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Holger Heitsch

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Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: holger.heitsch@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

The question for the capacity of a given gas network, i.e., determining the maximal amount of gas that can be transported by a given network, appears as an essential question that network operators and political administrations are regularly faced with. In that context we present a novel mathematical approach to assist gas network operators in managing uncertainty with respect to the demand and in exposing free network capacities while increasing reliability of transmission and supply. The approach is based on the rigorous examination of optimization problems with nonlinear probabilistic constraints. As consequence we deal with solving an optimization problem with joint probabilistic constraints over an infinite system of random inequalities. We will show that the inequality system can be reduced to a finite one in the situation of considering a tree network topology. A detailed study of the problem of maximizing free booked capacities in a stationary gas network is presented that comes up with an algebraic model involving Kirchhoff's first and second laws. The focus will be on both the theoretical and numerical side. We are going to validate a kind of rank two constraint qualification implying the differentiability of the considered capacity problem. At the numerical side we are going to solve the problem using a projected gradient decent method, where the function and gradient evaluations of the probabilistic constraints are performed by the approach of spheric-radial decomposition applied for multivariate Gaussian random variables and more general distributions.

1 Introduction

In the context of the liberalization paradigm, regulatory authorities have separated the natural gas transmission from production and services. Accordingly, the network operators are solely responsible for the transportation of gas, and gas traders only need to specify or nominate where they want to inject gas, at so-called entry points, or extract gas (loads), at so-called exit points. As a consequence, new mathematical challenges for the gas network operators have been introduced.

Presently, the reliability of the gas network operator depends on the accuracy of calculating the transport capacity and on the security of supply. This concern is called *nomination validation*, i.e., determine whether the given nominations of all entry and exit flows are technically and physical feasible under the available infrastructure [11]. This challenge is further complicated by the uncertainty in the feasibility check due to the coverage of future load. When ensuring security of gas supply for end consumers, network operators have to quantify the coverage of uncertain future loads. The amount of gas that

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enters the network depends on volatile prices, and the amount of gas that exits is influenced by ambient temperature changes. Nevertheless, it is possible to model the amount of future load by means of a stochastic distribution based on historical data.

In the research literature, there is a more in-depth study of nomination validation in [14]. The robustness of natural gas flows is examined in [5], and [10] gives a explicit characterization of gas flow feasibility and considers the stochastic nature of exit loads. The present paper develops a novel algorithm to enable a network operator to both locating and maximizing free available network capacities while keeping a high probability to satisfy the demands.

We consider a passive stationary gas network, which for simplicity will be assumed to be a tree. It is supposed that there exists one entry point coinciding with the root of the tree and supplying a set of exit points with random loads. Exits can nominate their loads only according to given booked capacities. In principle, the network owner has to make sure that all nominations complying with the booked capacities can be satisfied by a feasible flow through the network satisfying given lower and upper pressure bounds at its nodes. Since several nomination patterns may turn out to be highly unlikely, he may content himself with guaranteeing this feasibility only with a certain high probability p, being aware that rare infeasibilities in the stationary model can be compensated for by appropriate measures in the dispatch mode such as exploiting interruptible contracts (for details see [11]). This probabilistic relaxation of an originally worst-case-type requirement for feasibility, gives the network owner the chance of offering significantly larger booked capacities. For the given values, it may be the case that the probability of nominations being technically feasible is larger than the value p desired by the network owner. This degree of freedom can be used then, in order to extend the currently booked capacities by a value which still allows one to keep the desired probability level p no matter what additional nominations in the extended range have been chosen. The resulting optimization problem for the network owner will be presented in Sect. 3. The problem turns out to be of a new class of joint probabilistic/robust optimization models that has been introduced in [9] first. A proper substitution of the robust part allows to rewrite the problem of maximizing booked capacities as a stochastic optimization problem with probabilistic constraints.

The paper is organized as follows. A brief discussion of probabilistic problems is given in the following Sect. 2. After representation of the booked capacity problem in Sect. 3 the structure and analytical properties of the resulting optimization model, in particular, the validation of some constraint qualification is in focus of Sect. 4. What follows is Sect. 5 concerning all computational questions, namely, how to compute function values and gradients of the involved probability function (see below), where the approach of spheric-radial decomposition is applied. The final Sect. 6 concludes the theoretical part by a numerical study that includes solving the booked capacity problem for a reasonable large sized gas net adapted from real gas transportation networks under Gaussian-like random demand.

2 Optimization problems with probabilistic constraints

Data uncertainty prevails in many real world optimization models, where it typically enters the inequality constraints describing the set of feasible decisions

$$g_i(x,z) \geq 0 \quad (i = 1,\dots,k)\,. \tag{1}$$

Here $x \in \mathbb{R}^n$ is a decision vector, $z \in \mathbb{R}^m$ is an uncertain parameter and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ refers to a constraint mapping. Overlooking the aspect of uncertainty would result in optimal decisions which are notoriously non-robust with respect to deviations from the assumed deterministic data. When modeling uncertainty two situations typically occur: in the first one, access to historical observations is given such that uncertainty can be modeled by means of a random vector ξ obeying a certain estimated multivariate distribution. This allows one to turn (1) into a so-called (joint) probabilistic constraint

$$\mathbb{P}\big(g(x,\xi) \geq 0\big) \geq p \in (0,1) \tag{2}$$

(note that the first '$\geq$' sign is to be understood component-wise). The meaning of (2) is as follows: a decision x is declared to be feasible if and only if the original random inequality system (1) is satisfied with at least probability p, a level usually chosen close to but not identical to one in order to guarantee sufficient robustness without excessive costs. For a standard reference on probabilistic (or chance) constraints we refer to the monograph [15] by Prékopa.

A general optimization problem using the probabilistic constraints can be formulated as a generic, in general nonsmooth, optimization problem of the form

$$\big\{f(x) \,\big|\, \varphi(x) \geq p\big\}, \tag{3}$$

where $\varphi(x) := \mathbb{P}(g(x,\xi) \geq 0)$ is the so-called *probability function*. Both numerical and analytical properties of the optimization problem with probabilistic constraints strongly depend on the smoothness properties of the probability function $\varphi(\cdot)$. Unfortunately, simple examples show that even if the right hand side $g(\cdot,\cdot)$ in (3) is nice, i.e. smooth, we cannot expect smoothness of the probability function. However, under certain regularity assumptions sub-gradients (in the sense of Clarke or Mordukhovich) and even gradients of the probability function might be available [2]. Therefore, the validation of constraint qualifications in the context of gas transportation problems, and, in particular, for the problem of maximizing booked capacity is considered in Sect. 4.

3 The problem of maximizing booked capacity

In this section we want to describe the optimization problem of maximizing booking capacities under uncertain demand that comes up as a highly relevant optimization challenge for network operators. The presented approach of rigorous examination of the underlying optimization problem with nonlinear probabilistic constraints is novel in that context and it focuses on the booking capacities on the exit side in a classic exit-entry model. Alongside, in [4] the authors of that article pick up the booked capacity problem but they do without deeper justification, they rather try to extend the model to the entry side as well. Such extensions do not allow to reduce the mixed probabilistic and robust optimization model to a probabilistic one, in general. Such reductions we will discuss here.

3.1 General formulation as probabilistic/robust problem

As noted in the introduction, we consider a passive and stationary gas network with tree structure. The root of the tree refers to a single entry node, labeled by zero, supplying the remaining nodes, the exit nodes, labeled by $1, \dots, |\mathcal{V}|$, with gas. Let $G = (\mathcal{V}^+, \mathcal{E})$ represent the tree network graph, trivially a spanning tree of itself, where $\mathcal{V}^+$ denotes the set of both exit points and entry. Without loss of generality we direct all edges in $\mathcal{E}$ away from the root. Using depth-first search, number the nodes so that numbers increase along any path from the root to one of the leaves. For $k, \ell \in \mathcal{V}$, denote $k \succeq \ell$ if, in G, the unique directed path from the root to k, denoted $\Pi(k)$, passes through ℓ. Moreover, let $\pi(e)$ denote the head of edge e, i.e., $\pi(e) := \ell$ for $e = (k, \ell)$.

According to [10, Corollary 2], a vector of exit loads z in this configuration is technically feasible, whenever the inequality system

$$\begin{aligned} \min_{k=1,\dots,|\mathcal{V}|} \Big\{ (p_k^{max})^2 + h_k(z) \Big\} - (p_0^{min})^2 &\geq 0 \\ (p_0^{max})^2 - \max_{k=1,\dots,|\mathcal{V}|} \Big\{ (p_k^{min})^2 + h_k(z) \Big\} &\geq 0 \\ \min_{k=1,\dots,|\mathcal{V}|} \Big\{ (p_k^{max})^2 + h_k(z) \Big\} - \max_{k=1,\dots,|\mathcal{V}|} \Big\{ (p_k^{min})^2 + h_k(z) \Big\} &\geq 0 \end{aligned} \tag{4}$$

is satisfied, where the functions $h_k(\cdot)$ are given by

$$h_k(z) := \begin{cases} \sum\limits_{e \in \Pi(k)} \phi_e \Big(\sum\limits_{t \succeq \pi(e)} z_t \Big)^2 & \text{if } k \geq 1\,, \\ 0 & \text{if } k = 0\,. \end{cases} \tag{5}$$

Here, p_k^{min} and p_k^{max} refer to lower and upper pressure limits at the nodes of the network and represent, as well as certain positive roughness coefficients ϕ_e along edges $e \in \mathcal{E}$, fixed net parameters. By elimination of minima and maxima, the inequality system (4) can be represented equivalently in closed form by a number of $|\mathcal{V}|^2 + |\mathcal{V}|$ constraints of the form

$$g_{k,l}(z) \; := \; \Big((p_k^{max})^2 + h_k(z) \Big) - \Big((p_l^{min})^2 + h_l(z) \Big) \; \geq 0\,, \tag{6}$$

for all $k, l = 0, \dots, |\mathcal{V}|$ and $k \neq l$. Note, the number of inequalities reduces significantly in the event of considering constant upper and constant lower pressure limits at all nodes. In that case, if $p_k^{max} \equiv p^{max}$ and $p_k^{min} \equiv p^{min}$ for all $k = 0, \dots, |\mathcal{V}|$, by eliminating all redundant inequalities from (6), we obtain a system of only $|\mathcal{V}|$ inequalities

$$(p^{max})^2 - (p^{min})^2 - h_k(z) \; \geq 0\,, \tag{7}$$

$k = 1, \dots, |\mathcal{V}|$, to describe technical feasibility in a tree network.

According to the capacity problem we actually focus on, a particular nomination vector of certain exit demand is assumed to be given as the sum $\xi + y$, for a random demand $\xi \geq 0$ satisfying existing current capacities, and, a second vector $y \in [0, x]$, where the values x_k can be viewed as additional free booking capacities, say for new customers at the exit nodes k, $k = 1, \dots, |\mathcal{V}|$. The motivation for modeling ξ as a random vector comes from the fact that a sufficiently large data basis for load nominations according to former booked capacities may be given, which would allow one to approximate a multivariate distribution of ξ (see [11]). While this stochastic information enables the network owner to relax the technical feasibility of exit nominations in a probabilistic sense, nothing is known in contrast about the future nomination pattern of the new customer, so that one has to be prepared principally for every possible nomination $y \in [0, x]$. This constellation leads the network owner to define a capacity extension x as feasible, whenever the constraint

$$\mathbb{P}\big(g_{k,l}\,(\xi + y) \geq 0 \quad \forall y \in [0, x] \quad \forall k, l = 0, \ldots, |\mathcal{V}|\big) \geq p \tag{8}$$

is satisfied with that x. The meaning of this constraint is as follows. The capacity extension x is feasible if and only if, with probability larger than $p \in [0, 1)$, the sum $\xi + y$ of the original random nomination vector and of a new nomination vector can be technically realized for every such new nomination vector in the limits $[0, x]$. By its structure, (8) is a probabilistic constraint, but it is a nonstandard one in that it contains a robust (worst case) ingredient which makes the given random inequality system an infinite one. As mentioned in the introduction, such joint probabilistic/robust constraints have been considered first in the context of gas networks in [9].

By regulatory law, the network owner is invited to maximize the capacity which can be booked. This leads him to the consideration of the following optimization problem

$$\text{maximize } c^T x \quad \text{subject to (8)}, \tag{9}$$

where c is a weighted preference vector for capacity maximization, for example, $c = (1, \ldots, 1)$ in the case of no preferences among exit nodes.

3.2 Reformulation of the problem by probabilistic constraints only

In order to apply theory and methodology of optimization problems with probabilistic (chance) constraints, it might be essential to reduce the infinite system of constraints in (8) to a finite one. To this end we make use of the equivalence

$$g_{k,l}(z + y) \geq 0 \quad \forall y \in [0, x] \quad \Leftrightarrow \quad \min_{y \in [0,x]} g_{k,l}(z + y) \geq 0\,, \tag{10}$$

where $k, l = 0, \ldots, |\mathcal{V}|$ and $k \neq l$. Let be

$$\tilde{g}_{k,l}(x, z) := \min_{y \in [0,x]} g_{k,l}(z + y)\,, \quad k, l = 0, \ldots, |\mathcal{V}|, \tag{11}$$

the minimum function depending on both x and z. An explicit representation of the minimum function $\tilde{g}_{k,l}(x, z)$ can be obtained by taking a closer look to the constraint functions $g_{k,l}(\cdot)$. Inserting equation (5) into formula (6) leads to

$$\begin{aligned} g_{k,l}(z) &= \Big((p_k^{max})^2 + h_k(z)\Big) - \Big((p_l^{min})^2 + h_l(z)\Big) \\ &= (p_k^{max})^2 + \sum_{e \in \Pi(k) \setminus \Pi(l)} \phi_e \Bigg(\sum_{t \succeq \pi(e)} z_t\Bigg)^2 - (p_l^{min})^2 - \sum_{e \in \Pi(l) \setminus \Pi(k)} \phi_e \Bigg(\sum_{t \succeq \pi(e)} z_t\Bigg)^2. \end{aligned} \tag{12}$$

Hence, the minimum of (11) is observed as the minimum of the latter equation (12) with respect to y after replacing z by $z+y$. We have

$$
\begin{aligned}
\tilde{g}_{k,l}(x,z) &= \min_{y\in[0,x]} g_{k,l}(z+y) \\
&= \min_{y\in[0,x]} \left\{ (p_k^{max})^2 + \sum_{e\in\Pi(k)\setminus\Pi(l)} \phi_e\left(\sum_{t\succeq\pi(e)} z_t + y_t\right)^2 - (p_l^{min})^2 - \sum_{e\in\Pi(l)\setminus\Pi(k)} \phi_e\left(\sum_{t\succeq\pi(e)} z_t + y_t\right)^2 \right\} \\
&= (p_k^{max})^2 + \sum_{e\in\Pi(k)\setminus\Pi(l)} \phi_e\left(\sum_{t\succeq\pi(e)} z_t\right)^2 - (p_l^{min})^2 - \sum_{e\in\Pi(l)\setminus\Pi(k)} \phi_e\left(\sum_{t\succeq\pi(e)} z_t + x_t\right)^2, \qquad (13)
\end{aligned}
$$

where $k,l = 0,\dots,|\mathcal{V}|$ and $k \neq l$. Note, the latter equation is due to the fact that all edges $e \in \Pi(k)\setminus\Pi(l)$ and $e' \in \Pi(l)\setminus\Pi(k)$ are pairwise disjoint. Therefore, the optimization problem of maximizing booking capacities turns into a classical probabilistic problem with a finite number of probabilistic constraints. The reformulation of (9) reads

$$
\begin{gathered}
\text{maximize } c^T x \quad \text{subject to} \\
\mathbb{P}\big(\tilde{g}_{k,l}(x,\xi) \geq 0 \quad \forall\, k,l = 0,\dots,|\mathcal{V}|\big) \geq p\,,
\end{gathered} \qquad (14)
$$

where for all $k,l = 0,\dots,|\mathcal{V}|$ with $k \neq l$ the constraint mappings $\tilde{g}_{k,l}(\cdot,\cdot)$ are obtained by the analytical representation given in (13).

4 The validation of constraint qualifications

The analytical properties of an optimization problem strongly depend on whether it satisfies certain regularity conditions which are given by different types of constraint qualifications. We follow the approach of considering the Rank 2 constraint qualification (R2CQ) (cf. [2]) as a sufficient criterion for differentiability of the probability function of the probabilistic constraints.

To discus the constraint qualification for the constraint mappings in the context of gas transmission we start with the general inequality system $g_{k,l}(z)$ for $k,l = 0,\dots,|\mathcal{V}|$ and $k \neq l$. To simplify the representation we are going to introduce the following definitions and notations.

Definition 1. *For some given pair of nodes $k,l \in \mathcal{V}$ we set:*

(i) $\Pi_{kl}^+ := \Pi(k)\setminus\Pi(l)$ *and* $\Pi_{kl}^- := \Pi(l)\setminus\Pi(k)$,
the disjunctive subpaths *w.r.t.* $\Pi(k)$ *and* $\Pi(l)$,

(ii) $b_{kl} := \begin{cases} \max\{\pi(e) \mid e \in \Pi(k)\cap\Pi(l)\} & \text{if } \Pi(k)\cap\Pi(l) \neq \emptyset\,, \\ 0 & \text{otherwise}\,, \end{cases}$
the bifurcation node *of paths* $\Pi(k)$ *and* $\Pi(l)$*, and,*

(iii) $d_{kl}^+ := \min\{\pi(e) \mid e \in \Pi_{kl}^+\}$, $d_{kl}^- := \min\{\pi(e) \mid e \in \Pi_{kl}^-\}$,
the first direction nodes *for nonempty subpaths* Π_{kl}^+ *and* Π_{kl}^-*, respectively.*

To verify the constraint qualification (R2QC), we have to pairwise compare gradients of active constraints. The following Lemma displays the analytical representations in order to compute the gradients of the constraint mappings.

Lemma 1. *For the constraint mappings $g_{k,l}(\cdot)$ in (6) we obtain that*

$$[\nabla_z g_{k,l}(z)]_i = \begin{cases} \sum\limits_{e\in\Pi^+_{kl}\cap\Pi(i)} 2\phi_e \sum\limits_{t\succeq\pi(e)} z_t & \text{if } \Pi^+_{kl}\cap\Pi(i)\neq\emptyset, \\ -\sum\limits_{e\in\Pi^-_{kl}\cap\Pi(i)} 2\phi_e \sum\limits_{t\succeq\pi(e)} z_t & \text{if } \Pi^-_{kl}\cap\Pi(i)\neq\emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

for all $k,l=0,\dots,|\mathcal{V}|$ with $k\neq l$, for all $i=1,\dots,|\mathcal{V}|$, and, any z.

Proof: For any fixed k,l with $k\neq l$ and arbitrary $i\in\mathcal{V}$ from (12) we observe

$$[\nabla_z g_{k,l}(z)]_i = \sum_{e\in\Pi^+_{kl}\cap\Pi(i)} 2\phi_e \sum_{t\succeq\pi(e)} z_t - \sum_{e\in\Pi^-_{kl}\cap\Pi(i)} 2\phi_e \sum_{t\succeq\pi(e)} z_t$$

for any z. Hence, the gradient value of the ith component of $g_{k,l}(z)$ depends on the relation between the unique path $\Pi(i)$ with respect to the positive and negative subpaths Π^+_{kl} and Π^-_{kl}, respectively. But both paths are disjoint, and therefore, path $\Pi(i)$ can only intersect either the positive or negative path. Thus, for any fixed i,k,l gradient formula (15) is a consequence of the above equation. □

Another structural result is given by the following observation for active constraints.

Lemma 2. *Let be k,l with $k\neq l$ and $z\geq 0$ given such that $g_{k,l}(z)=0$. If $p^{\max}_k > p^{\min}_l$ then it holds*

a) $\Pi^-_{kl}\neq\emptyset$, *b)* $d^-_{kl}\neq 0$, *c)* $[\nabla_z g_{k,l}(z)]_{d^-_{kl}} < 0$.

Proof: Due to the assumption $p^{\max}_k > p^{\min}_l$, and, due to $g_{k,l}(\cdot)$ is active constraint in z, by (12) we obtain that path Π^-_{kl} is nonempty. The latter implies $d^-_{kl}\neq 0$ (see Definition 1 (iii) above). Moreover, the path $\Pi(d^-_{kl})$ intersects the nonempty path Π^-_{kl} by the arc $\left(b_{kl}, d^-_{kl}\right)\in\mathcal{E}$, and, by applying Lemma 1 we conclude with $z\geq 0$ that

$$[\nabla_z g_{k,l}(z)]_{d^-_{kl}} = -\sum_{e\in\Pi^-_{kl}\cap\Pi(d^-_{kl})} 2\phi_e \sum_{t\succeq\pi(e)} z_t = -2\phi_{\left(b_{kl},d^-_{kl}\right)} \sum_{t\succeq d^-_{kl}} z_t < 0,$$

where b_{kl} is the bifurcation node w.r.t. $\Pi(k)$ and $\Pi(l)$ (see Definition 1 (ii)). □

The following structural results focus on the connection between the constraints describing feasibility of gas flows in general (6) and the ones involved in the capacity problem (13).

Definition 2. *For every $k,l\in\mathcal{V}$ with $k\neq l$ and any $x\geq 0$ we define a mapping $\varphi_{k,l}:\mathbb{R}^{\mathcal{V}}\to\mathbb{R}^{\mathcal{V}}$*

$$\varphi_{k,l}(x) := \begin{cases} x_t & \text{if } t\succeq d^-_{kl}, \\ 0 & \text{otherwise}. \end{cases}$$

The functionals $\varphi_{k,l}(\cdot)$ represent the solution mappings for the generalized constraints in the capacity problem, as shown next.

Lemma 3. *For any $k, l \in \mathcal{V}$ ($k \neq l$) and any $x, z \geq 0$ we have*

$$\min_{y \in [0,x]} g_{k,l}(z+y) = g_{k,l}(z + \varphi_{k,l}(x))\,.$$

Proof: The result is a direct consequence of formula (13). □

Now we are prepared to state the first main result concerning the constraint mapping involved by the problem of maximizing booked capacities in a stationary gas transport network, in order to derive constraint qualifications for this type of problem.

Theorem 1. *For given $x \geq 0$, $z \geq 0$ let be $\alpha := \varphi_{k,l}(x)$, $\beta = \varphi_{m,n}(x)$, $k \neq l$ and $m \neq n$. If it holds $g_{k,l}(z+\alpha) = g_{m,n}(z+\beta) = 0$ and if $p_s^{\max} > p_t^{\min}$ for all $s, t \in \mathcal{V}^+$, then we have that (at least) one of the following statements is satisfied:*

(1) The gradients $\nabla_z g_{k,l}(z+\alpha)$ and $\nabla_z g_{m,n}(z+\beta)$ are linear independent.

(2) It exist indices $i, j \in \mathcal{V}$ with $z_i = z_j = 0$ and $i \neq j$.

(3) There is redundancy, i.e., $g_{k,l}(z) \geq g_{m,n}(z)$ or $g_{m,n}(z) \geq g_{k,l}(z)$ for all $z \geq 0$.

Proof: We want to prove the statement by a case study with respect to the bifurcation nodes observed from the paths with respect to the index pairs (k,l) and (m,n), respectively.

1) Case $b_{kl} \neq b_{mn}$:

If the bifurcation nodes do not coincide at least one of the relations $b_{kl} \not\succ b_{mn}$ or $b_{mn} \not\succ b_{kl}$ must be satisfied. Without loss of generality we assume $b_{kl} \not\succ b_{mn}$. We consider the negative path Π_{kl}^- and its first direction node d_{kl}^-. Clearly, that node can now either be the bifurcation node b_{mn} itself, or it is even not involved in the paths Π_{mn}^+ and Π_{mn}^-. However, in both cases it follows that we have

$$\Pi_{mn}^+ \cap \Pi(d_{kl}^-) = \emptyset \quad \text{and} \quad \Pi_{mn}^- \cap \Pi(d_{kl}^-) = \emptyset\,.$$

Thus, due to (15) we obtain

$$[\nabla_z g_{m,n}(z+\beta)]_{d_{kl}^-} = 0\,.$$

On the other hand, due to the assumptions $g_{kl}(z+\alpha) = 0$, $g_{mn}(z+\beta) = 0$ and $p_k^{\max} > p_l^{\min}$, by Lemma 2 we have that

$$[\nabla_z g_{k,l}(z+\alpha)]_{d_{kl}^-} < 0 \quad \text{and} \quad [\nabla_z g_{m,n}(z+\beta)]_{d_{mn}^-} < 0\,.$$

Thus, the gradients $\nabla_z g_{k,l}(z+\alpha)$ and $\nabla_z g_{m,n}(z+\beta)$ are linear independent.

2) Case $b_{kl} = b_{mn}$:

In the case of same bifurcation nodes we first want to consider the event that

a) $\Pi^-_{kl} \cap \left(\Pi^+_{mn} \cup \Pi^-_{mn}\right) = \emptyset$ or $\Pi^-_{mn} \cap \left(\Pi^+_{kl} \cup \Pi^-_{kl}\right) = \emptyset$:

Assuming the first expression, completely analog to 1) it follows that

$$[\nabla_z g_{m,n}(z+\beta)]_{d^-_{kl}} = 0, \quad [\nabla_z g_{k,l}(z+\alpha)]_{d^-_{kl}} < 0 \quad \text{and} \quad [\nabla_z g_{m,n}(z+\beta)]_{d^-_{mn}} < 0\,,$$

which implies linear independence of the considered gradients. Clearly, the same result we obtain for the second expression just by interchanging the role of α, β and indices (k,l), (m,n).

The next case we want to consider is

b) $\Pi^-_{kl} \cap \Pi^-_{mn} \neq \emptyset$:

Because of same bifurcation node we observe $d^-_{kl} = d^-_{mn}$. Note that $(b_{kl}, d^-_{kl}) \in \Pi^-_{kl} \cap \Pi^-_{mn}$ here, and thus, due to formula (15) we further obtain

$$[\nabla_z g_{k,l}(z+\alpha)]_{d^-_{kl}} = [\nabla_z g_{m,n}(z+\beta)]_{d^-_{kl}} = -2\phi_{\left(b_{kl}, d^-_{kl}\right)} \sum_{t \succeq d^-_{kl}} (z_t + x_t) < 0\,,$$

because $\alpha_t = \beta_t = x_t$ for all $t \succeq d^-_{kl}$ (see Definition 2). It follows that the gradients are co-linear, if and only if, all their components coincide. Let's assume co-linearity: In that case , by (15) again, on the one hand we have

$$[\nabla_z g_{k,l}(z+\alpha) - \nabla_z g_{m,n}(z+\beta)]_k \;=\; 2 \sum_{e \in \Pi^+_{kl} \setminus \Pi^+_{mn}} \phi_e \sum_{t \succeq \pi(e)} z_t \;=\; 0\,, \tag{16}$$

$$[\nabla_z g_{k,l}(z+\alpha) - \nabla_z g_{m,n}(z+\beta)]_m \;=\; 2 \sum_{e \in \Pi^+_{mn} \setminus \Pi^+_{kl}} \phi_e \sum_{t \succeq \pi(e)} z_t \;=\; 0\,, \tag{17}$$

on the other hand we obtain that

$$[\nabla_z g_{k,l}(z+\alpha) - \nabla_z g_{m,n}(z+\beta)]_l \;=\; -2 \sum_{e \in \Pi^-_{kl} \setminus \Pi^-_{mn}} \phi_e \sum_{t \succeq \pi(e)} (z_t + x_t) \;=\; 0\,, \tag{18}$$

$$[\nabla_z g_{k,l}(z+\alpha) - \nabla_z g_{m,n}(z+\beta)]_n \;=\; 2 \sum_{e \in \Pi^-_{mn} \setminus \Pi^-_{kl}} \phi_e \sum_{t \succeq \pi(e)} (z_t + x_t) \;=\; 0\,. \tag{19}$$

What follows is, in all cases two indices $i, j \in \{k, l, m, n\}$ with $z_i = z_j = 0$ and $i \neq j$ can be identified, unless $k = m$ and $l \succ n$ ($n \succ l$), or, $l = n$ and $k \succ m$ ($m \succ k$), respectively. The latter exception appears if three of the above sums vanish due to cancellation of paths. However, if we assume $k = m$ and $l \succ n$ (the other cases are analog) we might compute the difference of the active constraints $g_{k,l}(z+\alpha) - g_{m,n}(z+\beta)$ and by using (12) we obtain

$$-(p^{\min}_l)^2 + (p^{\min}_n)^2 - \sum_{e \in \Pi^-_{kl} \setminus \Pi^-_{mn}} \phi_e \left(\sum_{t \succeq \pi(e)} (z_t + x_t) \right)^2 = 0\,.$$

Due to (18) the involving sum needs to equal zero, hence, we have $(p^{\min}_l)^2 = (p^{\min}_n)^2$. But, as consequence of that observation from (12) it follows with $m = k$ that

$$g_{m,n}(z) \geq g_{k,l}(z) \quad \forall z \geq 0\,,$$

and, hence, inequality $g_{m,n}$ is redundant.

Finally, it remains to show the claim of the theorem for the final case

c) $\Pi^-_{kl} \cap \Pi^+_{mn} \neq \emptyset$ and $\Pi^+_{kl} \cap \Pi^-_{mn} \neq \emptyset$:

In this case, first of all, we define non-negative numbers

$$a := 2\Phi_{(b_{kl},d^+_{kl})} \sum_{t \succeq d^+_{kl}} z_t \geq 0\,, \qquad c := 2\Phi_{(b_{kl},d^+_{kl})} \sum_{t \succeq d^+_{kl}} x_t \geq 0\,,$$
$$b := 2\Phi_{(b_{kl},d^-_{kl})} \sum_{t \succeq d^-_{kl}} z_t \geq 0\,, \qquad d := 2\Phi_{(b_{kl},d^-_{kl})} \sum_{t \succeq d^-_{kl}} x_t \geq 0\,.$$

Assuming c) the numbers are well-defined. Moreover, we observe $d^+_{kl} = d^-_{mn}$ and $d^-_{kl} = d^+_{mn}$. By (15) combined with Definition 2 it is easy to show that

$$\begin{array}{rclrcl} [\nabla_z g_{k,l}(z+\alpha)]_{d^+_{kl}} &=& a\,, & [\nabla_z g_{m,n}(z+\beta)]_{d^+_{kl}} &=& -a-c\,,\\ [\nabla_z g_{k,l}(z+\alpha)]_{d^-_{kl}} &=& -b-d\,, & [\nabla_z g_{m,n}(z+\beta)]_{d^-_{kl}} &=& b\,. \end{array}$$

From Lemma 2 we conclude that $a + c > 0$ and $b + d > 0$. Thus, if $a = 0$ or $b = 0$ the considered gradients are linear independent. Let's assume $ab \neq 0$. Considering the ratio equation

$$\frac{a}{b+d} = \frac{a+c}{b}$$

shows that the gradients are linear independent as $d > 0$ and $c > 0$, respectively. It remains to show linear independence for the event that $c = d = 0$. The latter implies that $x_t = 0$ for all $t \succeq d^-_{kl}$ as well as for all $t \succeq d^-_{mn}$. It is sufficient to show that there is one component $t_0 \in \mathcal{V}$ such that $[\nabla_z g_{k,l}(z+\alpha)]_{t_0} + [\nabla_z g_{m,n}(z+\beta)]_{t_0} \neq 0$. To this end we compute the sum of the constraints $g_{k,l}(z+\alpha)$ and $g_{m,n}(z+\beta)$ which results, by using (12) and the assumption that the constraints are active, in

$$\begin{aligned} 0 \;=\;& (p^{\max}_k)^2 + (p^{\max}_m)^2 - (p^{\min}_l)^2 + (p^{\min}_n)^2 \\ &+ \sum_{e \in \Pi^+_{kl} \setminus \Pi^-_{mn}} \phi_e \Bigg(\sum_{t \succeq \pi(e)} z_t \Bigg)^2 + \sum_{e \in \Pi^+_{mn} \setminus \Pi^-_{kl}} \phi_e \Bigg(\sum_{t \succeq \pi(e)} z_t \Bigg)^2 \\ &- \sum_{e \in \Pi^-_{kl} \setminus \Pi^+_{mn}} \phi_e \Bigg(\sum_{t \succeq \pi(e)} z_t \Bigg)^2 - \sum_{e \in \Pi^-_{mn} \setminus \Pi^+_{kl}} \phi_e \Bigg(\sum_{t \succeq \pi(e)} z_t \Bigg)^2 . \end{aligned}$$

Note, joint paths disappear by canceling out here, due to the fact that $x_t = 0$ along the involved paths. In particular, it follows that

$$\sum_{e \in \Pi^-_{kl} \setminus \Pi^+_{mn}} \phi_e \Bigg(\sum_{t \succeq \pi(e)} z_t \Bigg)^2 + \sum_{e \in \Pi^-_{mn} \setminus \Pi^+_{kl}} \phi_e \Bigg(\sum_{t \succeq \pi(e)} z_t \Bigg)^2 > 0\,, \tag{20}$$

i.e., there exist $\hat{e}$ with $\hat{e} \in \Pi^-_{kl} \setminus \Pi^+_{mn}$ or $\hat{e} \in \Pi^-_{mn} \setminus \Pi^+_{kl}$, respectively, and $\hat{t}$ with $\hat{t} \succeq t_0 := \pi(\hat{e})$ such that $z_{\hat{t}} > 0$. However, in both cases with (15) we conclude that

$$\left| \left[\nabla_z g_{k,l}(z)\right]_{t_0} + \left[\nabla_z g_{m,n}(z)\right]_{t_0} \right| \geq 2\phi_{\hat{e}} \sum_{t \succeq t_0} z_t \geq 2\phi_{\hat{e}} z_{\hat{t}} > 0\,.$$

Thus, in case c) the gradients of the constraints are always linear independent.
In any cases we have proven that either two active constraints have linear independent gradients, or there exist at least two distinct indices i, j with $z_i = z_j = 0$, or one of the two active constraints is redundant at all. □

In fact, property (3) of Theorem 1 does not appear if we remove all redundant inequalities from the inequality system $g(z) \geq 0$ first. However, contained redundant inequalities do not affect the feasibility set and the analytical properties of the probability function. Of a somewhat different nature is property (2) of the Theorem. The following example shows that this property actually can appear even in a small network.

Example 1. *We consider a gas network consisting of one entry and three exit nodes as displayed in Figure 1. From the system of feasibility constraints* $g_{k,l}(\cdot)$ *in* (6) *we select* $g_{1,3}(\cdot)$ *and* $g_{2,3}(\cdot)$ *only. For*

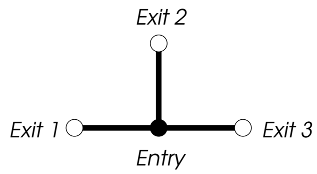

Figure 1: Small example network.

any $x \in \mathbb{R}^3$ *with* $x \geq 0$*, according to Definition 2, we have for* $z \in \mathbb{R}^3$

$$\begin{aligned} g_{1,3}(z + \varphi_{1,3}(x)) &= (p_1^{max})^2 - (p_3^{min})^2 + \phi_{(0,1)} z_1^2 - \phi_{(0,3)}(z_3 + x_3)^2 &\geq 0\,, \\ g_{2,3}(z + \varphi_{2,3}(x)) &= (p_2^{max})^2 - (p_3^{min})^2 + \phi_{(0,2)} z_2^2 - \phi_{(0,3)}(z_3 + x_3)^2 &\geq 0\,. \end{aligned}$$

As gradients we obtain

$$\nabla_z g_{1,3}(z+\varphi_{1,3}(x)) = \begin{pmatrix} 2\phi_{(0,1)} z_1 \\ 0 \\ -2\phi_{(0,3)}(z_3 + x_3) \end{pmatrix}, \ \nabla_z g_{2,3}(z+\varphi_{2,3}(x)) = \begin{pmatrix} 0 \\ 2\phi_{(0,2)} z_2 \\ -2\phi_{(0,3)}(z_3 + x_3) \end{pmatrix}.$$

In particular, assuming active constraints, both gradients are co-linear, if and only if, $z_1 = 0$ *and* $z_2 = 0$*. In that case it must hold* $p_1^{max} = p_2^{max}$ *and we observe that* x_3 *has to be small enough and* $z_3 = z^* := (((p_1^{max})^2 - (p_3^{min})^2)/\phi_{(0,3)} - x_3)^{1/2}$*. But as shown in the Theorem above, active constraints and co-linearity of the gradients, this fact implies that we can identify components being zero, here as shown,* z_1 *and* z_2*.*

Before proving the next result, we are going to simplify the notation. With respect to the formulation of the problem of maximizing booking capacities (14) let be

$$\mathcal{J} := \{j = (k,l) \,|\, k,l = 1,\ldots,|\mathcal{V}|;\ k \neq l\}$$

the index set of the feasibility constraints. We make use of the notation $\tilde{g}_j(\cdot,\cdot) \equiv \tilde{g}_{k,l}(\cdot,\cdot)$ for $j \in \mathcal{J}$ and $j = (k,l)$. Moreover, let be $\mathcal{J}^* \subseteq \mathcal{J}$ the index set of all nonredundant constraints (cf. Theorem 1, item *(3)*). With this notation, Theorem 1 implies the following Corollary.

Corollary 1. *For given $x, z \geq 0$ and $i, j \in \mathcal{J}^*$ let be $\tilde{g}_i(x,z) = \tilde{g}_j(x,z) = 0$ and $i \neq j$, where $\tilde{g}(\cdot,\cdot)$ given due to* (13)*. Under the assumptions of Theorem 1, i.e., if $p_s^{\max} > p_t^{\min}$ for all $s,t \in \mathcal{V}^+$, then one of the following statements is satisfied:*

(1) The gradient vectors $\nabla_z \tilde{g}_i(x,z)$ and $\nabla_z \tilde{g}_j(x,z)$ are linear independent.

(2) It exist indices $k,l \in \mathcal{V}$ with $z_k = z_l = 0$ and $k \neq l$.

The derived constraint qualifications for the considered feasibility constraints turn out to be sufficient to guarantee differentiability of the involved probability function, as we will show in the following. To this and, we first state the following Lemma.

Lemma 4. *For any fixed $x \geq 0$ we define*

$$S_j(x) := \{z \in \mathbb{R}^{|\mathcal{V}|} \,|\, \tilde{g}_j(x,z) = 0,\ \tilde{g}_i(x,z) \geq 0 \text{ for all } i \in \mathcal{J}^*\} \qquad j \in \mathcal{J}^*\,.$$

Then it holds for all $i \neq j$

$$\mathrm{mes}_{|\mathcal{V}|-1}\left(S_i(x) \cap S_j(x)\right) = 0\,,$$

where $\mathrm{mes}_{|\mathcal{V}|-1}(\cdot)$ denotes the surface Lebesgue measure in $\mathbb{R}^{|\mathcal{V}|}$.

Proof: Due to Corallary 1 we can decompose the intersection of two active constraints into two subsets $A, B \subseteq S_i(x) \cap S_j(x)$, where $A \cup B = S_i(x) \cap S_j(x)$, and, where we have that $z \in A$ implies $\mathrm{rank}\,\{\nabla_z \tilde{g}_i(x,z), \nabla_z \tilde{g}_j(x,z)\} = 2$; $z \in B$ implies that there exist zero components $z_k = z_l = 0$ ($k \neq l$). Clearly, it is evidently sufficient to show $\mathrm{mes}_{|\mathcal{V}|-1}(A) = 0$ and $\mathrm{mes}_{|\mathcal{V}|-1}(B) = 0$.

For the first equation, given x, we define a mapping $F(\cdot)$ such that

$$F(z) := \begin{pmatrix} \tilde{g}_i(x,z) \\ \tilde{g}_j(x,z) \end{pmatrix} \in \mathbb{R}^2, \qquad z \in \mathbb{R}^{|\mathcal{V}|}.$$

Hence, $F(\cdot)$ is continuously differentiable, and, for arbitrary $\bar{z} \in A$ we obtain $F(\bar{z}) = 0$. Moreover, due to the linear independence of the gradients, the Jacobian matrix D_F has rank 2 in $\bar{z}$. Thus, there exist indices k,l ($k \neq l$) such that the according Jacobian sub-matrix is invertible. Without loss of generality let assume $k = 1$ and $l = 2$. By the Implicit Function Theorem the equation $F(z) = 0$ can be resolved in a neighbourhood $U_{\bar{z}}$ of $\bar{z}$ equivalently as

$$z_1 = f_1(z_3,\ldots,z_{|\mathcal{V}|}) \quad \text{and} \quad z_2 = f_2(z_3,\ldots,z_{|\mathcal{V}|}) \qquad \forall (z_3,\ldots,z_{|\mathcal{V}|}) \in V_{\bar{z}}, \qquad (21)$$

where $V_{\bar{z}}$ is a well-defined neighbourhood of $(\bar{z}_3, \dots, \bar{z}_{|\mathcal{V}|})$. Moreover, the mapping

$$X_z(t_2, \dots, t_{|\mathcal{V}|}) := \big(f_1(t_3, \dots, t_{|\mathcal{V}|}) + t_2, f_2(t_3, \dots, t_{|\mathcal{V}|}) + t_2, t_3, \dots, t_{|\mathcal{V}|}\big)\,,$$

where $X_{\bar{z}} : \mathbb{R} \times V_{\bar{z}} \to \mathbb{R}^{|\mathcal{V}|}$, defines a parametrization of some surface S in $\mathbb{R}^{|\mathcal{V}|}$. Clearly, the set $\{z \in U_{\bar{z}} \mid F(z) = 0\}$ is a subset of the surface S and due to (21) we conclude that it holds

$$X_{\bar{z}}^{-1}(\{z \in U_{\bar{z}} \mid F(z) = 0\}) = \{0\} \times V_{\bar{z}} \quad \text{and} \quad \lambda_{|\mathcal{V}|-1}(\{0\} \times V_{\bar{z}}) = 0\,,$$

where $\lambda_{|\mathcal{V}|-1}$ is the Lebesgue measure in space $\mathbb{R}^{|\mathcal{V}|-1}$. In particular, for the according surface measure we obtain that $\mathrm{mes}_{|\mathcal{V}|-1}(\{z \in U_{\bar{z}} \mid F(z) = 0\})$ is zero. On the other hand, the union of the family of open sets $\{U_{\bar{z}}\}_{\bar{z} \in A}$ covers A. Because $\mathbb{R}^{|\mathcal{V}|}$ is separable, a countable selection $(\bar{z}_n)_{n \in \mathbb{N}}$ in A exists, where we obtain

$$A = \bigcup_{n \in \mathbb{N}} U_{\bar{z}_n} \cap A.$$

Due to the fact that $\mathrm{mes}_{|\mathcal{V}|-1}(U_{\bar{z}_n} \cap A) = 0$ ($n \in \mathbb{N}$), we found a union of countable many subsets of S of surface measure zero that covers A. Therefore, from [13, Proposition 4.32] we conclude that $\mathrm{mes}_{|\mathcal{V}|-1}(A) = 0$.

It remains to show that B has surface measure zero. But, subset B is included in the finite union of linear subspaces $U_{kl} := \{z \in \mathbb{R}^{|\mathcal{V}|} \mid z_k = 0,\, z_l = 0\}$, $k \neq l$, of co-dimension 2 ($k, l = 1, \dots, |\mathcal{V}|$). As a consequence, as well as subset A, also subset B has surface measure zero. This completes the proof. □

Note that Lemma 4 does not make use of the special structure of the constraints of the capacity problem and also remains valid in a more general context. It just requires a finite systems of continuously differentiable inequalities, where the claim of Corollary 1 is satisfied. The property, having surface measure zero of the intersection with respect to two active constraints, turns out to be the essential property when asking for differentiability of the probability function, as shown in [12]. Hence, with Lemma 4 we are prepared for the main result of this section, the differentiability of the capacity problem.

Theorem 2. *Let be given $\bar{x} \geq 0$ such that $\tilde{g}_j(\bar{x}, 0) > 0$ for all $j \in \mathcal{J}$. Then, the probability function $\varphi(x) := \mathbb{P}_{\xi \geq 0}\left(\tilde{g}_j(x, \xi) \geq 0\,, j \in \mathcal{J}\right)$ of the problem of maximizing booked capacities* (14) *is differentiable on some neighbourhood U of $\bar{x}$, if the distribution $\mathbb{P}$ of the random vector ξ has a continuous and bounded density on $\mathbb{R}^{|\mathcal{V}|}$.*

Proof: To prove the result of the Theorem we want to apply a general result regarding differentiability of probability functions in [12]. To this end, we first discuss the gradients of the constraints $\tilde{g}_j(x, \cdot)$. The special structure of the constraints allows to derive the following property. With the notation of Definition 1 and by applying Lemma 1 it is easy to show that for any $j \in \mathcal{J}$ the equation $\tilde{g}_j(\bar{x}, \bar{z}) = 0$ implies that

$$\|\nabla_z \tilde{g}_j(\bar{x}, \bar{z})\| \geq \left|\, [\nabla_z \tilde{g}_j(\bar{x}, \bar{z})]_{d_j^-} \,\right| \geq 2\phi^{min} \left(\frac{\Delta p}{|\mathcal{V}|\phi^{max}}\right)^{\frac{1}{2}} =: \gamma,$$

where $\phi^{max} := \max\{\phi_e \,|\, e \in \mathcal{E}\}$, $\phi^{min} := \min\{\phi_e \,|\, e \in \mathcal{E}\}$ denote maximal and minimal roughness coefficients, respectively, and $\Delta p := \min\{(p_k^{max})^2 - (p_l^{min})^2 \,|\, k, l \in \mathcal{V}^+\}$ denotes the minimal quadratic pressure difference. As consequence, due to continuity, we observe that

$$\|\nabla_z \tilde{g}_j(x, z)\| \geq \frac{\gamma}{2} > 0$$

on some neighbourhood U of $\bar{x}$ and V of $\bar{z}$ for any $j \in \mathcal{J}$. Secondly, we state that

$$\mathbb{P}_{\xi \geq 0}\left(\tilde{g}_j(x, \xi) \geq 0\,, j \in \mathcal{J}\right) \equiv \mathbb{P}\left(\tilde{g}_j(x, \xi) \geq 0\,, j \in \mathcal{J};\; \xi_k \geq 0,\; k \in \mathcal{V}\right)$$

and mention that the additional inequalities $\xi_k \geq 0$ ($k \in \mathcal{V}$) are compatible with the constraint $\tilde{g}(x, \xi) \geq 0$ in the sense that they do not destroy the recent properties. In particular, due to the assumption $\tilde{g}_j(\bar{x}, 0) > 0$, in fact, $\xi_k = 0$ and $\tilde{g}_j(\bar{x}, \xi) = 0$ implies that $\tilde{g}_j(\cdot, \cdot)$ depends on some $\xi_l \neq 0$ and, therefore, the gradients of both constraints are linear independent. Moreover, the additional non-negative constraints also satisfy the constraint qualifications of Corollary 1 and the norm of the gradients each equals one, no matter what ξ. As consequence, we apply Lemma 4 for the total system of inequalities, and, all requirements of [12, Theorem 2.4] are satisfied. We conclude that the probability function $\varphi(x)$ is differentiable for all $x \in U$. □

We want to complete this section by discussing Example 1 again. Therefore, we illustrate the constraint qualifications of Corollary 1 and the resulting surface measure condition for the intersection of the boundary of both involved constraints for special instances.

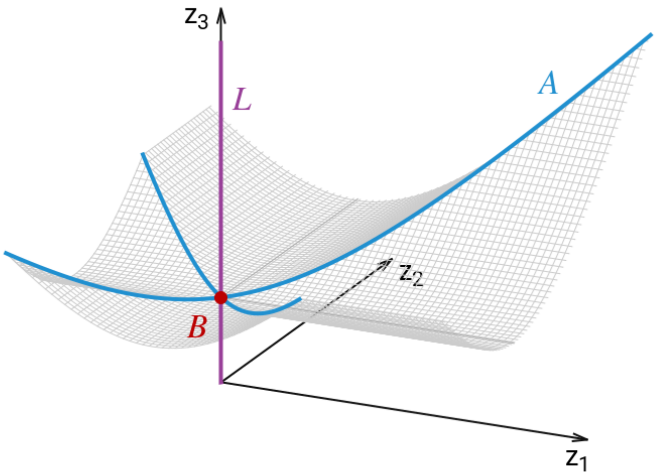

Figure 2: Boundary of the feasibility region obtained by a special instance (below) of Example 1.

As a special instance of the constraints in Example 1 we consider

$$\begin{aligned} \tilde{g}_1(\bar{x}, z) &= 1 + z_1^2 - (z_3 + \bar{x}_3)^2 \geq 0\,, \\ \tilde{g}_2(\bar{x}, z) &= 1 + z_2^2 - (z_3 + \bar{x}_3)^2 \geq 0\,, \end{aligned}$$

where $\bar{x}_3 = \frac{1}{2}$, i.e., for example $\bar{x} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and $z \in \mathbb{R}^3$. Particularly, as gradients we observe

$$\nabla_z \tilde{g}_1(\bar{x}, z) = \begin{pmatrix} 2z_1 \\ 0 \\ -2z_3 - 1 \end{pmatrix}, \quad \nabla_z \tilde{g}_2(\bar{x}, z) = \begin{pmatrix} 0 \\ 2z_2 \\ -2z_3 - 1 \end{pmatrix}.$$

The intersection $S := S_1(\bar{x}) \cap S_2(\bar{x})$ of the surfaces $S_j(\bar{x}) = \{z \,|\, \tilde{g}_j(\bar{x}, z) = 0\}$, where $j = 1, 2$, decomposes into two subsets, $A = \{z \in S \,|\, \text{rank 2 condition satisfied}\}$ and the set of singularities $B = \{z \in S \,|\, \text{rank 2 condition violated}\}$. In that example, for the latter set we obtain the singleton $B = \{(0, 0, \frac{1}{2})\}$. This set is included in the subspace $L = \{z \in \mathbb{R}^3 \,|\, z_1 = 0;\, z_2 = 0\}$ of co-dimension 2 (cf. Proof of Lemma 4). However, we have that $A \cup B = S$. Figure 2 shows the surfaces S_1 and S_2, defined by the active inequalities, as well as the intersection curve S represented by the subsets A and B. As shown in Lemma 4, it turns out that the 2-dimensional surface measure of S is zero. Indeed, we obtain $\mathrm{mes}_2(S) = 0$.

5 Algorithmic approach to solving the capacity problem

In this section we want to provide an algorithmic solution for the problem of maximizing booked capacities. In the previous sections we have shown that the capacity problem under weak conditions to the distribution of the random exit demand is differentiable with respect to the probabilistic constraints. Therefore, in principle any algorithm of nonlinear optimization that uses derivative information could be applied in order to solve the problem numerically. However, an efficient numerical solution is linked to an efficient computation of the involved probability function and its gradients.

Returning to the capacity problem (14), for any fixed decision x the set of feasible nominations is given by

$$M_x := \left\{ \xi \in \mathbb{R}^{|\mathcal{V}|} \;\middle|\; \tilde{g}_{k,l}(x, \xi) \geq 0;\ k, l = 0, \ldots, |\mathcal{V}| \right\}, \tag{22}$$

where $\tilde{g}_{k,l}(\cdot, \cdot)$ taken form (13). Now, that we are given an explicit description of the set M_x, we could use this finite inequality system in (22) in order to test the feasibility of simulated outcomes of the random demand ξ according to the given continuous distribution. The averaged number of feasible simulations would yield the Monte Carlo estimate for the desired probability $\mathbb{P}(\xi \in M_x)$. Such Monte Carlo approach has two drawbacks: first it may come with a comparatively large variance for the obtained probability estimation and, second, it does not provide us with information about the sensitivity of this probability with respect to changes of x. For this reason, we will alternatively make use of the so-called *spheric-radial decomposition* of Gaussian random vectors. In general, for a random vector $\xi(\omega)$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ the computation of the probability

$$\mathbb{P}\{\omega \in \Omega \,|\, \xi(\omega) \in M_x\} \tag{23}$$

amounts to the solution of a possibly high dimensional (number of exit nodes) multiple integral. A favorable situation to carry out this computation under Gaussian distribution occurs for polyhedral sets. For details, we refer to [8], which not only gives an excellent overview on this topic but also presents a very efficient algorithm developed by the author himself. Unfortunately, in our setting we cannot expect the feasible set M_x to be polyhedral, not even convex. Therefore, recourse to the mentioned algorithm is not possible.

5.1 Spheric-radial decomposition under Gaussian distribution

We shall rather propose here the so-called spheric-radial decomposition of a Gaussian distribution (e.g., [6]) as a promising alternative which not only may significantly reduce the variance of estimating (23) but, moreover, it offers the possibility of efficiently approximating gradients of (23) with respect to the external parameter x. The main variation of exit load data is temperature driven. However, even at fixed temperature, considerable random variation remains. That is why the exit demand may be characterized by a finite family of multivariate distributions, each of them referring to some (rather narrow) range of temperature and reflecting the joint distribution of loads at the given set of exit points, see [11, Chapter 13]. As recorded in the same reference [11, Table 13.3], these distributions are most likely to be Gaussian (possibly truncated) or lognormal. Our assumption to consider a multivariate Gaussian distribution for ξ can therefore be seen as a prototype setting which, using the spheric-radial decomposition presented next, maybe adapted without much effort to more realistic settings (multivariate log-normal distributions, probabilities with respect to several temperature classes simultaneously, etc.). The following result is well-known:

Theorem 3 (spheric-radial decomposition). *Let $\xi \sim \mathcal{N}(\mu, \Sigma)$ be some m-dimensional Gaussian distribution with mean vector μ and covariance matrix Σ. Then, for any Borel measurable subset $M \subseteq \mathbb{R}^m$ it holds that*

$$\mathbb{P}(\xi \in M) = \int_{\mathbb{S}^{m-1}} \mu_\chi\{r \geq 0 \,|\, rLv + \mu \in M\} d\mu_\eta(v),$$

where $\mathbb{S}^{m-1}$ is the $(m-1)$-dimensional sphere in $\mathbb{R}^m$, μ_η is the uniform distribution on $\mathbb{S}^{m-1}$, μ_χ denotes the χ-distribution with m degrees of freedom and L is such that $\Sigma = LL^T$ (e.g., Cholesky decomposition).

In order to evaluate the integrand in the spheric integral above, one has to be able to compute, for any fixed direction $v \in \mathbb{S}^{m-1}$, the χ-probability of the one-dimensional set

$$\{r \geq 0 \,|\, (rLv + \mu) \in M\}.$$

Since we are interested in the probability of the set M_x, this amounts by (22) to characterizing the set

$$\{r \geq 0 \,|\, \tilde{g}_{k,l}(x, rLv + \mu) \geq 0;\ k, l = 0, \ldots, |\mathcal{V}|\} \quad (v \in \mathbb{S}^{|\mathcal{V}|-1}). \tag{24}$$

Using the idea of spheric-radial decomposition presented in Theorem 3, we propose the following algorithm for computing the probability $\mathbb{P}(\xi \in M_x)$ for a fixed value x.

Algorithm 1 (Function evaluation). *Let be $x \geq 0$, $\xi \sim \mathcal{N}(\mu, \Sigma)$ and L such that $\Sigma = LL^T$:*

1 Sample N points $\{v^1, v^2, \ldots, v^N\}$ uniformly distributed on the sphere $\mathbb{S}^{|\mathcal{V}|-1}$.

2 $i := 0; S := 0.$

3 $i := i + 1$. *Find the zero's of the one-dimensional function (in r for x fixed)*

$$\theta_x^i(r) := \min_{k,l=0,\dots,|\mathcal{V}|} \tilde{g}_{k,l}(x, rLv^i + \mu)$$

with $\tilde{g}_{k,l}(\cdot,\cdot)$ defined in (13) *and represent the set $M_x^i := \{r \geq 0 \,|\, \theta_x^i(r) \geq 0\}$ corresponding to* (24) *as a disjoint union of intervals: $M_x^i = \cup_{j=1}^t [\alpha_j(x), \beta_j(x)]$, where $\alpha_j(x)$, $\beta_j(x)$ are the zero's obtained before and ordered appropriately.*

4 *Compute the χ-probability of M_x^i according to*

$$\mu_\chi(M_x^i) = \sum_j F_\chi(\beta_j(x)) - F_\chi(\alpha_j(x)), \tag{25}$$

where F_χ refers to the cumulative distribution function of the one-dimensional χ-distribution with $|\mathcal{V}|$ degrees of freedom. Put $S := S + \mu_\chi(M_x^i)$.

5 *Continue, if $i < N$, with step 3.*

6 *Finally, set $\mathbb{P}(\xi \in M_x) := S/N$.*

The above algorithm clearly provides an approximation to the spheric integral in Theorem 3 by means of a finite sum based on sampling of the sphere, and then, averaging the values of the integrand over all samples. Of course, this approximation will improve with the sampling size which may be large depending on the dimension $|\mathcal{V}|$ of the problem (i.e., exit nodes in the network) and on the desired precision for the probability.

Computing the zero's of the one-dimensional function $\theta_x^i(r)$ (step 3 of the algorithm) can be done analytically. As disclosed in formula (13) the constraint mappings $\tilde{g}_{k,l}(\cdot,\cdot)$ provide a particular quadratic structure such that $\theta_x^i(r)$, $i = 1,\dots,N$, turn out to be a piecewise quadratic functions as well.

Finally, we recall that the uniform distribution on the sphere $\mathbb{S}^{|\mathcal{V}|-1}$ can be represented as the distribution of $\eta/\|\eta\|$ (Euclidean norm), where η has a standard Gaussian distribution in $\mathbb{R}^{|\mathcal{V}|}$, i.e., $\eta \sim \mathcal{N}(0, I)$. Then, the simplest idea to sample points v^i on the sphere as in step 1 of the algorithm would be to independently sample $|\mathcal{V}|$ values w_j of a one-dimensional standard normal distribution by using standard random generators and then putting $v^i := w/\|w\|$ for $w := (w_1, \dots, w_{|\mathcal{V}|})$. When replacing such Monte Carlo sampling of the normal distribution (based on random number generators) by Quasi-Monte Carlo sampling (based on deterministic low discrepancy sequences), one observes a dramatic improvement in the precision of the result. For the problem of nomination validation in gas networks, this first was revealed in [10].

5.2 Computing gradients of the probability function

As well as the function evaluations also gradient computations in view of the probabilistic constraints within the capacity problem are needed in order to solve the problem efficiently. The above spheric-radial decomposition approach has the advantage that in many situations derivatives with respect to involved parameters can be computed without additional effort by using nearly the same approximation scheme. As shown in [1, 2], gradients can be represented as spheric integrals as well, just with different integrands. The basis for computing derivatives is the gradient formula for the probability function

$$\varphi(x) := \mathbb{P}\left(g(x,\xi) \geq 0\right)$$

(see Sect. 2) formulated in [2, Theorem 4.1]. With the notation of Theorem 3 and under some regularity conditions, requiring differentiability in both and convexity in the second argument for the constraint mapping $g(\cdot,\cdot)$, if $g(x,\mu) \geq 0$, and, in the Gaussian case $\xi \sim \mathcal{N}(\mu,\Sigma)$, then the gradient of $\varphi(\cdot)$ can be represented in the form

$$\nabla\varphi(x) = \int\limits_{\substack{v\in\mathbb{S}^{m-1}\\ \#J(x,v)=1}} -\frac{\chi(\rho(x,v))}{\langle\nabla_\xi g_{j(v)}(x,\rho(x,v)Lv+\mu), Lv\rangle}\nabla_x g_{j(v)}(x,\rho(x,v)Lv+\mu)d\mu_\eta(v), \quad (26)$$

where χ denotes the density of the χ-distribution, $\rho(x,v) := \max\{r \geq 0 \,|\, g(x,rLv+\mu) \geq 0\}$ and $J(x,v) := \{j \in \{1,\ldots,k\} \,|\, g_j(x,\rho(x,v)Lv+\mu) = 0\}$. Moreover, the index $j(v)$ is the unique index $j \in \{1,\ldots,k\}$ satisfying $g_j(x,\rho(x,v)Lv+\mu) = 0$. Unfortunately, we cannot expect convexity in connection with the capacity problem. Anyway, we are going to use gradient formula (26) in order to provide an algorithmic computation of the gradient similar to Algorithm 1. One reason for doing so is that in our case we consider bounded feasibility sets M_x only, i.e., the above convexity condition is quite strong condition just to ensure that the radius function $\rho(x,v)$ is well-defined for any radial $v \in \mathbb{S}^{m-1}$. What follows, the same gradient formula can be achieved by replacing the convexity condition by a much weaker requirement of starshapeness with respect to the feasibility set. Even though starshapeness of feasibility sets is hardly to verify, nevertheless, it is a reasonable condition in the context of gas transportation networks.

However, by applying formula (26) in a more general case we want to adapt Algorithm 1 in order to compute the gradient of the probability function, approximately. Therefore, the zero's $\alpha_j(x)$, $\beta_j(x)$ of the functional $\theta_x^i(r)$ in Algorithm 1 play the role of the radius function $\rho(x,v)$ in (26) for any direction $v = v^i$, $i = 1,\ldots,N$. By inclusion of the partial derivatives (gradients) of the constraints $\tilde{g}_{k,l}(\cdot,\cdot)$ we provide the following algorithm.

Algorithm 2 (Gradient evaluation)**.** *Let be $x \geq 0$, $\xi \sim \mathcal{N}(\mu,\Sigma)$ and L such that $\Sigma = LL^T$:*

1 *Sample N points $\{v^1, v^2, \ldots, v^N\}$ uniformly distributed on the sphere $\mathbb{S}^{|\mathcal{V}|-1}$.*

2 $i := 0; S' := 0.$

3 $i := i+1$. *Find the zero's of the one-dimensional function (in r for x fixed)*

$$\theta_x^i(r) := \min_{k,l=0,\ldots,|\mathcal{V}|} \tilde{g}_{k,l}(x, rLv^i + \mu)$$

with $\tilde{g}_{k,l}(\cdot,\cdot)$ defined in (13) *and represent the set $M_x^i := \{r \geq 0 \,|\, \theta_x^i(r) \geq 0\}$ corresponding to* (24) *as a disjoint union of intervals: $M_x^i = \cup_{j=1}^t [\alpha_j(x), \beta_j(x)]$, where $\alpha_j(x)$, $\beta_j(x)$ are the zero's obtained before and ordered appropriately.*

4 *To any of the zero's $\alpha_j(x)$, $\beta_j(x)$ select the active constraints, i.e., assign index mappings $\tau_\alpha(j), \tau_\beta(j) \in \{0, \ldots, |\mathcal{V}|\}^2$ such that*

$$\tilde{g}_{\tau_\alpha(j)}(x, \alpha_j(x)Lv + \mu) = 0 \quad \textit{and} \quad \tilde{g}_{\tau_\beta(j)}(x, \beta_j(x)Lv + \mu) = 0. \tag{27}$$

Compute the derivative of the χ-probability of M_x^i according to

$$\begin{aligned} D_j^\alpha(x) &= \frac{f_\chi(\alpha_j(x))}{\langle \nabla_\xi g_{\tau_\alpha(j)}(x, \alpha_j(x)Lv^i + \mu), Lv^i \rangle} \nabla_x g_{\tau_\alpha(j)}(x, \alpha_j(x)Lv^i + \mu), \\ D_j^\beta(x) &= \frac{f_\chi(\beta_j(x))}{\langle \nabla_\xi g_{\tau_\beta(j)}(x, \beta_j(x)Lv^i + \mu), Lv^i \rangle} \nabla_x g_{\tau_\beta(j)}(x, \beta_j(x)Lv^i + \mu), \\ \nabla_x \left(\mu_\chi(M_x^i)\right) &= \sum_j D_j^\alpha(x) - D_j^\beta(x), \end{aligned} \tag{28}$$

where f_χ refers to the probability density function of the one-dimensional χ-distribution with $|\mathcal{V}|$ degrees of freedom. Put $S' := S' + \nabla_x(\mu_\chi(M_x^i))$.

5 *Continue, if $i < N$, with step 3.*

6 *Finally, set $\nabla_x(\mathbb{P}(\xi \in M_x)) := S'/N$.*

Before concluding this section some remarks to the stated Algorithm 1 and 2 are appropriate. The update formula (28) for the derivative of the probability function with respect to the parameter x in step 4 of Algorithm 2 can be considered as rigorous differentiation of the respective formula for the probability (25) within step 4 of Algorithm 1. Since we have

$$\nabla_x (F_\chi(\beta_j(x)) - F_\chi(\alpha_j(x))) = f_\chi(\beta_j(x)) \nabla_x \beta_j(x) - f_\chi(\alpha_j(x)) \nabla_x \alpha_j(x),$$

formula (28) in Algorithm 2 appears when inserting the gradients $\nabla_x \alpha_j(x)$ and $\nabla_x \beta_j(x)$ which are obtained by total differentiation of the equations in (27) with respect to x and resolving them for $\nabla_x \alpha_j(x)$ and $\nabla_x \beta_j(x)$, respectively. Moreover, note that both algorithm are compatible in a sense that, after computing the sampling scheme on the unique sphere, one and the same sample v^i can be employed in order to update values and gradients of the involved probability function. Also the needed zero's $\alpha_j(x)$ and $\beta_j(x)$ are the same here. In general, determining these zero's corresponds to the most expensive parts. On the other hand, due to the analytical representation (13) the partial derivatives of the mappings $\tilde{g}_{k,l}(\cdot,\cdot)$ can easily determined analytically (similar to Lemma 1). Therefore, in order to compute both function end gradient evaluations, almost no additional effort for computing gradients is needed when computing function values and performing Algorithm 1 and 2 simultaneously.

The strategy of computating function values and gradients of the probabilistic constraints of the capacity problem by Algorithms 1 and 2 can be embedded into a simple projected gradient method. Clearly, due to the non-convexity of the model, performing a projected gradient method causes a termination at local minima, in general. Therefore, the accuracy strongly depend on finding reasonable starting points heuristically. However, the practicability of the approach is shown in the next section, where a numerical study related to realistic network data is presented.

6 Numerical study

In this section we finally want to test the performance of the presented methodology for solving the problem of maximizing booked capacities. Clearly, here we use the reformulation in terms of the classical probabilistic constrained optimization problem obtained in (14). In order to solve the underlying non-linear optimization problem we designed a straight forward decent method based on projected gradients. The core of the method consists in local linearizations of the feasibility set and the projection of the negative objective gradient onto these linearizations. Whenever this projection is non-zero a new decent step can be performed which results in a feasible point with improved objective. Because the decent direction could point away from the feasibility set, if necessary, a redirection back to the feasibility set must be performed, where the gradient information of the constraints may be used. The method terminates in a stationary point, where the projected objective gradient is zero.

The described decent method actually aims to solve a minimum problem that can be obtained just by switching the sign of the objective function in (14). All needed to perform this method are function and gradient evaluations for both the objective and constraint function. Function values and derivatives of the objective are computed analytically. Because we do not assume any preferences in the allocation of new capacities, the weight vector in the objective of problem (14) is chosen just as $c^T = (1, \dots, 1)$.

For the probabilistic constraint, represented by the probability function, the spheric-radial decomposition is applied. More precisely, Algorithm 1 and Algorithm 2 from the previous section are used in order to compute function values and gradients. Therefore, we employed Quasi-Monte Carlo (QMC) sampling on the bases of Sobol sequences as a special case of low-discrepancy sequences that are included in the category of (t, m, d)-nets and (t, d) sequences [7]. A QMC sample of 10 000 scenarios was created according to a standard Gaussian distribution (zero mean and identity covariance matrix). Normalizing each scenario to unit length provides a sample of the uniform distribution on the sphere as required in the simultaneous update of values and gradients of the probability function within Algorithm 1 and 2.

The appropriate choice of model parameters is a crucial step in numerical experiments. In the view of exit load nominations, as already mentioned [11, Chapter 13] provides a wide study concerning the statistical analysis of gas demand data in real gas networks. The approach is based on analyzing historical data with respect to different temperature classes and in identifying multivariate distributions coming up into consideration. According to the results, random gas demand can often be described by combinations of Gaussian-like multivariate distributions (Gaussian, truncated Gaussian, lognormal). Distribution parameters like mean, standard deviation and correlations can be estimated statistically from historical data, where the network owner may benefit from a long term data record.

6.1 Multivariate Gaussian distribution

We start our numerical experiences with the an example wherein assuming a multivariate Gaussian distributed exit demand as in Theorem 3 with parameters (mean μ, covariance Σ) chosen in a way to represent real-life data. The parameters are in fact slightly modified distribution parameters obtained from a real gas networks and adapted to a artificial example network containing one entry node, a number of 26 exit nodes and representing a tree. All remaining net parameters, particularly roughness, lower and upper pressure bounds are chosen in range of typical values for existing gas networks.

Probability	Average demand [kW]	Free capacity [kW]	Decent steps	Computing time [s]
0.95	27 797.76	541.37	27	45.92
0.90	27 797.76	1 495.01	44	159.48
0.85	27 797.76	2 130.38	63	210.96
0.80	27 797.76	2 631.46	78	252.78

Table 1: Results for the network example of medium size. Displayed are the obtained free booked capacities (total sum for all exits) compared to the average of the total gas demand at all exits computed by solving problem (14) for different chosen probability levels and fixed underlying multivariate normal distribution for the exit demand.

Results for solving the problem of maximizing booked capacities (14) for Gaussian distributed exit demand are displayed in Table 1. In addition, Fig. 3 visualizes the network topology and the allocated free capacities at the exit nodes for the selected probability levels $p = 0.90$ and $p = 0.80$, respectively. Clearly, a decreasing probability level for technical feasibility of random demand yields an increasing free capacity left in the network.

In Fig. 4 we perform a posterior check of the computed solution for the probability level $p = 0.80$. By a simulation of 4 sets of exit loads situations according to the chosen Gaussian distribution we check the feasibility of the computed solution of allocated capacities against the particular exit demand in the robust sense of (10). Feasibility is displayed by green circles indicating that the computed capacity as solution of (14) could even increased by upscaling while remaining feasible with respect to the simulated scenario. On the other hand, if the allocated capacity exceeds the possible technical feasibility in the simulated situation, this is displayed by red circles according to a needed downscale of the solution in order to become feasible with respect to the robust condition (10). As seen in Fig. 4, three out of four simulated exit demand situations turn out to be feasible whereas in one case the solution do not satisfy the simulated demand. However, when simulating a large set of such scenarios, say 1 000, it would turn out that according to the probability level $p = 0.80$ approximately 800 are feasible, while 200 are infeasible.

6.2 Extension to more general distributions

As discussed, according to [11] Gaussian and Gaussian-like distributions are mostly relevant for describing random demand in gas transportation networks. But in fact, the described methodology to treat optimization problems with probabilistic constraints via spheric-radial decomposition can be extended even to more general distributions. In [3] the class of elliptical distributions is considered, where the approach is used for the investigation of probability functions acting on nonlinear systems wherein the random vector can follow an elliptically symmetric distribution. Beside the Gaussian distribution the Student's distribution would be another example for an elliptically symmetric distribution.

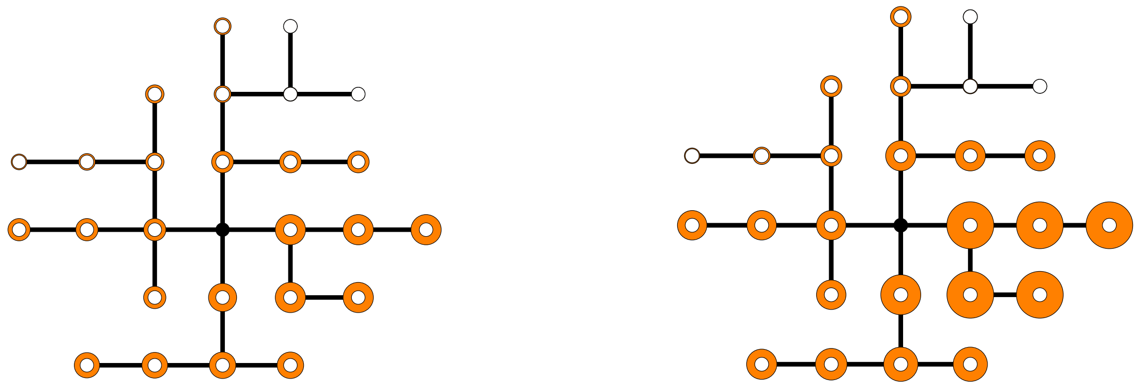

Figure 3: Network topology of a medium sized network for the example of Gaussian exit demand. Illustration of the solution of the capacity maximization problem at exit points for different probability levels $p = 0.90$ (left) and $p = 0.80$ (right). The entry and exit points are displayed in black (entry) and white (exit), respectively. A decreasing probability level allows for a higher allocation of capacity in certain regions of the network highlighted by colored circles of different size.

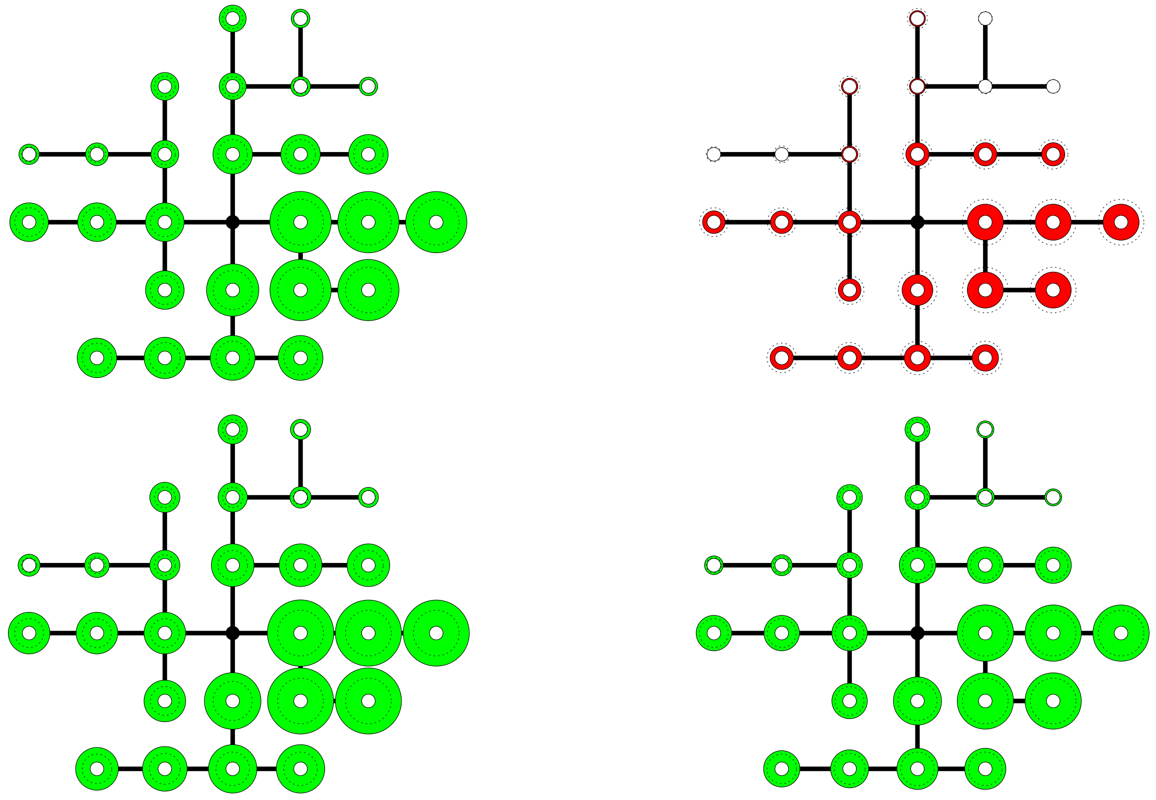

Figure 4: Four simulated exit demand realizations according to the chosen multivariate Gaussian distribution and the respective available free capacity compared to the allocated capacity provided by the numerical solution for the medium network for the probability level $p = 0.80$. Feasible and infeasible situations are displayed in green and red, respectively.

In the context of capacity optimization in a gas transportation network, we want to discuss a slightly more realistic situation, where gas nominations are in fact regulated by contracts between the network owner and the customers. Such contracts usually provide upper limits for the quantity of gas that could be delivered to the customers. Therefore, in the second numerical example for the problem of maximizing booked capacities we will suppose that the stochastic exit demand vector ξ follows a truncated multivariate Gaussian distribution

$$\xi \sim \mathcal{TN}(\mu, \Sigma, [0, L]). \tag{29}$$

More precisely, the distribution of ξ is obtained by truncating a $|\mathcal{V}|$-dimensional Gaussian distribution with mean μ and covariance matrix Σ to an $|\mathcal{V}|$-dimensional rectangle $[0, L]$ with upper limits L_k at exit node k. Therefore, the vector L represents booking limits given by former contracts. Clearly, the network owner is aiming to extend these limits by the allocation of free network capacities according to the solution of (14).

We want to proceed with the same methodology from Sec. 5, in particular, we want to apply Algorithm 1 and 2 based on spheric-radial decomposition in order to solve the capacity problem (14), but under truncated instead of Gaussian distribution. Therefore, a transformation back to a normal distribution can be discovered as follows. By definition of the truncated normal distribution, (29) is equivalent to the property

$$\mathbb{P}\left(\xi \in A\right) = \frac{\mathbb{P}\big(\tilde{\xi} \in A \cap [0, L]\big)}{\mathbb{P}\big(\tilde{\xi} \in [0, L]\big)}$$

for all Borel measurable subsets A of $\mathbb{R}^{|\mathcal{V}|}$, and, where $\tilde{\xi}$ is the associated Gaussian random vector with $\tilde{\xi} \sim \mathcal{N}(\mu, \Sigma)$. Hence, in order to determine probabilities under a truncated Gaussian distribution, it is sufficient to be able to determine probabilities under the Gaussian distribution itself. Applying this observation to the probabilistic constraint of the capacity problem, the equivalent representation to the reformulation (14) of the problem of maximizing booked capacities with truncated Gaussian exit load distribution $\mathcal{TN}(\mu, \Sigma, [0, L])$ reads

$$\text{maximize } c^T x \quad \text{subject to}$$
$$\mathbb{P}\left(\left\{\begin{array}{rcll} \tilde{g}_{k,l}(x, \tilde{\xi}) & \geq & 0 & (k, l = 0, \dots, \mathcal{V}) \\ \tilde{\xi}_k & \geq & 0 & (k = 1, \dots, \mathcal{V}) \\ \tilde{\xi}_k & \leq & L_k & (k = 1, \dots, \mathcal{V}) \end{array}\right\}\right) \geq p \cdot \mathbb{P}\big(\tilde{\xi} \in [0, L]\big)\,, \tag{30}$$

where $\tilde{\xi} \sim \mathcal{N}(\mu, \Sigma)$ is the Gaussian distribution with mean μ and covariance matrix Σ and $\tilde{g}_{k.l}(\cdot, \cdot)$ corresponds to the system of inequalities obtained in (13). Hence, the modified problem formulation (30) arises from (14) only by adding additional box constraints to the system of random inequalities, and, by scaling the given probability accordingly. The probability value $\mathbb{P}\big(\tilde{\xi} \in [0, L]\big)$ can easily computed by the spheric-radial decomposition, or alternatively, by other efficient computation schemes for the probability of rectangles when dealing with multivariate normal distributions [8].

The following numerical results are obtained for a larger example network containing 1 entry and 43 exit nodes. The more realistic sized network could be viewed as topological extension of the medium size network before. Although this network is academical constructed as well, the network parameters are adapted from real networks in the same way as before. The initial multivariate truncated Gaussian distribution again involves correlations between the exit points and the truncation limits are chosen in a way that one obtains an initial probability level of approximately $p = 0.98$ for the technical feasibility of the random demand (with no capacity extension). The truncation probability in (30), i.e., the Gaussian probability of the rectangle $[0, L]$, turns out to be $\mathbb{P}(\tilde{\xi} \in [0, L]) = 0.71$. However, the high initial probability level allows for allocating free capacities when decreasing the prescribed probability as shown in Table 2.

Probability	Average demand [kW]	Free capacity [kW]	Decent steps	Computing time [s]
0.95	28 870.74	727.08	20	212.66
0.90	28 870.74	1 271.07	32	277.58
0.85	28 870.74	1 654.31	53	362.09
0.80	28 870.74	1 941.04	45	363.83

Table 2: Numerical results for the network example of large size. Displayed are the obtained free booked capacities (total sum for all exits) compared to the average of the total gas demand at all exits computed by solving problem (14) for different chosen probability levels and fixed underlying multivariate truncated normal distribution for the exit demand.

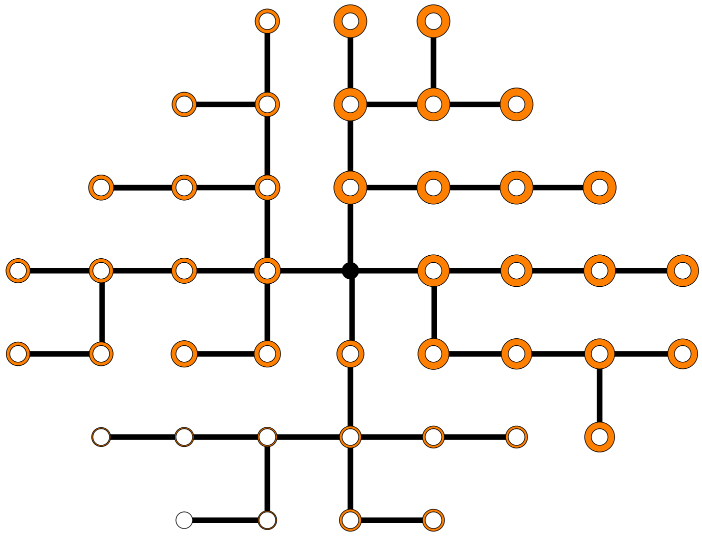

Figure 5: Network topology of the large sized network for the example of truncated Gaussian exit demand. Illustration of the solution of the capacity maximization problem at exit points for the probability level of $p = 0.80$. The entry and exit points are displayed as before. The picture shows the allocated free capacities obtained at the particular exit nodes of the network. Quantities are highlighted by colored circles of different size.

In Fig. 5, a visualization of the network topology and the allocated free capacities at the exit nodes for the second example under truncated Gaussian distribution and for a selected probability levels of $p = 0.80$ is given. It turns out that, although no preferences such as certain weights are assigned to the different exit points, the total amount of allocated free capacity is not uniformly distributed at the whole network. In fact, network and distribution specifics play the major role when answering the question of maximizing free booked capacities.

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