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Andrea N. Ceretani, Carlos N. Rautenberg

Abstract

A stationary Boussinesq system for an incompressible viscous fluid in a bounded domain with a nontrivial condition at an open boundary is studied. We consider a novel non-smooth boundary condition associated to the heat transfer on the open boundary that involves the temperature at the boundary, the velocity of the fluid, and the outside temperature. We show that this condition is compatible with two approaches at dealing with the "do-nothing" boundary condition for the fluid: 1) the "directional do-nothing" condition and 2) the "do-nothing" condition together with an integral bound for the backflow. Well-posedness of variational formulations is proved for each problem.

1 Introduction

We address a heat transfer problem for an incompressible viscous fluid in a room, which is allowed to flow freely through some part of the boundary. The case we consider is motivated by a situation that naturally occurs when modeling or controlling energy systems in buildings, see e.g. [13, 21]. We assume the room is heated through some part of its boundary, and the walls are thermally insulated. Further, we assume the fluid can enter the room through an inlet, and flow without any restriction across an outlet. The ambient temperature outside the room is assumed to be zero. Two common configurations are represented in Figure 1.

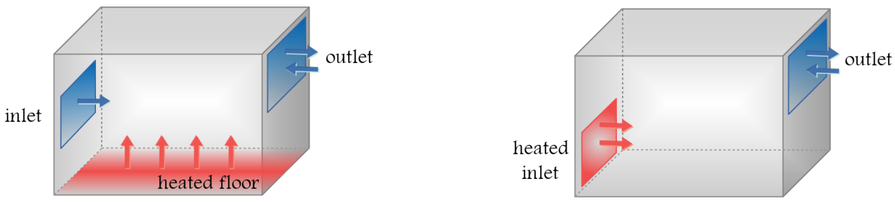

Figure 1: Rooms with a heated floor (left), and a heated inlet (right).

In addition to diffusing, the fluid temperature is affected by an advective heat transport, originated by the moving fluid. Further, the fluid velocity is influenced by the buoyancy effects due to temperature changes. This coupling between the temperature and the velocity will be studied under the Boussinesq approximation [10, 27]. Mixed boundary conditions arise naturally from the particular behavior of the fluid at the domain boundary. The outlet, across which the fluid flows without restrictions, is usually known as an *open boundary*. Coupled Navier-Stokes/energy systems for incompressible heat conducting fluids in domains with open boundaries were recently studied in [3–5, 30, 31] (steady problems), and [4, 7, 8] (unsteady problems). Boussinesq systems with other type of boundary conditions are also an active area of research, see e.g. [22, 23].

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It is still a matter of debate which boundary condition represent the unrestricted fluid flow through the outlet. Several works have been done in this regard. As a classic reference, we mention the work of Heywood et al. [20]. A survey on different mathematical treatments for open boundaries can be found in the recent article [9], where the authors also present reference problems, and compare the numerical performance of different approaches. In this work, the natural flow at the outlet will be represented in the spirit of the "do-nothing" condition, introduced by Gresho in [19]. Denoting the velocity field by $\mathbf{v}$, the pressure by p, and the outlet by $\partial\Omega_{out}$, the do-nothing condition is given by

$$\frac{1}{\mathrm{Re}}\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n} = 0 \quad \text{on } \partial\Omega_{out}, \tag{DN}$$

where $\mathrm{Re} > 0$ is the Reynolds number of the system, and $\mathbf{n}$ is the outer normal to the boundary. At present, (DN) is a well-established boundary condition to represent natural outflows (see e.g., [6] and the references therein). For a recent discussion on the derivation and physical meaning of the do-nothing condition (DN), we refer to [26].

One problem concerning (DN) is that it allows the fluid to re-entry the domain at possibly unrestrained rates. When they are high, the kinetic energy of the system can be severely affected, and energy estimates might not be possible to obtain for arbitrary data. Aiming to overcome this situation, which may significantly affect both the theoretical and numerical treatment of the problem, some authors have proposed to modify (DN) by imposing some restriction on the backflows. Here, we follow two different approaches. The first one consists in perturbing the do-nothing condition (DN) by some factor that affects the incoming flows, and enhances the stability of the problem. This approach was introduced by Bruneau and Fabrie in [12], and then followed by several authors, see e.g. [2, 11, 17, 28]. The other, is due to Kračmar and Neustupa, and consists in supplementing the do-nothing condition by a bound for the backflows, see [24–26].

When the first approach is considered in the framework of Navier-Stokes systems, bounds for the kinetic energy can be obtained for any Reynolds number Re; see [11]. The same occurs with the second approach, provided some smallness-type condition is assumed on the Reynolds number; [25]. Nevertheless, in presence of buoyancy effects, the kinetic energy is also influenced by changes on the fluid temperature. Further, the thermal energy is affected by the temperatures driven by the fluid at the outlet. This scenario requires particular attention on the relation between the velocity and the temperature at the open boundary. In this work, we propose a nonlinearly coupled boundary condition for the heat transfer at the outlet that is derived from physical assumptions which are consistent with the restrictions on the backflows imposed by the modified do-nothing conditions. In particular, this boundary condition allows to find energy estimates. To the authors' knowledge, the only precedent work involving coupled boundary conditions for the heat transfer at the open boundary is due to Pérez et al., see [30]. In that work, the authors consider an *ad-hoc* modification of a *zero flux density condition* at the outlet, which produces the coupled condition, and allows them to prove existence and uniqueness of weak solutions to a 2D steady Boussinesq system with variable thermophysical parameters. The zero flux density condition establishes pure convective heat flux at the outlet, and it was also used in several articles dealing with the Boussinesq equations in domains with open boundaries, see e.g. [3–5, 7, 8].

Chan and Tien performed numerical and experimental investigations for the non-isothermal fluid flow in an open cavity, see [14, 15]. They considered a domain with a heated vertical inlet, horizontal insulated walls, a vertical outlet, and ambient temperature equal to zero. Their results show positive temperatures at the outlet when the fluid is leaving the domain. For heat transfer processes dominated by conduction (i.e. for systems with a Rayleigh number Ra below the critical value), they also find this feature when the fluid is re-entering. The Figure 9 presented in [14] exhibits the aforementioned fluid behavior. These results suggest the existence of a conductive heat transfer process at the open boundary, which affects the temperatures driven by the incoming and outgoing fluid. The boundary condition proposed here aims to consider this phenomena.

The article is organized as follows. In Section 2, we derive a coupled boundary condition for the heat transfer at the outlet, and present the two Boussinesq systems under study: the problem ($\mathbb{P}$), which includes a perturbed do-nothing condition; and the problem ($\mathbb{Q}$), in which a bound for the backflow is assumed. Section 3 is devoted to introduce notation and known results. Finally, Sections 4 and 5 are dedicated to the study of problems ($\mathbb{P}$) and ($\mathbb{Q}$), respectively. We present variational formulations for the two problems, and prove existence of weak solutions under some restrictions on data.

2 The problems ($\mathbb{P}$) and ($\mathbb{Q}$)

The room is represented by a a simply connected bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, where $d = 2$ or $d = 3$. The velocity field $\mathbf{v} : \Omega \to \mathbb{R}^d$, the temperature $u : \Omega \to \mathbb{R}$, and the pressure $p : \Omega \to \mathbb{R}$ of the fluid, are assumed to satisfy the steady Boussinesq equations on Ω,

$$\begin{aligned} &(1) \qquad \mathbf{v}\cdot\nabla\mathbf{v} - \frac{1}{\mathrm{Re}}\Delta\mathbf{v} + \nabla p = \frac{\mathrm{Gr}}{\mathrm{Re}^2}u\mathbf{e} + \mathbf{f}_1 \\ &(2) \qquad \operatorname{div}\mathbf{v} = 0 \\ &(3) \qquad \mathbf{v}\cdot\nabla u - \frac{1}{\mathrm{RePr}}\Delta u = f_2, \end{aligned} \qquad (\mathbb{B})$$

where $\mathbf{f}_1 : \Omega \to \mathbb{R}^d$ and $f_2 : \Omega \to \mathbb{R}$ are external body forces, $\mathbf{e}$ is the unit normal in the vertical direction, and Re, Pr, Gr are the Reynolds, Prandtl, and Grashof numbers. They represent the ratio between inertial to viscous forces, kinematic viscosity to thermal diffusivity, and buoyancy to viscous forces, respectively [27]. Low values of Re are associated to laminar flows, and high values corresponds to turbulent regimes. Small values of Pr express the heat transfer is primarily due to conductive transport, while large Pr means that convective transport dominates. Finally, Gr has a similar meaning to Re, and the ratio between Gr and Re^2 gives a measure of the relative influence of natural and forced convection. The former is negligible when $\mathrm{Gr} \ll \mathrm{Re}^2$, and the latter when $\mathrm{Gr} \gg \mathrm{Re}^2$.

We consider two decompositions of $\partial\Omega$:

$$\mathcal{D}_1 = \{\partial\Omega_{in}, \partial\Omega_{walls}, \partial\Omega_{out}\}, \qquad \mathcal{D}_2 = \{\partial\Omega_D, \partial\Omega_N, \partial\Omega_{out}\}.$$

The sets involved in each of them are assumed to be pairwise disjoint open subsets of $\partial\Omega$, all of them of non-zero measure, the union of their closures being the whole boundary $\partial\Omega$. Further, we assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are related by

$$\partial\Omega_{in} \subset \partial\Omega_D, \qquad \partial\Omega_N \subset \partial\Omega_{walls}.$$

We define $\partial\Omega_{out}^c = \partial\Omega_D \cup \partial\Omega_{walls}$. Figure 2 illustrates the decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$ for a two-dimensional room with a heated floor (see Figure 1).

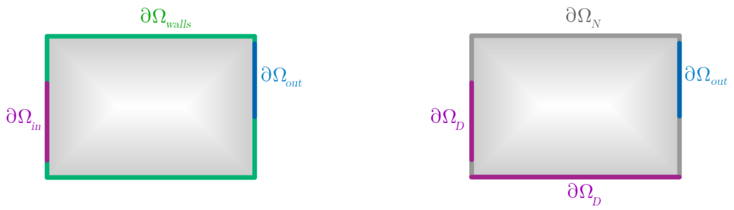

Figure 2: Decompositions $\mathcal{D}_1$ (left) and $\mathcal{D}_2$ (right) of the boundary $\partial\Omega$ for a 2D room with a heated floor (see Figure1).

Dirichlet conditions are considered for $\mathbf{v}$ and u on the inlet, and for u on the heated part of the boundary. On the walls, we use a Neumann condition for u to represent insulation, and a non-slip condition for $\mathbf{v}$. We write

$$\begin{aligned} &\mathbf{v} = \mathbf{v}_{in} \quad \text{on } \partial\Omega_{in}, && u = u_D \quad \text{on } \partial\Omega_D, \\ &\mathbf{v} = 0 \quad \text{on } \partial\Omega_{walls}, && \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega_N, \end{aligned} \qquad (\mathbb{BC}_{\partial\Omega_{out}^c})$$

where $\mathbf{v}_{in} : \partial\Omega_{in} \to \mathbb{R}^d$, $u_D : \partial\Omega_D \to \mathbb{R}$ are given functions, and $\mathbf{n}$ is the outer normal to the boundary.

As it was already mentioned, the unrestricted fluid flow across the outlet will be represented in the spirit of the do-nothing condition,

$$\frac{1}{\mathrm{Re}}\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n} = 0 \quad \text{on } \partial\Omega_{out}. \qquad (\mathrm{DN})$$

First, we follow the idea introduced by Bruneau and Fabrie in [12], see also [2, 11, 17, 28]. It consists in perturbing (DN) according to

$$\frac{1}{\mathrm{Re}}\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n} + \frac{1}{2}\mathbf{v}(\mathbf{v}\cdot\mathbf{n})_- = 0 \quad \text{on } \partial\Omega_{out}, \qquad (\mathrm{DDN})$$

where $(\mathbf{v}\cdot\mathbf{n})_-$ denotes the negative part of $\mathbf{v}\cdot\mathbf{n}$. Since the perturbation is mainly related with the influence of the incoming flows, it has received the name of "directional do-nothing" condition [11]. When there is no backflow, i.e. $(\mathbf{v}\cdot\mathbf{n})_- = \mathbf{0}$, (DDN) reduces to (DN). By contrast, in presence of incoming flows, its influence is stabilized by the term $\frac{1}{2}\mathbf{v}(\mathbf{v}\cdot\mathbf{n})_-$. We refer to [11] for a detailed discussion about the directional do-nothing condition (DDN). Second, we follow the idea introduced by Kračmar and Neustupa in [24], see also [25,26]. In this case, the do-nothing condition is supplemented with a bound for the incoming flows across the outlet,

$$\frac{1}{\mathrm{Re}}\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n} = 0 \quad \text{on } \partial\Omega_{out}, \qquad \int_{\partial\Omega_{out}} |(\mathbf{v}\cdot\mathbf{n})_-|^a d\sigma \le c_0, \qquad (\mathrm{DN{+}Bound})$$

where $a \in (2, 4)$, and $c_0 \ge 0$. When there is no backflow, the integral condition becomes superfluous, so only the do-nothing condition is imposed on $\partial\Omega_{out}$.

The heat transfer at the outlet might be affected by three mechanisms: conduction, natural convection (due to buoyancy effects), and advection (due to the incoming and outgoing fluid). The experimental and numerical results reported by Chan and Tien in [14, 15] suggest the existence of mutually dependent conductive and advective processes. These authors considered the heat transfer problem for an open cavity with insulated horizontal walls, a heated inlet opposite to the opening, and ambient temperature equal to zero. The numerical results from [15], show that the outlet can present positive temperatures, even when the fluid is re-entering the domain (see Figure 9 in [15]). Further, from the experimental investigations reported in [14], the authors conclude that the fluid flow through the open boundary is affected by the heating conditions inside and outside the domain. We will now derive a boundary condition, aiming to include this phenomena. Along the derivation, the outside ambient temperature is assumed to be given by a constant value u_∞ (which is not necessarily zero).

We make the following assumption *at the outlet*:

$(\mathrm{H1})$ Natural convection is negligible.

This hypothesis $(\mathrm{H1})$ is plausible in geometries with vertical outlets, or for systems with $\mathrm{Gr} \ll \mathrm{Re}^2$, and implies that the heat transfer at the outlet is mainly due to conductive and advectice transport.

From now on, starred letters will denote dimensional quantities. We denote the thermal conductivity by k^*, the specific heat capacity by c^*, and the density by ρ^*. We recall that we are considering k^* and c^* as constants. Further, we will assume that ρ^* is constant at the outlet, by virtue of assumption $(\mathrm{H1})$.

Let $\partial\Omega_{out}^{*+}$ be the subset of $\partial\Omega_{out}^{*}$ where $u^* > u^*_\infty$. The heat transfer process at the outlet generates a "mushy region" outside $\partial\Omega_{out}^{*+}$, with temperatures between u^*_∞ and u^*. The influence of this region on the heat transfer at the outlet will be represented by an "effective temperature" transported by the fluid across the opening. We denote this effective temperature as u^*_e, and define it as a weighted sum of the ambient and outlet temperatures, according to a velocity dependent parameter that distinguishes the fluid flow direction. Further, we limit the influence of the incoming and outgoing flows on the heat transfer at the outlet, by giving a more relative importance to the temperatures in the flow direction. More precisely, we set

$$u^*_e = u^*\beta(\mathbf{v}^* \cdot \mathbf{n}^*) + u^*_\infty(1 - \beta(\mathbf{v}^* \cdot \mathbf{n}^*)) \qquad \text{on } \partial\Omega_{out}^{*+}, \tag{1}$$

where $\beta : \mathbb{R} \to \mathbb{R}$ is a function with the properties

a) $\beta(s) \in [0, 1/2]$ if $s \geq 0$, and $\beta(s) \in [1/2, 1]$ if $s < 0$.

(2)

b) β is continuous, except maybe at the origin.

Thus, the velocity dependent weights $\beta(\mathbf{v}^* \cdot \mathbf{n}^*)$ and $(1 - \beta(\mathbf{v}^* \cdot \mathbf{n}^*))$ in (1) act in the following way. When the fluid is leaving the domain ($\mathbf{v}^* \cdot \mathbf{n}^* > 0$), the effective temperature u^*_e is given by a convex sum of the outlet and the external temperatures, where the latter dominates. Similarly, when the fluid is re-entering ($\mathbf{v}^* \cdot \mathbf{n}^* < 0$), u^*_e is defined as the above weighted average while dominated by the temperature at the outlet. The effective temperature on the part of the boundary where $u^* < u^*_\infty$ is defined in a similar manner. Finally, the part of the outlet where $u^* = u^*_\infty$ does not exhibit effective surrounding mushy region, thus we have $u^*_e = u^*_\infty$.

The heat flux at the outlet due to the advective transport of the effective temperature is then given by

$$\mathrm{q}^*_{\mathbf{n},\mathrm{adv}} = \rho^* c^* u^*_e(\mathbf{v}^* \cdot \mathbf{n}^*) = \rho^* c^* \left(u^*\beta(\mathbf{v}^* \cdot \mathbf{n}^*) + u^*_\infty(1 - \beta(\mathbf{v}^* \cdot \mathbf{n}^*))\right)(\mathbf{v}^* \cdot \mathbf{n}^*). \tag{3}$$

We use the Fourier's law to represent the conductive heat transfer in the normal direction to the outlet, so it is defined by

$$\mathrm{q}^*_{\mathbf{n},\mathrm{cond}} = -k^* \frac{\partial u^*}{\partial \mathbf{n}^*}. \tag{4}$$

Then, the *effective* heat flux normal to the outlet is given by $\mathrm{q}^*_{\mathbf{n}} = \mathrm{q}^*_{\mathbf{n},\mathrm{adv}} + \mathrm{q}^*_{\mathbf{n},\mathrm{cond}}$. The second assumption that we make *at the outlet* is the following:

$(\mathrm{H2})$ The effective heat flux has vanishing average on the open boundary.

This is a consequence of assuming that a local energy balance holds at any neighborhood of the outlet, which is consistent with the restrictions imposed before on the backflow.

According to the assumption $(\mathrm{H2})$, we can write

$$\int_{\partial\Omega^*_{out}} \mathrm{q}^*_{\mathbf{n}} \, d\sigma^* = 0. \tag{5}$$

Taking into consideration the definitions of $\mathrm{q}^*_{\mathbf{n},\mathrm{adv}}$ and $\mathrm{q}^*_{\mathbf{n},\mathrm{cond}}$ given by (3) and (4), we write (5) as

$$\int_{\partial\Omega^*_{out}} \left(-k^* \frac{\partial u^*}{\partial \mathbf{n}^*} + \rho^* c^* \left(u^* \beta(\mathbf{v}^* \cdot \mathbf{n}^*) + u^*_\infty (1 - \beta(\mathbf{v}^* \cdot \mathbf{n}^*)) \right) (\mathbf{v}^* \cdot \mathbf{n}^*) \right) d\sigma^* = 0. \tag{6}$$

Thus, we consider the following boundary condition for the heat transfer at the outlet:

$$k^* \frac{\partial u^*}{\partial \mathbf{n}^*} - \rho^* c^* \left(u^* \beta(\mathbf{v}^* \cdot \mathbf{n}^*) + u^*_\infty (1 - \beta(\mathbf{v}^* \cdot \mathbf{n}^*)) \right) (\mathbf{v}^* \cdot \mathbf{n}^*) = f^* \qquad \text{on } \partial\Omega^*_{out},$$

where f^* is a given function with vanishing average on $\partial\Omega^*_{out}$, which depends on the particular problem under study.

For simplicity, from now on we assume that $u^*_\infty = 0$ and $f^* = 0$. Then, in dimensionless form, the above condition becomes

$$\frac{1}{\mathrm{RePr}} \frac{\partial u}{\partial \mathbf{n}} - u\beta(\mathbf{v} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}) = 0 \qquad \text{on } \partial\Omega_{out}. \tag{HT}$$

This is the boundary condition of interest in our framework.

Example 2.1. *The simplest case of such boundary condition is given by what follows. If β is the piecewise constant function defined by*

$$\beta(s) = \begin{cases} 1/2 + \epsilon & \text{if} \quad s < 0 \\ 1/2 - \epsilon^\star & \text{if} \quad s \geq 0, \end{cases}$$

with $\epsilon, \epsilon^\star \in [0, 1/2]$, then the boundary condition (HT) *becomes*

$$\frac{1}{\mathrm{RePr}} \frac{\partial u}{\partial \mathbf{n}} + u \left(\frac{1}{2} + \epsilon \right) (\mathbf{v} \cdot \mathbf{n})_- - u \left(\frac{1}{2} - \epsilon^\star \right) (\mathbf{v} \cdot \mathbf{n})_+ = 0 \qquad \text{on } \partial\Omega_{out},$$

where $(\mathbf{v} \cdot \mathbf{n})_+$ and $(\mathbf{v} \cdot \mathbf{n})_-$ denote the positive and negative parts of $\mathbf{v} \cdot \mathbf{n}$. Values of ϵ close to 1/2 means that the effective temperature is close to the outlet temperature when the fluid is re-entering the domain. Something similar occurs for values of $\epsilon^\star$ close to 0. Thus, such values are associated to the mildest effects that the boundary condition (HT) *allows for the incoming and outgoing flows on the advective heat transport at the open boundary.*

Summarizing, we will study the following two problems ($\mathbb{P}$) and ($\mathbb{Q}$):

$$\text{First model:} \qquad (\mathbb{B}) + (\mathbb{BC}_{\partial\Omega^c_{out}}) + (\mathbb{BC}^{\mathrm{I}}_{\partial\Omega_{out}}) \tag{$\mathbb{P}$}$$

$$\text{Second model:} \qquad (\mathbb{B}) + (\mathbb{BC}_{\partial\Omega^c_{out}}) + (\mathbb{BC}^{\mathrm{II}}_{\partial\Omega_{out}}), \tag{$\mathbb{Q}$}$$

where

$$(\mathbb{BC}^{\mathrm{I}}_{\partial\Omega_{out}}) = (\mathrm{DDN}) + (\mathrm{HT}), \qquad\qquad (\mathbb{BC}^{\mathrm{II}}_{\partial\Omega_{out}}) = (\mathrm{DN+Bound}) + (\mathrm{HT}).$$

A few words on the coupled boundary condition considered by Pérez et al. in [30] are in order. In that work, the authors consider a 2D steady Boussinesq system with variable thermophysical properties in a domain with an open boundary. The conditions at the outlet are given by a do-nothing-type condition for the fluid flow, and a *zero density flux* condition for the heat transfer. By following a fixed point strategy, the authors prove well-posedness for a *modified* variational formulation of the original problem (provided a smallness-type condition on data holds). When the thermophysical properties are constant, the alternative variational formulation is formally associated with the directional do-nothing condition (DDN), and the following coupled condition for the heat transfer at the outlet:

$$\frac{1}{\mathrm{RePr}}\frac{\partial u}{\partial \mathbf{n}}+\frac{1}{2}u(\mathbf{v}\cdot\mathbf{n})_-=0 \qquad \partial\Omega_{out}. \tag{HT*}$$

The latter is a special case of (HT), corresponding to the function β given in the Example 2.1, with $\epsilon=0$ and $\epsilon^\star=1/2$. According to the above discussion, the effective temperature driven by the fluid across the outlet is then defined by: a) the average between the ambient and outlet temperatures when the fluid is re-entering the domain (this corresponds to $\epsilon=0$), b) the outside ambient temperature when the fluid is going outside (this corresponds to $\epsilon^\star=1/2$). We recall that the latter is associated to the mildest effects of the ouflows on the advective heat transfer process that condition (HT) allows.

When $\mathrm{Gr}=0$, the unknowns $\mathbf{v}$ and u in the Boussinesq equations ($\mathbb{B}$), are decoupled. For this special case, the analysis of problems ($\mathbb{P}$) or ($\mathbb{Q}$) reduces to study an incompressible Navier-Stokes system for the velocity and the pressure, and a convection-diffusion system for the temperature. The Navier-Stokes systems associated to each problem are, precisely, the systems studied in the articles [11] and [25], which motivate the present work. By contrast, when $\mathrm{Gr}\neq 0$, the flow under study takes into consideration buoyancy effects coming from changes in the temperature distribution, and the unknowns $\mathbf{v}$, u in the Boussinesq equations ($\mathbb{B}$) cannot be treated separately.

3 General considerations

Norms in L^p and $W^{m,p}$ on the domain Ω will be denoted by $||\cdot||_p$ and $||\cdot||_{m,p}$, respectively. When considering them on the boundary $\partial\Omega$, or on some part of it, the domain will be specifically indicated (e.g., $||\cdot||_{p,\partial\Omega_{out}}$ or $||\cdot||_{m,p,\partial\Omega_D}$). The inner product in L^2 will be denoted by $(\cdot,\cdot)$.

The weak solutions to problems ($\mathbb{P}$) and ($\mathbb{Q}$) will be obtained in the form

$$\mathbf{v}=\mathbf{v}_0+\mathbf{V}, \qquad\qquad u=u_0+U,$$

where $\mathbf{V}:\Omega\to\mathbb{R}^d$, $U:\Omega\to\mathbb{R}$ are known functions such that

$$\begin{aligned} &\mathbf{V}=\mathbf{v}_{in} \text{ a.e. on } \partial\Omega_{in}, && \mathbf{V}=0 \text{ a.e. on } \partial\Omega_{walls},\\ &U=u_D \text{ a.e. on } \partial\Omega_D, && \operatorname{div}\mathbf{V}=0 \text{ a.e. in } \Omega. \end{aligned} \tag{7}$$

The following lemma establishes conditions on $\mathbf{v}_{in}$, u_D, and Ω that ensure the existence of the functions $\mathbf{V}$ and U. For the proof of the first part, we refer to [25]. The second part is a well known result, see e.g. [1].

Lemma 3.1.

1. If $\mathbf{v}_{in}$ can be extended to $\partial\Omega$ by a function $\mathbf{v}_{in}^{ext}\in W^{1/2,2}(\partial\Omega)^d$ that satisfies

$$\int_{\partial\Omega}\mathbf{v}_{in}^{ext}\cdot\mathbf{n}\,d\sigma=0, \quad \mathbf{v}_{in}^{ext}=0 \quad \text{on } \partial\Omega_{walls}\cup\{\partial\Omega_{out}\setminus\mathcal{S}\}, \quad \text{and} \quad \mathbf{v}_{in}=k\mathbf{n} \quad \text{on } \mathcal{S}, \tag{8}$$

for some non-negative scalar function k, and some open set $\mathcal{S}$ contained in $\partial\Omega_{out}$, then there exists a vector function $\mathbf{V}\in W^{1,2}(\Omega)^d$ which satisfies the first three conditions in (7). Moreover, $\mathbf{V}$ can be chosen with the following properties:

a. $\displaystyle\int_{\partial\Omega_{out}} (\mathbf{V}\cdot\mathbf{n})_- d\sigma = 0,$

b. for any divergence-free vector functions $\mathbf{w}, \widetilde{\mathbf{w}} \in W^{1,2}(\Omega)^d$ *with vanishing traces on* $\partial\Omega_{in} \cup \partial\Omega_{walls}$*, we observe*

$$|(\mathbf{w}\cdot\nabla\mathbf{V}, \widetilde{\mathbf{w}})| \leq \frac{1}{2\mathrm{Re}}||\nabla\mathbf{w}||_2\,||\nabla\widetilde{\mathbf{w}}||_2.$$

2. If $u_D \in W^{1/2,2}(\partial\Omega_D)$*, then there exists a function* $U \in W^{1,2}(\Omega)$ *which satisfies the last condition in (7). Moreover,*

$$||U||_{1,2} \leq c^*||u_D||_{1/2,2,\partial\Omega_D}, \tag{9}$$

for some positive constant $c^* = c^*(d, \partial\Omega_D)$*.*

We denote by V_1 and V_2 to the function spaces defined by

$$V_1 = \overline{E_1(\Omega)}^{W^{1,2}(\Omega)^d}, \qquad \text{and} \qquad V_2 = \overline{E_2(\Omega)}^{W^{1,2}(\Omega)}.$$

That is, V_1 and V_2 are the closure of $E_1(\Omega)$ and $E_2(\Omega)$ with respect to the norms of $W^{1,2}(\Omega)^d$ and $W^{1,2}(\Omega)$, respectively, and where

$$E_1(\Omega) = \left\{\mathbf{w}\in C^\infty(\overline{\Omega})^d \,:\, \operatorname{div}\mathbf{w} = 0,\, \overline{\operatorname{supp}\mathbf{w}} \cap \{\partial\Omega_{in}\cup\partial\Omega_{walls}\} = \emptyset\right\}$$

$$E_2(\Omega) = \left\{w \in C^\infty(\overline{\Omega}) \,:\, \overline{\operatorname{supp} w} \cap \partial\Omega_D = \emptyset\right\},$$

and $C^\infty(\overline{\Omega})$ is the set of restrictions to $\overline{\Omega}$ of infinitely differentiable functions $C^\infty(\mathbb{R}^d)$. We recall that V_1 and V_2 are Hilbert spaces with the inner product $((\cdot,\cdot))$ defined by

$$((\varphi,\psi)) = \int_\Omega \nabla\varphi\cdot\nabla\psi\, dx,$$

where φ, ψ belongs to either V_1 or V_2. The norm induced by $((\cdot,\cdot))$ on V_1 and V_2 will be denoted as $||\cdot||$. Under certain regularity assumptions on $\partial\Omega$, $\partial\Omega_{in}\cup\partial\Omega_{walls}$, and $\partial\Omega_D$, the spaces V_1 and V_2 admit the characterizations

$$\begin{aligned} V_1 &= \left\{\mathbf{w}\in W^{1,2}(\Omega)^d \,:\, \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega,\ \mathbf{w}|_{\partial\Omega_{in}\cup\partial\Omega_{walls}} = 0 \text{ in the trace sense}\right\}, \\ V_2 &= \left\{w\in W^{1,2}(\Omega) \,:\, w|_{\partial\Omega_D} = 0 \text{ in the trace sense}\right\}. \end{aligned} \tag{10}$$

The density result implying the equivalence of both definitions of V_2 was proved in [16], and the class of the domains Ω for which it holds is denoted as $\widetilde{C}^{0,1}$, a subset of the Lipschitz domain class $C^{0,1}$. The class $\widetilde{C}^{0,1}$ involves domains Ω which Lipschitz boundary consisting on a finite number of smooth parts with finite number of relative maxima, minima, and inflexion points, and in three dimensions, also a finite number of saddle points. Additionally, the Dirichlet boundary $\partial\Omega_D$ consists on a finite number of of relative open parts in $\partial\Omega$, that in the case $d = 3$ possess a projective boundary Lipschitz regularity condition; see [16]. The analogous result for V_1 follows under identical assumptions over Ω. We assume throughout the paper that the domain Ω is of class $\widetilde{C}^{0,1}$.

Following standard notation from the literature of Navier-Stokes equations, we also introduce the tri-linear forms b_1 and b_2, defined by

$$b_1(\mathbf{w}, \widetilde{\mathbf{w}}, \widehat{\mathbf{w}}) = \int_\Omega (\mathbf{w} \cdot \nabla \widetilde{\mathbf{w}}) \cdot \widehat{\mathbf{w}}\, dx \qquad \mathbf{w}, \widetilde{\mathbf{w}}, \widehat{\mathbf{w}} \in W^{1,2}(\Omega)^d$$

$$b_2(\mathbf{w}, \widetilde{w}, \widehat{w}) = \int_\Omega (\mathbf{w} \cdot \nabla \widetilde{w}) \widehat{w}\, dx \qquad \mathbf{w} \in W^{1,2}(\Omega)^d, \widetilde{w}, \widehat{w} \in W^{1,2}(\Omega).$$

We recall that the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ and Hölder's inequality, yield

$$|b_1(\mathbf{w}, \widetilde{\mathbf{w}}, \widehat{\mathbf{w}})| \leq ||\mathbf{w}||_4 ||\nabla \widetilde{\mathbf{w}}||_2 ||\widehat{\mathbf{w}}||_4, \qquad |b_2(\mathbf{w}, \widetilde{w}, \widehat{w})| \leq ||\mathbf{w}||_4 ||\nabla \widetilde{w}||_2 ||\widehat{w}||_4, \tag{11}$$

for any $\mathbf{w}, \widetilde{\mathbf{w}}, \widehat{\mathbf{w}} \in W^{1,2}(\Omega)^d$ and $\widetilde{w}, \widehat{w} \in W^{1,2}(\Omega)$. Then, the forms b_1 and b_2 are well defined on the aforementioned spaces. In particular, when $\widetilde{\mathbf{w}} \in W^{1,2}(\Omega)^d$ is divergence-free, and $\mathbf{w} \in W^{1,2}(\Omega)^d$, $w \in W^{1,2}(\Omega)$, the forms b_1, b_2 satisfy

$$b_1(\widetilde{\mathbf{w}}, \mathbf{w}, \mathbf{w}) = \frac{1}{2} \int_{\partial\Omega} |\mathbf{w}|^2 (\widetilde{\mathbf{w}} \cdot \mathbf{n}) d\sigma, \qquad b_2(\widetilde{\mathbf{w}}, w, w) = \frac{1}{2} \int_{\partial\Omega} w^2 (\widetilde{\mathbf{w}} \cdot \mathbf{n}) d\sigma. \tag{12}$$

Finally, the letters c, $\widetilde{c}$ and $\widehat{c}$ will denote positive constants such that

$$||\varphi||_{1,2} \leq c||\varphi||, \qquad ||\psi||_4 \leq \widetilde{c}||\psi||_{1,2}, \qquad ||\psi||_{q,\partial\Omega_{out}} \leq \widehat{c}||\psi||_{1,2} \quad (q = 2, 4),$$

for any φ which belongs to either V_1 or V_2, and any ψ which belongs to either $W^{1,2}(\Omega)^d$ or $W^{1,2}(\Omega)$. Their existence is ensured by the embeddings $V_1 \hookrightarrow W^{1,2}(\Omega)^d$ and $V_2 \hookrightarrow W^{1,2}(\Omega)$ (existence of $c > 0$), $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ (existence of $\widetilde{c} > 0$), and $W^{1,2}(\Omega) \hookrightarrow L^q(\partial\Omega_{out})$ for $1 \leq q \leq 4$ (existence of $\widehat{c} > 0$). Moreover, c, $\widetilde{c}$ and $\widehat{c}$ can be chosen to be such that $c = c(d, \Omega)$, $\widetilde{c} = \widetilde{c}(d, \Omega)$, and $\widehat{c} = \widehat{c}(d, \partial\Omega_{out})$.

We consider the following assumptions on the data of the problem:

$$\mathbf{f}_1 \in L^2(\Omega)^d, \quad f_2 \in L^2(\Omega), \quad \mathbf{v}_{in} \in W^{1/2,2}(\partial\Omega_{in})^d, \quad u_D \in W^{1/2,2}(\partial\Omega_D). \tag{A}$$

There exists an extention $\mathbf{v}_{in}^{ext} \in W^{1,2}(\partial\Omega)^d$ of $\mathbf{v}_{in}$, with the properties (8).

4 Weak solutions to problem $(\mathbb{P})$

Throughout this section, we assume that (A) holds true. Multiplying equations (1) and (3) of $(\mathbb{B})$ by test functions $\mathbf{w} \in V_1$ and $w \in V_2$, respectively, integrating both expressions over Ω, and taking into consideration the boundary conditions $(\mathbb{BC}^{\mathrm{I}}_{\partial\Omega_{out}})$, we find

$$\int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \mathbf{w} dx + \frac{1}{\mathrm{Re}} \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{w} dx + \frac{1}{2} \int_{\partial\Omega_{out}} (\mathbf{v} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{n})_- d\sigma = \frac{\mathrm{Gr}}{\mathrm{Re}^2} \int_\Omega u\mathbf{e} \cdot \mathbf{w} dx + \int_\Omega \mathbf{f}_1 \cdot \mathbf{w} dx,$$

$$\int_\Omega (\mathbf{v} \cdot \nabla u) w dx + \frac{1}{\mathrm{RePr}} \int_\Omega \nabla u \cdot \nabla w dx - \int_{\partial\Omega_{out}} uw\beta(\mathbf{v} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}) d\sigma = \int_\Omega f_2 w dx.$$

Thus, we consider the following weak formulation for problem ($\mathbb{P}$):

Problem ($\mathbb{P}_{\mathrm{w}}$): Find $(\mathbf{v}_0, u_0) \in V_1 \times V_2$ such that

$$\begin{aligned} & b_1(\mathbf{v},\mathbf{v},\mathbf{w}) + \frac{1}{\mathrm{Re}}((\mathbf{v},\mathbf{w})) - \frac{\mathrm{Gr}}{\mathrm{Re}^2}(u\mathbf{e},\mathbf{w}) + \frac{1}{2}\int_{\partial\Omega_{out}} (\mathbf{v}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{n})_- d\sigma = (\mathbf{f}_1,\mathbf{w}), \\ & \text{for all } \mathbf{w} \in V_1, \text{ and} \\ & b_2(\mathbf{v},u,w) + \frac{1}{\mathrm{RePr}}((u,w)) - \int_{\partial\Omega_{out}} uw\beta(\mathbf{v}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})d\sigma = (f_2,w), \\ & \text{for all } w \in V_2, \text{ where } \mathbf{v} = \mathbf{v}_0 + \mathbf{V}, \text{ and } u = u_0 + U. \end{aligned} \tag{$\mathbb{P}_{\mathrm{w}}$}$$

The existence of solutions to ($\mathbb{P}_{\mathrm{w}}$) is proved via Galerkin's method. Let $\mathcal{B}_1 = \{\mathbf{w}_1, \mathbf{w}_2, \dots\}$ and $\mathcal{B}_2 = \{w_1, w_2, \dots\}$ be orthonormal basis for V_1 and V_2, respectively. For each $n \in \mathbb{N}$, let V_1^n and V_2^n be the finite-dimensional spaces spanned by $\mathcal{B}_1^n = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ and $\mathcal{B}_2^n = \{w_1, w_2, \dots, w_n\}$, respectively. We consider the following finite dimensional problem associated to ($\mathbb{P}_{\mathrm{w}}$):

Problem ($\mathbb{P}_{\mathrm{w}}^n$): Find $(\mathbf{v}_n, u_n) \in V_1^n \times V_2^n$ such that

$$\begin{aligned} & b_1(\mathbf{v}^n,\mathbf{v}^n,\mathbf{w}) + \frac{1}{\mathrm{Re}}((\mathbf{v}^n,\mathbf{w})) - \frac{\mathrm{Gr}}{\mathrm{Re}^2}(u^n\mathbf{e},\mathbf{w}) + \frac{1}{2}\int_{\partial\Omega_{out}} (\mathbf{v}^n\cdot\mathbf{w})(\mathbf{v}^n\cdot\mathbf{n})_- d\sigma = (\mathbf{f}_1,\mathbf{w}) \\ & \text{for all } \mathbf{w} \in V_1^n, \text{ and} \\ & b_2(\mathbf{v}^n,u^n,w) + \frac{1}{\mathrm{RePr}}((u^n,w)) - \int_{\partial\Omega_{out}} u^n w\beta(\mathbf{v}^n\cdot\mathbf{n})(\mathbf{v}^n\cdot\mathbf{n})d\sigma = (f_2,w) \\ & \text{for all } w \in V_2^n, \text{ where } \mathbf{v}^n = \mathbf{v}_n + \mathbf{V}, \text{and } u^n = u_n + U. \end{aligned} \tag{$\mathbb{P}_{\mathrm{w}}^n$}$$

Since the variational equations in ($\mathbb{P}_{\mathrm{w}}^n$) are linear on $\mathbf{w} \in V_1^n$ and $w \in V_2^n$, the representations

$$\mathbf{w} = \sum_{k=1}^{n} a_{1k}\mathbf{w}_k, \qquad \text{and} \qquad w = \sum_{k=1}^{n} a_{2k}w_k, \tag{13}$$

with $a_{11}, \cdots, a_{1n} \in \mathbb{R}$ and $a_{21} \cdots, a_{2n} \in \mathbb{R}$ enable us to write the following

Equivalent formulation of problem ($\mathbb{P}_{\mathrm{w}}^n$): Find $(\mathbf{v}_n, u_n) \in V_1^n \times V_2^n$ such that

$$\begin{aligned} b_1(\mathbf{v}^n,\mathbf{v}^n,\mathbf{w}_k) + \frac{1}{\mathrm{Re}}((\mathbf{v}^n,\mathbf{w}_k)) - \frac{\mathrm{Gr}}{\mathrm{Re}^2}(u^n\mathbf{e},\mathbf{w}_k) + \frac{1}{2}\int_{\partial\Omega_{out}} (\mathbf{v}^n\cdot\mathbf{w}_k)(\mathbf{v}^n\cdot\mathbf{n})_- d\sigma \\ = (\mathbf{f}_1,\mathbf{w}_k) \end{aligned}$$

and

$$b_2(\mathbf{v}^n,u^n,w_k) + \frac{1}{\mathrm{RePr}}((u^n,w_k)) - \int_{\partial\Omega_{out}} u^n w_k\beta(\mathbf{v}^n\cdot\mathbf{n})(\mathbf{v}^n\cdot\mathbf{n})d\sigma = (f_2,w_k),$$

for all $k = 1, \dots, n$, where $\mathbf{v}^n = \mathbf{v}_n + \mathbf{V}$, $u^n = u_n + U$.

At this point, we observe that the existence of solutions to problem ($\mathbb{P}^n_{\rm w}$) is equivalent to the existence of zeros of the operator $P : V_1^n \times V_2^n \to V_1^n \times V_2^n$ defined by

$$P(\mathbf{w}, w) = (\mathbf{P}_1(\mathbf{w}, w), P_2(\mathbf{w}, w)),$$

where

$$\begin{aligned}\mathbf{P}_1(\mathbf{w}, w) = &\sum_{k=1}^{n} \Big\{ -(\mathbf{f}_1, \mathbf{w}_k) + b_1(\mathbf{w}+\mathbf{V}, \mathbf{w}+\mathbf{V}, \mathbf{w}_k) + \frac{1}{\mathrm{Re}}((\mathbf{w}+\mathbf{V}, \mathbf{w}_k)) \\ &+\frac{1}{2}\int_{\partial\Omega_{out}} ((\mathbf{w}+\mathbf{V})\cdot\mathbf{w}_k)\,((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})_-\,d\sigma - \frac{\mathrm{Gr}}{\mathrm{Re}^2}((w+U)\mathbf{e}, \mathbf{w}_k) \Big\} \mathbf{w}_k,\end{aligned} \tag{14}$$

and

$$\begin{aligned}P_2(\mathbf{w}, w) = &\sum_{k=1}^{n} \Big\{ -(f_2, w_k) + b_2(\mathbf{w}+\mathbf{V}, w+U, w_k) + \frac{1}{\mathrm{RePr}}((w+U, w_k)) \\ &- \int_{\partial\Omega_{out}} w_k(w+U)\beta((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})\,((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})\,d\sigma \Big\} w_k.\end{aligned} \tag{15}$$

In the light of this observation, we make use of the following result. The proof can be found in [18].

Lemma 4.1. *Let X be a finite dimensional Hilbert space with inner product $[\cdot,\cdot]$ and norm $[\cdot]$. Assume that $P : X \to X$ is a continuous mapping with the following property:*

$$[P(\xi), \xi] > 0 \text{ for all } \xi \in X \text{ such that } [\xi] = k, \text{ for some } k > 0. \tag{16}$$

Then there exists $\xi^* \in X$ such that $P(\xi^*) = 0$. Moreover, $[\xi^*] \le k$.

We note that $X = V_1^n \times V_2^n$ is a finite dimensional Hilbert space with the inner product $[\cdot,\cdot]$ defined by

$$[(\mathbf{w}, w), (\widetilde{\mathbf{w}}, \widetilde{w})] = ((\mathbf{w}, \widetilde{\mathbf{w}})) + ((w, \widetilde{w})). \tag{17}$$

Note that the map $s \mapsto \beta(s)s$ is continuous by hypothesis (see (2)-b), and additionally $|\beta(s)s| \le \|\beta\|_{L^\infty}|s|$. Then, it follows that $\Psi(s) = \beta(s)s$, the Nemytskii operator $\Psi : L^q(\partial\Omega_{out}) \to L^q(\partial\Omega_{out})$ is continuous for $1 \le q < +\infty$; see [32]. It follows that $P : X \to X$ is a continuous operator in the norm induced by $[\cdot,\cdot]$. Then, in order to apply Lemma 4.1, only remains to analyze the inner product $[P(\mathbf{w}, w), (\mathbf{w}, w)]$. Exploiting the representation given in (13) for $\mathbf{w} \in V_1$, and taking into consideration that $((\mathbf{w}_j, \mathbf{w}_k)) = \delta_{j,k}$ for all $j, k = 1, \cdots, n$, we find

$$\begin{aligned}((\mathbf{P}_1(\mathbf{w}, w), \mathbf{w})) = &\sum_{k=1}^{n}\sum_{j=1}^{n} a_{1j} \Big\{ -(\mathbf{f}_1, \mathbf{w}_k) + b_1(\mathbf{w}+\mathbf{V}, \mathbf{w}+\mathbf{V}, \mathbf{w}_k) + \frac{1}{\mathrm{Re}}((\mathbf{w}+\mathbf{V}, \mathbf{w}_k)) \\ &+\frac{1}{2}\int_{\partial\Omega_{out}} ((\mathbf{w}+\mathbf{V})\cdot\mathbf{w}_k)\,\{(\mathbf{w}+\mathbf{V})\cdot\mathbf{n}\}_-\,d\sigma - \frac{\mathrm{Gr}}{\mathrm{Re}^2}((w+U)\mathbf{e}, \mathbf{w}_k) \Big\} ((\mathbf{w}_k, \mathbf{w}_j)),\end{aligned}$$

that is

$$\begin{aligned}((\mathbf{P}_1(\mathbf{w}, w), \mathbf{w})) = &-(\mathbf{f}_1, \mathbf{w}) + b_1(\mathbf{w}+\mathbf{V}, \mathbf{w}+\mathbf{V}, \mathbf{w}) + \frac{1}{\mathrm{Re}}((\mathbf{w}+\mathbf{V}, \mathbf{w})) \\ &+ \frac{1}{2}\int_{\partial\Omega_{out}} \{(\mathbf{w}+\mathbf{V})\cdot\mathbf{w}\}\,((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})_-\,d\sigma - \frac{\mathrm{Gr}}{\mathrm{Re}^2}((w+U)\mathbf{e}, \mathbf{w}).\end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned}((P_2(\mathbf{w},w),w)) = &-(f_2,w) + b_2(\mathbf{w}+\mathbf{V}, w+U, w) + \frac{1}{\mathrm{RePr}}((w+U,w)) \\ &- \int_{\partial\Omega_{out}} w(w+U)\beta((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})\,((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})\,d\sigma.\end{aligned} \tag{19}$$

The following two Lemmas collect some preparatory results to estimate the quantity $[P(\mathbf{w},w),(\mathbf{w},w)]$.

Lemma 4.2. *For any $\mathbf{w}\in V_1$, $w \in V_2$, the forms b_1 and b_2 satisfy*

$$\begin{aligned}b_1(\mathbf{w}+\mathbf{V},\mathbf{w}+\mathbf{V},\mathbf{w}) \geq &-\frac{1}{2}\int_{\partial\Omega_{out}} |\mathbf{w}|^2\,((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})_-\,d\sigma \\ &-\left(\frac{1}{2\mathrm{Re}}||\mathbf{w}|| + c\,\widetilde{c}^{\,2}||\mathbf{V}||_{1,2}^2\right)||\mathbf{w}||,\end{aligned} \tag{20}$$

and

$$\begin{aligned}b_2(\mathbf{w}+\mathbf{V},w+U,w) \geq &\frac{1}{2}\int_{\partial\Omega_{out}} w^2\,((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})\,d\sigma \\ &- c\,\widetilde{c}^{\,2}\,(c||\mathbf{w}|| + ||\mathbf{V}||_{1,2})\,||U||_{1,2}||w||.\end{aligned} \tag{21}$$

Proof. To prove (20), we estimate separately $b_1(\mathbf{w}+\mathbf{V},\mathbf{w},\mathbf{w})$, $b_1(\mathbf{w},\mathbf{V},\mathbf{w})$ and $b_1(\mathbf{V},\mathbf{V},\mathbf{w})$. First, from $(12)_1$, we find

$$b_1(\mathbf{w}+\mathbf{V},\mathbf{w},\mathbf{w}) = \frac{1}{2}\int_{\partial\Omega_{out}} |\mathbf{w}|^2\,((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})\,d\sigma \geq -\frac{1}{2}\int_{\partial\Omega_{out}} |\mathbf{w}|^2\,((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})_-\,d\sigma.$$

Second, from Lemma 3.1-1(b), we obtain

$$b_1(\mathbf{w},\mathbf{V},\mathbf{w}) \geq -|b_1(\mathbf{w},\mathbf{V},\mathbf{w})| \geq -\frac{1}{2\mathrm{Re}}||\mathbf{w}||^2.$$

Finally, from (11), we find

$$b_1(\mathbf{V},\mathbf{V},\mathbf{w}) \geq -|b_1(\mathbf{V},\mathbf{V},\mathbf{w})| \geq -c\,\widetilde{c}^{\,2}||\mathbf{V}||_{1,2}^2||\mathbf{w}||. \tag{22}$$

Now, (20) follows from the linearity of $\widetilde{\mathbf{w}} \mapsto b_1(\widetilde{\mathbf{w}},\cdot,\cdot)$ and $\widetilde{\mathbf{w}} \mapsto b_1(\cdot,\widetilde{\mathbf{w}},\cdot)$. Similarly, to prove (21) we analyze separately $b_2(\mathbf{w}+\mathbf{V},w,w)$ and $b_2(\mathbf{w}+\mathbf{V},U,w)$. We note that $\mathbf{w}+\mathbf{V}$ has vanishing trace on $\partial\Omega_N$ since $\partial\Omega_N \subset \partial\Omega_{walls}$. Then, from $(12)_2$, we obtain

$$b_2(\mathbf{w}+\mathbf{V},w,w) = \frac{1}{2}\int_{\partial\Omega_{out}} w^2\,((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})\,d\sigma.$$

The Minkowski inequality and (11) yield

$$\begin{aligned}b_2(\mathbf{w}+\mathbf{V},U,w) &\geq -c\,\widetilde{c}^{\,2}||\mathbf{w}+\mathbf{V}||_{1,2}||U||_{1,2}||w|| \\ &\geq -c^2\,\widetilde{c}^{\,2}||U||_{1,2}||w||\,||\mathbf{w}|| - c\,\widetilde{c}^{\,2}||\mathbf{V}||_{1,2}||U||_{1,2}||w||.\end{aligned}$$

Hence (21) follows from the linearity of $\widetilde{w} \mapsto b_2(\cdot,\widetilde{w},\cdot)$. □

Lemma 4.3. *For any $\mathbf{w} \in V_1$, $w \in V_2$, the following inequalities hold true:*

$$\int_{\partial\Omega_{out}} (\mathbf{V}\cdot\mathbf{w})\left((\mathbf{w}+\mathbf{V})\cdot\mathbf{n}\right)_{-}\,d\sigma \geq -c\,\widehat{c}^{3}\left(c||\mathbf{w}|| + ||\mathbf{V}||_{1,2}\right)||\mathbf{V}||_{1,2}||\mathbf{w}||, \tag{23}$$

and

$$\int_{\partial\Omega_{out}} wU\beta((\mathbf{w}+\mathbf{V})\cdot\mathbf{n})\left((\mathbf{w}+\mathbf{V})\cdot\mathbf{n}\right)d\sigma \geq -\beta^{*}c\,\widehat{c}^{3}\left(c||\mathbf{w}|| + ||\mathbf{V}||_{1,2}\right)||U||_{1,2}||w||, \tag{24}$$

where $0 \leq \beta^{*} \leq 1$ denotes the L^{∞}-norm of β in $\mathbb{R}$.

Proof. It is a direct consequence of the Hölder and Minkowski inequalities. □

Lemma 4.4. *For any $(\mathbf{w}, w) \in X$, the following inequalities hold true:*

$$((\mathbf{P}_1(\mathbf{w},w),\mathbf{w})) \geq \frac{1}{2}\left(\frac{1}{\mathrm{Re}} - c^{2}\,\widehat{c}^{3}||\mathbf{V}||_{1,2}\right)||\mathbf{w}||^{2} - c^{2}\frac{\mathrm{Gr}}{\mathrm{Re}^{2}}||w||\,||\mathbf{w}|| - A_1||\mathbf{w}||, \tag{25}$$

and

$$((P_2(\mathbf{w},w),w)) \geq \frac{1}{\mathrm{RePr}}||w||^{2} - c^{2}c^{*}\left(\beta^{*}\,\widehat{c}^{3} + \widetilde{c}^{2}\right)||u_D||_{1/2,2,\partial\Omega_D}||w||\,||\mathbf{w}|| - A_2||w||, \tag{26}$$

where A_1 and A_2 are non-negative constants which depend on the data of problem $(\mathbb{P})$.

Proof. From (18) and (20), we find

$$\begin{aligned}((\mathbf{P}_1(\mathbf{w},w),\mathbf{w})) \geq &-\left(\frac{1}{2\mathrm{Re}}||\mathbf{w}|| + c\,\widetilde{c}^{2}||\mathbf{V}||_{1,2}^{2}\right)||\mathbf{w}|| + \frac{1}{\mathrm{Re}}||\mathbf{w}||^{2} + \frac{1}{\mathrm{Re}}((\mathbf{V},\mathbf{w}))\\ &+\int_{\partial\Omega_{out}}(\mathbf{V}\cdot\mathbf{w})\left((\mathbf{w}+\mathbf{V})\cdot\mathbf{n}\right)_{-}d\sigma - \frac{\mathrm{Gr}}{\mathrm{Re}^{2}}((w+U)\mathbf{e},\mathbf{w}) - (\mathbf{f}_1,\mathbf{w}).\end{aligned}$$

Combining this with (23), we find (25) with

$$A_1 = c\left(\frac{1}{\mathrm{Re}}||\mathbf{V}||_{1,2} + \left(\widetilde{c}^{2} + \frac{\widehat{c}^{3}}{2}\right)||\mathbf{V}||_{1,2}^{2} + \frac{\mathrm{Gr}}{\mathrm{Re}^{2}}||U||_{1,2} + ||\mathbf{f}_1||_2\right).$$

Similarly, from (19), (21), and (24), we find

$$((P_2(\mathbf{w},w),w)) \geq \frac{1}{\mathrm{RePr}}||w||^{2} - c^{2}\left(\beta^{*}\,\widehat{c}^{3} + \widetilde{c}^{2}\right)||U||_{1,2}||w||\,||\mathbf{w}|| - A_2||w||, \tag{27}$$

where

$$A_2 = c\left(\frac{1}{\mathrm{RePr}}||U||_{1,2} + \left(\widetilde{c}^{2} + \beta^{*}\,\widehat{c}^{3}\right)||U||_{1,2}||\mathbf{V}||_{1,2} + ||f_2||_2\right).$$

This, in combination with with (9), yields (26). Note that to derive (27), it was taken into account that

$$\begin{aligned}\int_{\partial\Omega_{out}} w^{2}\left(\frac{1}{2} - \beta(\widetilde{\mathbf{w}}\cdot\mathbf{n})\right)(\widetilde{\mathbf{w}}\cdot\mathbf{n})\,d\sigma &= \int_{\partial\Omega_{out},\widetilde{\mathbf{w}}\cdot\mathbf{n}>0} w^{2}\left(\frac{1}{2} - \beta((\widetilde{\mathbf{w}}\cdot\mathbf{n})_{+})\right)(\widetilde{\mathbf{w}}\cdot\mathbf{n})_{+}\,d\sigma\\ &- \int_{\partial\Omega_{out},\widetilde{\mathbf{w}}\cdot\mathbf{n}<0} w^{2}\left(\frac{1}{2} - \beta(-(\widetilde{\mathbf{w}}\cdot\mathbf{n})_{-})\right)(\widetilde{\mathbf{w}}\cdot\mathbf{n})_{-}\,d\sigma \geq 0,\end{aligned}$$

for any $\widetilde{\mathbf{w}} \in W^{1,2}(\Omega)^{d}$, due to the properties of the function β (see (2)-a). □

We are now in shape to prove the following result based on Lemma 4.1.

Theorem 4.1. *Assume that*

$$c^2 \frac{\mathrm{Gr}}{\mathrm{Re}^2} < \min \left\{ \frac{1}{2} \left(\frac{1}{\mathrm{Re}} - c^2 \,\widehat{c}^3 ||\mathbf{V}||_{1,2} \right), \frac{1}{\mathrm{RePr}} \right\} - \eta ||u_D||_{1/2,2,\partial\Omega_D}$$

and $(\mathrm{A}_{\mathbb{P}})$

$$c^2 \,\widehat{c}^3 ||\mathbf{V}||_{1,2} < \frac{1}{\mathrm{Re}},$$

hold true, where

$$\eta = c^2 c^* \left(\beta^* \,\widehat{c}^3 + \widetilde{c}^2 \right) > 0, \tag{28}$$

for β^* the L^∞-norm of β. Then problem ($\mathbb{P}^n_{\mathrm{w}}$) has a solution $(\mathbf{v}_n, u_n) \in V_1^n \times V_2^n$. Moreover,

$$||\mathbf{v}_n|| \leq k, \qquad\qquad ||u_n|| \leq k, \tag{29}$$

for some positive constant k, which does not depend on n.

Proof. Let $(\mathbf{w}, w) \in X$ be given. By adding the estimations for $((\mathbf{P}_1(\mathbf{w}, w), \mathbf{w}))$ and $((P_2(\mathbf{w}, w), w))$ given in Lemma 4.4, and taking into consideration $(\mathrm{A}_{\mathbb{P}})_2$, we obtain

$$\begin{aligned}[P(\mathbf{w}, w), (\mathbf{w}, w)] \geq & \min \left\{ \frac{1}{2} \left(\frac{1}{\mathrm{Re}} - c^2 \,\widehat{c}^3 ||\mathbf{V}||_{1,2} \right), \frac{1}{\mathrm{RePr}} \right\} \left(||\mathbf{w}||^2 + ||w||^2 \right) \\ & - \left(c^2 \frac{\mathrm{Gr}}{\mathrm{Re}^2} + \eta ||u_D||_{1/2,2,\partial\Omega_D} \right) ||w|| \, ||\mathbf{w}|| - A_1 ||\mathbf{w}|| - A_2 ||w||. \end{aligned} \tag{30}$$

Let k be some positive given number. Assume that $[(\mathbf{w}, w)] = k$, i.e. $||\mathbf{w}||^2 + ||w||^2 = k^2$. Noting that $||\mathbf{w}|| \leq k$ and $||w|| \leq k$, it follows from (30) that

$$[P(\mathbf{w}, w), (\mathbf{w}, w)] \geq (A_0 k - (A_1 + A_2))\, k, \tag{31}$$

where

$$A_0 = \min \left\{ \frac{1}{2} \left(\frac{1}{\mathrm{Re}} - c^2 \,\widehat{c}^3 ||\mathbf{V}||_{1,2} \right), \frac{1}{\mathrm{RePr}} \right\} - \eta ||u_D||_{1/2,2,\partial\Omega_D} - c^2 \frac{\mathrm{Gr}}{\mathrm{Re}^2}.$$

Assumptions $(\mathrm{A}_{\mathbb{P}})$ imply that $A_0 > 0$. Thus, $K = \frac{A_1 + A_2}{A_0}$ is non-negative and the above computations ensure that P satisfies (16) for any $k \geq K$. Finally, we observe that k can be chosen to be independent on n (e.g. $k = K + 1$). □

We are now in a position to formulate the main result of this Section.

Theorem 4.2. *Suppose $(\mathrm{A}_{\mathbb{P}})$ holds true. Then problem ($\mathbb{P}_{\mathrm{w}}$) has a solution $(\mathbf{v}_0, u_0) \in V_1 \times V_2$.*

Proof. For each $n \in \mathbb{N}$, let $(\mathbf{v}_n, u_n) \in V_1^n \times V_2^n$ be a solution to ($\mathbb{P}^n_{\mathrm{w}}$). From Theorem 4.1, we find that $(\mathbf{v}_n)$ and (u_n) are bounded sequences in V_1 and V_2, respectively. Hence, they admit weakly convergent subsequences, which we will also denote by $(\mathbf{v}_n)$ and (u_n). Let $\mathbf{v}_0 \in V_1$ and $u_0 \in V_2$ be the weak limits of $(\mathbf{v}_n)$ and (u_n), respectively. We write $\mathbf{v}_n \rightharpoonup \mathbf{v}_0$ and $u_n \rightharpoonup u_0$. We show now that $(\mathbf{v}_0, u_0) \in V_1 \times V_2$ is a solution to ($\mathbb{P}_{\mathrm{w}}$) by taking the limit for $n \to +\infty$ in both equations of problem ($\mathbb{P}^n_{\mathrm{w}}$).

First, we will analyze the convergence of the surface integrals. We begin by proving that

$$\int_{\partial\Omega_{out}} (\mathbf{v}^n\cdot\mathbf{w})(\mathbf{v}^n\cdot\mathbf{n})_- d\sigma \to \int_{\partial\Omega_{out}} (\mathbf{v}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{n})_- d\sigma, \tag{32}$$

where $\mathbf{w}\in V_1$. Recall that $\mathbf{v}^n = \mathbf{v}_n + \mathbf{V}$, and $\mathbf{v} = \mathbf{v}_0 + \mathbf{V}$. From the embedding $V_1 \hookrightarrow W^{1,2}(\Omega)^d$ and the compact embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^q(\partial\Omega_{out})$ for $1\le q<4$ ([29]), we find $V_1 \hookrightarrow\hookrightarrow L^q(\partial\Omega_{out})^d$ for $1\le q<4$. Then, $\mathbf{v}^n\to\mathbf{v}$ in $L^q(\partial\Omega_{out})^d$, if $1\le q<4$. From this, we also find the strong convergence $(\mathbf{v}^n\cdot\mathbf{n})_-\to(\mathbf{v}\cdot\mathbf{n})_-$ in $L^q(\partial\Omega_{out})$ when $1\le q<4$, since $L^q(\partial\Omega_{out})^d\ni\mathbf{v}\mapsto(\mathbf{v}\cdot\mathbf{n})_-\in L^q(\partial\Omega_{out})$ is continuous in the strong topology. Therefore, $\mathbf{v}^n(\mathbf{v}^n\cdot\mathbf{n})_-\to\mathbf{v}(\mathbf{v}\cdot\mathbf{n})_-$ strongly in $L^q(\partial\Omega_{out})^d$ for any $1\le q<2$. Now, (32) follows from Hölder inequality,

$$\begin{aligned}\left|\int_{\partial\Omega_{out}} (\mathbf{v}^n(\mathbf{v}^n\cdot\mathbf{n})_- - \mathbf{v}(\mathbf{v}\cdot\mathbf{n})_-)\cdot\mathbf{w}d\sigma\right| \\ \le ||\mathbf{v}^n(\mathbf{v}^n\cdot\mathbf{n})_- - \mathbf{v}(\mathbf{v}\cdot\mathbf{n})_-||_{q,\partial\Omega_{out}}||\mathbf{w}||_{q^*,\partial\Omega_{out}},\end{aligned}$$

with $q=2-\xi$, $q^*=\frac{q}{q-1}$ and $0<\xi<\frac{2}{3}$.

We now look at the convergence

$$\int_{\partial\Omega_{out}} wu^n\beta(\mathbf{v}^n\cdot\mathbf{n})(\mathbf{v}^n\cdot\mathbf{n})d\sigma \to \int_{\partial\Omega_{out}} wu\beta(\mathbf{v}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})d\sigma, \tag{33}$$

where $w\in V_2$. Taking into account that $\beta:\mathbb{R}\to\mathbb{R}$ is bounded, and the only point where it is allowed to have a discontinuity is the origin (see (2)), we find $s\mapsto\beta(s)s$ is continuous, and $|\beta(s)s|\le\beta^*|s|$. Then, for $\Psi(s)=\beta(s)s$, the Nemytskii operator $\Psi: L^q(\partial\Omega_{out})\to L^q(\partial\Omega_{out})$ is continuous for $1\le q<+\infty$; see [32]. By the embeddings in the above paragraph, we know that $(\mathbf{v}^n\cdot\mathbf{n})\to(\mathbf{v}\cdot\mathbf{n})$ in $L^q(\partial\Omega_{out})$ for $1\le q<4$. From this and the continuity of Ψ, it follows that $\beta(\mathbf{v}^n\cdot\mathbf{n})(\mathbf{v}^n\cdot\mathbf{n})\to\beta(\mathbf{v}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})$ in $L^q(\partial\Omega_{out})$ when $1\le q<4$. Recalling the embedding $V_2\hookrightarrow W^{1,2}(\Omega)$, we also find that $V_2\hookrightarrow\hookrightarrow L^q(\partial\Omega_{out})$ for $1\le q<4$, so $u^n\to u$ in $L^q(\partial\Omega_{out})$ when $1\le q<4$. Thus, $u^n\beta(\mathbf{v}^n\cdot\mathbf{n})(\mathbf{v}^n\cdot\mathbf{n})\to u\beta(\mathbf{v}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})$ in $L^q(\partial\Omega_{out})$ for $1\le q<2$. Now (33) follows from the Hölder inequality, as before.

Secondly, we will analyze the convergences of terms involving the trilinear forms b_1 and b_2. We begin by proving that

$$b_1(\mathbf{v}^n,\mathbf{v}^n,\mathbf{w})\to b_1(\mathbf{v},\mathbf{v},\mathbf{w}), \tag{34}$$

where $\mathbf{w}\in V_1$. Due to the structure of b_1, it is enough to prove that $\int_\Omega z^n\frac{\partial z^n}{\partial x_j}w\to\int_\Omega z\frac{\partial z}{\partial x_j}w$ provided that $z^n\rightharpoonup z$ in $W^{1,2}(\Omega)$, $j\in\{1,\dots,d\}$, and $w\in W^{1,2}(\Omega)$. From the compact embedding $W^{1,2}(\Omega)\hookrightarrow\hookrightarrow L^4(\Omega)$, we have that $z^n\to z$ in $L^4(\Omega)$, and $wz^n\to wz$ in $L^2(\Omega)$. Since $\frac{\partial z^n}{\partial x_j}\rightharpoonup\frac{\partial z}{\partial x_j}$ in $L^2(\Omega)$, then $\int_\Omega z^n\frac{\partial z^n}{\partial x_j}w\to\int_\Omega z\frac{\partial z}{\partial x_j}w$ follows from the strong-weak continuity of the $L^2(\Omega)$ inner product $(\cdot,\cdot)$. The convergence $b_2(\mathbf{v}^n,u^n,w)\to b_2(\mathbf{v},u,w)$ for any $w\in V_2$ it is proved similarly.

Finally, $((\mathbf{v}^n,\mathbf{w}))\to((\mathbf{v},\mathbf{w}))$, $(u^n\mathbf{e},\mathbf{w})\to(u\mathbf{e},\mathbf{w})$ and $((u^n,w))\to((u,w))$ are direct consequences of the weak convergences $\mathbf{v}_n\rightharpoonup\mathbf{v}_0$ in V_1, and $u_n\rightharpoonup u_0$ in V_2. □

Remark 1.

1 *When* $\mathrm{Gr}=0$*, the problem* $(\mathbb{P})$ *is decoupled into a Navier-Stokes system for* $\mathbf{v}$ *and* p*, and a convection-diffusion system for* u*. The former was studied by Braack and Mucha for the case* $\mathbf{v}_{in}=0$ *in [11], where they proved existence of weak solution without any condition on the*

Reynolds number. This result is recovered here. In fact, when $\mathbf{v}_{in} = 0$, $\mathbf{V}$ can be chosen to be the zero function. Then, conditions $(\mathrm{A}_{\mathbb{P}})$ reduce to

$$\eta||u_D||_{1/2,2,\partial\Omega_D} < \frac{1}{Re}\min\left\{\frac{1}{2}, \frac{1}{Pr}\right\},$$

which only affects the convection-diffusion system.

2 *When $d = 2$, and β is the function given in the Example 2.1 with $\epsilon = 0$ and $\epsilon^\star = 1/2$, the problem $(\mathbb{P})$ was studied by Pérez et al. in [30], see also [31]. They consider the more general case in which the thermophysical properties are allowed to vary with the temperature, and assume no internal heat generation exists (i.e. $f_2 = 0$). Under certain restriction on data, they are able to prove existence and uniqueness of weak solutions by decoupling the problem, and then following a fixed point strategy. In particular, they assume that $\mathrm{Gr} \ll \mathrm{Re}^2$ (for physical consistence with the 2D setting), and small temperature variations (to prove a result of existence by a fixed point theorem, see (H2) in [30]). Further, when the thermophysical parameters are assumed to be constant, the first condition is quantified in terms of (see Theorem 4.1 in [30])*

$$\widetilde{C}\frac{\mathrm{Gr}}{\mathrm{Re}^2} < 1, \tag{35}$$

where $\widetilde{C}$ is a positive constant that depends on data. By following the guideline given in [30] to obtain $\widetilde{C}$ (see Proposition 4.1 in [30]), we found $\widetilde{C} = c^2c^*\max\{1, \widehat{c}^3/2\} + c^*$. Then, condition $(\mathrm{A}_{\mathbb{P}})_1$ is less restrictive than (35) since it allows to control the value of $\mathrm{Gr}/\mathrm{Re}^2$ by the boundary data, and the Prandtl number of the system.

The following corollary establishes conditions on the Reynolds, Prandtl, and Grashof numbers in order to obtain solutions to problem $(\mathbb{P}_\mathrm{w})$. In particular, it shows that if Re and Gr are known and satisfy some smallness-type conditions, then solutions to problem $(\mathbb{P}_\mathrm{w})$ can be obtained for arbitrarily small Pr.

Corollary 4.1.

1 *Assume that $\mathbf{v}_{in} = 0$ and $u_D = 0$. Then there exists a positive constant $\delta = \delta(d, \Omega)$, such that if*

$$0 < \delta\mathrm{Gr} < \mathrm{Re} \qquad \text{and} \qquad 0 < \mathrm{Pr} \leq 2, \tag{36}$$

then problem $(\mathbb{P}_\mathrm{w})$ has a solution.

2 *Assume that $\mathbf{v}_{in} \neq 0$ or $u_D \neq 0$. Then there exist positive constants $\delta_i = \delta_i(d, \Omega, \mathbf{v}_{in}, u_D, \partial\Omega_{out}, \partial\Omega_D)$, $i = 0, \cdots, 3$, such that if*

$$0 < \mathrm{Gr} \leq \delta_0, \qquad \delta_1 < \mathrm{Re} < \delta_2, \qquad \text{and} \qquad 0 < \mathrm{Pr} \leq \delta_3, \tag{37}$$

then problem $(\mathbb{P}_\mathrm{w})$ has a solution.

Proof.

1 The proof follows directly from Theorem 4.2, considering $\delta = 2c^2$.

2 Aiming to keep notation simple, we set $\alpha = c^2\,\widehat{c}^{\,3}||\mathbf{V}||_{1,2} + 2\eta||u_D||_{1/2,2,\partial\Omega_D}$. Observe that $\alpha > 0$ since $||\mathbf{V}||_{1,2}$ and $||u_D||_{1/2,2,\partial\Omega_D}$ are not both zero. Let δ_0 be such that

$$0 < \delta_0 \leq \frac{1}{8c^2\alpha},$$

and assume that $0 < \mathrm{Gr} \leq \delta_0$. Let δ_1 and δ_2 be the positive numbers defined by

$$\delta_1 = \frac{1}{2\alpha}\left(1 - \sqrt{1 - 8c^2\alpha\delta_0}\right) \qquad \text{and} \qquad \delta_2 = \frac{1}{2\alpha}\left(1 + \sqrt{1 - 8c^2\alpha\delta_0}\right),$$

and assume that $\delta_1 < \mathrm{Re} < \delta_2$. Then, condition $(\mathrm{A}_{\mathbb{P}})_2$ holds. In fact, if $\mathbf{v}_{in} = 0$, there is nothing to prove. If $\mathbf{v}_{in} \neq 0$, we find

$$\delta_2 < \frac{1}{\alpha} \leq \frac{1}{c^2\,\widehat{c}^{\,3}||\mathbf{V}||_{1,2}}.$$

Then, $1 - c^2\,\widehat{c}^{\,3}||\mathbf{V}||_{1,2}\delta_2 > 0$. This, in conjunction with $\mathrm{Re} < \delta_2$, gives $(\mathrm{A}_{\mathbb{P}})_2$. Let ρ_1 and ρ_2 be the roots of the equation $\alpha x^2 - x + 2c^2\mathrm{Gr} = 0$. The assumption on the Grashof number implies that $\rho_1 \leq \delta_1 \leq \delta_2 \leq \rho_2$. Then,

$$\alpha\mathrm{Re}^2 - \mathrm{Re} + 2c^2\mathrm{Gr} < 0. \tag{38}$$

Let δ_3 be the real number defined by

$$\delta_3 = \frac{1}{1 - c^2\,\widehat{c}^{\,3}||\mathbf{V}||_{1,2}\delta_1}.$$

If $\mathbf{v}_{in} = 0$, $\delta_3 = 1$, so it is positive. If $\mathbf{v}_{in} \neq 0$, we find

$$\delta_1 < \frac{1}{\alpha} \leq \frac{1}{c^2\,\widehat{c}^{\,3}||\mathbf{V}||_{1,2}}.$$

Thus, δ_3 is positive. Now, assume that $0 < \mathrm{Pr} \leq \delta_3$. We find that (38) is equivalent to $(\mathrm{A}_{\mathbb{P}})_1$. In fact, the assumptions on Re and Pr imply that

$$\mathrm{Pr} \leq \frac{1}{1 - c^2\,\widehat{c}^{\,3}||\mathbf{V}||_{1,2}\mathrm{Re}}.$$

Then

$$\min\left\{\frac{1}{2}\left(\frac{1}{\mathrm{Re}} - c^2\,\widehat{c}^{\,3}||\mathbf{V}||_{1,2}\right), \frac{1}{\mathrm{RePr}}\right\} = \frac{1}{2}\left(\frac{1}{\mathrm{Re}} - c^2\,\widehat{c}^{\,3}||\mathbf{V}||_{1,2}\right).$$

Noting that (38) can be written as

$$c^2\frac{\mathrm{Gr}}{\mathrm{Re}^2} < \frac{1}{2\mathrm{Re}} - \frac{\alpha}{2},$$

the equivalence between (38) and $(\mathrm{A}_{\mathbb{P}})_1$ follows directly from the definition of α. The existence of solution to problem $(\mathbb{P}_{\mathrm{w}})$ is then ensured by Theorem 4.2. Only remains to observe that the constants δ_i defined in this proof can be selected to be dependent on d, Ω, $\mathbf{v}_{in}$, u_D, $\partial\Omega_{out}$ and $\partial\Omega_D$ only.

□

We end this Section by showing that weak solutions to problem ($\mathbb{P}_{\rm w}$) are indeed strong solutions, provided some additional regularity is assumed for them.

Theorem 4.3. *Suppose* $(\mathbf{v}_0, u_0)$ *solve* ($\mathbb{P}_{\rm w}$) *and* $\mathbf{v}_0 \in W^{2,2}(\Omega)^d \cap V_1$, $u_0 \in W^{2,2}(\Omega) \cap V_2$, *with* $\mathbf{V} \in W^{2,2}(\Omega)^d$, $U \in W^{2,2}(\Omega)$. *Then there exists a function* $p \in W^{1,2}(\Omega)$ *such that* $\mathbf{v} = \mathbf{v}_0 + \mathbf{V}$, $u = u_0 + U$ *and* p *satisfy problem* ($\mathbb{P}$) *almost everywhere.*

Proof. First, we prove that there exists $p \in W^{1,2}(\Omega)$ such that $\mathbf{v} = \mathbf{v}_0 + \mathbf{V}$, $u = u_0 + U$ and p satisfy the equation (1) of ($\mathbb{B}$) a.e. in Ω. We set $\mathcal{V}_1 = \left\{ \mathbf{w} \in C_0^\infty(\Omega)^d \,:\, \operatorname{div} \mathbf{w} = 0 \right\}$. Since any function $\mathbf{w} \in \mathcal{V}_1$ vanishes on $\partial\Omega$, from the first equation in problem ($\mathbb{P}_{\rm w}$), we find

$$b_1(\mathbf{v}, \mathbf{v}, \mathbf{w}) + \frac{1}{\rm Re}((\mathbf{v}, \mathbf{w})) - \frac{\rm Gr}{\rm Re^2}(u\mathbf{e}, \mathbf{w}) = (\mathbf{f}_1, \mathbf{w}), \tag{39}$$

for all $\mathbf{w} \in \mathcal{V}_1$. Exploiting the Gauss-Green formula,

$$\int_\Omega \nabla\mathbf{v} \cdot \nabla\mathbf{w} dx = \int_{\partial\Omega} \frac{\partial\mathbf{v}}{\partial\mathbf{n}} \cdot \mathbf{w} d\sigma - \int_\Omega \Delta\mathbf{v} \cdot \mathbf{w} dx, \tag{40}$$

with $\mathbf{w} \in V_1$, and noting that the surface integral in its r.h.s. vanishes when $\mathbf{w} \in \mathcal{V}_1$, we write (39) as

$$\left(\mathbf{v} \cdot \nabla\mathbf{v} - \frac{1}{\rm Re}\Delta\mathbf{v} - \frac{\rm Gr}{\rm Re^2} u\mathbf{e} - \mathbf{f}_1, \mathbf{w} \right) = 0,$$

for all $\mathbf{w} \in \mathcal{V}_1$. From this, we obtain the existence of a function $p \in W^{1,2}(\Omega)$ as desired [18].

Secondly, we examine the directional do-nothing condition in ($\mathbb{BC}^{\rm I}_{\partial\Omega_{out}}$). Adding the term $\int_\Omega \nabla p \cdot \mathbf{w} dx$ side by side of the first equation in problem ($\mathbb{P}_{\rm w}$), and noting that

$$\int_\Omega \nabla p \cdot \mathbf{w} dx = \int_{\partial\Omega_{out}} p\mathbf{n} \cdot \mathbf{w} d\sigma,$$

for $\mathbf{w} \in V_1$, we find

$$\begin{aligned} b_1(\mathbf{v}, \mathbf{v}, \mathbf{w}) + \frac{1}{\rm Re}((\mathbf{v}, \mathbf{w})) + \int_{\partial\Omega_{out}} \left(\frac{1}{2}\mathbf{v}(\mathbf{v} \cdot \mathbf{n})_- - p\mathbf{n} \right) \cdot \mathbf{w} d\sigma - \frac{\rm Gr}{\rm Re^2}(u\mathbf{e}, \mathbf{w}) \\ + \int_\Omega \nabla p \cdot \mathbf{w} dx = (\mathbf{f}_1, \mathbf{w}), \qquad \forall\, \mathbf{w} \in V_1, \end{aligned} \tag{41}$$

Using again (40), and taking into consideration that the surface integral in its r.h.s. reduces to an integral over $\partial\Omega_{out}$ when $\mathbf{w} \in V_1$, we write (41) as

$$\begin{aligned} &\left(\mathbf{v} \cdot \nabla\mathbf{v} - \frac{1}{\rm Re}\Delta\mathbf{v} - \frac{\rm Gr}{\rm Re^2} u\mathbf{e} - \mathbf{f}_1 + \nabla p, \mathbf{w} \right) + \int_{\partial\Omega_{out}} \left(\frac{1}{\rm Re}\frac{\partial\mathbf{v}}{\partial\mathbf{n}} + \frac{1}{2}\mathbf{v}(\mathbf{v} \cdot \mathbf{n})_- - p\mathbf{n} \right) \cdot \mathbf{w} d\sigma \\ &\quad = 0, \end{aligned}$$

for all $\mathbf{w} \in V_1$. Since the first term in the l.h.s. of the above expression vanishes, we find

$$\int_{\partial\Omega_{out}} \left(\frac{1}{\rm Re}\frac{\partial\mathbf{v}}{\partial\mathbf{n}} + \frac{1}{2}\mathbf{v}(\mathbf{v} \cdot \mathbf{n})_- - p\mathbf{n} \right) \cdot \mathbf{w} d\sigma = 0,$$

for all $\mathbf{w} \in V_1$. Thus, $\mathbf{v}$ and p satisfy the directional do-nothing condition a.e. on $\partial\Omega_{out}$.

By proceeding as in the beginning of this proof, we also find

$$\left(\mathbf{v}\cdot\nabla u-\frac{1}{\mathrm{RePr}}\Delta u-f_2,w\right)=0,$$

for all $w\in C_0^\infty(\Omega)$, from which follows that $\mathbf{v}$ and u satisfy the equation (3) of ($\mathbb{B}$) a.e. in Ω.

Finally, we examine the Neumann condition in ($\mathbb{BC}_{\partial\Omega_{out}^c}$), and the heat transfer condition in ($\mathbb{BC}^{\mathrm{I}}_{\partial\Omega_{out}}$). As before, we find that the first equation of problem ($\mathbb{P}_{\mathrm{w}}$) can be written as

$$\left(\mathbf{v}\cdot\nabla u-\frac{1}{\mathrm{RePr}}\Delta u-f_2,w\right)+\int_{\partial\Omega_{out}}w\left(\frac{1}{\mathrm{RePr}}\frac{\partial u}{\partial\mathbf{n}}-u\beta(\mathbf{v}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})\right)d\sigma$$
$$+\frac{1}{\mathrm{RePr}}\int_{\partial\Omega_N}w\frac{\partial u}{\partial\mathbf{n}}d\sigma=0,$$

for all $w\in V_2$, which reduces to

$$\int_{\partial\Omega_{out}}w\left(\frac{1}{\mathrm{RePr}}\frac{\partial u}{\partial\mathbf{n}}-u\beta(\mathbf{v}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})\right)wd\sigma+\frac{1}{\mathrm{RePr}}\int_{\partial\Omega_N}w\frac{\partial u}{\partial\mathbf{n}}d\sigma=0,$$

for all $w\in V_2$, since the first term in its l.h.s. vanishes. In particular, we find

$$\int_{\partial\Omega_{out}}w\left(\frac{1}{\mathrm{RePr}}\frac{\partial u}{\partial\mathbf{n}}-u\beta(\mathbf{v}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})\right)d\sigma=0,$$

for all $\mathbf{w}\in V_1$ with vanishing trace on $\partial\Omega_N$. Since $\partial\Omega_N$ and $\partial\Omega_{out}$ are disjoint sets, the above expression implies that the heat transfer condition holds a.e. on $\partial\Omega_{out}$. Similarly, we find

$$\int_{\partial\Omega_N}w\frac{\partial u}{\partial\mathbf{n}}d\sigma=0,$$

for all $\mathbf{w}\in V_1$ with vanishing trace on $\partial\Omega_{out}$, which implies the Neumann condition for u holds a.e. on $\partial\Omega_N$.

The remaining conditions in problem ($\mathbb{P}$) hold straightforward from the definition of the spaces V_1 and V_2. □

5 Weak solutions to problem ($\mathbb{Q}$)

All through this section, assumptions (A) hold true. We begin by noting that the integral condition in ($\mathbb{BC}^{\mathrm{II}}_{\partial\Omega}$) implies that $\mathbf{v}_0$ must satisfy

$$\int_{\partial\Omega_{out}}|\left((\mathbf{v}_0+\mathbf{V})\cdot\mathbf{n}\right)_-|^a d\sigma\leq c_0.$$

Thus, we cannot search for $\mathbf{v}_0$ in the whole space V_1. Following the ideas in [25], we will formulate a variational inequality for $\mathbf{v}_0$ on the subset $K(\mathbf{V})$ of V_1 defined by

$$K(\mathbf{V})=\overline{F(\mathbf{V})}^{V_1},$$

that is, $K(\mathbf{V})$ is the closure of $F(\mathbf{V})$ in V_1, where

$$F(\mathbf{V})=\left\{\mathbf{w}\in E_1(\Omega)\,:\,\int_{\partial\Omega_{out}}|\left((\mathbf{w}+\mathbf{V})\cdot\mathbf{n}\right)_-|^a d\sigma\leq c_0\right\}.$$

$K(\mathbf{V})$ is a non-empty convex set of V_1, which admits the characterization

$$K(\mathbf{V}) = \left\{ \mathbf{w} \in V_1 \ : \ \int_{\partial\Omega_{out}} |\left((\mathbf{w}+\mathbf{V})\cdot\mathbf{n}\right)_-|^a d\sigma \leq c_0 \right\}.$$

We refer to [25] for details on the properties of $K(\mathbf{V})$. The variational inequality is obtained similarly to the first equation in problem ($\mathbb{P}_{\mathrm{w}}$). The difference is that now we multiply the equation (1) of ($\mathbb{B}$) by $\mathbf{w}-\mathbf{v}_0$, where $\mathbf{w}$ is a test function in $F(\mathbf{V})$, and relax the resulting equation by considering an inequality. More precisely, we consider the following weak formulation for problem ($\mathbb{Q}$):

Problem ($\mathbb{Q}_{\mathrm{w}}$): Find $\mathbf{v}_0 \in K(\mathbf{V})$ and $u_0 \in V_2$ such that

$$\begin{aligned} &b_1(\mathbf{v},\mathbf{v},\mathbf{w}-\mathbf{v}_0) + \frac{1}{\mathrm{Re}}((\mathbf{v},\mathbf{w}-\mathbf{v}_0)) - \frac{\mathrm{Gr}}{\mathrm{Re}^2}(u\mathbf{e},\mathbf{w}-\mathbf{v}_0) - (\mathbf{f}_1,\mathbf{w}-\mathbf{v}_0) \geq 0, \\ &\text{for all } \mathbf{w} \in F(\mathbf{V}), \text{ and} \\ &b_2(\mathbf{v},u,w) + \frac{1}{\mathrm{RePr}}((u,w)) - \int_{\partial\Omega_{out}} uw\beta(\mathbf{v}\cdot\mathbf{n})\,(\mathbf{v}\cdot\mathbf{n})\,d\sigma = (f_2,w), \\ &\text{for all } w \in V_2, \text{ where } \mathbf{v} = \mathbf{v}_0 + \mathbf{V}, \text{ and } u = u_0 + U. \end{aligned} \tag{$\mathbb{Q}_{\mathrm{w}}$}$$

Following [25], we will prove the existence of solutions to problem ($\mathbb{Q}_{\mathrm{w}}$) through the method of Galerkin in combination with the method of penalization. The finite dimensional problems will be defined on the same spaces V_1^n and V_2^n considered in Section 4. The penalization is introduced to deal with the variational inequality through a variational equation, in the finite dimensional setting. We consider the projector operator π from V_1 to $K(\mathbf{V})$, and define the penalization operator $\theta : V_1 \to V_1$ by $\theta\mathbf{w} = \mathbf{w} - \pi\mathbf{w}$.

More precisely, for each $n \in \mathbb{N}$, we consider the following finite dimensional problem associated to ($\mathbb{Q}_{\mathrm{w}}$):

Problem ($\mathbb{Q}_{\mathrm{w}}^n$): Find $(\mathbf{v}_n, u_n) \in V_1^n \times V_2^n$ such that

$$\begin{aligned} &b_1(\pi\mathbf{v}_n + \mathbf{V},\mathbf{v}^n,\mathbf{w}) + \frac{1}{\mathrm{Re}}((\mathbf{v}^n,\mathbf{w})) - \frac{\mathrm{Gr}}{\mathrm{Re}^2}(u^n\mathbf{e},\mathbf{w}) - (\mathbf{f}_1,\mathbf{w}) + n((\theta\mathbf{v}_n,\mathbf{w})) = 0, \\ &\text{for all } \mathbf{w} \in V_1^n, \text{ and} \\ &b_2(\mathbf{v}^n,u^n,w) + \frac{1}{\mathrm{RePr}}((u^n,w)) - \int_{\partial\Omega_{out}} u^n w\beta(\mathbf{v}^n\cdot\mathbf{n})\,(\mathbf{v}^n\cdot\mathbf{n})\,d\sigma = (f_2,w), \\ &\text{for all } w \in V_2^n, \text{ where } \mathbf{v}^n = \mathbf{v}_n + \mathbf{V}, \text{ and } u^n = u_n + U. \end{aligned} \tag{$\mathbb{Q}_{\mathrm{w}}^n$}$$

The existence of solutions to problem ($\mathbb{Q}_{\mathrm{w}}^n$) can be proved by following the same steps in the proof of the existence of solutions to problem ($\mathbb{P}_{\mathrm{w}}$). In the following, we briefly present them, making an emphasis on the few differences.

In this case, we consider the operator $P^* : V_1^n \times V_2^n \to V_1^n \times V_2^n$ defined by

$$P^*(\mathbf{w},w) = \left(\mathbf{P}_1^*(\mathbf{w},w), P_2(\mathbf{w},w)\right),$$

where

$$\begin{aligned} \mathbf{P}_1^*(\mathbf{w},w) = \sum_{k=1}^{n} \Big\{ &b_1(\pi\mathbf{w}+\mathbf{V},\mathbf{w}+\mathbf{V},\mathbf{w}_k) + \frac{1}{\mathrm{Re}}((\mathbf{w}+\mathbf{V},\mathbf{w}_k)) \\ &-\frac{\mathrm{Gr}}{\mathrm{Re}^2}((w+U)\mathbf{e},\mathbf{w}_k) - (\mathbf{f}_1,\mathbf{w}_k) + n((\theta\mathbf{w},\mathbf{w}_k)) \Big\}\,\mathbf{w_k}, \end{aligned} \tag{42}$$

and P_2 is given by (15). As before, we write $X = V_1^n \times V_2^n$, and note that $P^* : X \to X$ is a continuous operator when X is endowed with the norm induced by the inner product $[\cdot,\cdot]$ defined in (36).

When computing $((\mathbf{P}_1^*(\mathbf{w}, w), \mathbf{w}))$, we obtain

$$((\mathbf{P}_1^*(\mathbf{w}, w), \mathbf{w})) = b_1(\pi\mathbf{w} + \mathbf{V}, \mathbf{w} + \mathbf{V}, \mathbf{w}) + \frac{1}{\mathrm{Re}}((\mathbf{w} + \mathbf{V}, \mathbf{w})) \\ - \frac{\mathrm{Gr}}{\mathrm{Re}^2}((w + U)\mathbf{e}, \mathbf{w}) - (\mathbf{f}_1, \mathbf{w}) + n((\theta\mathbf{w}, \mathbf{w})). \tag{43}$$

Let b be the Hölder conjugate of a, i.e. $\frac{1}{a} + \frac{1}{b} = 1$. Since $2 < 2b < 4$, the embedding $W^{1,2}(\Omega) \hookrightarrow L^{2b}(\partial\Omega_{out})$ holds. Then, there exists a positive constant $c^\star = c^\star(d, \partial\Omega_{out})$, such that $||\mathbf{w}||_{2b,\partial\Omega_{out}} \leq c^\star||\mathbf{w}||_{1,2}$ for all $\mathbf{w} \in W^{1,2}(\Omega)^d$.

Lemma 5.1. *For any $\mathbf{w} \in V_1$, the form b_1 satisfies*

$$b_1(\pi\mathbf{w} + \mathbf{V}, \mathbf{w} + \mathbf{V}, \mathbf{w}) \geq -\frac{1}{2}\left(\frac{1}{\mathrm{Re}} + c_0^{1/a}(cc^\star)^2\right)||\mathbf{w}||^2 \\ - c\widetilde{c}^2||\mathbf{V}||_{1,2}^2||\mathbf{w}||. \tag{44}$$

Proof. We estimate $b_1(\pi\mathbf{w} + \mathbf{V}, \mathbf{w}, \mathbf{w})$ and $b_1(\pi\mathbf{w}, \mathbf{V}, \mathbf{w})$ separately, and use the estimation for $b_1(\mathbf{V}, \mathbf{V}, \mathbf{w})$ given in (22). Noting that $\pi\mathbf{w} + \mathbf{V}$ is divergence free, we use $(12)_1$, and obtain

$$\begin{aligned} b_1(\pi\mathbf{w} + \mathbf{V}, \mathbf{w}, \mathbf{w}) &= \frac{1}{2}\int_{\partial\Omega_{out}} |\mathbf{w}|^2\left((\pi\mathbf{w} + \mathbf{V})\cdot\mathbf{n}\right)d\sigma \\ &\geq -\frac{1}{2}\int_{\partial\Omega_{out}} |\mathbf{w}|^2\left((\pi\mathbf{w} + \mathbf{V})\cdot\mathbf{n}\right)_- d\sigma \\ &\geq -\frac{1}{2}\left(\int_{\partial\Omega_{out}} |\mathbf{w}|^{2b}d\sigma\right)^{1/b}\left(\int_{\partial\Omega_{out}} |\left((\pi\mathbf{w} + \mathbf{V})\cdot\mathbf{n}\right)_-|^a d\sigma\right)^{1/a} \\ &\geq -\frac{1}{2}c_0^{1/a}||\mathbf{w}||_{2b,\partial\Omega_{out}}^2 \geq -\frac{1}{2}c_0^{1/a}c^{\star 2}||\mathbf{w}||_{1,2}^2 \geq -\frac{1}{2}c_0^{1/a}(cc^\star)^2||\mathbf{w}||^2. \end{aligned}$$

From Lemma 3.1-1(b), we have

$$b_1(\pi\mathbf{w}, \mathbf{V}, \mathbf{w}) \geq -\frac{1}{2\mathrm{Re}}||\pi\mathbf{w}||\,||\mathbf{w}|| \geq -\frac{1}{2\mathrm{Re}}||\mathbf{w}||^2,$$

since the projection operator π is a contraction. Now (44) follows from the linearity properties of b_1. □

Lemma 5.2. *For any $(\mathbf{w}, w) \in X$, the following inequality holds true:*

$$((\mathbf{P}_1^*(\mathbf{w}, w), \mathbf{w})) \geq \frac{1}{2}\left(\frac{1}{\mathrm{Re}} - c_0^{1/a}(cc^\star)^2\right)||\mathbf{w}||^2 - c^2\frac{\mathrm{Gr}}{\mathrm{Re}^2}||w||\,||\mathbf{w}|| - A_1^*||\mathbf{w}||, \tag{45}$$

where A_1^* is a non-negative constant which depends on d, Ω, $\mathbf{v}_{in}$, $\mathbf{f}_1$, Re and Gr only.

Proof. It is similar to the proof of (25) in Lemma 4.4. □

Theorem 5.1. *Assume that*

$$c^2\frac{\mathrm{Gr}}{\mathrm{Re}^2} < \min\left\{\frac{1}{2}\left(\frac{1}{\mathrm{Re}} - c_0^{1/a}(cc^\star)^2\right), \frac{1}{\mathrm{RePr}}\right\} - \eta||u_D||_{1/2,2,\partial\Omega_D},$$

and $(\mathrm{A}_{\mathbb{Q}})$

$$c_0^{1/a}(cc^\star)^2 < \frac{1}{\mathrm{Re}},$$

hold true, where η is defined by (28). Then problem ($\mathbb{Q}_{\mathrm{w}}^n$) has a solution $(\mathbf{v}_n, u_n) \in V_1^n \times V_2^n$. Moreover,

$$||\mathbf{v}_n|| \le k, \qquad ||u_n|| \le k, \tag{46}$$

for some positive constant k, which does not depend on n.

Proof. It is analogous to the proof of Theorem 4.1. □

Theorem 5.2. *Assume $(\mathrm{A}_{\mathbb{Q}})$ holds true. Then problem ($\mathbb{Q}_{\mathrm{w}}$) has a solution $\mathbf{v}_0 \in K(\mathbf{V})$, $u_0 \in V_2$.*

Proof. For $n \in \mathbb{N}$, let $(\mathbf{v}_n, u_n) \in V_1^n \times V_2^n$ denote a solution to ($\mathbb{Q}_{\mathrm{w}}^n$). From Theorem 5.1, we have that $(\mathbf{v}_n)$ and (u_n) are bounded sequences in V_1 and V_2, respectively. Hence, they admit weakly convergent subsequences, which we will also denote by $(\mathbf{v}_n)$ and (u_n). Let $\mathbf{v}_0 \in V_1$ and $u_0 \in V_2$ be the weak limits of $(\mathbf{v}_n)$ and (u_n), respectively. From Theorem 4.2, we know that $\mathbf{v}_0 \in V_1$ and $u_0 \in V_2$ satisfy the variational equation in problem ($\mathbb{Q}_{\mathrm{w}}$). Following the same arguments presented in the Section 4 of [25], it can be shown that $\mathbf{v}_0 \in V_1$ and $u_0 \in V_2$ also satisfy the variational inequality in problem ($\mathbb{Q}_{\mathrm{w}}$) (not reproduced here). Finally, we observe that the proof of $\mathbf{v}_0 \in K(\mathbf{V})$ given in [25] also applies in our case, so the proof is finished. □

Remark 2. *When $\mathrm{Gr} = 0$, problem ($\mathbb{Q}$) decouples into a Navier-Stokes, and a convection diffusion system. The former was studied by Kračmar and Neustupa in [25]. They proved existence of solution to the variational inequality in problem ($\mathbb{Q}_{\mathrm{w}}$), provided Re, c_0 and a satisfy a relation of the form $c_0 < \left(\frac{c_5}{\mathrm{Re}}\right)^a$, where c_5 is a positive constant which depends on d, Ω, $\mathbf{v}_{in}$, $\mathbf{f}_1$, and $\partial\Omega_{out}$. This result is recovered here. In fact, when $\mathrm{Gr} = 0$, the condition $(\mathrm{A}_{\mathbb{Q}})_1$ reduces to*

$$\eta||u_D||_{1/2,2,\partial\Omega_D} < \min\left\{\frac{1}{2}\left(\frac{1}{\mathrm{Re}} - c_0^{1/a}(cc^\star)^2\right), \frac{1}{\mathrm{RePr}}\right\},$$

which only affects the convection-diffusion system, and $(\mathrm{A}_{\mathbb{Q}})_2$ becomes a condition like the one in [25].

The following result is the analogue to Corollary 4.1, now for problem ($\mathbb{Q}_{\mathrm{w}}$).

Corollary 5.1. *There exist positive constants $\delta_i^* = \delta_i^*(c_0, a, d, \Omega, u_D, \partial\Omega_{out}, \partial\Omega_D)$, $i = 0, \cdots, 3$, such that if*

$$0 < \mathrm{Gr} \le \delta_0^*, \qquad \delta_1^* < \mathrm{Re} < \delta_2^*, \qquad \text{and} \qquad 0 < \mathrm{Pr} \le \delta_3^*, \tag{47}$$

then problem ($\mathbb{Q}_{\mathrm{w}}$) has a solution.

Proof. It is similar to the proof of Corollary 4.1. □

We end this Section with two results regarding strong properties of weak solutions to problem ($\mathbb{Q}_{\mathrm{w}}$) when some additional assumptions are considered. Since their proofs can be obtained by a direct combination of the arguments in the demonstration of Theorem 4.3, with Theorems 1 and 2 from [25], we do not present them here.

Theorem 5.3. *Suppose* $\mathbf{v}_0 \in W^{2,2}(\Omega)^d \cap K(\mathbf{V})$*,* $u_0 \in W^{2,2}(\Omega) \cap V_2$ *solve* $(\mathbb{Q}_{\mathrm{w}})$*, with* $\mathbf{V} \in W^{2,2}(\Omega)^d$*,* $U \in W^{2,2}(\Omega)$*. Then there exists a function* $p \in W^{1,2}(\Omega)$ *such that* $\mathbf{v} = \mathbf{v}_0 + \mathbf{V}$*,* $u = u_0 + U$ *and* p *satisfies problem* $(\mathbb{Q})$ *almost everywhere, with the do-nothing condition in* $(\mathbb{BC}^{\mathrm{II}}_{\partial\Omega_{out}})$ *replaced by*

$$\int_{\partial\Omega_{out}} \left(\frac{1}{\mathrm{Re}} \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n} \right) \cdot (\mathbf{w} - \mathbf{v}_0) d\sigma \geq 0 \qquad \forall\, \mathbf{w} \in K(\mathbf{V}).$$

Theorem 5.4. *Consider the same assumptions of Theorem 5.3. In addition, assume that there exists a neighborhood* E *of zero in* V_1 *such that* $\mathbf{v}_0 + \mathbf{w} \in K(\mathbf{V})$ *for all* $\mathbf{w} \in E$*. Then* $\mathbf{v} = \mathbf{v}_0 + \mathbf{V}$ *and the function* $p \in W^{1,2}(\Omega)$ *given by Theorem 5.3, satisfy the do-nothing condition in* $(\mathbb{BC}^{\mathrm{II}}_{\partial\Omega_{out}})$ *a.e. on* $\partial\Omega_{out}$*.*

6 Conclusion

We studied a steady Boussinesq system with mixed boundary conditions that arises when modeling energy systems in buildings. The heat transfer problem takes place in a room that presents an outlet where the fluid is allowed to flow without restrictions, an inlet through which the fluid only can enter, a heat source at some part of its boundary, and insulated walls. The heat transfer at the open boundary was considered by a nonlinear condition that involves the temperature at the outlet, the velocity of the fluid, and the ambient temperature. It was derived from physical assumptions, encouraged by numerical and experimental studies for open cavities, which are consistent with the boundary condition imposed on the fluid flow at the outlet. The latter was considered from a twofold approach based on the so called “do-nothing” condition. The two of them were introduced for Navier-Stokes systems, aiming to control the kinetic energy from the effects of the re-entering fluid flow. First, we considered the “directional do-nothing” condition, which introduces a perturbation on the do-nothing condition in terms of the backflow. Second, the do-nothing condition was considered jointly with a bound for the incoming flows. We presented variational formulations for the two problems, and proved that they are well-posed. Further, we investigated strong properties of the weak solutions when additional regularity assumptions are made on them. The present theoretical results encourage future numerical studies of the Boussinesq system with the novel non-smooth boundary condition for the heat transfer at the open boundary presented here.

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