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Abstract

We consider planar autonomous systems $dx/dt = P(x,y,\lambda),\ dy/dt = Q(x,y,\lambda)$ depending on a scalar parameter λ. We present conditions on the functions P and Q which imply that there is a parameter value λ_0 such that for $\lambda > \lambda_0$ this system has a unique limit cycle which is hyperbolic and stable. Dulac-Cherkas functions, rotated vector fields and singularly perturbed systems play an important role in the proof.

1 Introduction

We consider planar autonomous differential systems

$$\frac{dx}{dt} = P(x,y,\lambda),\ \frac{dy}{dt} = Q(x,y,\lambda) \tag{1.1}$$

depending on a scalar parameter $\lambda \in \mathbb{R}$. Our goal is to derive conditions on P and Q such that there is a $\lambda_0 \in \mathbb{R}$ with the property that for all $\lambda > \lambda_0$ system (1.1) has a unique limit cycle in the phase plane which is hyperbolic and stable.
Our approach to treat this problem is based on the bifurcation theory of planar autonomous systems. We recall that $\lambda = \lambda_b$ is said to be a bifurcation point of system (1.1) if there is some sufficiently small open interval Λ having λ_b as boundary point such that the phase portraits of system (1.1) for $\lambda = \lambda_b$ and $\lambda \in \Lambda$ are topologically different.
The underlying idea of our approach can be formulated as follows: We assume that $\lambda = \lambda_0$ and $\lambda = +\infty$ are bifurcation points of system (1.1) connected with the appearance of a limit cycle which is hyperbolic and stable, and we suppose that the interval $(\lambda_0, +\infty)$ does not contain any bifurcation point of system (1.1).
Our main goal in that paper is to derive conditions on the functions P and Q such that the assumptions of our underlying idea are fulfilled. The class of Dulac-Cherkas functions, the theory of one-parameter families of rotated vector fields and singularly perturbed systems are key ingredients in our approach. In the Appendices 1 - 3, their basic properties are summarized. Finally, we illustrate our approach by an example.

There is a vast literature about local and global bifurcation scenarios of limit cycles in planar systems, see for examples [3, 4, 5, 9, 10] and the cited literature therein. The seminal book "Theory of bifurcations of smooth dynamic systems on the plane" by A.A. Andronov and his coworkers [1] which has been published in Russian in 1967 contains the corner stones of the local bifurcation theory of planar dynamic systems, including all basic results about the bifurcation of limit cycles. If we consider the bifurcation of planar vector fields as a nonlinear eigenvalue problem in some function space, then the first global bifurcation results has been obtained by P.H. Rabinowitz [8].

In the next section we formulate our conditions on the functions P and Q under which our approach works and we present our main result.

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2 Assumptions. Main result

Consider system (1.1) under the following assumptions:

(A_1). $P, Q : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are sufficiently smooth.*

(A_2). *System* (1.1) *has* $\forall \lambda \in \mathbb{R}$ *a unique equilibrium* $E(\lambda)$ *in the finite part of the phase plane.*

Without loss of generality we may suppose that $E(\lambda)$ is located at the origin $\forall \lambda$.

(A_3). *The origin changes its stability at* $\lambda = \lambda_0$ *and is unstable for* $\lambda > \lambda_0$.

(A_4). *There exists for* $\lambda > \lambda_0$ *a Dulac-Cherkas function* $\Psi(x, y, \lambda)$ *of system* (1.1) *in the phase plane such that the set* $\mathcal{W}_\lambda := \{(x, y) \in \mathbb{R}^2 : \Psi(x, y, \lambda) = 0\}$ *consists of a unique oval surrounding the origin.*

(A_5). *For* $\lambda > \lambda_0$ *there is a one-to-one mapping*

$$\bar{x} = \varphi_1(x, y, \lambda), \quad \bar{y} = \psi_1(x, y, \lambda)$$

such that system (1.1) will be transformed into the system

$$\frac{d\bar{x}}{dt} = \bar{P}(\bar{x}, \bar{y}, \lambda), \ \frac{d\bar{y}}{dt} = \bar{Q}(\bar{x}, \bar{y}, \lambda) \tag{2.1}$$

with the following properties:

(i). *The functions* $\bar{P}$ *and* $\bar{Q}$ *have for* $\lambda > \lambda_0$ *the same smoothness as the functions* P *and* Q.
(ii). *The origin is the unique equilibrium of system* (2.1) $\forall \lambda > \lambda_0$.
(iii). λ_0 *is a Hopf bifurcation point for system* (2.1) *connected with the bifurcation of a stable limit cycle* $\overline{\Gamma}_\lambda$ *from the origin for increasing* λ *which is positively (that is anti-clockwise) oriented.*
(iv). *System* (2.1) *represents for* $\lambda > \lambda_0$ *a one-parameter family of positively rotated vector fields.*

$(A_6.)$ *For* $\lambda > \lambda_0$ *there is a one-to-one mapping*

$$\tilde{x} = \varphi_2(x, y, \lambda), \quad \tilde{y} = \psi_2(x, y, \lambda), \quad \tau = \chi(t, \lambda),$$

where τ *increases with* t *for any* $\lambda > \lambda_0$*, such that system (1.1) will be transformed into the system*

$$\begin{aligned} \frac{d\tilde{x}}{d\tau} &= \tilde{P}(\tilde{x}, \tilde{y}, \varepsilon), \\ \varepsilon \frac{d\tilde{y}}{d\tau} &= \tilde{Q}(\tilde{x}, \tilde{y}, \varepsilon) \end{aligned} \tag{2.2}$$

with the following properties:

(i). *There is a smooth function* $\zeta : (\lambda_0, +\infty) \to \mathbb{R}^+$ *with* $\zeta(\lambda) \to 0$ *as* $\lambda \to +\infty$ *such that* $\varepsilon = \zeta(\lambda)$.
(ii). *The functions* $\tilde{P}$ *and* $\tilde{Q}$ *have for* $\varepsilon > 0$ *the same smoothness as the functions* P *and* Q.
(iii). *There is a sufficiently small positive number* δ *such that for* $\varepsilon \in (0, \delta)$ *system (2.2) has a family* $\{\tilde{\Gamma}_\varepsilon\}$ *of uniformly bounded hyperbolic stable limit cycles which surround the origin and are positively oriented.*

The following theorem is our main result.

Theorem 2.1. *Under the assumptions $(A_1) - (A_6)$ system (1.1) has for $\lambda > \lambda_0$ a unique family $\{\Gamma_\lambda\}$ of limit cycles which are hyperbolic, stable and positively oriented, and whose amplitudes are bounded on any bounded λ-interval.*

3 Proof of Theorem 2.1

From our assumptions it follows that the phase portraits of the systems (1.1), (2.1) and (2.2) are topologically equivalent for $\lambda > \lambda_0$. Under the assumptions $(A_1) - (A_4)$, Theorem 5.3 and Theorem 5.4 in Appendix 1 imply that for $\lambda > \lambda_0$ system (1.1) has at most one limit cycle $\Gamma(\lambda)$, and if it exists, $\Gamma(\lambda)$ is hyperbolic and stable and surrounds the origin. Taking into account assumption (A_5), we can conclude that for $\lambda - \lambda_0$ sufficiently small, system (2.1) has a unique limit cycle $\bar{\Gamma}(\lambda)$ which is hyperbolic, stable and surrounds the origin in positive sense. Assumption (A_6) implicates that system (2.1) has for sufficiently large λ a unique family $\{\bar{\Gamma}_\lambda\}$ of limit cycles which are hyperbolic stable and positively oriented. By assumption (A_5), system (2.1) is a one-parameter family of positively rotated vector fields. Thus, we can apply Theorem 6.3 in Appendix 2 and get that the limit cycle $\bar{\Gamma}_\lambda$ contracts monotonously for decreasing λ and exists as long as it does not coincide with the origin. By the assumptions (A_3) and (A_5) this happens for $\lambda = \lambda_0$. Thus, we can conclude that system (2.1) has for $\lambda > \lambda_0$ a family $\{\bar{\Gamma}_\lambda\}$ of limit cycles which are hyperbolic, stable and positively oriented. The topological equivalence of the structure of the trajectories of the systems (1.1) and (2.1) implies that also system (1.1) has for $\lambda > \lambda_0$ a family Γ_λ of limit cycles which are hyperbolic, stable and positively oriented. This completes the proof of Theorem 2.1.

In the next section we present an application of Theorem 2.1.

4 Example

We consider the Liénard system

$$\begin{aligned} \frac{dx}{dt} &= -\,y, \\ \frac{dy}{dt} &= x - \lambda(x^{2q} - 1)y \end{aligned} \tag{4.1}$$

with $q \in \mathbb{N}$. The corresponding second order equation reads

$$\frac{d^2x}{dt^2} + \lambda(x^{2q} - 1)\frac{dx}{dt} + x = 0. \tag{4.2}$$

For $q = 1$, equation (4.2) represents the famous van der Pol equation. We want to show that system (4.1) has the same properties as the van der Pol equation. The transformation $\lambda \to -\lambda, t \to -t$ leaves equation (4.2) invariant, for $\lambda = 0$, system (4.1) represents a linear conservative system. To establish that equation (4.2) has the same properties as the van der Pol equation we have to prove that (4.2) has for $\lambda > 0$ a unique stable limit cycle. For this purpose we prove that the assumptions $(A_1) - (A_6)$ are fulfilled for system (4.1).

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Assumptions (A_1) and (A_2) are obviously satisfied. If we linearize system (4.1) at the origin, the corresponding eigenvalues $a(\lambda) \pm ib(\lambda)$ read

$$a(\lambda) = \frac{\lambda}{2}, \quad b(\lambda) = \frac{\sqrt{4-\lambda^2}}{2}. \tag{4.3}$$

Thus, the origin changes its stability as λ crosses $\lambda = 0$, it is unstable for $\lambda > 0$. Hence, assumption (A_3) holds true for $\lambda_0 = 0$.

Lemma 4.1. *The function*

$$\Psi(x, y, \lambda) \equiv x^2 + y^2 - 1$$

is a Dulac-Cherkas function for system (4.1) in the phase plane for $\lambda > 0$.

Proof. We denote by $X(\lambda)$ the vector field defined by system (4.1). It holds

$$divX(\lambda) \equiv -\lambda(x^{2q} - 1), \quad (grad\Psi, X(\lambda)) \equiv 2\lambda(x^{2q} - 1)y^2.$$

Setting $\kappa = -2$ in (5.1) we get

$$\Phi(x, y, \lambda) := (grad\Psi, X(\lambda)) - 2\Psi divX(\lambda)$$

$$= 2\lambda(x^2 - 1)^2(1 + x^2 + x^4 + ... + x^{2q-2}) \geq 0$$

for $\lambda > 0$ and any $(x, y) \in \mathbb{R}^2$. According to Definition 5.1 and Remark 5.2 in Appendix 1, Lemma 4.1 is proved. □

Since the set $\mathcal{W}_\lambda := \{(x, y) \in \mathbb{R}^2 : \Psi(x, y, \lambda) = 0\}$ consists of the unit circle, assumption (A_4) is valid.

Using for $\lambda > 0$ the scaling

$$\bar{x} = \lambda^{\frac{1}{2q}} x, \quad \bar{y} = \lambda^{\frac{1}{2q}} y$$

system (4.1) takes the form

$$\begin{aligned} \frac{d\bar{x}}{dt} &= -\bar{y}, \\ \frac{d\bar{y}}{dt} &= \bar{x} + \lambda\bar{y} - \bar{x}^{2q}\bar{y}. \end{aligned} \tag{4.4}$$

It is obvious that for $\lambda > 0$ system (4.1) and system (4.4) have the same topological structure of their trajectories including their orientation. The origin is the unique equilibrium of system (4.4), the corresponding eigenvalues of its linearization at the origin coincide with those of the linearization of system (4.1). Thus, the origin is unstable for $\lambda > 0$.

Lemma 4.2. *The origin is a focus of multiplicity q of system (4.4) for $\lambda = 0$ and $q \in \mathbb{N}$.*

Proof. By means of the coordinate transformation

$$u = a(\lambda)\bar{x} + \bar{y}, \quad v = -b(\lambda)\bar{x}$$

system (4.4) takes the form

$$\begin{aligned}\frac{du}{dt} =& a(\lambda)u - b(\lambda)v - \frac{v^{2q}}{b(\lambda)^{2q}}\Big(u + \frac{a(\lambda)}{b(\lambda)}v\Big),\\ \frac{dv}{dt} =& b(\lambda)u + a(\lambda)v.\end{aligned} \tag{4.5}$$

Introducing polar coordinates

$$u = r\,\cos\varphi, \quad v = r\,\sin\varphi$$

system (4.5) is in a sufficiently small neighborhood of the origin and for $0 \le \lambda < 2$ equivalent to the differential equation

$$\frac{dr}{d\varphi} = \sum_{i=1}^{\infty} k_i(\varphi,\lambda)r^i, \tag{4.6}$$

where

$$k_1(\varphi,\lambda) = \frac{a(\lambda)}{b(\lambda)} = \frac{\lambda}{\sqrt{4-\lambda^2}}. \tag{4.7}$$

For $\lambda = 0$ we have

$$\frac{dr}{d\varphi} = -r^{2q+1}(\sin\varphi)^{2q}(\cos\varphi)^2 + O(r^{4q+1}) \tag{4.8}$$

which implies

$$k_1(\varphi,0) \equiv 0, k_2(\varphi,0) \equiv 0, ..., k_{2q+1}(\varphi,0) \equiv -(\sin\varphi)^{2q}(\cos\varphi)^2. \tag{4.9}$$

We denote by $r(\varphi,\lambda,r_0)$ the solution of (4.8) satisfying $r(0,\lambda,r_0) = r_0 > 0$. For sufficiently small r_0, this solution can be represented as

$$r(\varphi,\lambda,r_0) = \sum_{i=1}^{\infty} h_i(\varphi,\lambda)r_0^i. \tag{4.10}$$

Substituting (4.10) into (4.6), we get a system of differential equations for determining the functions h_i. Using the initial conditions $h_1(0,\lambda) = 1, h_i(0,\lambda) = 0$ for $i = 2, ...$, the functions $h_i(\varphi,\lambda)$ can be determined uniquely. We obtain

$$\frac{dh_1}{dt} = k_1(\varphi,\lambda)h_1,$$

which implies by (4.7)

$$h_1(\varphi,\lambda) = \exp\varphi\frac{\lambda}{\sqrt{4-\lambda^2}}. \tag{4.11}$$

For $\lambda = 0$ we have by (4.9)

$$\frac{dh_1}{dt} \equiv 0, \frac{dh_2}{dt} \equiv 0, ..., \frac{dh_{2q+1}}{dt} = k_{2q+1}(\varphi,0) = -(\sin\varphi)^{2q}(\cos\varphi)^2.$$

Thus, it holds

$$h_1(2\pi,0) = 1, h_2(2\pi,0) = 0, ..., h_{2q}(2\pi,0) = 0, h_{2q+1}(2\pi,0) = -\int_0^{2\pi}(\sin\varphi)^{2q}(\cos\varphi)^2 d\varphi < 0. \tag{4.12}$$

The number of positive zeros of the function $d(r_0, \lambda) := r(2\pi, \lambda, r_0) - r_0$ determines the number of the limit cycles of system (4.4) near the origin. For sufficiently small r_0, $d(r_0, \lambda)$ can be represented in the form

$$d(r_0, \lambda) = \sum_{i=1}^{\infty} \alpha_i(\lambda) r_0^i. \tag{4.13}$$

For system (4.4) we get by (4.11)

$$\alpha_1(\lambda) = h_1(2\pi, \lambda) - 1 = \exp 2\pi \frac{\lambda}{\sqrt{4-\lambda^2}} - 1. \tag{4.14}$$

The coefficients $\alpha_i(0)$ are called Lyapunov numbers of the origin. They are determined by the relations

$$\alpha_1(0) = h_1(2\pi, 0) - 1, \alpha_i(0) = h_i(2\pi, 0), i = 2, ... \tag{4.15}$$

The first non-vanishing Lyapunov number determines the multiplicity of the origin. For system (4.4) we have

$$\alpha_1(0) = 0, \alpha_2(0) = 0, ..., \alpha_{2q}(0) = 0, \alpha_{2q+1}(0) = -\int_0^{2\pi} (\sin\varphi)^{2q} (\cos\varphi)^2 d\varphi < 0, \tag{4.16}$$

that is, the origin is a focus of multiplicity q of system (4.4) for $\lambda = 0$. □

In general, from a focus of multiplicity m at most m limit cycles may bifurcate. In our case, by Lemma 4.1 and Theorem 5.4 at most one limit cycle may bifurcate. Using (4.12) - (4.16) we can conclude that system (4.4) has for sufficiently small positive λ a unique small amplitude limit cycle which is hyperbolic, stable and positively oriented.

From (4.4) we obtain

$$\Big(\frac{d\bar{x}}{dt}\Big)\frac{\partial}{\partial\lambda}\Big(\frac{d\bar{y}}{dt}\Big) - \Big(\frac{d\bar{y}}{dt}\Big)\frac{\partial}{\partial\lambda}\Big(\frac{d\bar{x}}{dt}\Big) = -\bar{y}^2 \le 0,$$

that is, by Definition 6.1 and Remark 6.2 in Appendix 2, system (4.4) represents a one-parameter family of negatively rotated vector fields. Therefore, assumption (A_5) is satisfied.

To verify assumption (A_6) we introduce a new independent variable τ by $t = \lambda\tau$ in equation (4.2) and set $\varepsilon := 1/\lambda^2$. We obtain

$$\varepsilon \frac{d^2x}{d\tau^2} + (x^{2q} - 1)\frac{dx}{d\tau} + x = 0$$

which can be expressed in the form

$$\frac{d}{d\tau}\Big[\varepsilon\frac{dx}{d\tau} - x + \frac{x^{2q+1}}{2q+1}\Big] + x = 0. \tag{4.17}$$

Using the transformation

$$\eta := x, \quad \xi := -\frac{y}{\lambda} - x + \frac{x^{2q+1}}{2q+1} = \varepsilon\frac{d\eta}{d\tau} - \eta + \frac{\eta^{2q+1}}{2q+1}$$

which is one to one for $\varepsilon > 0$, equation (4.17) is equivalent to the system

$$\begin{aligned} \frac{d\xi}{d\tau} &= -\eta, \\ \varepsilon\,\frac{d\eta}{d\tau} &= \xi + \eta - \frac{\eta^{2q+1}}{2q+1}. \end{aligned} \tag{4.18}$$

Concerning system (4.18) we have the following properties:
The origin is the unique equilibrium for all $\varepsilon > 0$, it is unstable. The degenerate equation ($\varepsilon = 0$ in (4.18)) can be uniquely solved with respect to ξ

$$\xi = \varphi(\eta) := -\eta + \frac{\eta^{2q+1}}{2q+1}$$

satisfying $\varphi(0) = 0, \varphi'(0) = -1$. The equation $\varphi'(\eta) = 0$ has exactly two real roots $\eta = \pm 1$, where $\varphi''(-1) < 0, \varphi''(+1) > 0$. Thus, the conditions $(C_1) - (C_3)$ of Theorem 7.1 in Appendix 3 are valid such that assumption (A_6) holds true. Therefore, we can apply Theorem 2.1 and get the result

Theorem 4.3. *System (4.1) has for all $\lambda > 0$ ($\lambda < 0$) a unique limit cycle which is hyperbolic stable (unstable) and positively oriented.*

5 Appendix 1

Suppose that P, Q satisfy assumption (A_1). We denote by $X(\lambda)$ the vector field defined by system (1.1), by Λ some λ-interval and by Ω some region in $\mathbb{R}^2$.

Definition 5.1. *A function $\Psi : \Omega \times \Lambda \to \mathbb{R}$ with the same smoothness as P, Q is called a Dulac-Cherkas function of system (1.1) in Ω for $\lambda \in \Lambda$ if there exists a real number $\kappa \neq 0$ such that*

$$\Phi := (grad\ \Psi, X(\lambda)) + \kappa\Psi\ div\ X(\lambda) > 0 \quad (< 0) \quad \text{for} \quad (x, y, \lambda) \in \Omega \times \Lambda. \tag{5.1}$$

Remark 5.2. *Condition (5.1) can be relaxed by assuming that Φ may vanish in Ω on a set of measure zero, and that no closed curve of this set is a limit cycle of (1.1).*

The following two theorems can be found in [2].

Theorem 5.3. *Let Ψ be a Dulac-Cherkas function of (1.1) in Ω for $\lambda \in \Lambda$. Then any limit cycle Γ_λ of (1.1) in Ω is hyperbolic and its stability is determined by the sign of the expression $\kappa\Phi\Psi$ on Γ_λ.*

Theorem 5.4. *Let Ω be a p-connected region, let Ψ be a Dulac-Cherkas function of (1.1) in Ω such that the set $\mathcal{W}_\lambda := \{(x, y) \in \Omega : \Psi(x, y, \lambda) = 0\}$ consists of s ovals in Ω. Then system (1.1) has at most $p - 1 + s$ limit cycles in Ω.*

6 Appendix 2

The following facts can be found in [7].

Definition 6.1. *Let the assumption (A_1) be satisfied. System (1.1) is said to define a one-parameter family of negatively (positively) rotated vector fields for $\lambda \in \Lambda$ if for $\lambda \in \Lambda$ the equilibria of system (1.1) are isolated and at all ordinary points it holds*

$$\Delta(x,y,\lambda) := P(x,y,\lambda)\frac{\partial Q(x,y,\lambda)}{\partial \lambda} - Q(x,y,\lambda)\frac{\partial P(x,y,\lambda)}{\partial \lambda} < 0 \quad (> 0).$$

Remark 6.2. *This condition can be relaxed by assuming that Δ vanishes on a set of measure zero and that no closed curve of this set is a limit cycle of (1.1).*

Theorem 6.3. *Suppose that the assumptions (A_1) and (A_2) are satisfied and that system (1.1) represents a one-parameter family of negatively (positively) rotated vector fields. Let $\{\Gamma_\lambda\}$ be a family of hyperbolic stable limit cycles of system (1.1) with positive orientation. Then the amplitude of Γ_λ decreases monotonically with decreasing (increasing) λ, and the family terminates at $\lambda = \lambda_*$ when Γ_{λ_*} represents an equilibrium.*

7 Appendix 3

Consider the singularly perturbed system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y), \\ \varepsilon\,\frac{dy}{dt} &= g(x,y) \end{aligned} \tag{7.1}$$

under the following assumptions
(C_1). *$f, g : \mathbb{R}^2 \to \mathbb{R}$ are sufficiently smooth, ε is a small positive parameter.*
(C_2). *The origin is the unique equilibrium of system* (7.1) *in the finite part of the phase plane. It is unstable for $\varepsilon > 0$. The trajectories are positively oriented near the origin.*
(C_3). *$g(x,y) = 0$ has the unique simple solution $x = \varphi(y)$, where φ is sufficiently smooth and satisfies*

$$\varphi(0) = 0, \varphi'(0) < 0.$$

$\varphi'(y) = 0$ has exactly two real roots y_- and y_+ satisfying

$$y_- < 0,\ \varphi''(y_-) < 0,\ y_+ > 0,\ \varphi''(y_+) > 0.$$

Using assumption (C_3) we can define a closed curve $\mathcal{Z}_0$ in the phase plane consisting of two finite segments of the curve $x = \varphi(y)$ bounded by the points $D = (y_{--}, \varphi(y_+)), A = (y_-, \varphi(y_-))$ and $C = (y_+, \varphi(y_+)), B = (y_{++}, \varphi(y_-))$ (see Fig.1) and of two finite segments of the straight lines $x = \varphi(y_-)$ and $x = \varphi(y_+)$ bounded by the points A, B and D, C respectively (see Fig.1).

The following theorem is a special case of a more general theorem by E.F. Mishchenko and N. Kh. Rozov in [6].

Theorem 7.1. *Under the assumptions $(C_1) - (C_3)$, system (7.1) has for sufficiently small ε a unique limit cycle Γ_ε in a small neighborhood of $\mathcal{Z}_0$ which is stable and positively oriented.*

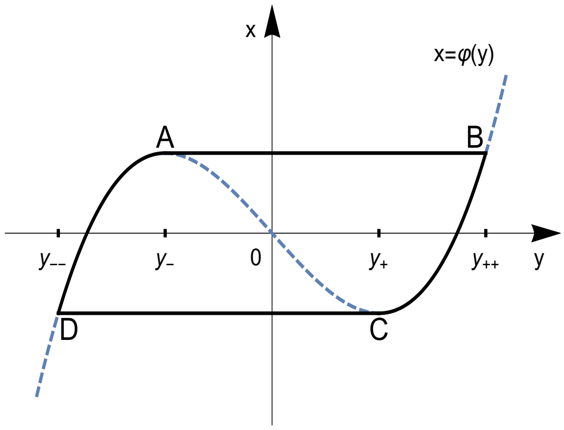

Fig.1. Closed curve $\mathbb{Z}_0$.

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