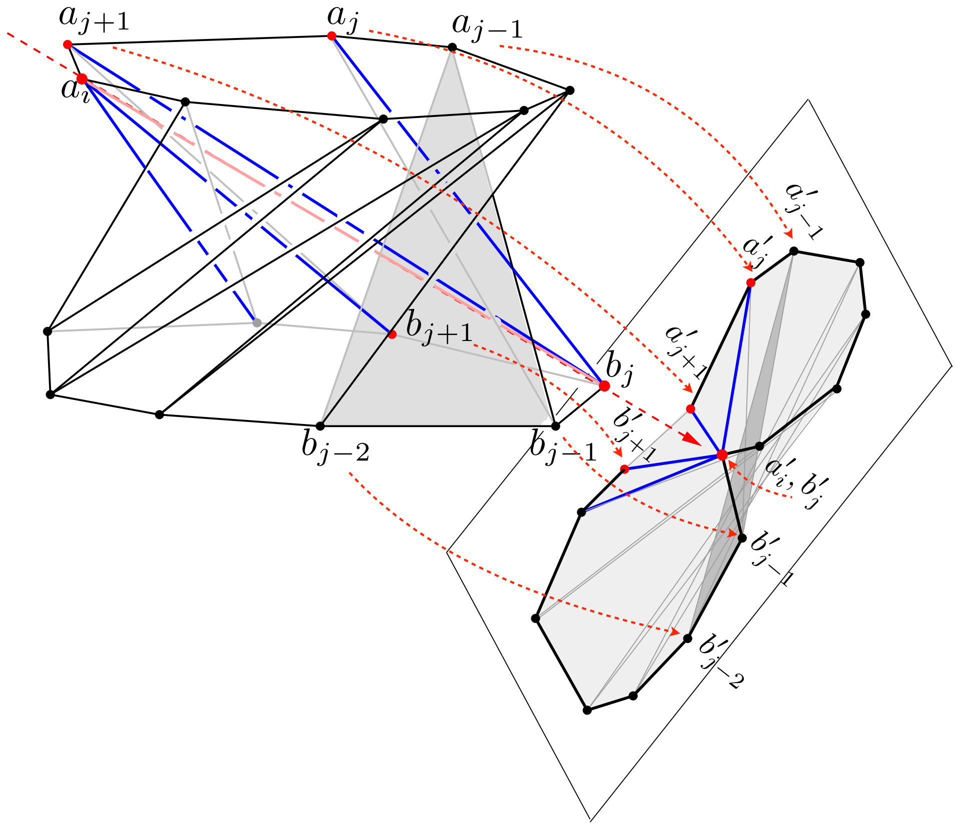

FIGURE 10. Projecting the twisted prism P onto a plane orthogonal to the line direction of the edge $\{a_i, b_j\}$.

We project the prism P along the line containing the edge $a_i b_j$ onto a plane further than b_j, see Figure 10. This plane's normal is defined by the edge vector. Denote a'_i be the projection of a_i in this plane, and the same for other vertices of P. This projection of P (in the plane) has the following properties:

- P is projected into a (non-convex) region, denoted as R, in this plane, i.e, the shaded area in Figure 10.
- The projection of the edge $\{a_i, b_j\}$ is coincident at one point in R. The four faces in Table (3) are projected into the four edges, shown in blue in Figure 10.
- Each side facet of P is projected into either a triangle or a line segment in R, an example of the facet $\{a_{j-2}, b_{j-2}, b_{j-1}\}$ and its projection $\{a'_{j-2}, b'_{j-2}, b'_{j-1}\}$ is highlighted in Figure 10. In particular, a facet is projected into a line segment if it is parallel to the edge $\{a_i, b_j\}$.

By this particular projection, we can verify the geometric fact: if the projection of a side facet of P in R does not cross the projection of the four faces, then they do not intersect in their interior in $\mathbb{R}^3$. Observing the image of the projection of P (Figure 10 Right). The four edges $\{a'_j, b'_j\}$, $\{a'_{j-1}, b'_j\}$, $\{a'_i, b'_j\}$, and $\{a'_{i+1}, b'_j\}$ are not overlapping any other projected triangles in the plane. This phenomenon implies that the interior of these two faces do not intersect any other boundary facets of P in $\mathbb{R}^3$. From this observation, let us formally prove this fact.

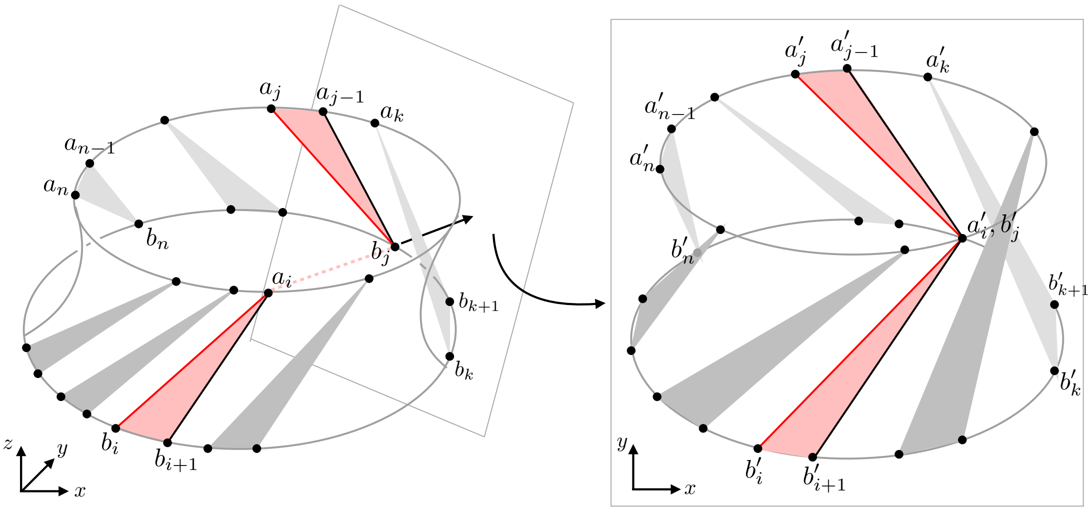

FIGURE 11. Proof of Lemma 6. Projecting a pure non-convex twisted prism (left) onto a plane orthogonal to the line direction of the edge $\{a_i, b_j\}$ (right).

The projections of the top and bottom polygons of P in the plane are two convex polygons which must intersect each other. In general, there are two intersection points, one of them must be the double-point, a'_i and b'_j, which is the projection of the edge $\{a_i, b_j\}$, see Figure 11. It is possible that these two polygons intersect at only one point. In this case, this point must be the double-point, see Figure 10. Based on this double-point, a'_i and b'_j, we can divide the projected vertices of P into four sets:

$$\begin{aligned}
A_1 &:= \{a'_j, a'_{j+1}, \cdots, a'_{n-1}, a'_n, \cdots, a'_i\};\\
A_2 &:= \{a'_i, a'_{i+1}, \cdots, a'_{j-1}\};\\
B_1 &:= \{b'_j, b'_{j+1}, \cdots, b'_{n-1}, b'_n, \cdots, b'_i\};\\
B_2 &:= \{b'_{i+1}, b'_{i+2}, \cdots, b'_j\};
\end{aligned}$$

We prove that any projected boundary facet of P must not cross the four edges $\{a'_i, b'_i\}$, $\{a'_i, b'_{i+1}\}$, $\{a'_j, b'_j\}$, and $\{a'_j, b'_{j-1}\}$. Let t be a boundary triangular facet of P, and t' be the projected triangle (may be an edge) in the plane. The vertices of t' must be one of the following cases:

(1) The vertices of t' are in $A_1 \cup B_1$. All vertices are on the projected base polygons of P. Due to the convexity of the base polygons of P, t' must not cross any of the four edges.
(2) The vertices of t' are in $A_1 \cup B_2$. This case is not possible. Without less of generality, let $t' := \{a'_u, b'_u, b'_{u-1}\}$, see Figure 12. Since $a'_u \in A_1$, then $j \le u' \le i + n$. Since $b'_u \in B_2$, then $i \le u' \le j$. A contradiction.
(3) The vertices of t' are in $A_2 \cup B_1$. This case is not possible due to the same reason as case (2).
(4) The vertices of t' are in $A_2 \cup B_2$. Assume t' is a triangle (not an edge). We show that t' must not intersect any of these four edges. Assume the contrary, t' does cross these edges. Without loss of generally, let $t' := \{a'_v, b'_v, b'_{v+1}\}$, where $a'_v \in A_2$ and $b'_v, b'_{v+1} \in B_2$, and assume t' intersect the edges $\{b'_{i+1}, a'_i\}$ and $\{a'_{j-1}, b'_j\}$, see Figure 13. If this happens, there must exists a boundary facet of P which cuts the edge $\{a_i, b_j\}$, which implies that $\{a_i, b_j\}$ is not an interior edge of P, a contradiction.

Hence the projection of these two faces in the plane must be two edges which are not crossing by any other projected facets of P in this plane. By the property (iii) of an $S_{n,m}$, i.e., P is embedded in $\mathbb{R}^3$, it contains no self-intersected boundary faces. This shows that the two faces $\{a_i, a_j, b_j, \}$ and $\{a_i, a_{j-1}, b_j, \}$ do not intersect other side facets of P in their interiors. This proves (i).

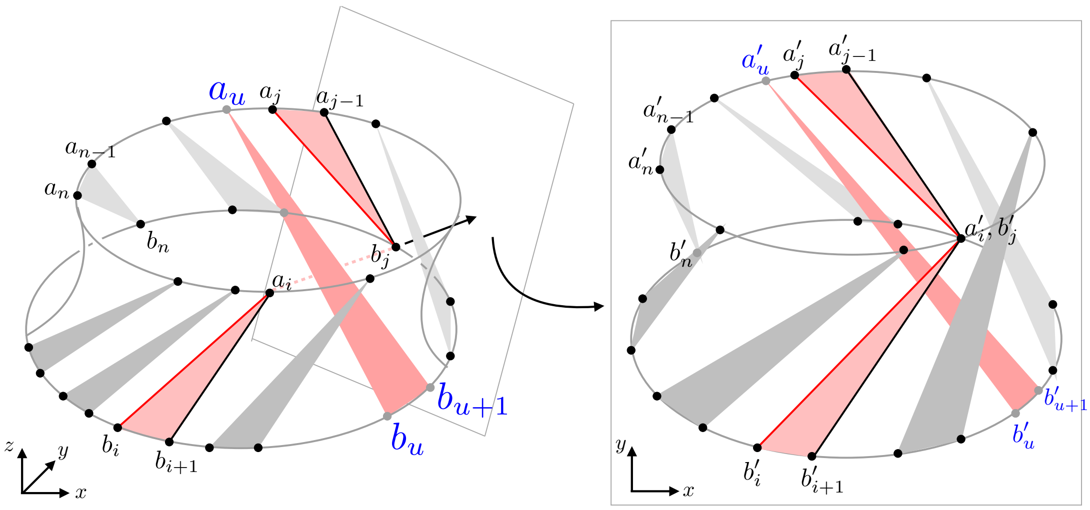

FIGURE 12. Proof of Lemma 6. The facet $\{a_u, b_u, b_{u+1}\}$ cannot exist in the boundary of P.

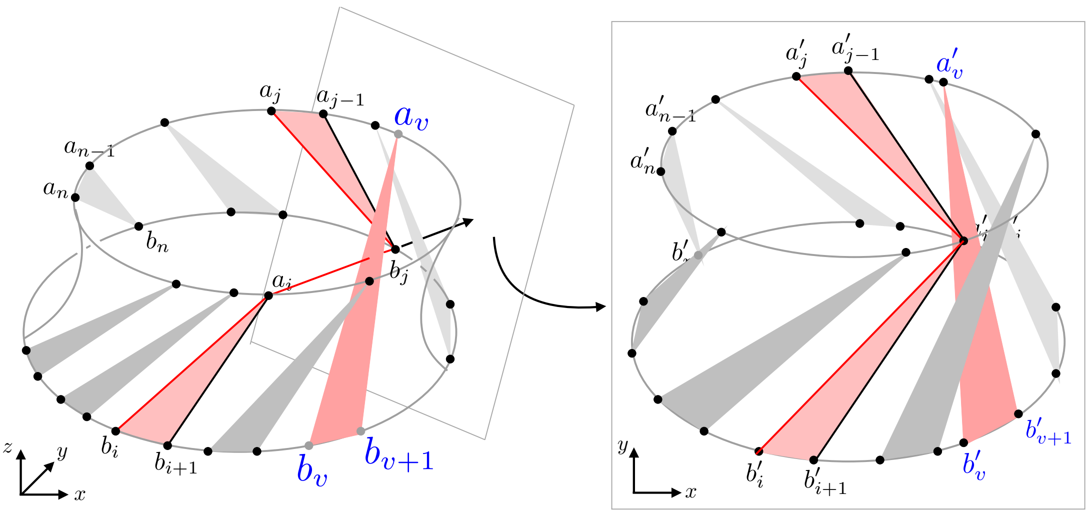

FIGURE 13. Proof of Lemma 6. Since $\{a_i, b_j\}$ is an interior edge of P , the facet $\{a_v, b_v, b_{v+1}\}$ whose interior intersects $\{a_i, b_j\}$ cannot exist in the boundary of P.

Observe that the edges $\{a'_j, b'_j\}$ and $\{a'_{j-1}, b'_j\}$ lie inside the image of the projection of P. This shows that all interior points of the faces $\{a_i, a_j, b_j\}$ and $\{a_i, a_{j+1}, b_j\}$ must lie inside P. This proves (ii).

Therefore the four faces in table (3) must be interior faces of P.

Indeed, we also proved that the interiors of the two triangles $\{a'_i, b'_i, b'_{i+1}\}$ and $\{a'_j, a'_{j-1}, b'_j\}$ do not intersect any projected triangles of P. This shows that the interiors of the following two tetrahedra,

$$\{a_i, a_j, a_{j-1}, b_j\} \text{ and } \{a_i, b_i, b_{i+1}, b_j\},$$

must not intersect any boundary facets of P. Hence they can be removed from P. This shows that P is decomposable. However, our goal is to separate P into two prismatoids with convex bases.

Now it is easy to show that any combination of two faces, one from F_{a_i} and one from F_{b_j} will dissect the prism P into two smaller prismatoids. For example, by choosing the pair of faces:

$$\{a_i, a_u, b_j\} \in F_{a_i}, \text{ and } \{a_i, b_v, b_j\} \in F_{b_j},$$

where $u = \{j-1, j\}$ and $v = \{i, i+1\}$. The edge $\{a_i, a_u\}$ will divide the top facet (a n-gon) into two convex polygons, an n_1-gon and an n_2-gon. The edge $\{b_v, b_j\}$ will divide the bottom facet (a n-gon) into another two convex polygons, an n_1-gon and an n_2-gon. Therefore, the original prism is cut into two smaller twisted prismatoids, an S_{n_1,m_1} and an S_{n_2,m_2}, see Figure 14 for an example. Without loss of generality, assume $i < u$ and $v < j$, then $n_1, n_2, m_1, m_2 \leq n$ can be calculated:

$$\begin{aligned} n_1 &:= u - i + 1 \\ n_2 &:= n - n_1 + 2 \\ m_1 &:= j - v + 1 \\ m_2 &:= n - m_1 + 2 \end{aligned} \tag{4}$$

where $u = \{j-1, j\}$ and $v = \{i, i+1\}$.

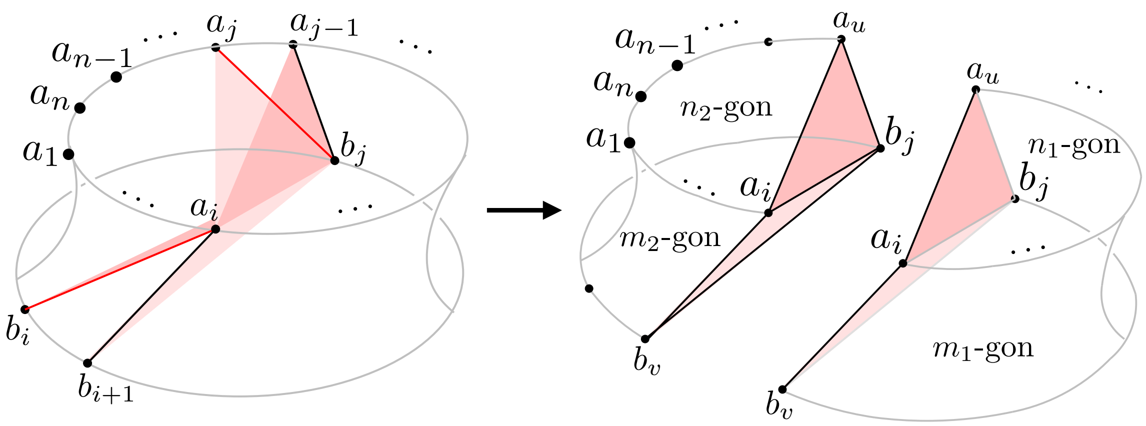

FIGURE 14. Decompose a non-convex twisted prism, an $S_{n,n}$ (left), along an interior edge $\{a_i, b_j\}$. The result (right) is two prismatoids, an S_{n_1,m_1} and an S_{n_2,m_2}, respectively.

Since there are total 4 possible combinations of faces from F_{a_i} and F_{b_j}, therefore there are four possible dissections of this prism.

REFERENCES

[1] F. Bagemihl. On indecomposable polyhedra. *The American Mathematical Monthly*, 55(7):411–413, 1948.
[2] A. Bezdek and B. Carrigan. On nontriangulable polyhedra. *Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry)*, 57(1):51–66, 2016.
[3] B. Chazelle. Convex partition of polyhedra: a lower bound and worst-case optimal algorithm. *SIAM Journal on Computing*, 13(3):488–507, 1984.
[4] P. L. Chew. Constrained Delaunay triangulation. *Algorithmica*, 4:97–108, 1989.
[5] F. Hurtado, M. Noy, and J. Urrutia. Flipping edges in triangulations. *Discrete & Computational Geometry*, 22(3):333–346, 1999.
[6] B. Jessen. Orthogonal icosahedra. *Nordisk Mat. Tidskr*, 15:90–96, 1967.
[7] Willis F. Kern and James R. Bland. *Solid Mensuration*. John Wiley & Sons, Ltd, London: Chapman & Hall, 1934.
[8] D. T. Lee and A. K. Lin. Generalized Delaunay triangulations for planar graphs. *Discrete and Computational Geometry*, 1:201–217, 1986.
[9] G. H. Meisters. Polygons have ears. *The American Mathematical Monthly*, 82(6):648–651, 1975.
[10] J. Rambau. On a generalization of Schönhardt's polyhedron. In J. E. Goodman, J. Pach, and E. Welzl, editors, *Combinatorial and Computational Geometry*, volume 52, pages 501–516. MSRI publications, 2005.

[11] J. Ruppert and R. Seidel. On the difficulty of triangulating three-dimensional nonconvex polyhedra. *Discrete & Computational Geometry*, 7:227–253, 1992.

[12] E. Schönhardt. Über die zerlegung von dreieckspolyedern in tetraeder. *Mathematische Annalen*, 98:309–312, 1928.

[13] H. Si and N. Goerigk. Generalised Bagemihl polyhedra and a tight bound on the number of interior Steiner points. *Computer-Aided Design*, 103:92 – 102, 2018.

[14] Wikipedia contributors. Prismatoid, 2019. [Online; accessed 20-May-2019].