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Dissipative and non-dissipative evolutionary quasi-variational inequalities with gradient constraints

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Abstract

Evolutionary quasi-variational inequality (QVI) problems of dissipative and non-dissipative nature with pointwise constraints on the gradient are studied. A semi-discretization in time is employed for the study of the problems and the derivation of a numerical solution scheme, respectively. Convergence of the discretization procedure is proven and properties of the original infinite dimensional problem, such as existence, extra regularity and non-decrease in time, are derived. The proposed numerical solver reduces to a finite number of gradient-constrained convex optimization problems which can be solved rather efficiently. The paper ends with a report on numerical tests obtained by a variable splitting algorithm involving different nonlinearities and types of constraints.

1 Introduction

The notion of a Quasi-Variational Inequality (QVI) was introduced by Lions and Bensoussan in [6] and [34] in connection with impulse control problems and in a general setting for obstacle-type problems. For this problem class, the state of the underlying system is charaterized by a variational inequality involving a set-valued constraint mapping, which again depends on the state variable. For the aforementioned obstacle-type constraint, for instance, the QVI setting imposes state-dependent upper (and/or lower) bounds on the state variable. We note that QVIs represent generalizations of Variational Inequalities (VIs) and arise as mathematical models of various phenomena. Indeed, instances of QVIs can be found in game theory, solid and continuum mechanics or electrostatics, to mention only a few. For further examples of QVI-models and associated analytical investigations, we refer here to [7, 13, 18, 33, 35, 38, 41] and the monographs [2, 32].

A very interesting QVI model involving pointwise constraints on the gradient of the state variable in a parabolic setting is related to superconductivity. The QVI arises here as an equivalent reformulation of Bean's critical state model; see, e.g., [41, 45, 3, 36]. General existence results, approximation techniques, and numerical solution procedures for this and related gradient constrained problems can be found in the work by Rodrigues and Santos in [45] and the first two authors of this paper in [23]. More specifically, in [45] an approximation technique replacing the QVI by a sequence of quasi-linear partial differential equations (PDEs) is utilized. On the other hand, in [23] a semi-group approach is employed for proving existence of a solution and its discrete approximations. Alternatively, the gradient constrained QVI can be re-written as a generalized equation rendering the QVI problem a particular instance of a yet more general problem class; see, e.g., [28, 29]. Additional work of Kenmochi and collaborators can be found in [27, 15, 26, 16].

The first physical applications of non-dissipative (in the sense that the spatial part of the partial differential operator vanishes) QVIs with gradient constraints were studied by Prigozhin in [39] when modeling the surface growth of a cohesionless granular material that is poured on a supporting structure.

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Subsequently, several physical models for the growth of sandpiles as well as of river networks were developed by resorting to VI and QVI problems with gradient type constraints. This is evidenced by a series of seminal papers by Prigozhin [4, 43, 42, 40]. In this body of work, not only theoretical aspects are studied but also the numerical simulation is considered. It is worth mentioning that some of the aforementioned models are within the scope of the general results in [46], which involve quasi-linear first-order QVI problems.

Interestingly, despite their wide applicability the literature on solution algorithms for QVIs with gradient constraints is rather scarce. Some of the few papers on numerical solvers include [22] in the elliptic case, and [23, 3, 4, 24] in the time evolution case. This scarcity of solvers is mainly due to the highly nonlinear and nonsmooth nature of the problems and the fact that QVIs (in the elliptic setting) are typically *not* related to first-order conditions of constrained energy minimization. For the iterative solution, in some cases these challenges may be overcome by considering fixed point iterations which, however, require rather strong assumptions for convergence.

In view of the above discussion, this paper extends the current state of the art in two directions: (i) In both, a dissipative and a non-dissipative, settings we are interested in obtaining existence and extra-regularity results, as well as qualitative properties of solutions such as the non-decrease in time. (ii) We establish a solution algorithm involving only a finite number $N \in \mathbb{N}$ of convex sub-problems in a time discrete setting and where convergence of the discrete solutions u^N to the solution of the original problem u is guaranteed. While, in the dissipative setting, these results apply to problems in transient electrostatics or thermo-plasticity, the non-dissipative setting is of interest in the modeling of growth behavior of granular materials.

The rest of the paper is organized as follows. In section 2 we provide the notation used throughout the paper and elementary results involving variational inequalities such as stability of solutions with respect to Mosco convergence of constraint sets. The problem formulation and its semi-discrete counterparts are given in section 3. The main results concerning the non-dissipative problem are the subject of section 4 and the ones for the dissipative problem can be found in section 5. In these sections, we provide existence and regularity results for the original evolutionary QVIs and properties concerning their time discrete approximations. The paper ends by a report on numerical tests in section 6, where we show that a variable splitting approach with rather simple subproblems can be used as a solver.

2 Notation and Preliminaries

The sets of natural and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively, and positive real numbers by $\mathbb{R}_+$. The Euclidian norm in $\mathbb{R}^n$ is written as $|\cdot|$, and the positive and negative parts for $x \in \mathbb{R}$ as $x^+ := \max(x, 0)$ and $x^- := -\min(x, 0)$, respectively. Further, for a measurable set $\Omega \subset \mathbb{R}^n$, we denote its measure by $|\Omega|$.

For $\nu \in \mathbb{R}_+$, the set $L^\infty_\nu(\Omega)$ is defined as $L^\infty_\nu(\Omega) := \{v \in L^\infty(\Omega) : v \geq \nu \text{ a.e.}\}$ where where "a.e." stands for "almost everywhere". Additionally, "for a.e." stands for "for almost every". Further, $L^\infty_+(\Omega)$ corresponds to the cone of a.e. non-negative functions in $L^\infty(\Omega)$. We denote by $H^1_0(\Omega)$ the usual Sobolev space of $L^2(\Omega)$ functions with weak derivatives also in $L^2(\Omega)$ and zero on $\partial\Omega$ (in the sense of the trace), and we write $\langle\cdot,\cdot\rangle : H^{-1}(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ for the usual duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Further, $W^{1,\infty}(\Omega)$ is the space of uniformly Lipschitz continuous functions over $\overline{\Omega}$.

A function $F : (0, T) \to X$, where $T > 0$ and X is a Banach space, is called Bochner measurable if there exists a sequence $\{F_n\}$ of simple X-valued functions such that $\lim_{n\to\infty} F_n(t) = F(t)$ in X and for a.e. $t \in (0, T)$ (see [19]). We denote by $L^p(0, T; X)$ the (Lebesgue-Bochner) space of

Bochner measurable X-valued functions with domain $(0,T)$ such that $\int_0^T |F(t)|_X^p \mathrm{d}t < +\infty$ where the integral is taken in Lebesgue's sense. Further, the space of Lipschitz continuous X-valued maps on $[0,T]$ is denoted by $C^{0,1}([0,T];X)$.

Let $f \in L^2(\Omega)$, $\mathbf{K}$ be a closed, convex and nonempty subset of $H_0^1(\Omega)$, and suppose the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ induces a continuous coercive bilinear form $\langle Au, v\rangle := a(u,v)$ with $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$. Then, we denote by $\mathbb{S}(A,f,\mathbf{K})$ the unique solution (see [30] for the existence and uniqueness proof) of the problem:

$$\text{Find } u \in \mathbf{K} : \langle Au - f, v - u\rangle \geq 0, \quad \forall v \in \mathbf{K}. \tag{1}$$

It is well-known (see also [30]) that $(u,v) \mapsto a(u,v)$ is a bilinear coercive form, if and only if, the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is linear, continuous and uniformly monotone, i.e., if there is $c > 0$ such that

$$\langle Av - Aw, v - w\rangle \geq c|v - w|^2_{H_0^1(\Omega)}, \quad \forall v, w \in H_0^1(\Omega). \tag{2}$$

It should be noted that if $\mathbf{K}$ is a *bounded*, closed, convex and nonempty subset of $H_0^1(\Omega)$ and $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ linear, continuous and strictly monotone, i.e., A satisfies (2) with $c = 0$ and $\langle Aw, w\rangle = 0 \iff w = 0$, then $\mathbb{S}(A,f,\mathbf{K})$ is also uniquely defined for each $f \in L^2(\Omega)$ (see [30])

We make use of the concept of a *lower solution* of a variational inequality which was initially developed by Bensoussan.

Definition 1 (LOWER SOLUTIONS)**.** *We say that $z \in \mathbf{K}$ is a lower solution for the triple $(A,f,\mathbf{K})$, if $\langle Az - f, \phi\rangle \leq 0$ for all $\phi \in H_0^1(\Omega)$ such that $\phi \geq 0$ a.e. in Ω.*

In the case where $\mathbf{K} = \{v \in H_0^1(\Omega) : v \leq \varphi \text{ a.e. in } \Omega\}$, with $\varphi \in L^\infty_+(\Omega)$, we have that $\mathbb{S}(A,f,\mathbf{K})$ is a lower solution, and for any lower solution z, we have that $z \leq \mathbb{S}(A,f,\mathbf{K})$ (see proposition 5 in the appendix A).

Some of the subsequent results concern convergence of closed, convex and non-empty subsets of a reflexive Banach space. For this matter, we make use of Mosco convergence (see [37, 44]):

Definition 2 (MOSCO CONVERGENCE)**.** *Let $\mathbf{K}$ and $\mathbf{K}_n$, for each $n \in \mathbb{N}$, be non-empty, closed and convex subsets of X, a reflexive Banach space. We say that the sequence $\{\mathbf{K}_n\}$ converges to $\mathbf{K}$ in the sense of Mosco as $n \to \infty$ if:*

i. $\forall v \in \mathbf{K}, \exists v_n \in \mathbf{K}_n : v_n \to v$ *in X.*

ii. *If $v_n \in \mathbf{K}_n$ and $v_n \rightharpoonup v$ in X along a subsequence, then $v \in \mathbf{K}$.*

In this case we write $\mathbf{K}_n \xrightarrow{\mathrm{M}} \mathbf{K}$, as $n \to \infty$.

An important consequence of Mosco convergence in our context for $X = H_0^1(\Omega)$, is given by the fact that the map $\mathbf{K} \mapsto \mathbb{S}(A,f,\mathbf{K})$ is continuous in $H_0^1(\Omega)$ with respect to the topology induced by Mosco convergence. In other words, $\mathbf{K}_n \xrightarrow{\mathrm{M}} \mathbf{K}$ implies $\mathbb{S}(A,f,\mathbf{K}_n) \to \mathbb{S}(A,f,\mathbf{K})$ in $H_0^1(\Omega)$ as $n \to \infty$. For a proof, we refer to [44], for instance.

3 Problem Formulation

We are interested in the following problems (P_0) and (P_1), which we refer to as the non-dissipative and the dissipative problem, respectively. Both problems are special cases of the following general formulation:

Problem (P). *Find $u \in L^2(0,T;H_0^1(\Omega))$, with $u(0) = u_0 \in H_0^1(\Omega)$ and $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ such that for a.e. $t \in (0,T)$, $u(t) \in \mathbf{K}(\Phi(t,u(t)))$ and that for every $v \in L^2(0,T;H_0^1(\Omega))$, with $v(t) \in \mathbf{K}(\Phi(t,u(t)))$ for a.e. $t \in (0,T)$, it holds that*

$$\int_0^T \langle \partial_t u(t) + Au(t) - \Theta(t,u(t)) - f(t), v(t) - u(t)\rangle \,\mathrm{d}t \geq 0, \tag{3}$$

where, for a non-negative ϕ, $\mathbf{K}(\phi) \subset H_0^1(\Omega)$ is defined as

$$\mathbf{K}(\phi) := \{v \in H_0^1(\Omega) : |\nabla v| \leq \phi \text{ a.e. in } \Omega\}. \tag{4}$$

The tools for analyzing problem (P) vary significantly with respect to the choice of A. Therefore, we distinguish the following two different problems announced above:

Problem (P_0). *Solve problem (P) with $A \equiv 0$.*

Problem (P_1). *Solve problem (P) when $A \not\equiv 0$ is a monotone operator.*

Applications for these two problems are diverse. For example, problem (P_0) arises in the mathematical modelling of surface growth for granular cohesionless materials and in the determination of lakes and river networks, while problem (P_1) is used in superconductivity for certain geometries, as a model for the magnetic field (see [41, 3, 4, 43, 42, 40]).

The requirements on Θ, Φ and f are different for the two cases above and they are made explicit in the beginning of section 4 and section 5 below.

It should be noted that if $\phi \in L^2(\Omega)$ is non-negative, then $\mathbf{K}(\phi) \subset H_0^1(\Omega)$ is closed, convex, bounded and $0 \in \mathbf{K}(\phi)$. In addition to $\mathbf{K}(\phi)$, we are also interested in two other types of set-valued mappings: For w non-negative, these are $\mathbf{K}^+(w) \subset H_0^1(\Omega)$ and $\mathbf{K}^\pm(w) \subset H_0^1(\Omega)$ defined by

$$\begin{aligned}\mathbf{K}^+(w) &: = \{v \in H_0^1(\Omega) : v(x) \leq w(x)\mathrm{dist}(x,\partial\Omega) \quad \text{for a.e. } x \in \Omega\}, \quad \text{and}\\ \mathbf{K}^\pm(w) &: = \{v \in H_0^1(\Omega) : |v(x)| \leq w(x)\mathrm{dist}(x,\partial\Omega) \quad \text{for a.e. } x \in \Omega\},\end{aligned}$$

where dist$(x,\partial\Omega)$ is the distance of $x \in \Omega$ to the boundary $\partial\Omega$ of Ω, respectively.

The following sequence of approximating problems represents a specific semi-discretization of (P) in time. It can be described as an implicit Euler integration scheme where the nonlinearities associated with Θ and Φ are lagged behind in the discretization.

Problem (P^N). *Let $N \in \mathbb{N}$, $k := T/N$, $t_n := nk$ and $\mathrm{I}_n := [t_{n-1}^N, t_n^N)$ with $n = 0,1,\ldots,N$. Find $\{u_n^N\}_{n=0}^N$ with $u_0^N = u_0$, $u_n^N \in \mathbf{K}(\Phi(t_{n-1}^N, u_{n-1}^N))$, and for which*

$$\left\langle \frac{u_n^N - u_{n-1}^N}{k} + A(u_n^N) - \Theta(t_{n-1}^N, u_{n-1}^N) - f_n^N, v - u_n^N \right\rangle \geq 0, \tag{5}$$

for all $v \in \mathbf{K}(\Phi(t_{n-1}^N, u_{n-1}^N))$ with

$$f^N = \sum_{n=1}^N f_n^N \chi_{[t_{n-1}^N, t_n^N)} \qquad \text{and} \qquad f_n^N = \frac{1}{k}\int_{t_{n-1}^N}^{t_n^N} f(t)\,\mathrm{d}t.$$

Problem (P^N) is equivalent to solving N variational inequalities with gradient constraints. Hence, it is not only useful to find properties of the solution to problem (P) (as we observe in what follows), but it is also suitable for numerical implementation as we shown in section 6. Analogously as with (P_0) and (P_1), we distinguish the following two different problems:

Problem (P_0^N)**.** *Solve problem* (P^N) *with* $A \equiv 0$.

Problem (P_1^N)**.** *Solve problem* (P^N) *when* $A \not\equiv 0$ *is a monotone operator.*

3.1 The instantaneous problem

A typical variation of problem (P) arises when the integral inequality in (3) is replaced by a non-negativity requirement for the integrand for a.e. $t \in (0,T)$. This problem is termed the *instantaneous* problem:

Problem (iP)**.** *Find* $u \in L^2(0,T;H_0^1(\Omega))$*, with* $u(0) = u_0 \in H_0^1(\Omega)$ *and* $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ *such that for a.e.* $t \in (0,T)$*,* $u(t) \in \mathbf{K}(\Phi(t,u(t)))$ *and*

$$\langle \partial_t u(t) + A(u(t)) - \Theta(t,u(t)) - f(t), v - u(t)\rangle \geq 0, \tag{6}$$

for all $v \in \mathbf{K}(\Phi(t,u(t)))$.

Further the instantaneous versions of problems (P_0) and (P_1) are referred to as problems (iP_0) and (iP_1), respectively. In order to provide a link between (P) and (iP) we need to define versions of the constraints $\mathbf{K}, \mathbf{K}^+$, and $\mathbf{K}^\pm$ on the cylinder $(0,T)\times\Omega$. In fact, provided that $\varphi \in L^2(0,T;L^\infty_\nu(\Omega))$ we define

$$\mathscr{K}(\varphi) := \{v \in L^2(0,T;H_0^1(\Omega)) : v(t) \in \mathbf{K}(\varphi(t)) \text{ for a.e. } t \in (0,T)\}, \tag{7}$$

and analogously for $\mathscr{K}^+(\varphi)$ and $\mathscr{K}^\pm(\varphi)$.

Under certain conditions problems (P) and (iP) are equivalent. The proof of this assertion is based on the application of the following result.

Proposition 1. *For* $\Gamma \in L^2(0,T;L^2(\Omega))$ *with* $\Gamma(t) := \gamma(t)\varphi$*,* $\gamma \in C([0,T],\mathbb{R}_+)$ *and* $\varphi \in L^\infty_\nu(\Omega)$*, suppose that* $u \in L^2(0,T;H_0^1(\Omega))$*,* $u(t) \in \mathbf{K}(\Gamma(t))$ *for a.e.* $t \in (0,T)$*, and* $F \in L^\infty(0,T;H^{-1}(\Omega))$*. Then, the following holds true:*

$$\langle F(t), v - u(t)\rangle \geq 0,\ \forall v \in \mathbf{K}(\Gamma(t)),\ \text{for a.e. } t \in (0,T), \tag{8}$$

if and only if,

$$\int_0^T \langle F(\tau), w(\tau) - u(\tau)\rangle \,\mathrm{d}\tau \geq 0,\ \forall w \in \mathscr{K}(\Gamma). \tag{9}$$

Proof. Let $w \in \mathscr{K}(\Gamma)$ be arbitrary. Then, by definition $w(t) \in \mathbf{K}(\Gamma(t))$ for a.e. $t \in (0,T)$, and if (8) holds true, then it follows that (9) is satisfied by time integration of the initial inequality.

Next we prove the reverse implication "(9)$\Longrightarrow$ (8)". Let $\tau \in (0,T)$ and $v \in \mathbf{K}(\Gamma(\tau))$ be arbitrary. For sufficiently small $\epsilon > 0$, define

$$w_\epsilon^\tau(t) := v\frac{\gamma(t)}{\sup_{s\in(\tau-\epsilon,\tau+\epsilon)}\gamma(s)}\chi_{(\tau-\epsilon,\tau+\epsilon)}(t) + u(t)\chi_{(0,T)\setminus(\tau-\epsilon,\tau+\epsilon)}(t),$$

where χ_O denotes the characteristic function of the set O. It follows that $w_\epsilon^\tau \in L^2(0,T;H_0^1(\Omega))$ and $|\nabla w_\epsilon^\tau(t)| = |\nabla u(t)| \leq \Gamma(t)$ if $t \in (0,T) \setminus (\tau-\epsilon, \tau+\epsilon)$ and if $t \in (\tau-\epsilon,\tau+\epsilon)$ we have

$$|\nabla w_\epsilon^\tau(t)| = \frac{\gamma(t)}{\sup_{s\in(\tau-\epsilon,\tau+\epsilon)}\gamma(s)}|\nabla v| \leq \frac{\gamma(t)}{\sup_{s\in(\tau-\epsilon,\tau+\epsilon)}\gamma(s)}\gamma(\tau)\varphi \leq \gamma(t)\varphi = \Gamma(t).$$

Using $k_\epsilon^\tau(t) := \gamma(t)/\sup_{s\in(\tau-\epsilon,\tau+\epsilon)}\gamma(s)$, we infer

$$\begin{aligned} 0 \leq \frac{1}{2\epsilon}\int_0^T \langle F(t), w_\epsilon^\tau(t) - u(t)\rangle \,\mathrm{d}t &= \frac{1}{2\epsilon}\int_{\tau-\epsilon}^{\tau+\epsilon} \langle F(t), k_\epsilon^\tau(t)v - u(t)\rangle \,\mathrm{d}t \\ &= \frac{1}{2\epsilon}\int_{\tau-\epsilon}^{\tau+\epsilon} \langle F(t), (k_\epsilon^\tau(t)-1)v\rangle \,\mathrm{d}t + \frac{1}{2\epsilon}\int_{\tau-\epsilon}^{\tau+\epsilon}\langle F(t), v-u(t)\rangle \,\mathrm{d}t. \end{aligned}$$

Since $F \in L^\infty(0,T;H^{-1}(\Omega))$ and $\lim_{\epsilon\downarrow 0}\sup_{\tau-\epsilon\leq t\leq\tau+\epsilon}|k_\epsilon^\tau(t)-1| = 0$, we have

$$\begin{aligned} &\lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\left|\int_{\tau-\epsilon}^{\tau+\epsilon}\langle F(t), (k_\epsilon^\tau(t)-1)v\rangle \,\mathrm{d}t\right| \\ &\qquad\leq |F|_{L^\infty(0,T;H^{-1}(\Omega))}|v|_{H_0^1(\Omega)}\left(\lim_{\epsilon\downarrow 0}\sup_{\tau-\epsilon\leq t\leq\tau+\epsilon}|k_\epsilon^\tau(t)-1|\right) = 0. \end{aligned}$$

Moreover, $t \mapsto (F(t), v-u(t))$ belongs to $L^1(0,T)$ and hence, almost every point t is a Lebesgue point (see [12] for a proof). As a consequence, we obtain

$$\lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_{\tau-\epsilon}^{\tau+\epsilon}\langle F(t), v-u(t)\rangle \,\mathrm{d}t = \langle F(\tau), v-u(\tau)\rangle \geq 0,$$

for a.e. $\tau \in (0,T)$. □

4 The non-dissipative problem (P_0)

In this section we consider problem (P_0). For its investigation, throughout this section we rely on the following assumptions on f, u_0, Θ and Φ.

Assumption 1.

i. $f \in L^\infty(0,T;L^2(\Omega))$ *is non-negative, i.e.,* $f(t) \geq 0$ *a.e. in* Ω*, for a.e.* $t \in (0,T)$.

ii. *The initial condition* $u_0 \in H_0^1(\Omega)$ *satisfies* $|\nabla u_0| \leq \Phi(0,u_0)$ *a.e. in* Ω.

iii. $\Theta : [0,T]\times L^2(\Omega) \to L^2(\Omega)$ *is uniformly continuous and satisfies* $\Theta(t,v) \geq 0$ *a.e. if* $v \geq u_0$ *a.e. in* Ω*, for a.e.* $t \in [0,T]$. *It is further assumed that* Θ *has* α*-order of growth:*

$$\exists\,\alpha > 0, L_\Theta > 0: \quad |\Theta(t,v)|_{L^2(\Omega)} \leq L_\Theta |v|^\alpha_{L^2(\Omega)}, \quad \forall t \in [0,T], \forall v \in L^2(\Omega). \tag{10}$$

iv. *The operator* $\Phi : [0,T]\times L^2(\Omega) \to L^\infty(\Omega)$ *is uniformly continuous and* $\Phi(t,v) \geq \nu > 0$ *a.e. in* Ω*, for a.e.* $t \in [0,T]$ *and all* $v \in L^2(\Omega)$. *We also assume that* Φ *is non-decreasing:*

$$0 \leq t_1 \leq t_2 \leq T,\ \ u_0 \leq v_1 \leq v_2 \ \text{ a.e.} \quad \Longrightarrow \quad \Phi(t_1,v_1) \leq \Phi(t_2,v_2) \ \text{ a.e.},$$

and that $v \mapsto \Phi(T,v)$ *maps bounded sets in* $L^2(\Omega)$ *into bounded sets in* $L^\infty(\Omega)$.

It should be noted that even in the case where $\Phi(t,v)=\phi\in L^\infty_\nu(\Omega)$ for all $(t,v)\in[0,T]\times L^2(\Omega)$ so that $\mathbf{K}(\Phi(t,v))=\mathbf{K}(\phi)$ is a constant set, no assumptions on the monotonicity of $-\Theta$ are made and as a consequence the standard theory for parabolic variational inequalities can not be applied here. Note also that since $0<\nu\le\Phi(t,v)\le\Phi(T,v)$ a.e. in Ω, for a.e. $t\in[0,T]$ and all $v\in L^2(\Omega)$, we actually observe that for each $t\in[0,T]$, $v\mapsto\Phi(t,v)$ maps bounded sets in $L^2(\Omega)$ into bounded sets in $L^\infty(\Omega)$.

The main result of this section is stated next.

Theorem 1. *Let $\alpha\in[0,1]$ in (10). Then there exists a solution u^* to problem (P_0) that satisfies:*

(i)
$$u^*\in L^\infty(0,T;W^{1,\infty}_0(\Omega))\cap C^{0,1}([0,T];L^2(\Omega)),\qquad \partial_t u^*\in L^\infty(0,T;L^2(\Omega)).$$

(ii) It is non-decreasing, i.e., if $0\le t_1\le t_2\le T$ then $u_0\le u^*(t_1)\le u^*(t_2)\le u^*(T)$ a.e. in Ω.

(iii) Solves in addition problem (iP_0) if $\Phi(t,u^*(t))=\gamma(t)\varphi$ for $t\in[0,T]$ for some $\gamma\in C([0,T],\mathbb{R}^+)$ and $\varphi\in L^\infty_\nu(\Omega)$.

(iv) The sequence $\{\tilde{u}^N\}$ defined by

$$\tilde{u}^N(t)=u_0+\int_0^t\sum_{n=1}^N\frac{u_n^N-u_{n-1}^N}{k}\chi_{[t_{n-1},t_n)}(s)\,\mathrm{d}s,$$

where $\{u_n^N\}_{n=0}^N$ solves (P_0^N), satisfies

$$\tilde{u}^N\to u^*\text{ in }C([0,T];L^2(\Omega))\qquad\text{and}\qquad\partial_t\tilde{u}^N\rightharpoonup\partial_t u^*\text{ in }L^2(0,T;L^2(\Omega)),$$

along a subsequence.

Furthermore, if $\alpha>1$, then the same holds true provided that

$$|u_0|_{L^2(\Omega)}+L_\Theta|u_0|^\alpha_{L^2(\Omega)}+T^{1/2}|f|_{L^2(0,T;L^2(\Omega))}<\frac{1}{((\alpha-1)L_\Theta T)^{\frac{1}{\alpha-1}}}.\tag{11}$$

Remark 1. It should be noted that since $\partial_t u^*\in L^\infty(0,T;L^2(\Omega))$ the representation $u^*(t)=u_0+\int_0^t\partial_t u^*(s)\mathrm{d}s$ immediately implies that u^* is Lipschitz continuous in $L^2(\Omega)$.

Remark 2. In the case where $\alpha>1$, the condition (11) is a type of "small data" assumption. However, this condition does not imply that the solution u^* remains inactive over $[0,T]$, i.e., that $|\nabla u^*(t)|<\Phi(t,u^*(t))$ a.e. in Ω, for a.e. $t\in(0,T)$. This can be seen from the fact that for arbitrary u_0 and f, there will be a solution provided that L_Θ is small enough.

Note that if u^* solves (P_0), then the fact that it additionally solves (iP_0) if $\Phi(t,u^*(t))=\gamma(t)\varphi$ for $t\in[0,T]$ for some $\gamma\in C([0,T],\mathbb{R}^+)$ and $\varphi\in L^\infty_\nu(\Omega)$ follows by direct application of proposition 1. Further, in the trivial case $|u_0|_{L^2(\Omega)}=|f|_{L^2(0,T;L^2(\Omega))}=0$ we have $u_n^N=0$ for all $0\le n\le N$ and all $N\in\mathbb{N}$ and it is elementary to check that the solution $u^*=0$ satisfies the conditions of the previous theorem. Henceforth, we will assume throughout this section that $|u_0|_{L^2(\Omega)}+|f|_{L^2(0,T;L^2(\Omega))}>0$.

In order to prove theorem 1 we consider the following propositions and lemmas.

Proposition 2. *The solution $\{u_n^N\}_{n=0}^N$ to $(\mathrm{P}_0^\mathrm{N})$ is well-defined and the following assertions hold true:*

i. *For each $N \in \mathbb{N}$, $n \to u_n^N$ is non-decreasing, i.e., $u_{n-1}^N \leq u_n^N$ a.e. in Ω with $n = 1, 2, \dots, N$.*

ii. *If $\alpha \in [0, 1]$, then there is a constant $C_1 > 0$ such that*

$$|\nabla u_n^N| \leq C_1, \quad \text{a.e. on } \Omega, \tag{12}$$

uniformly in $n = 0, 1, \dots, N$ and $N \in \mathbb{N}$. If $\alpha > 1$, then the same holds true, provided that

$$|u_0|_{L^2(\Omega)} + L_\Theta |u_0|_{L^2(\Omega)}^\alpha + T^{1/2} |f|_{L^2(0,T;L^2(\Omega))} < \frac{1}{((\alpha - 1) L_\Theta T)^{\frac{1}{\alpha - 1}}}. \tag{13}$$

iii. *If $\alpha \in [0, 1]$ (or if $\alpha > 1$ and* (13) *holds), then there exists a constant $C_2 > 0$ such that*

$$\left| u_n^N - u_{n-1}^N \right|_{L^2(\Omega)} \leq C_2 k, \tag{14}$$

uniformly in $n = 1, 2, \dots, N$ and $N \in \mathbb{N}$.

Proof. Let $A_k : H_0^1(\Omega) \to H^{-1}(\Omega)$ be defined as

$$\langle A_k(w), v \rangle := \frac{1}{k} \int_\Omega w(x) v(x) \, \mathrm{d}x.$$

Then, A_k is strictly monotone over $H_0^1(\Omega)$, i.e., $\langle A_k(w), w \rangle = \frac{1}{k} |w|_{L^2(\Omega)}^2 \geq 0$ and $\langle A_k(w), w \rangle = 0$ implies $w = 0$. Also, A_k is linear, continuous and $\mathbf{K}(\Phi(t_{n-1}^N, u_{n-1}^N))$ is a closed, convex, bounded (because $\Phi(t, v) \in L_\nu^\infty(\Omega)$) and non-empty set. Hence, for any $g \in L^2(\Omega)$ the problem

$$\text{Find } u \in \mathbf{K}(\Phi(t_{n-1}^N, u_{n-1}^N)) : (A_k(u) - g, v - u) \geq 0, \quad \forall v \in \mathbf{K}(\Phi(t_{n-1}^N, u_{n-1}^N)),$$

admits a unique solution. This result follows from [30]. Then, for $u_{n-1}^N \in H_0^1(\Omega)$ and taking $g := \frac{1}{k} u_{n-1}^N + \Theta(t_{n-1}^N, u_{n-1}^N)$, $u_n^N \in \mathbf{K}(\Phi(t_{n-1}^N, u_{n-1}^N))$ is well defined by (5).

We concentrate first on **i** and proceed by induction. For u_1^N in (5) we consider $v := u_1^N + (u_0 - u_1^N)^+ = \max(u_0, u_1^N) \in H_0^1(\Omega)$ such that

$$\nabla v = \begin{cases} \nabla u_0, & u_0 \leq u_1^N; \\ \nabla u_1^N, & u_0 > u_1^N. \end{cases}$$

Since $|\nabla u_0| \leq \Phi(0, u_0)$ a.e. by assumption, $|\nabla v| \leq \Phi(0, u_0)$ a.e., and using this v in (5), we have

$$\left(\frac{u_1^N - u_0}{k} - \Theta(0, u_0) - f_1^N, (u_0 - u_1^N)^+ \right) \geq 0.$$

Also, $\Theta(0, u_0) \geq 0$ and $f_1^N \geq 0$ a.e., and hence

$$0 \geq -k(\Theta(0, u_0) + f_1^N, (u_0 - u_1^N)^+) \geq \left(u_0 - u_1^N, (u_0 - u_1^N)^+ \right),$$

which implies that $(u_0 - u_1^N)^+ = 0$. Hence $u_0 \leq u_1^N$ and $|\nabla u_1^N| \leq \Phi(0, u_0) \leq \Phi(t_1^N, u_1^N)$ a.e. in Ω because Φ is non-decreasing in both variables according to assumption 1.

Suppose $u_0 \leq u_{n-1}^N$ and $|\nabla u_{n-1}^N| \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ a.e. and let $v := \max(u_n^N, u_{n-1}^N) = u_n^N + (u_{n-1}^N - u_n^N)^+$. Since u_n^N solves (5), we have $|\nabla u_n^N| \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ and therefore $|\nabla v| \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ a.e. in Ω. Using this v in (5), we obtain

$$\left(\frac{u_n^N - u_{n-1}^N}{k} - \Theta(t_{n-1}^N, u_{n-1}^N) - f_n^N, (u_{n-1}^N - u_n^N)^+ \right) \geq 0.$$

This implies

$$\big(u^N_{n-1} - u^N_n, (u^N_{n-1} - u^N_n)^+\big) \le -k(\Theta(t^N_{n-1}, u^N_{n-1}) + f^N_n, (u^N_{n-1} - u^N_n)^+).$$

Since $u_0 \le u^N_{n-1}$ a.e., we observe that $\Theta(t^N_{n-1}, u^N_{n-1}) \ge 0$ a.e. and also, by assumption, $f^N_n \ge 0$. Therefore, $(u^N_{n-1} - u^N_n)^+ = 0$, i.e., $u^N_{n-1} \le u^N_n$ a.e., and by the fact that Φ is non-decreasing in both variables, we have $|\nabla u^N_n| \le \Phi(t^N_{n-1}, u^N_{n-1}) \le \Phi(t^N_n, u^N_n)$.

Next we focus on **ii**. Let $v = 0$ in (5). Reordering terms, it follows that $|u^N_n|^2_{L^2(\Omega)} \le (u^N_{n-1}, u^N_n) + k(\Theta(t^N_{n-1}, u^N_{n-1}) + f^N_n, u^N_n)$. Since $u_0 \le u^N_{n-1} \le u^N_n$ a.e. and $|\Theta(t,v)|_{L^2(\Omega)} \le L_\Theta |v|^\alpha_{L^2(\Omega)}$ for all $t \in [0,T], v \in L^2(\Omega)$, we infer

$$|u^N_n|_{L^2(\Omega)} - |u^N_{n-1}|_{L^2(\Omega)} \le L_\Theta k |u^N_{n-1}|^\alpha_{L^2(\Omega)} + k|f^N_n|_{L^2(\Omega)}. \tag{15}$$

Summation over n yields

$$\begin{aligned} |u^N_m|_{L^2(\Omega)} &\le |u_0|_{L^2(\Omega)} + L_\Theta \sum_{n=1}^m k|u^N_{n-1}|^\alpha_{L^2(\Omega)} + \sum_{n=1}^m k|f^N_n|_{L^2(\Omega)} \\ &= (|u_0|_{L^2(\Omega)} + kL_\Theta |u_0|^\alpha_{L^2(\Omega)}) + \sum_{n=1}^m k|f^N_n|_{L^2(\Omega)} + L_\Theta \sum_{n=1}^{m-1} k|u^N_n|_{L^2(\Omega)}. \end{aligned} \tag{16}$$

Recalling that $f^N_n = \frac{1}{k}\int_{t^N_{n-1}}^{t^N_n} f(t)\,\mathrm{d}t$, and $|\mathrm{I}_m| = t^N_m - t^N_{m-1} = k$ we obtain the bound

$$\sum_{n=1}^m k|f^N_n|_{L^2(\Omega)} \le \int_0^T |f(t)|_{L^2(\Omega)}\,\mathrm{d}t \le T^{1/2}|f|_{L^2(0,T;L^2(\Omega))}.$$

Further, considering $u^N := \sum_{n=1}^N u^N_n \chi_{[t^N_{n-1}, t^N_n)}$, we have for $t \in [t^N_{m-1}, t^N_m)$

$$\sum_{n=1}^{m-1} k|u^N_n|^\alpha_{L^2(\Omega)} \le \int_0^t |u^N(\tau)|^\alpha_{L^2(\Omega)}\,\mathrm{d}\tau.$$

Therefore, the inequality in (16) implies that

$$|u^N(t)|_{L^2(\Omega)} \le M_0 + L_\Theta \int_0^t |u^N(\tau)|^\alpha_{L^2(\Omega)}\,\mathrm{d}\tau,$$

with $M_0 := (|u_0|_{L^2(\Omega)} + L_\Theta|u_0|^\alpha_{L^2(\Omega)}) + T^{1/2}|f|_{L^2(0,T;L^2(\Omega))}$. Here we have used that $k = 1/N \le 1$. Now we consider three different cases: $0 \le \alpha < 1$, $\alpha = 1$ and $1 < \alpha$, respectively.

For $\alpha = 1$, by Gronwall's inequality, we have $|u^N(t)|_{L^2(\Omega)} \le M_0 e^{L_\Theta t}$ and

$$|u^N_n|_{L^2(\Omega)} \le \big((1 + L_\Theta)|u_0|_{L^2(\Omega)} + T^{1/2}|f|_{L^2(0,T;L^2(\Omega))}\big)\, e^{L_\Theta T} =: M.$$

For $0 \le \alpha < 1$ and $1 < \alpha$, Gronwall's inequality can not be applied, but the generalization by Willet and Wong (see Theorem 2 in [51]) is applicable. In the case $\alpha > 1$, condition (13) is equivalent (in terms of M_0) to $M_0^{1-\alpha} + (1-\alpha)L_\Theta T > 0$, and hence for $\alpha \in [0,1) \cup (1,\infty)$, $|u^N(t)|_{L^2(\Omega)} \le (M_0^{1-\alpha} + (1-\alpha)L_\Theta t)^{\frac{1}{1-\alpha}}$. As a consequence, we get

$$|u^N_n|_{L^2(\Omega)} \le \Big((|u_0|_{L^2(\Omega)} + L_\Theta|u_0|^\alpha_{L^2(\Omega)} + T^{1/2}|f|_{L^2(0,T;L^2(\Omega))})^{1-\alpha} + (1-\alpha)L_\Theta T\Big)^{\frac{1}{1-\alpha}} =: M.$$

Therefore, for all cases we obtain that $|u_n^N|_{L^2(\Omega)} \leq M$ holds uniformly. Since $|\nabla u_n^N| \leq \Phi(t_{n-1}^N, u_{n-1}^N) \leq \Phi(T, u_{n-1}^N)$ a.e. (because Φ is non-decreasing in both variables) and $\Phi(T,\cdot)$ maps bounded sets in $L^2(\Omega)$ into bounded sets in $L^\infty(\Omega)$, we have

$$|\nabla u_n^N| \leq \sup_{v\in L^2(\Omega):|v|_{L^2(\Omega)}\leq M} |\Phi(T,v)|_{L^\infty(\Omega)} =: C_1$$

where C_1 does not depend on n nor N.

Finally, we focus on **iii**. Since $n \mapsto u_n^N$ is non-decreasing, Φ is non-decreasing in both variables and $|\nabla u_{n-1}^N| \leq \Phi(t_{n-2}^N, u_{n-2}^N)$, we have $|\nabla u_{n-1}^N| \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ a.e. in Ω. Choosing $v = u_{n-1}^N$ in (5), we obtain

$$\left(\frac{u_n^N - u_{n-1}^N}{k}, u_n^N - u_{n-1}^N\right) \leq (\Theta(t_{n-1}^N, u_{n-1}^N) + f_n^N, u_n^N - u_{n-1}^N),$$

from which we infer

$$|u_n^N - u_{n-1}^N|_{L^2(\Omega)} \leq (L_\Theta |u_{n-1}^N|_{L^2(\Omega)}^\alpha + |f_n^N|_{L^2(\Omega)})k \leq C_2 k,$$

for $C_2 := L_\Theta M^\alpha + \text{ess-sup}_{t\in(0,T)}|f(t)|_{L^2(\Omega)}$. Since C_2 is independent of n and N, this finalizes the proof. ☐

Remark 1. In order for the previous result to hold, weaker conditions (than the ones assumed in the introduction) on the operators Θ and Φ can be considered. In fact, the uniform continuity of both operators is superfluous and if $\Phi : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ is non-decreasing in both variables, $\Phi(t,v) \geq 0$ for all $t \in [0,T]$ and all $v \in L^2(\Omega)$ and for each $t \in [0,T]$, $v \mapsto \Phi(t,v)$ maps bounded sets in $L^2(\Omega)$ into bounded sets in $L^2(\Omega)$, then the previous proposition also holds. However, for the following results the continuity assumption is heavily invoked.

We define u^N, u_-^N and $\tilde{u}^N$, which correspond to functions in $L^2(0,T;H_0^1(\Omega))$ constructed with different arrangements of $\{u_n^N\}_{n=0}^N$ and that are used throughout the paper for the characterization of the solutions to problem (P). In fact, we define

$$u^N(t) := \sum_{n=1}^N u_n^N \chi_{[t_{n-1},t_n)}(t), \qquad u_-^N(t) := \sum_{n=1}^N u_{n-1}^N \chi_{[t_{n-1},t_n)}(t), \tag{17}$$

and $\tilde{u}^N \in C([0,T];H_0^1(\Omega))$ as

$$\tilde{u}^N(t) := u_0 + \int_0^t \sum_{n=1}^N \frac{u_n^N - u_{n-1}^N}{k} \chi_{[t_{n-1},t_n)}(s)\,\mathrm{d}s. \tag{18}$$

For $t \in [t_{m-1}, t_m)$, the latter definition yields

$$\tilde{u}^N(t) = u_{m-1}^N + \frac{u_m^N - u_{m-1}^N}{k}(t - t_{m-1}) = \frac{t - t_{m-1}}{k} u_m^N + \left(1 - \frac{t - t_{m-1}}{k}\right) u_{m-1}^N.$$

If $n \mapsto u_n^N$ is non-decreasing, then we have that the three mappings u^N, u_-^N and $\tilde{u}^N$ are non-decreasing, as well. They also satisfy $u_0 \leq u_-^N(t) \leq \tilde{u}^N(t) \leq u^N(t)$ a.e. in Ω, for a.e. $t \in (0,T)$, and the following inequality holds:

$$|\tilde{u}^N(t) - u_-^N(t)|_{L^2(\Omega)} \leq |u^N(t) - u_-^N(t)|_{L^2(\Omega)} \leq kC_2, \quad \forall t \in [0,T]. \tag{19}$$

In particular, note that $\tilde{u}^N \in W(0,T)$ where

$$W(0,T) := \{v \in L^2(0,T;H_0^1(\Omega)) : \partial_t v \in L^2(0,T;L^2(\Omega))\}.$$

As $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;L^2(\Omega))$ are reflexive spaces, $W(0,T)$ is a reflexive Banach space (see [11] or [47]) endowed with the norm

$$|v|_{W(0,T)} := |v|_{L^2(0,T;H_0^1(\Omega))} + |\partial_t v|_{L^2(0,T;L^2(\Omega))}.$$

Now, consider the sequence of continuous functions $\{\tilde{u}^N\}_{N=1}^{\infty}$. Under the assumptions of the previous proposition, we next characterize the limiting behaviour of $N \mapsto \tilde{u}^N$.

Theorem 2. *Suppose that for $N \in \mathbb{N}$, $\{u_n^N\}_{n=0}^N$ satisfies the following assumptions:*

a. *The map $n \mapsto u_n^N$ is non-decreasing.*

b. *There exists $C_1 > 0$ such that $|\nabla u_n^N| \le C_1$ a.e. in Ω uniformly in $n = 0, 1, \ldots, N$ and $N \in \mathbb{N}$.*

c. *There exists $C_2 > 0$ such that $\left|u_n^N - u_{n-1}^N\right|_{L^2(\Omega)} \le kC_2$ a.e. in Ω, uniformly in $n = 1, 2, \ldots, N$ and $N \in \mathbb{N}$.*

Then, there exist a $u^* \in W(0,T)$ such that

$$\tilde{u}^N \to u^* \text{ in } C([0,T];L^2(\Omega)) \qquad \text{and} \qquad \partial_t \tilde{u}^N \rightharpoonup \partial_t u^* \text{ in } L^2(0,T;L^2(\Omega)),$$

along a subsequence of $\{\tilde{u}^N\}_{N=1}^{\infty}$ defined in (18)*. Furthermore, $u^* : [0,T] \to L^2(\Omega)$ is Lipschitz continuous, non-decreasing, it satisfies $u^* \in L^\infty(0,T;W_0^{1,\infty}(\Omega))$ and $\partial_t u^* \in L^\infty(0,T;L^2(\Omega))$ and, in addition, $|\nabla u^*(t)| \le \Phi(t, u^*(t))$ a.e. in Ω, for a.e. $t \in (0,T)$.*

Proof. A direct calculation yields $|\nabla \tilde{u}^N(t)|_{L^2(\Omega)} \le C_1$ and $|\partial_t \tilde{u}^N(t)|_{L^2(\Omega)} \le C_2$ a.e. in Ω. In particular, $\tilde{u}^N$ is bounded in $W(0,T)$. Since $W(0,T)$ is reflexive and is compactly embedded into $L^2(0,T;L^2(\Omega))$ (by the Lions-Aubin Lemma, see Proposition 1.3, Chapter III in [47] or Theorem 3.4.13 in [11]), we have $\tilde{u}^N \rightharpoonup u^*$ in $L^2(0,T;H_0^1(\Omega))$, $\tilde{u}^N \to u^*$ and $\partial_t \tilde{u}^N \rightharpoonup \partial_t u^*$ both in $L^2(0,T;L^2(\Omega))$ along a subsequence for some $u^* \in W(0,T)$. Since also the space $W(0,T)$ is continuously embedded into $C([0,T];L^2(\Omega))$ by virtue of the representation $u^*(t) = u(0) + \int_0^t \partial_t u^*(s)\mathrm{d}s$ (see the proof of Theorem 3.4.13 in [11]), we have that $u^* \in C([0,T];L^2(\Omega))$.

Since $|\nabla \tilde{u}^N(t)| \le C_1$ a.e. with C_1 independent of t and $N \in \mathbb{N}$, we have that $\{\tilde{u}^N(t)\}$ is uniformly (in $t \in [0,T]$ and $N \in \mathbb{N}$) bounded in $H_0^1(\Omega)$. For a fixed $t \in [0,T]$, consider the sequence $\{\tilde{u}^N(t)\}$ in $H_0^1(\Omega)$. Hence, by the Rellich-Kondrachov Theorem, $\tilde{u}^N(t) \to v$ in $L^2(\Omega)$ along a subsequence, so that

$$\{\tilde{u}^N(t) : N = 1, 2, \ldots\} \quad \text{is precompact in } L^2(\Omega). \tag{20}$$

Since $\tilde{u}^N(t) := u_0 + \int_0^t \partial_t \tilde{u}^N(s)\,\mathrm{d}s$ and $|\partial_t \tilde{u}^N(t)|_{L^2(\Omega)} \le C_2$, we observe that

$$|\tilde{u}^N(\theta) - \tilde{u}^N(\eta)|_{L^2(\Omega)} \le C_2|\theta - \eta|, \qquad \forall \theta, \eta \in [0,T],$$

i.e., $\{\tilde{u}^N(t)\}$ is equicontinuous in $L^2(\Omega)$. Then, the Arzelá-Ascoli Theorem (see Theorem 2.0.15 in [14]), implies $\tilde{u}^{N_i} \to u^*$ in $C([0,T];L^2(\Omega))$ for some subsequence $\{\tilde{u}^{N_i}\}$. Considering $\tilde{u}^{N_i}$ in the above inequality and taking the limit as $i \to \infty$, we find that $t \mapsto u^*(t)$ is Lipschitz continuous in $L^2(\Omega)$, i.e.,

$$|u^*(\theta) - u^*(\eta)|_{L^2(\Omega)} \le C_2|\theta - \eta|.$$

Since $u^* \in W(0,T)$, the strong derivative in the $L^2(\Omega)$-sense pointwise in time is well defined (see [47]), and therefore $|\partial_t u^*(s)|_{L^2(\Omega)} \le C_2$, i.e., $\partial_t u^* \in L^\infty(0,T;L^2(\Omega))$ and

$$|\partial_t u^*|_{L^\infty(0,T;L^2(\Omega))} \le C_2.$$

In addition, given that $\tilde{u}^{N_i}(t_1) \le \tilde{u}^{N_i}(t_2)$ a.e. for all $t_1 \le t_2$ and $i \in \mathbb{N}$, and in view of $\lim_{i\to\infty} \tilde{u}^{N_i} = u^*$ in $L^2(\Omega)$, we observe that $u^*(t_1) \le u^*(t_2)$, i.e.,

$$t \mapsto u^*(t) \text{ is non-decreasing over } [0,T].$$

Since $\tilde{u}^{N_i} \to u^*$ uniformly on $[0,T]$ in the $L^2(\Omega)$-sense and $t \mapsto u^*(t)$ is uniformly continuous in the $L^2(\Omega)$-norm (for being Lipschitz continuous), the estimate

$$|\tilde{u}^{N_i}(\tau_i) - u^*(\tau)|_{L^2(\Omega)} \le |\tilde{u}^{N_i}(\tau_i) - u^*(\tau_i)|_{L^2(\Omega)} + |u^*(\tau_i) - u^*(\tau)|_{L^2(\Omega)} \tag{21}$$

implies for $\tau_i \to \tau$ that $\lim_{i\to\infty} u^{N_i}(\tau_i) = u^*(\tau)$ in $L^2(\Omega)$. Therefore, given that $\Phi : [0,T] \times L^2(\Omega) \to L^\infty(\Omega)$ is continuous, we obtain

$$\Phi(\tau_i, \tilde{u}^{N_i}(\tau_i)) \to \Phi(\tau, u^*(\tau)) \text{ in } L^\infty(\Omega) \text{ as } i \to \infty. \tag{22}$$

Let Ω_0 be an open ball in Ω. Consider $F : \Omega \times \mathbb{R}^l \to \mathbb{R}$ defined as $F(x,\xi) = \frac{1}{|\Omega_0|}\chi_{\Omega_0}(x)|\xi| \ge 0$. Then, it follows that $\xi \mapsto F(x,\xi)$ is convex and continuous, and $x \mapsto F(x,\xi)$ is measurable (as a real valued function) for each $\xi \in \mathbb{R}^l$. Hence, the functional $J(v) = \int_\Omega F(x,\nabla v)\,\mathrm{d}x$ is weakly lower semicontinuos on $H_0^1(\Omega)$ (see Theorem 3.23 in [10]), i.e., if $v_j \rightharpoonup v$ in $H_0^1(\Omega)$, then

$$\int_\Omega F(x,\nabla v)\,\mathrm{d}x \le \liminf_{j\to\infty} \int_\Omega F(x,\nabla v_j)\,\mathrm{d}x. \tag{23}$$

Fix $\tau \in [0,T]$, then we have $|\nabla \tilde{u}^{N_i}(\tau)| \le C_1$, a.e. on Ω, and thus, $\tilde{u}^{N_{i_j}}(\tau) \rightharpoonup w(\tau)$ in $H_0^1(\Omega)$ for some subsequence $\{\tilde{u}^{N_{i_j}}\}$ of $\{\tilde{u}^{N_i}\}$. Furthermore, $w(\tau) = u^*(\tau)$: We have therefore proven that $\tilde{u}^{N_{i_j}}(\tau) \to u^*(\tau)$ in $L^2(\Omega)$ and since $H_0^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, we also have that $\tilde{u}^{N_{i_j}}(\tau) \to w(\tau)$ in $L^2(\Omega)$, so that $w(\tau) = u^*(\tau)$. Then, by (23) we have

$$\frac{1}{|\Omega_0|}\int_{\Omega_0} |\nabla u^*(\tau)|\,\mathrm{d}x \le \liminf_{j\to\infty} \frac{1}{|\Omega_0|}\int_{\Omega_0} |\nabla \tilde{u}^{N_{i_j}}(\tau)|\,\mathrm{d}x. \tag{24}$$

We recall that $|\nabla u_n^N| \le \Phi(t_{n-1}^N, u_{n-1}^N)$ and $\Phi(t_{n-1}^N, u_{n-1}^N) \le \Phi(t_n^N, u_n^N)$ a.e. by the fact that $\{u_n^N\}$ is non-decreasing, Φ is also non-decreasing (in both arguments) and also $\tilde{u}^N(t_n^N) = u_n^N$. Suppose $\tau \in [t_{m-1}^N, t_m^N)$, where $m = m(\tau, N)$. Then, we have

$$\begin{aligned}|\nabla \tilde{u}^N(\tau)| &\le \frac{\tau - t_{m-1}}{k}|\nabla u_m^N| + \left(1 - \frac{\tau - t_{m-1}}{k}\right)|\nabla u_{m-1}^N| \le \Phi(t_{m-1}^N, u_{m-1}^N)\\ &= \Phi(t_{m-1}^N, \tilde{u}^N(t_{m-1}^N)).\end{aligned}$$

If $\tau \in [t_{m-1}^{N_{i_j}}, t_m^{N_{i_j}})$, with $m = m(\tau, N_{i_j})$, then $\lim_{j\to\infty} t_{m-1}^{N_{i_j}} = \tau$, and $\Phi(t_{m-1}^{N_{i_j}}, \tilde{u}^{N_{i_j}}(t_{m-1}^{N_{i_j}})) \to \Phi(\tau, u^*(\tau))$ in $L^\infty(\Omega)$ (as proven in (22)) and hence from (24) we observe

$$\begin{aligned}\frac{1}{|\Omega_0|}\int_{\Omega_0} |\nabla u^*(\tau)|\,\mathrm{d}x &\le \liminf_{j\to\infty} \frac{1}{|\Omega_0|}\int_{\Omega_0} |\nabla \tilde{u}^{N_{i_j}}(\tau)|\,\mathrm{d}x\\ &\le \liminf_{j\to\infty} \frac{1}{|\Omega_0|}\int_{\Omega_0} \Phi(t_{m-1}^{N_{i_j}}, \tilde{u}^{N_{i_j}}(t_{m-1}^{N_{i_j}}))\,\mathrm{d}x \le \frac{1}{|\Omega_0|}\int_{\Omega_0} \Phi(\tau, u^*(\tau))\,\mathrm{d}x.\end{aligned}$$

Since Ω_0 was an arbitrary ball in Ω, by taking $\Omega_0 := B_r(x_0)$ with $x_0 \in \Omega$ and $r \downarrow 0$, we have that $\frac{1}{|\Omega_0|}\int_{\Omega_0} |\nabla g(x)|\,\mathrm{d}x \to g(x_0)$ for almost all x_0, if $g \in L^1(\Omega)$. Consequently,

$$|\nabla u^*(\tau)| \leq \Phi(\tau, u^*(\tau)).$$

Finally, as $t \mapsto u^*(t)$ is continuous in $L^2(\Omega)$, the set $\{u^*(s) : s \in [0,T]\}$ is bounded, and since Φ is non-decreasing (in both arguments) and $\Phi(T,\cdot)$ maps bounded sets in $L^2(\Omega)$ into bounded sets in $L^\infty(\Omega)$, we have

$$\Phi(\tau, u^*(\tau)) \leq \Phi(T, u^*(\tau)) \leq |\Phi(T, u^*(\tau))|_{L^\infty(\Omega)} \leq \sup_{s\in[0,T]} |\Phi(T, u^*(s))|_{L^\infty(\Omega)} < \infty,$$

a.e. in Ω. This implies $|\nabla u^*(\cdot)| \in L^\infty(0,T;L^\infty(\Omega))$, i.e., $u^* \in L^\infty(0,T;W_0^{1,\infty}(\Omega))$. □

Remark. The result $\tilde{u}^N \to u^*$ in $C([0,T];L^2(\Omega))$ along a subsequence of $\{\tilde{u}^N\}$ can also be obtained by the application of the Lions-Aubin-Simon Lemma (see [48], Corollary 4, page 85): For this purpose, let X_1, X_2 and X_3 be Banach reflexive spaces such that the embedding $X_1 \hookrightarrow X_2$ is compact and the embedding $X_2 \hookrightarrow X_3$ is continuous. Moreover, let $\mathbf{F}$ be a set of mappings from $[0,T]$ to X_1 such that

$$\mathbf{F} \text{ is bounded in } L^\infty(0,T;X_1) \qquad \text{and} \qquad \partial_t\mathbf{F} \text{ is bounded in } L^r(0,T;X_2) \quad \text{with } r>1.$$

Then $\mathbf{F}$ is relatively compact in $C([0,T];X_2)$ by the Lions-Aubin-Simon lemma. Choosing $X_1 = H_0^1(\Omega)$, $X_2 = L^2(\Omega)$ and $X_3 = H^{-1}(\Omega)$ in our context, the result is obtained.

The following result guarantees (among others) that, for a fixed $\tau \in [0,T]$, the sequence of sets $\mathbf{K}(\Phi(\tau,\tilde{u}^N(\tau)))$ (with $N = 1,2,\ldots$) satisfies **i.** in Definition 2, provided that $\tilde{u}^N \to u^*$ in $C([0,T];L^2(\Omega))$ as $N \to \infty$. Condition **ii.** of Definition 2 was actually proven in theorem 2. We delay the proof of the following result to appendix B.

Lemma 1. *Let u^* be given according to theorem 2 and define $\mathscr{K}_1(\cdot) := \mathscr{K}(\cdot)$, $\mathscr{K}_2(\cdot) := \mathscr{K}^+(\cdot)$, $\mathscr{K}_3(\cdot) := \mathscr{K}^\pm(\cdot)$, $\mathbf{K}_1(\cdot) := \mathbf{K}(\cdot)$, $\mathbf{K}_2(\cdot) := \mathbf{K}^+(\cdot)$, and $\mathbf{K}_3(\cdot) := \mathbf{K}^\pm(\cdot)$. Then the following two statements hold true.*

$\mathbf{a}$. *Let $\Psi = \Phi(\cdot, u^*(\cdot))$ and suppose that $w_i \in \mathscr{K}_i(\Psi)$ for $i = 1,2,3$ are arbitrarily fixed. Then, there exist sequences $\{w_i^N\}$, for $i = 1,2,3$, in $L^2(0,T;H_0^1(\Omega))$ such that*

$$w_i^N(t) \in \mathbf{K}_i(\Phi(t_{n-1}^N, u_{n-1}^N)),$$

where $t \in [t_{n-1}^N, t_n^N)$, for $1 \leq n \leq N$ and satisfy $w_i^N \to w$ in $L^2(0,T;H_0^1(\Omega))$ along a subsequence, for $i = 1,2,3$.

$\mathbf{b}$. *Let $\tau \in [0,T]$ be fixed, such that $\tau \in [t_{n-1}^N, t_n^N)$, and let $n = n(\tau, N)$. Define $\phi := \Phi(\tau, u^*(\tau))$ and $\phi^N := \Phi(t_{n-1}^N, u_{n-1}^N)$, and consider arbitrary $w_i \in \mathbf{K}_i(\phi)$ for $i = 1,2,3$. Then, there exist sequences $\{w_i^N\}$ in $H_0^1(\Omega)$, for $i = 1,2,3$, such that*

$$w_i^N \in \mathbf{K}_i(\phi^N),$$

and $w_i^N \to w$ in $H_0^1(\Omega)$ along a subsequence.

lemma 1 provides sufficient conditions for the existence of a recovery sequence in the definition of Mosco convergence for a variety of settings which involve problem (P). In light of this result, we are now in shape to provide the following result which finalizes the proof of theorem 1.

Proposition 3. *Let u^* be given according to theorem 2. Then, it solves problem* (P_0).

Proof. For the sake of brevity, let $\{\tilde{u}^N\}$ denote the subsequence according to theorem 2. It fulfils

$$\tilde{u}^N \to u^* \text{ in } C([0,T];L^2(\Omega)), \qquad \partial_t \tilde{u}^N \rightharpoonup \partial_t u^* \text{ in } L^2(0,T;L^2(\Omega)), \tag{25}$$

$u^* \in \mathbf{K}(\Phi(t, u^*(t)))$ a.e. for $t \in [0,T]$, and $u^* \in W(0,T)$.

Suppose that $w \in L^2(0,T;H_0^1(\Omega))$ satisfies $|\nabla w(\tau)| \leq \Phi(\tau, u^*(\tau))$ a.e. in Ω and for a.e. $\tau \in [0,T]$. By lemma 1, there exists $\{w^N\}$ such that $|\nabla w^N(t)| \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ a.e. with $t \in [t_{n-1}^N, t_n^N)$ for $n = 1,2,\ldots,N$ and $w^N \to w$ in $L^2(0,T;H_0^1(\Omega))$.

We define

$$\hat{\Theta}(\tau, u_-^N(\tau)) := \sum_{m=1}^{N} \Theta(t_{m-1}^N, u_-^N(\tau))\chi_{[t_{m-1}^N, t_m^N)}(\tau) = \sum_{m=1}^{N} \Theta(t_{m-1}^N, u_{m-1}^N)\chi_{[t_{m-1}^N, t_m^N)}(\tau).$$

Since $\Theta : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ is uniformly continuous (by the exact same argument as in the proof of lemma 1) we have

$$\lim_{N\to\infty} \sup_{\tau\in[0,T]} |\hat{\Theta}(\tau, u_-^N(\tau)) - \Theta(\tau, u^*(\tau))|_{L^2(\Omega)} = 0,$$

or

$$\hat{\Theta}(\cdot, u_-^N(\cdot)) \to \Theta(\cdot, u^*(\cdot)), \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{26}$$

as $N \to \infty$. Also, defining $f^N = \sum_{n=1}^N f_n^N \chi_{[t_{n-1}^N, t_n^N)}$ with $f_n^N = \frac{1}{k}\int_{t_{n-1}^N}^{t_n^N} f(t)\,\mathrm{d}t$ we get

$$f^N \to f \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } N \to \infty, \tag{27}$$

(see for example [17] or page 21 in [23])

Then, by definition of $\{u_n^N\}_{n=0}^N$ from (5), u^N and $\tilde{u}^N$, the following holds:

$$(\partial_t \tilde{u}^N(\tau) - \hat{\Theta}(\tau, u_-^N(\tau)) - f^N(\tau), w^N(\tau) - u^N(\tau)) \geq 0, \quad \forall \tau \in (0,T),$$

and hence, by integration over $(0,T)$ we obtain

$$\int_0^T (\partial_t \tilde{u}^N(\tau) - \hat{\Theta}(\tau, u_-^N(\tau)) - f^N(\tau), w^N(\tau) - u^N(\tau))\,\mathrm{d}\tau \geq 0. \tag{28}$$

Finally, using (25), (26), (27) in the inequality (28), by taking the limit $N \to \infty$, we infer

$$\int_0^T (\partial_t u^*(\tau) - \Theta(\tau, u^*(\tau)) - f(\tau), w(\tau) - u^*(\tau))\,\mathrm{d}\tau \geq 0. \tag{29}$$

Since $u^*(\tau) \in \mathbf{K}(\Phi(\tau, u^*(\tau)))$ for all $\tau \in [0,T]$ by theorem 2, and additionally $w \in L^2(0,T;H_0^1(\Omega))$ satisfies $|\nabla w(\tau)| \leq \Phi(\tau, u^*(\tau))$ a.e. in Ω, for a.e. $\tau \in [0,T]$, but otherwise is arbitrary, the assertion is proven. $\square$

5 The dissipative problem (P_1)

In this section we focus on the dissipative problem (P_1) where prototypical operator A is given by $-\Delta$, where Δ is the Laplacian. Although for this problem we obtain an analogous result to theorem 1, this is possible only under more restrictive conditions than in the previous section. For the rest of the paper we suppose that $\Omega \subset \mathbb{R}^l$ is open, bounded and convex which implies that the boundary $\partial\Omega$ is Lipschitz. Further, we make the following assumptions throughout this section on A, f, u_0, Θ and Φ.

Assumption 2.

i. *The operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is of the form $Av = -\sum_{n=1}^N \partial_{x_n} a_n \partial_{x_n} v$, that is*

$$\langle Av, w\rangle = \sum_{n=1}^N a_n \int_\Omega \frac{\partial v}{\partial x_n}\frac{\partial w}{\partial x_n}\,\mathrm{d}x \quad \forall v, w \in H_0^1(\Omega) \tag{30}$$

with $a_n \geq a > 0$, $a_n \in \mathbb{R}$ for $n = 1, 2, \dots, N$. Therefore, A is linear, $|Av|_{H^{-1}(\Omega)} \leq M_A|v|_{H_0^1(\Omega)}$ with $M_A \geq 0$ and it is uniformly monotone.

ii. *$f \in L^\infty(0, T; \mathbb{R})$ is non-decreasing.*

iii. *The initial condition $u_0 \in H_0^1(\Omega)$ satisfies $A(u_0) \in L^2(\Omega)$,*

$$|\nabla u_0| \leq \Phi(0, u_0) \quad \text{and} \quad A(u_0) \leq \Theta(0, u_0) + \frac{1}{k}\int_0^k f(t)\,\mathrm{d}t,$$

a.e. in Ω, for all $k \in (0, \epsilon_k)$ and some $\epsilon_k > 0$.

iv. *$\Theta : [0, T] \times L^2(\Omega) \to \mathbb{R}$ is uniformly continuous, non-decreasing:*

$$0 \leq t_1 \leq t_2 \leq T,\ u_0 \leq v_1 \leq v_2 \text{ Êa.e.} \quad \Longrightarrow \quad \Theta(t_1, v_1) \leq \Theta(t_2, v_2) \text{ Êa.e.}, \tag{31}$$

and has α-order of growth:

$$\exists\, \alpha > 0, L_\Theta > 0 : \quad |\Theta(t, v)| \leq L_\Theta |v|^\alpha_{L^2(\Omega)}, \quad \forall t \in [0, T], \forall v \in L^2(\Omega). \tag{32}$$

v. *$\Phi : [0, T] \times L^2(\Omega) \to \mathbb{R}$ is uniformly continuous and $\Phi(t, v) \geq \nu > 0$ for all $t \in [0, T]$ and all $v \in L^2(\Omega)$. We also assume it is non-decreasing (as* (31)*) and that , $v \mapsto \Phi(T, v)$ maps bounded sets in $L^2(\Omega)$ into bounded sets in $\mathbb{R}$.*

Analogous to theorem 1 in the non-dissipative case, the following theorem is the main result for the dissipative problem and concerns existence, regularity and approximation of solutions.

Theorem 3. *Let $\alpha \in [0, 1]$ in* (32)*, then there is a solution u^* to problem (P_1) such that*

$$u^* \in L^\infty(0, T; W_0^{1,\infty}(\Omega)) \cap C^{0,1}([0, T]; L^2(\Omega)), \quad \text{and} \quad \partial_t u^* \in L^\infty(0, T; L^2(\Omega)).$$

Moreover, u^* is nondecreasing, i.e., if $0 \leq t_1 \leq t_2 \leq T$ then $u_0 \leq u^*(t_1) \leq u^*(t_2) \leq u^*(T)$ a.e. in Ω, and it satisfies

$$A(u^*) \in L^\infty(0, T; L^2(\Omega)), \tag{33}$$

and solves problem (P_1) *and* (iP_1) *when the map* $\mathbf{K}$ *is replaced by either* $\mathbf{K}^+$ *or* $\mathbf{K}^\pm$*.*

Furthermore, the sequence $\{\tilde{u}^N\}$ *defined as*

$$\tilde{u}^N(t) = u_0 + \int_0^t \sum_{n=1}^N \frac{u_n^N - u_{n-1}^N}{k} \chi_{[t_{n-1},t_n)}(s)\,\mathrm{d}s,$$

where $\{u_n^N\}_{n=0}^N$ *solves problem* (P_1^N)*, satisfies*

$$\tilde{u}^N \to u^* \text{ in } C([0,T];L^2(\Omega)) \qquad \text{and} \qquad \partial_t \tilde{u}^N \rightharpoonup \partial_t u^* \text{ in } L^2(0,T;L^2(\Omega)),$$

along a subsequence.

If $\alpha > 1$*, then the same holds true provided that*

$$|u_0|_{L^2(\Omega)} + L_\Theta |u_0|^\alpha_{L^2(\Omega)} + T^{1/2}|f|_{L^2(0,T;L^2(\Omega))} < \frac{1}{((\alpha-1)L_\Theta T)^{\frac{1}{\alpha-1}}}. \tag{34}$$

The first step for proving theorem 3 is to provide the necessary conditions for applying theorem 2. This is the purpose of the following proposition.

Proposition 4. *Problem* $(\mathrm{P}_1^{\mathrm{N}})$ *is well-posed and there exists* $N^* \in \mathbb{N}$ *such that its solution* $\{u_n^N\}_{n=0}^N$*, for* $N \geq N^*$*, satisfies* $\mathbf{i}$*,* $\mathbf{ii}$ *and* $\mathbf{iii}$ *of proposition 2 and in addition*

iv. *there exists a constant* $C_3 > 0$*,*

$$|A(u_n^N)|_{L^2(\Omega)} \leq C_3,$$

uniformly in $n = 1, 2, \ldots, N$ *and* $N \in \mathbb{N}$*.*

Further, we have that $u_n^N \in \mathbf{K}^+(\Phi(t_{n-1}^N, u_{n-1}^N))$ *and*

$$\left\langle \frac{u_n^N - u_{n-1}^N}{k} + A(u_n^N) - \Theta(t_{n-1}^N, u_{n-1}^N) - f_n^N, v - u_n^N \right\rangle \geq 0, \tag{35}$$

for all $v \in \mathbf{K}^+(\Phi(t_{n-1}^N, u_{n-1}^N))$*, and the same holds when* $\mathbf{K}^+$ *is replaced by* $\mathbf{K}^\pm$*.*

Proof. Let $A_k : H_0^1(\Omega) \to H^{-1}(\Omega)$ be defined as $A_k := I/k + A$, i.e.,

$$\langle A_k(w), v\rangle := \frac{1}{k}\int_\Omega w(x)v(x)\,\mathrm{d}x + \sum_{n=1}^N a_n \int_\Omega \frac{\partial v}{\partial x_n}(x)\frac{\partial w}{\partial x_n}(x)\,\mathrm{d}x, \quad \forall w, v \in H_0^1(\Omega).$$

It follows that it is uniformly monotone over $H_0^1(\Omega)$, i.e., $\langle A_k(w), v\rangle \geq \frac{1}{k}|w|^2_{L^2(\Omega)} + a|\nabla w|^2_{L^2(\Omega)} \geq 0$ for some $a > 0$. Also, A_k is continuous and $\mathbf{K}(t_{n-1}^N, \Phi(u_{n-1}^N))$ is a closed, convex and non-empty subset of $H_0^1(\Omega)$. Hence, the problem

$$\text{Find } u \in \mathbf{K}(\Phi(t_{n-1}^N, u_{n-1}^N)) : \langle A_k(u) - g, v - u\rangle \geq 0, \quad \forall v \in \mathbf{K}(\Phi(t_{n-1}^N, u_{n-1}^N)),$$

admits an unique solution for any $g \in L^2(\Omega)$ (see [30]). Then, provided $u_{n-1}^N \in H_0^1(\Omega)$ and taking $g = \frac{1}{k}u_{n-1}^N + \Theta(t_{n-1}^N, u_{n-1}^N)$, it follows that $u_n^N \in \mathbf{K}(\Phi(t_{n-1}^N, u_{n-1}^N))$ is well defined by (5).

We concentrate first on $\mathbf{i}$. For each $1 \leq n \leq N$, we have that $u_n^N \in H_0^1(\Omega)$, $|\nabla u_n^N| \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ and

$$\left\langle \left(\frac{I}{k} - A\right) u_n^N - \left(\Theta(t_{n-1}^N, u_{n-1}^N) + f_n^N + \frac{u_{n-1}^N}{k}\right), v - u_n^N \right\rangle \geq 0,$$

for all $v \in H_0^1(\Omega)$ such that $|\nabla v| \leq \Phi(t_{n-1}^N, u_{n-1}^N)$. Since $\Phi(t_{n-1}^N, u_{n-1}^N) \geq \nu > 0$, we define

$$\bar{u}_n^N := \frac{u_n^N}{\Phi(t_{n-1}^N, u_{n-1}^N)} \qquad \text{and} \qquad \bar{f}_n^N := \frac{\Theta(t_{n-1}^N, u_{n-1}^N) + f_n^N}{\Phi(t_{n-1}^N, u_{n-1}^N)} + \frac{u_{n-1}^N}{\Phi(t_{n-1}^N, u_{n-1}^N)k}.$$

Then, we have that $\bar{u}_n^N$ solves

$$\text{Find } u \in \mathbf{K}(1) : \left\langle \left(\frac{I}{k} - A\right) u - \bar{f}_n^N, \bar{v} - u \right\rangle \geq 0, \ \forall \bar{v} \in \mathbf{K}(1),$$

where $\mathbf{K}(1) = \{v \in H_0^1(\Omega) : |\nabla v| \leq 1 \text{ a.e. in } \Omega\}$.

Then, for $n \geq 2$ and because $|\nabla u_{n-1}^N| \leq \Phi(t_{n-2}^N, u_{n-2}^N)$ a.e., the following statement holds true:

$$\Phi(t_{n-2}^N, u_{n-2}^N) \leq \Phi(t_{n-1}^N, u_{n-1}^N) \qquad \Longrightarrow \qquad \frac{|\bar{f}_n^N|_{W^{1,\infty}(\Omega)}}{1/k} \leq 1.$$

Also, since $|\nabla u_0| \leq \Phi(0, u_0)$ a.e. by assumption, for $n = 1$ we also have that $\frac{|\bar{f}_1^N|_{W^{1,\infty}(\Omega)}}{1/k} \leq 1$. Therefore, for $n = 1$ and provided that $u_{n-2}^N \leq u_{n-1}^N$ a.e. for $n \geq 2$ (which implies $\Phi(t_{n-2}^N, u_{n-2}^N) \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ a.e. since Φ is non-decreasing in both variables), by the the equivalence result of Brézis-Sibony (see [8]), $\bar{u}_n^N$ solves

$$\text{Find } u \in \mathbf{K}^{\pm}(1) : \quad \left\langle \left(\frac{I}{k} - A\right) u - \bar{f}_n^N, \bar{v} - u \right\rangle \geq 0, \quad \forall \bar{v} \in \mathbf{K}^{\pm}(1).$$

It is straightforward to observe that this implies that u_n^N belongs to

$$\mathbf{K}_{n-1}^{\pm} := \{v \in H_0^1(\Omega) : |v(x)| \leq \Phi(t_{n-1}^N, u_{n-1}^N)\text{dist}(x, \partial\Omega) \text{ for a.e. } x \in \Omega\},$$

and solves the problem

$$\text{Find } u \in \mathbf{K}_{n-1}^{\pm} : \quad \left\langle \left(\frac{I}{k} + A\right) u - F_n^N, v - u \right\rangle \geq 0, \quad \forall v \in \mathbf{K}_{n-1}^{\pm}, \tag{36}$$

with $F_n^N := \Theta(t_{n-1}^N, u_{n-1}^N) + f_n^N + \frac{u_{n-1}^N}{k}$.

We now proceed by induction: we first prove that $u_0 \leq u_1^N$ a.e. in Ω. We know that $u_1^N = \mathbb{S}(A_k, F_1^N, \mathbf{K}_0^{\pm})$. Consider $\mathbf{K}_{n-1}^{+}$ defined as

$$\mathbf{K}_{n-1}^{+} := \{v \in H_0^1(\Omega) : v(x) \leq \Phi(t_{n-1}^N, u_{n-1}^N)\text{dist}(x, \partial\Omega) \text{ for a.e. } x \in \Omega\}.$$

For u_0, due to assumption 2 we have that $A(u_0) \leq \Theta(0, u_0) + f_1^N$ a.e. for N larger than some N^* and this implies

$$\left\langle \left(\frac{I}{k} + A\right) u_0 - \left(\Theta(0, u_0) + f_1^N + \frac{u_0}{k}\right), \phi \right\rangle \leq 0,$$

for all $\phi \in H_0^1(\Omega), \phi \geq 0$ a.e. in Ω. Also by our initial assumption, $|\nabla u_0| \leq \Phi(0, u_0)$ a.e. and $u_0 \in \mathbf{K}_0^{\pm} \subset \mathbf{K}_0^{+}$. Hence, we have that u_0 is a lower solution of the triple $(A_k, F_1^N, \mathbf{K}_0^{+})$ (see

Definition 1). Then, by proposition 5, we have that $u_0 \leq \mathbb{S}(A_k, F_1^N, \mathbf{K}_0^+)$ a.e. in Ω. The latter implies that

$$-\Phi(0, u_0)\text{dist}(x, \partial\Omega) \leq \mathbb{S}(A_k, F_1^N, \mathbf{K}_0^+)(x)$$

for a.e. $x \in \Omega$ and hence, $\mathbb{S}(A_k, F_1^N, \mathbf{K}_0^+) \in \mathbf{K}_0^\pm$. Since the solutions $\mathbb{S}(A_k, F_1^N, \mathbf{K}_0^+)$ and $\mathbb{S}(A_k, F_1^N, \mathbf{K}_0^\pm)$ are uniquely defined and $\mathbf{K}_{n-1}^\pm \subset \mathbf{K}_{n-1}^+$, we have

$$\mathbb{S}(A_k, F_1^N, \mathbf{K}_0^+) = \mathbb{S}(A_k, F_1^N, \mathbf{K}_0^\pm) = u_1^N, \tag{37}$$

and hence $u_0 \leq u_1^N$ a.e. in Ω. In addition, the latter also implies that u_2^N satisfies (36) (for $n = 2$).

We now prove that $\mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^+) = \mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^\pm) = u_n^N$, provided that $u_{n-2}^N \leq u_{n-1}^N$ a.e. and $\mathbb{S}(A_k, F_{n-1}^N, \mathbf{K}_{n-2}^+) = \mathbb{S}(A_k, F_{n-1}^N, \mathbf{K}_{n-2}^\pm) = u_{n-1}^N$. The latter condition implies that $F_{n-1}^N \leq F_n^N$ and $\mathbf{K}_{n-2}^+ \subset \mathbf{K}_{n-1}^+$ (these follow since $n \mapsto f_n^N$, and the maps Θ and Φ, in both variables, are non-decreasing). Then, by proposition 5, we have that $\mathbb{S}(A_k, F_{n-1}^N, \mathbf{K}_{n-2}^+) \leq \mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^+)$ a.e. in Ω. However, $\mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-2}^+) = \mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-2}^\pm)$ and therefore

$$-\Phi(t_{n-1}^N, u_{n-1}^N)\text{dist}(x, \partial\Omega) \leq -\Phi(t_{n-2}^N, u_{n-2}^N)\text{dist}(x, \partial\Omega) \leq \mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^+)(x),$$

for a.e. $x \in \Omega$, which implies $\mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^+) \in \mathbf{K}_{n-1}^\pm$. Since $\mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^+)$ and $\mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^\pm)$ are uniquely defined, and $\mathbf{K}_{n-1}^\pm \subset \mathbf{K}_{n-1}^+$, we observe that $\mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^+) = \mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^\pm)$.

In particular, since we have that $u_0 \leq u_1^N$ a.e. and (37), this implies that $u_2^N = \mathbb{S}(A_k, F_2^N, \mathbf{K}_1^\pm) = \mathbb{S}(A_k, F_2^N, \mathbf{K}_1^+)$.

Now, suppose that $\mathbb{S}(A_k, F_{n-1}^N, \mathbf{K}_{n-2}^+) = u_{n-1}^N$ and $u_{n-2}^N \leq u_{n-1}^N$ a.e. in Ω. Therefore u_n^N satisfies (36) and $\mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^+) = u_n^N$. We also have that $F_{n-1}^N \leq F_n^N$, given the fact that $n \mapsto f_n^N$ and Θ (in both variables) are non-decreasing. Furthermore, since Φ (in both variables) is non-decreasing, it follows that $|\nabla u_{n-1}^N| \leq \Phi(t_{n-2}^N, u_{n-2}^N) \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ a.e., which implies that $u_{n-1}^N \in \mathbf{K}_{n-1}^+$. These facts yield that u_{n-1}^N is a lower solution of the triple $(A_k, F_n^N, \mathbf{K}_{n-1}^+)$: By definition $u_{n-1}^N = \mathbb{S}(A_k, F_{n-1}^N, \mathbf{K}_{n-2}^+)$ is a lower solution of $(A_k, F_{n-1}^N, \mathbf{K}_{n-2}^+)$, but $u_{n-1}^N \in \mathbf{K}_{n-1}^+$ and $F_{n-1}^N \leq F_n^N$ imply that

$$\langle A_k(u_{n-1}^N) - F_n^N, \phi\rangle \leq (F_{n-1}^N - F_n^N, \phi) \leq 0,$$

for all $\phi \in H_0^1(\Omega)$ such that $\phi \geq 0$ a.e. in Ω. Hence, by definition u_{n-1}^N is a lower solution of the triple $(A_k, F_n^N, \mathbf{K}_{n-1}^+)$.

From proposition 5, we infer that $u_{n-1}^N \leq \mathbb{S}(A_k, F_n^N, \mathbf{K}_{n-1}^+) = u_n^N$. In turn, this implies that u_{n+1}^N satisfies (36) (for n replaced by $n+1$) and $u_{n+1}^N = \mathbb{S}(A_k, F_n^N, \mathbf{K}_n^+)$.

The application of the above argument by means of induction proves **i.** and the equivalent formulation in (35) with the exchange of $\mathbf{K}^\pm$ by $\mathbf{K}^+$.

Next, we focus on $\mathbf{ii}$. Using $v = 0$ in (5), we obtain

$$|u_n^N|_{L^2(\Omega)}^2 + \langle A(u_n^N), u_n^N\rangle \leq (u_{n-1}^N, u_n^N) + k(\Theta(t_{n-1}^N, u_{n-1}^N) + f_n^N, u_n^N).$$

Since $\langle A(u_n^N), u_n^N\rangle \geq a|\nabla u_n^N|_{L^2(\Omega)}^2 \geq 0$, we have

$$|u_n^N|_{L^2(\Omega)}^2 \leq (u_{n-1}^N, u_n^N) + k(\Theta(t_{n-1}^N, u_{n-1}^N) + f_n^N, u_n^N), \tag{38}$$

from which, since $|\Theta(t, v)|_{L^2(\Omega)} \leq L_\Theta |v|_{L^2(\Omega)}^\alpha$ for all $t \in [0, T], v \in L^2(\Omega)$, we obtain

$$|u_n^N|_{L^2(\Omega)} - |u_{n-1}^N|_{L^2(\Omega)} \leq L_\Theta k |u_{n-1}^N|_{L^2(\Omega)}^\alpha + k|f_n^N|_{L^2(\Omega)}.$$

This is the same inequality as in (15) in the proof of **ii** in proposition 2, thus the same conclusion holds true, i.e., there exist $M > 0$ and $C_1 > 0$ such that

$$|u_n^N|_{L^2(\Omega)} \leq M \qquad \text{and} \qquad |\nabla u_n^N| \leq \sup_{v\in L^2(\Omega):|v|_{L^2(\Omega)}\leq M} |\Phi(T,v)|_{L^\infty(\Omega)} =: C_1,$$

a.e. in Ω and uniformly for $1 \leq n \leq N$ and $N \in \mathbb{N}$.

We consider now $\mathbf{iii}$. Since $n \mapsto u_n^N$ is non-decreasing, Φ is non-decreasing in both variables and $|\nabla u_{n-1}^N| \leq \Phi(t_{n-2}^N, u_{n-2}^N)$ a.e., we have $|\nabla u_{n-1}^N| \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ a.e. in Ω. Choosing $v = u_{n-1}^N$ in (5), we have

$$\left\langle \frac{u_n^N - u_{n-1}^N}{k} + A(u_n^N) - \Theta(t_{n-1}^N, u_{n-1}^N) - f_n^N, u_{n-1}^N - u_n^N \right\rangle \geq 0. \tag{39}$$

We split the rest of proof in steps:

Step 1. We first prove $|(u_1^N - u_0)/k|_{L^2(\Omega)}$ is bounded uniformly for $N \in \mathbb{N}$ and $1 \leq n \leq N$. From the above inequality, in the case when $n = 1$, we have that

$$\left(\frac{u_1^N - u_0}{k}, u_1^N - u_0\right) + \langle A(u_0) - A(u_1^N), u_0 - u_1^N\rangle \leq (A(u_0) - \Theta(0,u_0) - f_1^N, u_0 - u_1^N).$$

Since A is monotone, $A(u_0) \in L^2(\Omega)$ and $|f_1^N|_{L^2(\Omega)} \leq \sup_{t\in[0,T]} |f(t)|_{L^2(\Omega)}$ we have

$$\left|u_1^N - u_0\right|_{L^2(\Omega)} \leq \left(|A(u_0)|_{L^2(\Omega)} + |\Theta(0,u_0)|_{L^2(\Omega)} + \sup_{t\in[0,T]} |f(t)|_{L^2(\Omega)}\right) k < \infty. \tag{40}$$

Step 2. We now prove that u_n^N are regular enough so that $A(u_n^N) \in L^2(\Omega)$. Define, $\bar{u}_n^N$ and $\bar{f}_n^N$ as

$$\bar{u}_n^N := \frac{u_n^N}{\Phi(t_{n-1}^N, u_{n-1}^N)} \qquad \text{and} \qquad \bar{f}_n^N := \frac{\Theta(t_{n-1}^N, u_{n-1}^N) + f_n^N}{\Phi(t_{n-1}^N, u_{n-1}^N)} + \frac{u_{n-1}^N - u_n^N}{\Phi(t_{n-1}^N, u_{n-1}^N)k}.$$

Therefore, $\bar{u}_n^N$ solves the problem

$$\text{Find } u \in \mathbf{K}(1) : \langle A(u) - \bar{f}_n^N, \bar{v} - u\rangle \geq 0, \qquad \forall \bar{v} \in \mathbf{K}(1).$$

Then, since by initial assumption we have that A is defined as (30), the domain Ω being open, bounded and convex, and $\bar{f}_n^N \in L^2(\Omega)$, we can apply the regularity result by Brézis-Stampacchia (see [9], $\mathcal{S}$ III) which implies that $A(\bar{u}_n^N) \in L^2(\Omega)$ and moreover, we have the bound $|A(\bar{u}_n^N)|_{L^2(\Omega)} \leq |\bar{f}_n^N|_{L^2(\Omega)}$ (see [9], page 170). Equivalently,

$$A(u_n^N) \in L^2(\Omega), \qquad |A(u_n^N)|_{L^2(\Omega)} \leq \left|\Theta(t_{n-1}^N, u_{n-1}^N) + f_n^N + \frac{u_{n-1}^N - u_n^N}{k}\right|_{L^2(\Omega)}. \tag{41}$$

Step 3: There is a uniform bound for $|(u_n^N - u_{n-1}^N)/k|_{L^2(\Omega)}$. Let $\hat{u}_n^N$, with $n \geq 1$, and $\hat{v}$ be defined as

$$\hat{u}_n^N := u_n^N - \sum_{m=1}^{n} k(\Theta(t_{m-1}^N, u_{m-1}^N) + f_m^N), \qquad \hat{v} := u_{n-1}^N + k(\Theta(t_{n-1}^N, u_{n-1}^N) + f_n^N),$$

and also $\hat{u}_0^N := u_0$. Then, by direct calculation, we have that

$$A(\hat{u}_n^N) = A(u_n^N) \qquad \text{and} \qquad |\nabla\hat{v}| = |\nabla u_{n-1}^N| \leq \Phi(t_{n-2}^N, u_{n-2}^N) \leq \Phi(t_{n-1}^N, u_{n-1}^N),$$

a.e. in Ω. Here we use that $n \mapsto u_n^N$ is non-decreasing and Φ is also non-decreasing, in both variables. Additionally, we have that

$$\frac{\hat{u}_n^N - \hat{u}_{n-1}^N}{k} + A(u_n^N) = \frac{u_n^N - u_{n-1}^N}{k} + A(u_n^N) - \Theta(t_{n-1}^N, u_{n-1}^N) - f_n^N, \tag{42}$$

$$\hat{u}_{n-1}^N - \hat{u}_n^N = \hat{v} - u_n^N, \tag{43}$$

$$\frac{\hat{u}_{n-2}^N - \hat{u}_{n-1}^N}{k} = \Theta(t_{n-2}^N, u_{n-2}^N) + f_{n-1}^N + \frac{u_{n-2}^N - u_{n-1}^N}{k}. \tag{44}$$

Using $v = \hat{v}$, (42) and (43) in (5), we therefore have that

$$\left\langle \frac{\hat{u}_n^N - \hat{u}_{n-1}^N}{k} + A(u_n^N), \hat{u}_{n-1}^N - \hat{u}_n^N \right\rangle \geq 0,$$

and (since $A(\hat{u}_{n-1}^N) = A(u_{n-1}^N)$) equivalently

$$\left(\frac{\hat{u}_n^N - \hat{u}_{n-1}^N}{k}, \hat{u}_n^N - \hat{u}_{n-1}^N \right) + \langle A(\hat{u}_{n-1}^N) - A(\hat{u}_n^N), \hat{u}_{n-1}^N - \hat{u}_n^N \rangle \leq \langle A(u_{n-1}^N), \hat{u}_{n-1}^N - \hat{u}_n^N \rangle.$$

Since A is monotone and $A(u_{n-1}^N) \in L^2(\Omega)$, as in (41), we obtain the inequality $|(\hat{u}_n^N - \hat{u}_{n-1}^N)/k|_{L^2(\Omega)} \leq |A(u_{n-1}^N)|_{L^2(\Omega)}$; or, by using the bound in (41) and (44),

$$\left|\hat{u}_n^N - \hat{u}_{n-1}^N\right|_{L^2(\Omega)} \leq \left|\hat{u}_{n-1}^N - \hat{u}_{n-2}^N\right|_{L^2(\Omega)}, \tag{45}$$

and hence

$$\left|\hat{u}_n^N - \hat{u}_{n-1}^N\right|_{L^2(\Omega)} \leq \left|\hat{u}_1^N - u_0\right|_{L^2(\Omega)} \leq \left|u_1^N - u_0\right|_{L^2(\Omega)} + (|\Theta(0, u_0)| + |f_1^N|_{L^2(\Omega)})k. \tag{46}$$

From (40), the fact hat Θ is non-decreasing in both variables, $|u_n^N|_{L^2(\Omega)} \leq M$ uniformly and (44), we infer

$$\begin{aligned}\left|u_n^N - u_{n-1}^N\right|_{L^2(\Omega)} &\leq \left(|A(u_0)|_{L^2(\Omega)} + 3 \sup_{t \in [0,T]} |f(t)|_{L^2(\Omega)} + 3 \sup_{\substack{v \,\in\, L^2(\Omega) \\ |v|_{L^2(\Omega)} \,\leq\, M}} |\Theta(T, v)| \right) k \\ &=: C_2 k.\end{aligned}$$

Finally, we focus on $\mathbf{iv}$. Using the above result and (41), we obtain that

$$|A(u_n^N)|_{L^2(\Omega)} \leq \sup_{t \in [0,T]} |f(t)|_{L^2(\Omega)} + \sup_{\substack{v \,\in\, L^2(\Omega) \\ |v|_{L^2(\Omega)} \,\leq\, M}} |\Theta(T, v)| + C_2 =: C_3, \tag{47}$$

where $C_3 > 0$ is independent of n and N. $\square$

We are now in shape to provide the proof of the main result of the section.

Theorem 3. The approximants $\{\tilde{u}^N\}$ satisfy

$$\tilde{u}^N \to u^* \text{ in } C([0,T]; L^2(\Omega)) \qquad \text{and} \qquad \partial_t \tilde{u}^N \rightharpoonup \partial_t u^* \text{ in } L^2(0,T; L^2(\Omega)), \tag{48}$$

along a subsequence as proven by theorem 2: Note that by proposition 4 we have that $n \mapsto u_n^N$ is non-decreasing and there are uniform bounds on $|\nabla u_n^N|_{L^2(\Omega)}$ and $|(u_n^N - u_{n-1}^N)/k|_{L^2(\Omega)}$. Additionally, the fact that $u^* : [0,T] \to L^2(\Omega)$ is Lipschitz continuous, non-decreasing, $u^* \in L^\infty(0,T;W_0^{1,\infty}(\Omega))$, $\partial_t u^* \in L^\infty(0,T;L^2(\Omega))$ and $|\nabla u^*(t)| \leq \Phi(t,u^*(t))$ a.e. in Ω for a.e. $t \in (0,T)$, follow again from the aforementioned theorem.

Let $\tau \in [0,T]$ be fixed. Denote by $\{\tilde{u}^N\}$ to the convergent subsequence obtained in (48). Then, in addition to $\tilde{u}^N \to u^*$ in $C([0,T];L^2(\Omega))$ we also have that $u^N \to u^*$ in $L^\infty(0,T;L^2(\Omega))$ by the inequality (19). Also, since $|\nabla u^N(\tau)| \leq C_1$ a.e., then $u^N(\tau) \rightharpoonup w(\tau)$ in $H_0^1(\Omega)$ along a subsequence. The embedding $L^2(\Omega) \hookrightarrow H_0^1(\Omega)$ is compact, and thus we have that $u^N(\tau) \to w(\tau)$ in $L^2(\Omega)$ along a further subquence, but $u^N \to u^*$ in $C([0,T];L^2(\Omega))$ which implies $w(\tau) = u^*(\tau)$. By the very same argument, any weakly convergent sequence in $H_0^1(\Omega)$ has the same limit. Therefore, the original sequence satisfies $u^N(\tau) \rightharpoonup u^*(\tau)$ in $H_0^1(\Omega)$.

By proposition 4, we have $|A(u^N(\tau))|_{L^2(\Omega)} \leq C_3$ and hence

$$\lim_{N\to\infty} |(A(u^N(\tau)), u^N(\tau) - u^*(\tau))| \leq C_3 \lim_{N\to\infty} |u^N(\tau) - u^*(\tau)|_{L^2(\Omega)} = 0.$$

Since $u^N(\tau) \rightharpoonup u^*(\tau)$ in $H_0^1(\Omega)$, we conclude (see Lemme I.2., page 156 in [9]) that $A(u^N(\tau)) \rightharpoonup A(u^*(\tau))$ in $H^{-1}(\Omega)$. Finally, from $u^N(\tau) \to u^*(\tau)$ in $L^2(\Omega)$, $u^N(\tau) \rightharpoonup u^*(\tau)$ in $H_0^1(\Omega)$ and $|A(u^N(\tau))|_{L^2(\Omega)} \leq C_3$, we infer (see Démonstration du théorème I.1. and Corollaire I.2'. in [9]) that

$$A(u^N(\tau)) \rightharpoonup A(u^*(\tau)) \quad \text{in } L^2(\Omega). \tag{49}$$

Note that the above limit holds for the entire sequence $\{A(u^N(\tau))\}$ and not only for a subsequence: This follows since every weakly convergent subsequence converges to the same limit. Additionally, by the lower semicontinuity of the norm, we observe that

$$|A(u^*(\tau))|_{L^2(\Omega)} \leq \varliminf_{N\to\infty} |A(u^N(\tau))|_{L^2(\Omega)} \leq C_3.$$

In order to show that $A(u^*) \in L^\infty(0,T;L^2(\Omega))$, we only need to prove the (Bochner) measurability of the map $A(u^*(\cdot)) : [0,T] \to L^2(\Omega)$; and since $L^2(\Omega)$ is separable, we only require weak measurability (see Corollary 1.1.2., page 8 in [1]), i.e., that $t \mapsto (g, A(u^*(t)))$ is measurable, as a real-valued function, for each $g \in L^2(\Omega)$. However, $t \mapsto (g, A(u^N(t)))$ is measurable for each $N \in \mathbb{N}$ for being a step function, and $t \mapsto \langle g, A(u^*(t))\rangle_{L^2(\Omega)}$ is the pointwise limit of the previous sequence and hence it is measurable Hence, $A(u^*) \in L^\infty(0,T;L^2(\Omega))$, holds.

Now, let $z \in L^2(0,T,L^2(\Omega))$, then $t \mapsto (A(u^N(t)), z(t))$ is integrable, bounded as $|(A(u^N(t)), z(t))| \leq C_3|z(t)|_{L^2(\Omega)}$ and also $\lim_{N\to\infty}(A(u^N(t)), z(t)) = (A(u^*(t)), z(t))$ for $t \in [0,T]$. The function $t \mapsto (A(u^*(t)), z(t))$ is also integrable, then by Lebesgue bounded convergence theorem we have

$$\lim_{N\to\infty} \int_0^T (A(u^N(t)), z(t))\,\mathrm{d}t = \int_0^T (A(u^*(t)), z(t))\,\mathrm{d}t,$$

i.e.,

$$A(u^N(\cdot)) \rightharpoonup A(u^*(\cdot)) \quad \text{in } L^2(0,T;L^2(\Omega)) \text{ as } N \to \infty. \tag{50}$$

Let $w \in L^2(0,T;H_0^1(\Omega))$ with $|\nabla w(\tau)| \leq \Phi(\tau, u^*(\tau))$ a.e. in Ω, for a.e. $\tau \in (0,T)$, be arbitrary. By lemma 1, there exists $\{w^N\}$ such that $|\nabla w^N(t)| \leq \Phi(t_{n-1}^N, u_{n-1}^N)$ with $t \in [t_{n-1}^N, t_n^N)$ for $n = 1, 2, \ldots, N$ and

$$w^N \to w \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ as } N \to \infty. \tag{51}$$

Since $f^N = \sum_{n=1}^N f_n^N \chi_{[t_{n-1}^N, t_n^N)}$ with $f_n^N = \frac{1}{k}\int_{t_{n-1}^N}^{t_n^N} f(t)\,\mathrm{d}t$, we have that

$$f^N \to f \quad \text{in } L^2(0,T;L^2(\Omega)) \text{ as } N \to \infty, \tag{52}$$

(see for example [17] or page 21 in [23]) and also, as proven in proposition 3 we observe that

$$\hat{\Theta}(\cdot, u_-^N(\cdot)) \to \Theta(\cdot, u^*(\cdot)), \quad \text{in } L^\infty(0,T;L^2(\Omega)) \text{ as } N \to \infty. \tag{53}$$

Then, by definition of $\{u_n^N\}_{n=0}^N$ in problem $(\mathrm{P}_1^{\mathrm{N}})$, u^N and $\tilde{u}^N$, we have

$$(\partial_t \tilde{u}^N(\tau) + A(u^N(\tau)) - \hat{\Theta}(\tau, u_-^N(\tau)) - f^N(\tau), w^N(\tau) - u^N(\tau)) \geq 0, \quad \forall \tau \in (0,T),$$

and hence by integration on $(0,T)$, we observe

$$\int_0^T (\partial_t \tilde{u}^N(\tau) + A(u^N(\tau)) - \hat{\Theta}(\tau, u_-^N(\tau)) - f^N(\tau), w^N(\tau) - u^N(\tau))\,\mathrm{d}\tau \geq 0. \tag{54}$$

Taking the limit as $N \to \infty$ in (54) (using (48), (50), (51), (52) and (53)) , we have

$$\int_0^T (\partial_t u^*(\tau) + A(u^*(\tau)) - \Theta(\tau, u^*(\tau)) - f(\tau), v - u^*(\tau))\,\mathrm{d}\tau \geq 0. \tag{55}$$

Further, $u^*(\tau) \in \mathbf{K}(\Phi(\tau, u^*(\tau)))$ a.e. in Ω, for a.e. $\tau \in [(0,T])$, as shown in the first paragraph of the proof. Since $w \in L^2(0,T;H_0^1(\Omega))$ with $|\nabla w(\tau)| \leq \Phi(\tau, u^*(\tau))$ a.e. in Ω, for a.e. $\tau \in (0,T)$ is arbitrary, then u^* solves Problem (P_1).

It follows immediately, from $|\nabla u^*(t)| \leq \Phi(t, u^*(t))$ a.e. Ω, for a.e. $t \in (0,T)$, that

$$-\Phi(t, u^*(t))\text{dist}(x, \partial\Omega) \leq u^*(t) \leq \Phi(t, u^*(t))\text{dist}(x, \partial\Omega), \tag{56}$$

for a.e. $x \in \Omega, t \in (0,T)$, i.e., $u(t) \in \mathbf{K}^\pm(\Phi(t, u(t)))$ and consequently $u(t) \in \mathbf{K}^+(\Phi(t, u(t)))$ for a.e. $t \in (0,T)$.

Let $y \in L^2(0,T;H_0^1(\Omega))$ such that $y \in \mathbf{K}^\pm(\Phi(\tau, u(\tau)))$ for a.e. $\tau \in [0,T]$ be arbitrary. By Lemma 1 there exists $\{y^N\}$ such that $|y^N(\tau)| \leq \Phi(t_{n-1}^N, u_{n-1}^N))\text{dist}(x, \partial\Omega)$ for a.e. $x \in \Omega$ with $\tau \in [t_{n-1}^N, t_n^N)$ a.e. and

$$y^N \to y \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ as } N \to \infty. \tag{57}$$

Then, by definition of $\{u_n^N\}_{n=0}^N$ from Problem $(\mathrm{P}_1^{\mathrm{N}})$ and the equivalence result of proposition 4 we have

$$(\partial_t \tilde{u}^N(\tau) + A(u^N(\tau)) - \hat{\Theta}(\tau, u_-^N(\tau)) - f^N(\tau), y^N(\tau) - u^N(\tau)) \geq 0, \quad \forall \tau \in (0,T),$$

and hence, integrating with respect to τ from 0 to T and subsequently taking the limit as $N \to \infty$ (using (48), (50), (57), (52) and (53))

$$\int_0^T (\partial_t u^*(\tau) + A(u^*(\tau)) - \Theta(\tau, u^*(\tau)) - f(\tau), v - u^*(\tau))\,\mathrm{d}\tau \geq 0. \tag{58}$$

Since $y \in L^2(0,T;H_0^1(\Omega))$ with $y(\tau) \in \mathbf{K}^\pm(\Phi(\tau, u^*(\tau)))$ for a.e. $\tau \in (0,T)$ is arbitrary, then u^* solves Problem (P_1) with $\mathbf{K}$ exchanged by $\mathbf{K}^\pm$.

An analogous argument and Lemma 1 proves that u^* also solves Problem (P_1) with $\mathbf{K}$ exchanged by $\mathbf{K}^+$. Finally, the fact that u^* solves Problem (iP_1) follows directly by application of proposition 1. $\square$

6 Numerical Tests

In this section we report on variable splitting type solution algorithms for $(\mathrm{P}_0^{\mathrm{N}})$ and $(\mathrm{P}_1^{\mathrm{N}})$, respectively. For $N \in \mathbb{N}$, the problems $(\mathrm{P}_0^{\mathrm{N}})$ and $(\mathrm{P}_1^{\mathrm{N}})$ reduce to finding $\{u_n^N\}_{n=1}^N$ where, for a fixed n and given u_{n-1}^N, u_n^N is the unique solution to the convex minimization problem:

Problem (P^n)**.**

$$\begin{aligned}
&\min \ \ \mathcal{J}_n^N(u) := \frac{1}{2k}|u-u_{n-1}^N|^2_{L^2(\Omega)} + \frac{1}{2}\langle Au,u\rangle - (\Theta(t_{n-1},u_{n-1}^N)+f(t_{n-1}),u)\\
&\textit{over} \quad u \in H_0^1(\Omega)\\
&\textit{subject to (s.t.)} \quad u \in \mathbf{K}(\Phi(t_{n-1},u_{n-1}^N)).
\end{aligned}$$

The initial state $u_0^N = u_0$ is given and an equidistant discretization in time with mesh size $k := T/N$ is used. Here $T > 0$ corresponds to the final time and $t_n := nk$. Furthermore, $A \equiv 0$ corresponds to $(\mathrm{P}_0^{\mathrm{N}})$ and $Av = \sum_{n=1}^{\ell} \frac{\partial}{\partial x_n} a_n \frac{\partial v}{\partial x_n}$ to $(\mathrm{P}_1^{\mathrm{N}})$. The computation of u_n^N, for fixed n, is performed by algorithm 1 and the overall sequence $\{u_n^N\}_{n=1}^N$ by algorithm 2.

Note that (P^n) is a gradient constrained optimization problem which can be solved by a variety of algorithms such as first-order descent, or semismooth Newton methods. Here, we provide a variable splitting approach which has the advantage of rather simple subproblem solves in its respective steps.

6.1 A Variable Splitting Approach

For $\Omega \subset \mathbb{R}^\ell$, we define the convex and closed set $\mathcal{K}_{n-1} \subset L^2(\Omega)^\ell$ by

$$\mathcal{K}_{n-1} := \{v \in L^2(\Omega)^\ell : |v| \le \Phi(t_{n-1},u_{n-1}^N) \text{ a.e. in } \Omega\}. \tag{59}$$

Note that if u solves (P^n) then $\nabla u \in \mathcal{K}_{n-1}$. Based on this, we introduce a new variable $p \in \mathcal{K}_{n-1}$ and penalize violations of $\nabla u - p = 0$ in $L^2(\Omega)^\ell$ via the following family of γ-parametrized approximating problems:

Problem (P_γ^n)**.**

$$\begin{aligned}
&\min \mathcal{J}_{n,\gamma}^N(u,p) := \mathcal{J}_n^N(u) + \tfrac{\gamma}{2}|\nabla u - p|^2_{L^2(\Omega)^\ell}\\
&\textit{over} \ \ (u,p) \in H_0^1(\Omega) \times L^2(\Omega)^\ell\\
&\textit{s.t.} \ \ p \in \mathcal{K}_{n-1}.
\end{aligned}$$

The existence of minimizers (u^*,p^*) of (P_γ^n) follows from standard arguments. In the case where A is a second-order elliptic operator, variable splitting methods for solving elliptic variational problems with gradient constraints have been investigated recently in [21]: In particular, for $\gamma \to \infty$ the convergence of solutions $\{u_\gamma, p_\gamma\}$ of (P_γ^n) to $\{u, \nabla u\}$, where u is the minimizer of (P^n), is established. Minor modifications of the arguments yield a similar consistency result for $A \equiv 0$.

For given $\gamma > 0$, a solution to (P_γ^n) is obtained via alternating minimization according to algorithm 1. Here, $I_{\mathcal{K}_{n-1}}$ denotes the indicator function of the constraint set defined by the iterate u_{n-1}^N, i.e.

$$I_{\mathcal{K}_{n-1}}(p) = \begin{cases} 0 & \text{if } p \in \mathcal{K}_{n-1},\\ +\infty & \text{else.}\end{cases}$$

Algorithm 1 Variable Splitting Algorithm

Data: $n \in \mathbb{N}, k, \gamma \in \mathbb{R}^+$, $u_{n-1}^N \in L^2(\Omega)$
1: Choose $u^{(0)} \in L^2(\Omega)$ and set $l = 0$.
2: **repeat**
3: Compute $p^{(l+1)} = \operatorname{argmin}_{p \in L^2(\Omega)^\ell} |p - \nabla u^{(l)}|^2_{L^2(\Omega)^\ell} + I_{\mathcal{K}_{n-1}}(p)$.
4: Compute $u^{(l+1)} = \operatorname{argmin}_{u \in H_0^1(\Omega)} \mathcal{J}_{n,\gamma}^N(u, p^{(l)})$.
5: Set $l = l + 1$.
6: **until** some stopping rule is satisfied.

The problem in step 3 of algorithm 1 has a unique solution in closed form. In fact, it is given by the projection of $\nabla u^{(l)}$ onto the set $\mathcal{K}_{n-1}$, i.e.,

$$p^{(l+1)} = P_{\mathcal{K}_{n-1}}(\nabla u^{(l)}) = \begin{cases} \nabla u^{(l)} \min\{1, \frac{\Phi(t_{n-1}, u_{n-1}^N)}{|\nabla u^{(l)}|}\}, & \text{if } |\nabla u^{(l)}| > 0, \\ 0 & \text{else.} \end{cases}$$

Note here that the $\min$-operation is pointwise. Further, for given $p^{(l)} \in L^2(\Omega)^\ell$, there exists a unique minimizer $u^* \in H_0^1(\Omega)$ of the problem in step 4 of algorithm 1. Consequently, the sequence $\{u^{(l)}\}$ obtained by algorithm 1 is generated as follows: Given $u^{(l)}$, $u^{(l+1)}$ is the unique solution of

$$\min \mathcal{J}_{n,\gamma}^N(u, P_{\mathcal{K}_{n-1}}(\nabla u^{(l)})), \quad \text{over } u \in H_0^1(\Omega). \tag{60}$$

Denoting the solution mapping $u^{(l)} \mapsto u^{(l+1)}$ in (60) by $T : H_0^1(\Omega) \to H_0^1(\Omega)$, we have $u^{(l+1)} = T(u^{(l)})$. For establishing convergence of the associated algorithm, we next study continuity properties of the map T. For this purpose, we observe that the first-order necessary optimality condition for (60) reads

$$\begin{aligned} &k^{-1}u^{(l+1)} + Au^{(l+1)} - \gamma\Delta u^{(l+1)} \\ &\qquad = k^{-1}u_{n-1}^N + \Theta(t_{n-1}, u_{n-1}^N) + f(t_{n-1}) - \gamma\nabla \cdot P_{\mathcal{K}_{n-1}}(\nabla u^{(l)}), \quad \text{in } H^{-1}(\Omega). \end{aligned} \tag{61}$$

Let $v, w \in H_0^1(\Omega)$ and define $V := T(v), W := T(w)$. Using $V - W$ as a test function in the corresponding equations (61) for $T(v)$ and $T(w)$, respectively, and subtracting the resulting equations, we obtain the estimate

$$\frac{|V - W|^2_{L^2(\Omega)}}{k|V - W|_{H_0^1(\Omega)}} + (\gamma + \eta_A)|V - W|_{H_0^1(\Omega)} \leq \gamma|v - w|_{H_0^1(\Omega)}, \tag{62}$$

where $\eta_A \geq 0$ is the uniform monotonicity constant of $A : H_0^1(\Omega) \to H^{-1}(\Omega)$. For this estimate we also use the non-expansiveness of the map $P_{\mathcal{K}_{n-1}} : L^2(\Omega)^\ell \to \mathcal{K}_{n-1}$, i.e., $|P_{\mathcal{K}_{n-1}}(q_1) - P_{\mathcal{K}_{n-1}}(q_2)|_{L^2(\Omega)^\ell} \leq |q_1 - q_2|_{L^2(\Omega)^\ell}$ for all $q_1, q_2 \in L^2(\Omega)^\ell$. Consequently, we find

$$|V - W|_{H_0^1(\Omega)} \leq \frac{\gamma}{\gamma + \eta_A}|v - w|_{H_0^1(\Omega)}. \tag{63}$$

In the case of problem (P_1), we have $\eta_A > 0$. Thus, T is a contractive mapping and for each $\gamma > 0$ there exists a unique fixed point u^γ due to Banach's fixed point theorem. Further, the pair $(u^\gamma, P_{\mathcal{K}_{n-1}}(\nabla u^\gamma))$ is a solution to (P_γ^n) and u^γ converges to the solution of (P^n) in $H_0^1(\Omega)$ as $\gamma \to \infty$. In the case of problem (P_0) (where $A \equiv 0$) we have $\eta_A = 0$ and only obtain non expansiveness of T. Here the existence of a fixed point (which is not necessarily unique) is ensured by the theorem of Browder-Göhde-Kirk (see [5, Chapter 4.3]). Moreover, let u^γ be one of these fixed points for each γ, then u^γ converges to the solution of P^n in $L^2(\Omega)$ as $\gamma \to \infty$.

6.2 Finite Element Discretization

Next we introduce the spatial discretization of the problem and restrict ourselves to the setting of polygonal and bounded subsets $\Omega \subset \mathbb{R}^2$. Let $\mathcal{T}$ be a shape regular, quasi uniform triangularization of Ω of mesh width h with shape parameter $C_{\mathcal{T}} = \max_{\tau \in \mathcal{T}} h_\tau/\rho_\tau$. Here, h_τ is the diameter of the triangle τ and ρ_τ the radius of the largest ball inscribed into it, respectively. The set of inner nodes is denoted by $\mathcal{N}$. For the discretization of the functions u_n^N we utilize $\mathcal{P}_{1,0}$; the space of globally continuous functions $v : \Omega \to \mathbb{R}$ with zero boundary conditions such that $v|_\tau$ is affine for $\tau \in \mathcal{T}$. The associated nodal basis is

$$\{\varphi_z \in \mathcal{P}_{1,0} : z \in \mathcal{N}, \forall \bar{z} \in \mathcal{N}, \varphi_z(\bar{z}) = \delta_{z,\bar{z}}\}$$

where $\delta_{z,\bar{z}}$ denotes the Kronecker-Delta with $\delta_{z,\bar{z}} = 1$ for $\bar{z} = z$ and $\delta_{z,\bar{z}} = 0$ otherwise. As a consequence, the gradient of the discrete approximations of u_n^N is a $\mathcal{T}$-piecewise constant vector.
The variable $p \in L^2(\Omega)^2$ is discretized by vectors of $\mathcal{T}$-piecewise constant functions, and the forcing terms $f(t_{n-1})$ and $\Theta(t_{n-1}, u_{n-1}^N)$, both elements of $L^2(\Omega)$, by $\mathcal{T}$-piecewise constant functions, as well. The gradient bound $\Phi(t_{n-1}, u_{n-1}^N)$ is discretized as a $\mathcal{T}$-piecewise constant function where we use averages on the elements of the discretization in case $\Phi(t_{n-1}, u_{n-1}^N)$ is a spatially distributed function (cf. [25]). In Examples 6.3.2 and 6.3.3 this average can be computed exactly for the discrete approximations of the state while in Example 6.3.4 we used a Gaussian quadrature rule with four evaluation points on the reference triangle. For more information on finite-element discretizations we refer to [49, 50]. Based on this discretization, the so called inverse inequality is available, providing (see, e.g. [50, Chapter 3.6]) the estimate $|v_h|^2_{H_0^1(\Omega)} \leq \tilde{\beta}(h)|v_h|^2_{L^2(\Omega)}$ for all $v_h \in \mathcal{P}_{1,0}$ with

$$\tilde{\beta}(h) = 2(3/2)^3(2+\sqrt{2})C_{\mathcal{T}} \max_{\tau \in \mathcal{T}}(h_\tau^{-2}).$$

Restricting arguments of T and solutions to (61) to the subspace $\mathcal{P}_{1,0} \subset H_0^1(\Omega)$, we can further refine (63) by utilizing the inverse inequality in (62). In fact, we obtain

$$|V_h - W_h|_{H_0^1(\Omega)} \leq \frac{\gamma}{\gamma + \eta_A + (\tilde{\beta}(h)k)^{-1}} |v_h - w_h|_{H_0^1(\Omega)}. \tag{64}$$

for $v_h, w_h \in \mathcal{P}_{1,0}$ with $V_h = T(v_h)$ and $W_h = T(w_h)$. Consequently, there exist a unique fixed point of the solution mapping T by the Banach Contraction Principle for the discretized versions of both problems, (P_0) and (P_1).

6.3 Overall Solution Algorithm

For each time step in (P_0) and (P_1), algorithm 2 states our overall numerical solution scheme. In this context, algorithm 1 is used in steps 3 and 4, respectively. Moreover, algorithm 2 consist of two parts: First, the time step $n = 1$ is considered and an approximate solution to (P^1) is computed by a γ-path following strategy via the family of problems (P^1_γ). The parameter γ is increased until it reaches a value γ_{max} where we accept the solution of (P^1_γ) as approximation to the solution of (P^1). In the second part, the remaining time steps $n = 2, \ldots, N$ are computed with fixed $\gamma \geq \gamma_{max}$.

We are left to specify the stopping rule for Algorithm 1 which is used in each time step in Algorithm 2. For this purpose, consider the following: Let $\mathcal{X}$ be a Banach space and $H : \mathcal{X} \to \mathcal{X}$ a contractive mapping with contraction rate $r \in (0, 1)$. Denote its unique fixed point by x^* and let $\{x^i\}_{i=0}^\infty$ be the

Algorithm 2 QVI Solution Algorithm

Data: $N \in \mathbb{N}$, $k, \gamma_0, \gamma_{\max} \in \mathbb{R}^+$, $\beta_\gamma > 1$ $u_0^N \in L^2(\Omega)$

1: Initialize $u^{(0)} = u_0^N$, and set $\gamma = \gamma_0$.
2: **repeat**
3: Compute $u_{1,\gamma}^N$ by Algorithm 1 initialized with $u^{(0)}$ for $n = 1, k, \gamma$ and u_0^N.
4: Set $u^{(0)} := u_{1,\gamma}^N$ and $\gamma := \beta_\gamma \gamma$.
5: **until** $\gamma \geq \gamma_{\max}$.
6: Set $u_1^N = u_{1,\gamma}^N$.
7: **For** $n = 2$ **to** N
8: Compute u_n^N by Algorithm 1 initialized with $u^{(0)} := u_{n-1}^N$ for n, k, γ and u_{n-1}^N.
9: **end**

sequence generated by $x^{i+1} = H(x^i)$ for a given starting point $x^0 \in \mathcal{X}$. From Banach's fixed point theorem we obtain

$$|x^i - x^*|_{\mathcal{X}} \leq \frac{1}{1-r}|x^{i+1} - x^i|_{\mathcal{X}}.$$

In light of (64), this yields a suitable way to estimate $|u_h^{(l)} - u_h^*|_{H_0^1(\Omega)}$ in terms of the distance of two consecutive iterates, with u_h^* denoting the fixed point of the discretized version of T, the solution mapping of (60). The contraction rate here is $r = \gamma(\gamma + \eta_A + (\tilde{\beta}(h)k)^{-1})^{-1}$ and, hence, algorithm 1 is stopped as soon as

$$|u_h^{(l+1)} - u_h^{(l)}|_{H_0^1(\Omega)} \leq TOL \frac{\eta_A + (\tilde{\beta}(h)k)^{-1}}{\gamma + \eta_A + (\tilde{\beta}(h)k)^{-1}} \tag{65}$$

is satisfied, which ensures $|u_h^{(l+1)} - u_h^*|_{H_0^1(\Omega)} \leq TOL$ for some user-specified stopping tolerance $TOL > 0$.

The value of $\gamma_{\max}$ in algorithm 2 is selected based on two considerations: (i) the discretization error of the finite element method, and (ii) the error introduced by the regularization of the state constraint $u \in \mathbf{K}(\Phi(t_{n-1}, u_{n-1}^N))$. In [20], a heuristic rule was developed for this purpose and evidence was found that the discretization error dominates if $\gamma \geq ch^{-4}$ for some constant $c > 0$. On the other hand, in numerical computations, the maximal value γ is limited by (65) and the limited accuracy of implementations on computers (double precision floating point representation in our case). Thus, we utilize $\gamma_{\max} = \max\{10^{-12}/TOL, ch^{-4}\}$ in all our numerical tests.

6.4 Examples

In all of our numerical tests we use $\Omega = (0,1) \times (0,1)$. We discretize Ω by a uniform grid with mesh size $h = 2^{-7}$ providing a partition into triangles, and the time step $k = T/N$ is chosen differently for each example. Here, we have $\beta := \tilde{\beta}(h) \approx 9 \times 10^5$. Note, that for each value of γ, the system matrix of the linear problems in step 4 of algorithm 1 is fixed. We exploit this property and solve the linear problems by a Cholesky factorization of the system matrix which has to be computed once for each value of γ. The update of this parameter uses $\gamma_0 = 1$, $\beta_\gamma = 4$ and $\gamma_{\max} = 10^6$ unless otherwise stated. The termination criterion of Algorithm 1 in (65) utilizes $TOL = 10^{-6}$, and in all of the following examples $u_0 \equiv 0$ is chosen. In addition to studying the behaviour of the solution to the QVI, we further investigate the active set $\mathcal{A}$ defined by

$$\mathcal{A}(t) := \{x \in \Omega : |\nabla u(t,x)| = \Phi(t, u(x))\}.$$

As the case $A \not\equiv 0$ appears more common in the literature, we consider $A = -\Delta$ only in the first example and focus on $A \equiv 0$ in the following three.

6.4.1 Example 1

For the first example we consider the dissipative case in the setting of (P_1) for a final time $T = 10^{-3}$ and time step $k = 10^{-4}$.

We utilize $f \equiv 1, \Theta \equiv 0$ and $A = -\Delta$. Further, the gradient bound is given by $\Phi(t,u) = \beta_1 |u|_{L^2(\Omega)} + \beta_2$ with $\beta_1 = 0.03$ and $\beta_2 = 0.001$. Since f is constant, and the bound on the gradient constraint is not spatially dependent, the problem is equivalent to a parabolic QVI of the double obstacle-type.

Figure 1 depicts the final state $u(T)$, and the active set $\mathcal{A}(T)$ comprises essentially the entire domain Ω.

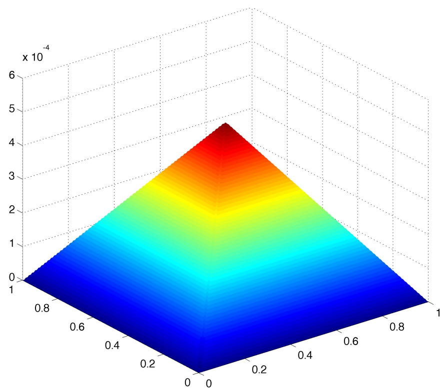

Figure 1: The state u at final time $T = 10^{-3}$.

6.4.2 Example 2: Growth of large sandpiles

The growth of sandpiles over a flat surface Ω where the sand is removed instantaneously on $\partial\Omega$ and where the intensity of material being poured per unit of time is given by f can be described by a variational inequality with a constant gradient constraint (see, e.g. [40, 43]). Specifically, the solution u to (P_0) for $\Theta = 0$ and $\Phi \equiv \tan(\theta)$ for θ the angle of repose of the material being poured onto the pile, represents the height of the surface determined by the outermost layer of the pile. It has recently been discovered that the angle of repose θ is actually a gravity dependent quantity (see [31]) and hence it should be taken as an increasing function of the height of the pile. This entails that the overall formulation of the problem, for piles which are relatively high, amounts to a quasi-variational inequality of class (P_0).

In order to model the above type of behaviour we consider $\Phi(u) = \beta_1 u + \beta_2$ with parameters $\beta_1 = 5, \beta_2 = 10^{-4}$. The choice of the parameters is made in order to capture interesting features of the behaviour of the problem: In particular, to observe large regions of the domain where $|\nabla u| = \beta_1 u + \beta_2$. We assume that material is allocated uniformly everywhere on the domain so that $f \equiv 1$ and further, we consider $\Theta \equiv 0, T = 0.001$, and $k = 10^{-5}$. In Figures 2(a) and 2(d) we have depicted the state at $t = 5 \cdot 10^{-5}$ and $t = T$, respectively. Lateral views on these graphs are shown in Figures 2(b) and 2(e) where the dependence of the gradient on the state is evident in regions of activity. The active sets at $t = 5 \times 10^{-5}$ is given in Figure 2(c), and the one at $t = T$ essentially comprises the entire Ω.

6.4.3 Example 3: Nonzero Θ and finite time blow-up

If $\alpha > 1$ with $|\Theta(t,v)|_{L^2(\Omega)} = L_\Theta |v|^\alpha_{L^2(\Omega)}$, then theorem 1 and theorem 3 only ensure the existence of a solution u up to a certain time T^* which depends on $\alpha > 0$, $|u_0|_{L^2(\Omega)}$ and $|f|_{L^2(0,T^*;L^2(\Omega))}$. This example is chosen to study the behavior of the numerical approximation of the solution in a case of finite time blow-up. We consider (P_0) with a piece-wise constant forcing term f which is independent

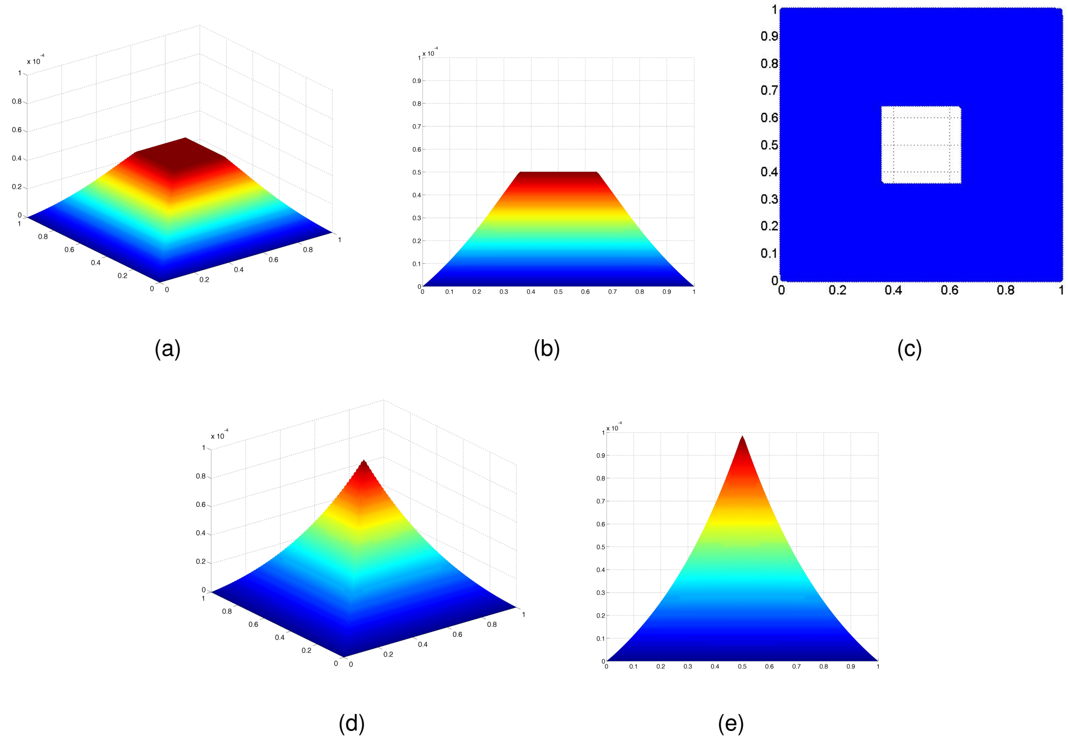

Figure 2: The state $u(t)$ at time $t = 5 \times 10^{-5}$ is depicted in figures 2(a), 2(b) and at $t = 10^{-3}$ in 2(d) and 2(e). The active set $\mathcal{A}(t)$ at $t = 5 \times 10^{-5}$ is given in 2(c)

of t and defined by

$$f(x) = \begin{cases} \frac{\sqrt{133}}{10}, & \text{if } x_2 \geq \frac{1}{12} + \frac{2}{3}x_1 \text{ and } x_2 \leq -\frac{1}{8} + \frac{3}{2}x_1; \\ \frac{\sqrt{13}}{100}, & \text{else.} \end{cases}$$

Moreover, we set $\Theta(t,u) := 2 \cdot 10^{12} |u|_{L^2(\Omega)} u$ and $\Phi(t,u) := \beta_1 u + \beta_2$ for $\beta_1 = 100, \beta_2 = 10^{-8}$. In this case, according to theorem 1, solutions are guaranteed to exist until a time T with

$$T < (2 \times 10^{12} |f|_{L^2(\Omega)})^{-1/2} = 10^{-6} =: T^*.$$

In our tests, we set the time step to $k = 10^{-8}$.

In Figures 3(a),3(b) and 3(c) we depict the solution at times $t = 10^{-7}$, $t = 5 \cdot 10^{-7}$ and $t = T^*$, while Figures 3(d), 3(e) and 3(f) show the corresponding active sets. The behaviour of $t \mapsto |u(t)|_{H_0^1(\Omega)}$ for $t > T^*$ is also studied and it is observed that the solutions seems to blow up for $t > T^*$, at $t \simeq 1.78 \cdot 10^{-6}$ (see section 6.4.3).

6.4.4 Example 4: Magnetization of a superconductor

The evolution of a magnetic field $u(t)$ inside a type-II-superconductor under the influence of an external magnetic field $b_e(t)$ can be described by a quasivariational inequality (see [3] and the references therein) of the type (P_0) where $f(t) = \partial_t b_e(t)$ and $\Theta \equiv 0$. Here the function characterizing the gradient bound is given by $\Phi(t,u) := a(a + |u + b_e(t)|)^{-1}$, as it can be found in Bean's critical state model. Note that the function $\Phi(t,u)$ does not meet the assumptions from section 4 since it fails to be increasing with respect to u, in fact, it is decreasing. However, we use the methodology presented in

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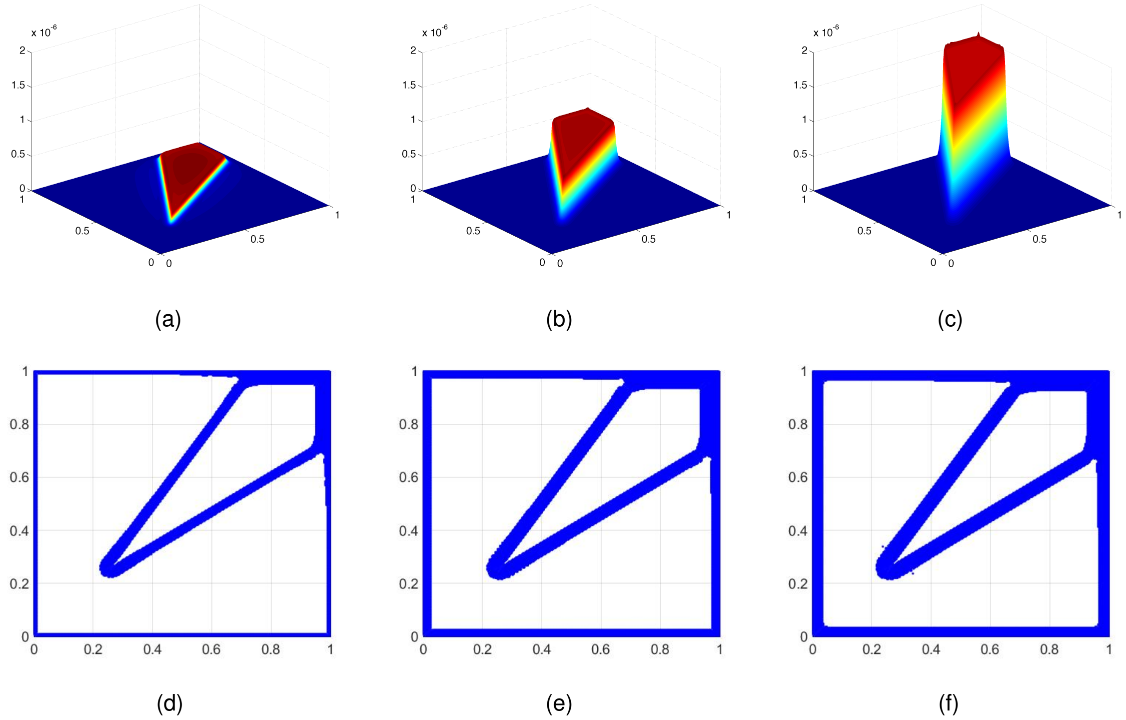

Figure 3: The state $u(t)$ at times $t = 10^{-7}, 5 \cdot 10^{-7}, 10^{-6}$ is depicted in Figures 3(a), 3(b) and 3(c), respectively. The corresponding active sets $\mathcal{A}(t)$ at those same times are given in 3(d), 3(e) and 3(f), respectively.

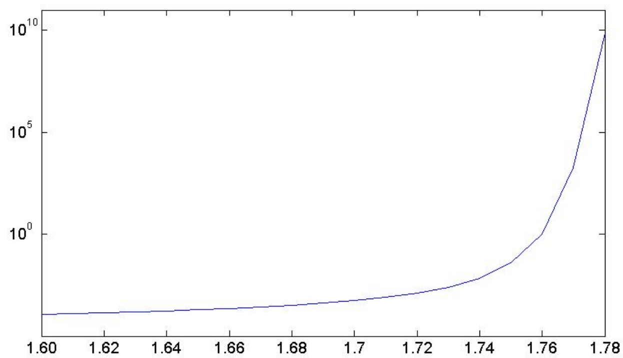

Figure 4: Plot of $t \mapsto |u(t)|_{H_0^1(\Omega)}$ where the x-axis represents $t \times 10^{-6}$ units.

this section and solve the problem for a final time $T = 0.08$ and a time step $k = 8 \times 10^{-4}$. As in [3], we choose $a = 0.02$ and $b_e(t) = t$.

It is remarkable that in this example, even for very small values of $\gamma_{\max}$, we obtain results that correspond to the real solution of the problem (see [3]). In fact, there seem to be no significant changes in the solution for $\gamma_{\max} > 10$: In Figures 5(a) and 5(c) we depict final states for $\gamma_{\max} = 10$ and $\gamma_{\max} = 100$ and in Figures 5(b) and 5(d) their corresponding active sets depicted, respectively.

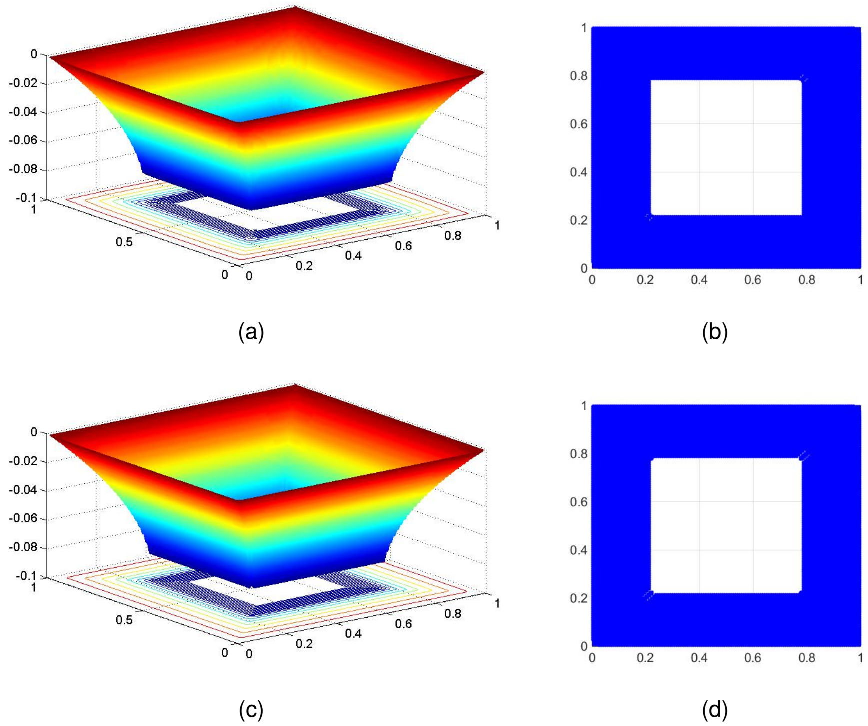

Figure 5: The final state and active set for $\gamma = 10$ are depicted in figures 5(a) and 5(b), respectively, and for $\gamma = 100$ in 5(c) and 5(d), respectively.

7 Conclusions

We have provided a general theoretical and numerical framework to deal with certain types of time-evolution quasi-variational inequalities, given by problems (P_0) and (P_1). A sequence of approximations is built from problems (P_0^N) and (P_1^N), which reduce to compute solutions to N convex optimization problems. This sequence of approximations is shown useful to provide an existence result, to extend the regularity and to prove the non-decreasing property of solutions. Further, the problems (P_0^N) and (P_1^N) are suitable for computer implementation and a simple algorithm involving a splitting method is shown to provide reasonable numerical approximations to solutions of (P_0) and (P_1).

A Lower Solutions for VIs

The following result is due (to the best of our knowledge) to Bensoussan and is included for the sake of completeness.

Proposition 5. *Let $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ be linear, bounded and uniformly monotone. Additionally, suppose that if $v \in H_0^1(\Omega)$, then $\langle Av^-, v^+\rangle \leq 0$. Let $\varphi_i \in L^\infty(\Omega)$ with $i = 1, 2$ be such $0 \leq \varphi_1 \leq \varphi_2$ a.e.,*

$$\mathbf{K}(\varphi_i) := \{v \in H_0^1(\Omega) : v \leq \varphi_i \;\text{Êa.e.}\; \},$$

and suppose that $f_i \in L^2(\Omega)$ with $i = 1, 2$ and $f_1 \leq f_2$ a.e.. Then, $y_1 \leq y_2$ a.e., where $y_i = \mathbb{S}(A, f_i, \mathbf{K}(\varphi_i))$.

Further, let $\varphi \in L_+^\infty(\Omega)$ and $f \in L^2(\Omega)$. If $z \in \mathbf{K}(\varphi)$ satisfies

$$\langle Az - f, \phi\rangle \leq 0, \quad \forall \phi \in H_0^1(\Omega) : \quad \phi \geq 0 \text{ a.e.}, \tag{66}$$

we say z is a lower solution for the triple $(A, f, \mathbf{K}(\varphi))$. For any lower solution z, we have that $z \leq \mathbb{S}(A, f, \mathbf{K}(\varphi))$.

Proof. Since $y_i \in \mathbf{K}(\varphi_i)$ for $i = 1, 2$, then $0 \leq v_1 := \min(y_1, y_2) = y_1 - (y_1 - y_2)^+ \leq \psi_1$ and $0 \leq v_2 := \max(y_1, y_2) = y_2 + (y_1 - y_2)^+ \leq \psi_2$. Hence, from $\langle Ay_i - f_i, v_i - y_i\rangle \geq 0$ for $i = 1, 2$, we obtain

$$\langle Ay_1 - f_1, -(y_1 - y_2)^+\rangle \geq 0, \qquad \text{and} \qquad \langle Ay_2 - f_2, -(y_1 - y_2)^+\rangle \leq 0. \tag{67}$$

Subtracting the second inequality from the first one, we observe

$$\langle A(y_1 - y_2), (y_1 - y_2)^+\rangle \leq (f_1 - f_2, (y_1 - y_2)^+) \leq 0,$$

since $f_1 - f_2 \leq 0$. Since A is uniformly monotone and $\langle Av^-, v^+\rangle \leq 0$, for all $v \in H_0^1(\Omega)$, we obtain the following chain of inequalities:

$$\begin{aligned}
c|(y_1 - y_2)^+|^2_{H_0^1(\Omega)} &\leq \langle A(y_1 - y_2)^+, (y_1 - y_2)^+\rangle \\
&\leq \langle A(y_1 - y_2)^+, (y_1 - y_2)^+\rangle - \langle A(y_1 - y_2)^-, (y_1 - y_2)^+\rangle \\
&= \langle A(y_1 - y_2), (y_1 - y_2)^+\rangle \leq 0.
\end{aligned}$$

Therefore, $(y_1 - y_2)^+ = 0$ a.e., that is, $y_1 \leq y_2$ a.e. in Ω.

Let $y = \mathbb{S}(A, f, \varphi)$, so $y \in \mathbf{K}(\varphi)$ and

$$\langle Ay - f, v - y\rangle \geq 0, \quad \forall y \in \mathbf{K}(\varphi). \tag{68}$$

Replacing $v = y - \phi$ with $\phi \in H_0^1(\Omega)$ and $\phi \geq 0$ a.e. in Ω, we observe that $y = \mathbb{S}(A, f, \mathbf{K}(\varphi))$ is a lower solution for the triple $(A, f, \mathbf{K}(\varphi))$. Now we prove that if z is an arbitrary lower solution, the $z \leq y$ a.e. in Ω. Let $\phi = (z - y)^+$ and $v = \max(y, z) = y + (z - y)^+$ on (66) and (68), respectively, then

$$\langle Az - f, -(z - y)^+\rangle \geq 0 \qquad \text{and} \qquad \langle Ay - f, -(z - y)^+\rangle \leq 0.$$

These are exactly the same inequalities as in (67). Therefore, we have that $(z - y)^+ = 0$, i.e., $z \leq y$ a.e. in Ω. □

B Proof of lemma 1

Proof. Consider first **a** and $i = 1$. Let $w \in \mathscr{K}(\Psi)$ and note that the condition "$w^N(t) \in \mathbf{K}(\Phi(t^N_{n-1}, u^N_{n-1}))$ with $t \in [t^N_{n-1}, t^N_n)$" is equivalent to

$$|\nabla w^N(\tau)| \leq \sum_{m=1}^{N} \Phi(t^N_{m-1}, u^N_-(\tau))\chi_{[t^N_{m-1}, t^N_m)}(\tau) =: \hat{\Phi}(\tau, u^N_-(\tau)), \quad \tau \in [0, T]. \tag{69}$$

Denote by $\{\tilde{u}^N\}$ the convergent subsequence obtained in theorem 2, i.e., $\tilde{u}^N \to u^*$ in $C([0, T]; L^2(\Omega))$. Then, by the inequality in (19) we also have that

$$\lim_{N\to\infty} |u^N_- - u^*|_{L^\infty(0,T;L^2(\Omega))} = \lim_{N\to\infty} \sup_{t\in[0,T]} |u^N_-(t) - u^*(t)|_{L^2(\Omega)} = 0.$$

By assumption 1, we have that $\Phi : [0,T] \times L^2(\Omega) \to L^\infty(\Omega)$ is uniformly continuous, i.e., for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$|t_1 - t_2| + |y_1 - y_2|_{L^2(\Omega)} < \delta(\epsilon) \quad \Longrightarrow \quad |\Phi(t_1, y_1) - \Phi(t_2, y_2)|_{L^\infty(\Omega)} < \epsilon.$$

Therefore, for sufficiently large N we have that

$$\frac{1}{N} + |u_-^N - u^*|_{C([0,T];L^2(\Omega))} < \delta(\epsilon) \quad \Longrightarrow \quad |\hat{\Phi}(\tau, u_-^N(\tau)) - \Phi(\tau, u^*(\tau))|_{L^\infty(\Omega)} < \epsilon, \forall \tau \in [0,T].$$

Recall that by assumption we have that the mapping Φ satisfies: 1) $\Phi(t,v) \geq \nu > 0$ a.e. in Ω, for a.e. $t \in [0,T]$ and all $v \in L^2(\Omega)$. 2) It is non-decreasing in both variables. 3) $T \mapsto \Phi(T,v)$ maps bounded sets in $L^2(\Omega)$ into bounded sets in $L^\infty(\Omega)$. Then, we define $\varphi_N(t,x) := \hat{\Phi}(t, u_-^N(t))(x)$ and $\varphi(t,x) := \Phi(t, u^*(t))(x)$ with $(t,x) \in Q := [0,T] \times \Omega$. It follows that $\varphi_N, \varphi \in L^\infty(Q)$ and also

$$\varphi_N, \varphi \geq \nu > 0 : \quad \varphi_N \to \varphi \text{ in } L^\infty(Q), \text{ as } N \to \infty. \tag{70}$$

Now, we prove that for any $\eta \in (0,1)$, there is an $N(\eta)$ such that

$$0 \leq \eta\varphi(z) \leq \varphi_N(z) \quad \text{a.e. } z \in Q,$$

for $N \geq N(\eta)$. In fact, let $\eta \in (0,1)$ be arbitrary, and consider the sets

$$Q_N := \{z \in Q : \eta\varphi(z) > \varphi_N(z) \text{ a.e.}\}.$$

Then, for almost all $z \in Q_N$, we have

$$|\varphi - \varphi_N|_{L^\infty(Q)} \geq \varphi(z) - \varphi_N(z) > (1-\eta)\varphi(z) \geq (1-\eta)\nu > 0.$$

But since $|\varphi - \varphi_N|_{L^\infty(Q)} \to 0$, there exists $N(\eta) \in \mathbb{N}$, such that $|Q_N| = 0$ for all $N \geq N(\eta)$.

Let $\{\eta_j\}$ be a monotonically increasing sequence in $(0,1)$ such that $\lim_{j\to\infty} \eta_j = 1$. Let $w \in L^2(0,T; H_0^1(\Omega))$ satisfy $|\nabla w(t)(x)| \leq \varphi(t,x)$. Then, $w^j := \eta_j w$ fulfils

$$|\nabla w^j(t)(x)| \leq \eta_j |\nabla w(t)(x)| \leq \eta_i \varphi(t,x) \leq \varphi_{N(\eta_j)}(t,x),$$

for almost all $(t,x) \in Q$. Finally, $|w^j - w|_{L^2(0,T;H_0^1(\Omega))} = (1-\eta_j)|w|_{L^2(0,T;H_0^1(\Omega))} \leq (1-\eta_j)|\varphi|_{L^\infty(Q)} \to 0$ as $j \to \infty$. This proves the statement concerning $w \in \mathscr{K}(\Psi)$.

Next, we focus on **a** and $i = 2$. For the same sequence $\{\eta_j\}$ as before, suppose $w \in L^2(0,T; H_0^1(\Omega))$ is arbitrary and such that $w \in \mathscr{K}^\pm(\Psi)$. Then $w^j(t) = \eta_j w(t)$ belongs to $\mathbf{K}^\pm(\varphi_{N(\eta_j)}(t,\cdot))$, i.e.,

$$-\varphi_{N(\eta_j)}\text{dist}(x,\partial\Omega) \leq -\eta_j\varphi\text{dist}(x,\partial\Omega) \leq \eta_j w \leq \eta_j\varphi\text{dist}(x,\partial\Omega) \leq \varphi_{N(\eta_j)}\text{dist}(x,\partial\Omega),$$

(where we have omitted "(t,x)" for the sake of brevity) for a.e. $t \in (0,T), x \in \Omega$. Further, it follows that $|w^j - w|_{L^2(0,T;H_0^1(\Omega))} = (1-\eta_j)|w|_{L^2(0,T;H_0^1(\Omega))} \to 0$ as $j \to \infty$, and hence proves this case $i = 2$ for the **a** statement and an analogous argument can be used to prove $i = 3$.

We now consider **b**. Since $\tau \in [t_{n-1}^N, t_n^N)$ is constant, $\lim_{N\to\infty} t_{n-1}^N = \tau$ and $\phi^N = \Phi(t_{n-1}^N, u_{n-1}^N) = \Phi(t_{n-1}^N, \tilde{u}^N(t_{n-1}^N))$. By (22) we have $\phi^N \to \phi$ in $L^\infty(\Omega)$ and in addition $\phi^N, \phi \geq \nu > 0$ a.e. in Ω. These are the conditions in (70) (with Q exchanged by Ω), and using the same argument we can prove that given a monotonically increasing sequence $\{\eta_j\}$ in $(0,1)$ with $\lim_{j\to\infty} \eta_j = 1$, then $w^j = \eta_j w$ satisfies $w^j \in \mathbf{K}(\phi^{N(\eta_j)})$, provided that $w \in \mathbf{K}(\phi)$, and $w^j \to w$ in $H_0^1(\Omega)$. This proves the $i = 1$ case and analogous modifications of the argument in **a.** proves the cases concerning $i = 2$ and $i = 3$. □

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