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Analysis of improved Nernst–Planck–Poisson models of compressible isothermal electrolytes.

Part III: Compactness and convergence

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Abstract

We consider an improved Nernst–Planck–Poisson model first proposed by Dreyer et al. in 2013 for compressible isothermal electrolytes in non equilibrium. The model takes into account the elastic deformation of the medium that induces an inherent coupling of mass and momentum transport. The model consists of convection–diffusion–reaction equations for the constituents of the mixture, of the Navier-Stokes equation for the barycentric velocity, and of the Poisson equation for the electrical potential. Due to the principle of mass conservation, cross–diffusion phenomena must occur and the mobility matrix (Onsager matrix) has a kernel. In this paper, which continues the investigations of [DDGG17a, DDGG17b], we prove the compactness of the solution vector, and existence and convergence for the approximation schemes. We point at simple structural PDE arguments as an adequate substitute to the Aubin–Lions compactness Lemma and its generalisations: These familiar techniques attain their limit in the context of our model in which the relationship between time derivatives (transport) and diffusion gradients is highly non linear.

1 Introduction

This paper is the third part after [DDGG17a, DDGG17b] of an investigation devoted to the mathematical analysis of an improved Nernst-Planck-Poisson system first proposed in [DGM13] and extended in [DGL14, DGM15]. In the first part of this investigation (see [DDGG17a]), we have exposed the model and presented a survey of the main results. The second part was concerned with *a priori* estimates for a larger class of regularised problem. In this last part we want to prove the existence and the convergence of approximate solutions. This implies as a main pillar the discussion of the compactness of the solution vector.

In this paper we finalise the proof of the main statements announced in [DDGG17a] providing proofs for:

- The existence of approximate solutions for the system with regularised free energy and mobility matrix;
- The compactness of the solution vector;
- The convergence toward a weak solution of the model.

In particular, the Aubin-Lions Lemma and its generalisation which are the traditional tools to pass to the limit in similar problems [CJL14] cannot be applied in the context of the model of [DGM13] which exhibits too complex a free energy function. We will show that purely structural PDE arguments in the spirit of [Hop51] provide an adequate substitute. Moreover we apply an original Galerkin method in order to construct the approximate solutions.

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The model We consider a bounded domain $\Omega \subset \mathbb{R}^3$ representing an electrolyte. The boundary of Ω possesses a disjoint decomposition $\partial\Omega = \Gamma \cup \Sigma$: The surface Γ represents an *active surface*, a one-sided interface between the electrolyte and an external material (electrode). The surface Σ is an inert outer wall. The electrolyte is a compressible mixture of $N \in \mathbb{N}$ species $\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{N}$ with mass densities $\rho_1, \ldots, \rho_N$. Each species A_i is a carrier of atomic mass $m_i \in \mathbb{R}_+$, charge $z_i \in \mathbb{Z}$ and possesses a reference specific volume $V_i \in \mathbb{R}_+$. We assume that the system is isothermal. Following [DDGG16, DDGG17a], the mixture obeys in $]0, T[\times\Omega$ the following system of partial differential equations

$$\frac{\partial \rho_i}{\partial t} + \mathrm{div}(\rho_i\, v + J^i) = r_i \qquad \text{for } i = 1, \ldots, N \tag{1}$$

$$\frac{\partial \varrho\, v}{\partial t} + \mathrm{div}(\varrho\, v \otimes v - \mathbb{S}^{\text{visc}}) + \nabla p = -n^F\, \nabla\phi \tag{2}$$

$$-\epsilon_0\,(1+\chi)\,\triangle\phi = n^F \qquad . \tag{3}$$

Here, v denotes the *barycentric velocity* of the mixture, while for $i = 1, \ldots, N$ the quantities J^i and r_i denote the dissipative diffusion flux, and the mass production due to chemical reactions for the ith constituent. In the momentum balance (2), we have introduced the total bulk mass density $\varrho := \sum_{i=1}^N \rho_i$, the viscous stress tensor $\mathbb{S}^{\text{visc}}$, the pressure p, and the Lorentz force $-n^F\,\nabla\phi$ for a quasi-static approximation of the electro-dynamical phenomena. The function n^F is the density of free charges. Moreover, ϵ_0 is the Gauss constant, while χ denote the dielectric susceptibility of the medium assumed constant as well.

In order to formulate constitutive equations for the quantities J, r and p, the free energy of the system must be specified. Following [DDGG17a] (see [DGM13] for the original breakthrough), we assume that its density $\varrho\psi$ is given in the form $\varrho\psi = h(\theta,\, \rho)$, where the function h is defined via

$$\begin{aligned} h(\theta,\, \rho) &= \sum_{i=1}^N \rho_i\, \mu_i^{\text{ref}} + h^{\text{mech}}(\rho) + h^{\text{mix}}(\theta,\, \rho) \\ h^{\text{mech}} &= K\;F(\sum_{i=1}^N n_i\, V_i) \\ h^{\text{mix}} &= k_B\;\theta\, \sum_{i=1}^N n_i\, \sum_{i=1}^N y_i\, \ln y_i \end{aligned} \tag{4}$$

Here μ_i^{ref} $(i = 1, \ldots, N)$ are constants related to certain reference states of the pure constituents. The *number densities* $n_1, \ldots, n_N$ of the constituents are defined via $n_i := \rho_i/m_i$ $(i = 1, \ldots, N)$. The mechanical free energy is an increasing function of the dimensionless quantity $\sum_{i=1}^N n_i\, V_i =: n \cdot V$ (a 'volume density' for the mixture). The constant $K > 0$ is the compression modulus of the mixture. In the definiton of the mixing-entropy, k_B denotes the Boltzmann constant and θ is the absolute temperature assumed constant. The quantity $\sum_{i=1}^N n_i$ is the *total number density* and $y_i := n_i/(\sum_{i=1}^N n_i)$ $(i = 1, \ldots, N)$ are the *number fractions* summing up to one.

The chemical potentials of the mixture are defined via

$$\mu_i = \partial_{\rho_i} h(\theta,\, \rho_1, \ldots, \rho_N) \text{ for } i = 1, \ldots, N\,. \tag{5}$$

Thus, under the particular constitutive assumption (4)

$$\mu_i = c_i + K\,\tfrac{V_i}{m_i}\,F'(n\cdot V) + \tfrac{k_B\,\theta}{m_i}\,\ln y_i \;\text{ for } i = 1, \ldots, N\,, \tag{6}$$

where $c_1, \ldots, c_N$ are certain constants. The following constitutive equations and definitions are assumed:

$$J^i = -\sum_{j=1}^{N} M_{i,j}\, D^j \quad \text{for } i = 1, \ldots, N\,, \tag{7a}$$

$$D^j := \nabla\left(\frac{\mu_j}{\theta}\right) + \frac{1}{\theta}\,\frac{z_j}{m_j}\,\nabla\phi \quad \text{for } j = 1, \ldots, N \tag{7b}$$

$$r_i = -\sum_{k=1}^{s} \partial_{D^{\mathsf{R}}_k} \Psi(D^{\mathsf{R}}_1, \ldots, D^{\mathsf{R}}_s)\, \gamma^k_i\,, \qquad D^{\mathsf{R}}_k := \gamma^k \cdot \mu \tag{7c}$$

$$\mathbb{S}^{\mathsf{visc}}(\nabla v) = \eta\, D(v) + \lambda\, \operatorname{div} v\, \mathsf{Id} \tag{7d}$$

$$p = -h(\theta,\, \rho) + \sum_{i=1}^{N} \mu_i\, \rho_i \tag{7e}$$

$$n^F = \sum_{i=1}^{N} \tfrac{z_i}{m_i}\, \rho_i \tag{7f}$$

In (7a), M is a symmetric, positive semi definite $N \times N$ matrix called the mobility matrix, while $D \in \mathbb{R}^{N\times 3}$ is the diffusion driving force. In (7c), $s \in \mathbb{N} \cup \{0\}$ is the number of chemical reactions. The vector $\gamma^k \in \mathbb{R}^N$ ($k = 1, \ldots, s$) does not as usual denote the stoichiometric vector $\gamma^{\mathsf{stoi},k} \in \mathbb{Z}^N$ associated with the reactions. For reasons of notation we set $\gamma^k := \gamma^{\mathsf{stoi},k}_i\, m_i$ for $i = 1, \ldots, N$ and $k = 1, \ldots, s$. The reaction potential Ψ is defined on $\mathbb{R}^s$ and assumed convex (see [DDGG17a] for plausible examples). The entries of the vector $D^{\mathsf{R}} \in \mathbb{R}^s$ are called reaction driving forces. The assumption (7d) is the usual expression for the Newtonian viscous stress tensor: Here $D(v) = (\partial_i v_j + \partial_j v_i)_{i,j=1,\ldots,3}$ while $\eta > 0$ and $\lambda + \frac{2}{3}\,\eta \geq 0$ are the coefficients of shear and bulk viscosity. The constitutive assumption (7e) for the pressure is called the Gibbs-Duhem equation, while (7f) is actually the definition of the free charge density.

The equations (1), (2), (3) with the constitutive equations (7) based on the choice (4) of the free energy density are the constituent parts of a generalised model of Poisson–Nernst–Planck type first proposed in [DGM13] and extensively developed in [DGL14], [DGM15] and [Guh14]. This model provides a general description of electrolytes in the presence of electrochemical interfaces for non equilibrium situations. In this paper, the focus is on mathematical analysis and we will consider for the system (1), (2), (3) simplified boundary conditions. At first we assume no velocity slip, and Dirichlet conditions for the electrical potential on the active boundary

$$v = 0 \text{ on }]0, T[\times \partial\Omega \tag{8}$$

$$\phi = \phi_0 \text{ on }]0, T[\times \Gamma, \quad \nabla\phi \cdot \nu = 0 \text{ on }]0, T[\times \Sigma\,. \tag{9}$$

At second, for the diffusion-reaction equations we assume for $i = 1, \ldots, N$ that

$$J^i \cdot \nu + \hat{r}_i = -J^0_i \tag{10a}$$

$$\hat{r}_i := \sum_{k=1}^{\hat{s}^\Gamma} \hat{R}^\Gamma_k(t,\, x,\, \hat{\gamma}^1 \cdot \mu, \ldots,\, \hat{\gamma}^{\hat{s}^\Gamma} \cdot \mu)\, \hat{\gamma}^k_i \tag{10b}$$

$$J^0_i := \sum_{k=1}^{\hat{s}^\Gamma} \jmath_k(t,\, x)\, \hat{\gamma}^k_i\,. \tag{10c}$$

The boundary conditions describe the reaction and adsorption of contituents on the active surface $]0,T[\times\Gamma$ in contact with an external bulk. The meaning of the number $\hat{s}^\Gamma \in \mathbb{N}\cup\{0\}$ and of the vectors $\hat{\gamma}^1,\ldots,\hat{\gamma}^{\hat{s}^\Gamma} \in \mathbb{R}^N$ have been explained in the modelling part of the paper [DDGG17a]. Both are related to the boundary reaction and adsorption phenomena. In particular, each vector $\hat{\gamma}^k$ satisfies $\sum_{i=1}^N \hat{\gamma}^k_i = 0$. In other words, it is orthogonal to the vector $\mathbb{1} = 1^N = (1,\,1,\ldots,1) \in \mathbb{R}^N$. The vector field $\hat{R}^\Gamma$ defining the reaction rates is derived from a potential $\hat{\Psi}^\Gamma :]0,T[\times\Gamma\times\mathbb{R}^{\hat{s}^\Gamma} \to \mathbb{R}_{0,+}$ via

$$\hat{R}^\Gamma(t,\,x,\,D) = \nabla_D \hat{\Psi}^\Gamma(t,\,x,\,D) \text{ for } (t,x) \in]0,T[\times\Gamma,\; D \in \mathbb{R}^{\hat{s}^\Gamma}\,.$$

Following [DDGG17a], the potential $\hat{\Psi}^\Gamma$ is convex in the D variable, and $\nabla_D\hat{\Psi}^\Gamma(t,\,x,\,0) = 0$. In (10), the coefficients $\jmath \in [0,T]\times\Gamma \to \mathrm{span}\{\hat{\gamma}^1,\ldots,\hat{\gamma}^{\hat{s}^\Gamma}\}$ are given.

2 Assumptions on the data and preliminaries.

Notations To get rid of overstressed indexing, we simplify the notation by making use of vectors. For instance we denote ρ the vector of mass densities, n the vector of number densities i.e.

$$\rho := (\rho_1,\rho_2,\ldots,\rho_N) \in \mathbb{R}^N, \quad n := (n_1,n_2,\ldots,n_N) \in \mathbb{R}^N\,.$$

Moreover we define the vector $\mathbb{1} := 1^N := (1,\,1,\ldots,1) \in \mathbb{R}^N$, and the vectors of quotients of charge and mass, and of volume and mass

$$\frac{z}{m} := (\tfrac{z_1}{m_1},\,\tfrac{z_2}{m_2},\ldots,\tfrac{z_N}{m_N}) \in \mathbb{R}^N, \quad \frac{V}{m} := (\tfrac{V_1}{m_1},\,\tfrac{V_2}{m_2},\ldots,\tfrac{V_N}{m_N}) \in \mathbb{R}^N\,.$$

Using these conventions, we have a. o. the identities

$$\varrho = \mathbb{1}\cdot\rho, \quad n^F = \frac{z}{m}\cdot\rho, \quad n\cdot V = \rho\cdot\frac{V}{m} \text{ etc.}$$

The diffusion fluxes $J^1,\ldots,J^N$ span a rectangular matrix $J = \{J^i_j\} \in \mathbb{R}^N\times\mathbb{R}^3$. The upper index corresponds to the lines of this matrix. Vectors of $\mathbb{R}^N$ are multiplicated from the left, as for instance in $\mathbb{1}\cdot J = \sum_{i=1}^N J^i$ which is an identity in $\mathbb{R}^3$.

The vectors $\gamma^1,\ldots,\gamma^s$ span a rectangular matrix $\gamma = \{\gamma^k_i\} \in \mathbb{R}^s\times\mathbb{R}^N$. The upper index corresponds to the line of the matrix. Vectors of $\mathbb{R}^s$ are multiplicated from the left, as for instance in the identity $r = R\cdot\gamma = \sum_{k=1}^s R_k\,\gamma^k$ in $\mathbb{R}^N$. Analogously the vectors $\hat{\gamma}^1,\ldots,\hat{\gamma}^{\hat{s}^\Gamma}$ span a rectangular matrix $\hat{\gamma} = \{\hat{\gamma}^k_i\} \in \mathbb{R}^{\hat{s}^\Gamma}\times\mathbb{R}^N$.

Since we assume overall that $\theta =$ const, we write $h(\rho)$ for $h(\theta,\,\rho)$.

The analysis presuposes restrictions of mathematical nature to the data.

(1) **Free energy:** In (4), we assume that the function F belongs to $C^2(\mathbb{R}_+)\cap C(\mathbb{R}_{0,+})$ and is convex. We assume that there are $\frac{3}{2} < \alpha < +\infty$ and constants $0 < c_0,\, c_1$ such that

$$F(s) \geq c_0\, s^\alpha - c_1 \quad \text{for all } s > 0\,. \tag{11}$$

In the neighbourhood of zero, we assume that $F(s)$ behaves like $s\ln s$: There are constants positive constants $k_0 < k_1$ and $s_0 > 0$ such that

$$\frac{k_0}{s} \leq F''(s) \leq \frac{k_1}{s} \quad \text{for all } s \in]0,s_0]\,. \tag{12}$$

As explained in the papers [DDGG16], [DDGG17a] we crucially need that $F' : \mathbb{R}_+ \to \mathbb{R}$ is a surjective map in order to obtain an unconstrained PDE system.

(2) **Mobility matrix:** We assume that the mobility matrix M is given by a mapping $\overline{M}(\rho)$ of the mass densities. The mapping $\overline{M}$ is defined on $\mathbb{R}^N_+$ and it maps into the set of symmetric, positive semi definite $N \times N$ matrices. Throughout the paper, we assume that $\overline{M}$ is mass conservative, that is

$$\overline{M}(\rho)\mathbb{1} = 0 \text{ for all } \rho \in \mathbb{R}^N_+ . \tag{13}$$

Moreover we assume that the entries of $\overline{M}(\rho)$ are continuous functions with at most linear–growth. In this paper we restrict ourselves to the assumption that M has rank $N-1$ independently on ρ: Denoting $0 = \lambda_1(M) \leq \lambda_2(M) \leq \ldots \leq \lambda_N(M)$ the eigenvalues of the matrix M, we assume that there are positive constants $0 < \underline{\lambda} \leq \overline{\lambda}$ such that

$$\underline{\lambda} \leq \lambda_i(\overline{M}(\rho)) \leq \overline{\lambda}\,(1+|\rho|) \quad \text{for all } i = 2, 3, \ldots, N,\ \ \rho \in \mathbb{R}^N_+ . \tag{14}$$

(3) **Reaction rates:** We assume that the reaction rates are derived from a strictly convex, non-negative potential $\Psi \in C^2(\mathbb{R}^s)$. Moreover, Ψ satisfies

$$\Psi(0) = 0, \qquad \frac{\Psi(D^{\mathsf{R}})}{|D^{\mathsf{R}}|} \to +\infty \text{ for } |D^{\mathsf{R}}| \to \infty . \tag{15}$$

Similarly, we require that the boundary reaction rates are derived from a strictly convex, non-negative potential $\hat{\Psi}^\Gamma \in L^\infty([0,T]\times\Gamma;\, C^2(\mathbb{R}^{\hat{s}^\Gamma}))$ such that

$$\hat{\Psi}^\Gamma(t,x,\,0) = 0 \ \text{ for (almost) all } (t,x) \in [0,T]\times\Gamma . \tag{16}$$

For simplicity we explicitly require at least linear growth of the reaction rates (uniformly quadratic growth of the potentials)

$$\inf_{D^{\mathsf{R}}\in\mathbb{R}^s} \lambda_{\min}(D^2\Psi(D^{\mathsf{R}})) > 0, \ \operatorname*{essinf}_{(t,x)\in[0,T]\times\Gamma}\ \inf_{D^{\Gamma,\mathsf{R}}\in\mathbb{R}^{\hat{s}^\Gamma}} \lambda_{\min}(D^2\Psi^\Gamma(t,\,x,\,D^{\Gamma,\mathsf{R}})) > 0 . \tag{17}$$

(4) **Domain:** The domain $\Omega \subset \mathbb{R}^3$ possesses a boundary of class $\mathcal{C}^{0,1}$. In connection with the optimal regularity of the solution to the Poisson equation with mixed-boundary conditions, we need to introduce a further exponent $r(\Omega,\,\Gamma)$ as the largest number in the range $]2,+\infty[$ such that

$$\begin{aligned} -\triangle u = f \text{ in } [W^{1,\beta'}_\Gamma(\Omega)]^* \text{ implies } u \in W^{1,\beta}_\Gamma(\Omega) \\ \text{for all } f \in [W^{1,\beta'}_\Gamma(\Omega)]^* \text{ and all } \beta \in]r',\,r[. \end{aligned} \tag{18}$$

With the α from (11), we require that

$$\alpha' := \frac{\alpha}{\alpha-1} < r . \tag{19}$$

(5) **Initial and boundary data:** We assume sufficient (not optimal) regularity

$$\begin{aligned} \rho^0 &\in L^\infty(\Omega;\,(\mathbb{R}_+)^N) \\ v^0 &\in L^\infty(\Omega;\,\mathbb{R}^3) \\ \phi_0 &\in L^\infty(0,T;\,W^{1,r}(\Omega)) \cap L^\infty(]0,T[\times\Omega) \\ \partial_t\phi_0 &\in W^{1,0}_2(]0,T[\times\Omega) \cap L^{\alpha'}(]0,T[\times\Omega) \\ \jmath &\in L^\infty(]0,T[\times\Gamma;\,\mathbb{R}^{\hat{s}^\Gamma}) . \end{aligned} \tag{20}$$

Moreover we assume as a compatibility condition the validity in the weak sense of

$$-\epsilon_0\,(1+\chi)\,\triangle\phi_0(0) = \tfrac{z}{m}\cdot\rho^0$$

(6) **Initial compatibility condition**: The linear space

$$W := \operatorname{span}\left\{\gamma^1, \ldots, \gamma^s, \hat{\gamma}^1, \ldots, \hat{\gamma}^{\hat{s}^\Gamma}\right\}, \tag{21}$$

plays a fundamental role in the estimate of the chemical potentials. Call *selection* S of cardinality $|S| \leq N$ a subset $\{i_1, \ldots, i_{|S|}\}$ of $\{1, \ldots, N\}$ such that $i_1 \leq \ldots \leq i_{|S|}$. For every selection, we introduce the corresponding projector $P_S : \mathbb{R}^N \to \mathbb{R}^N$ via $P_S(\xi)_i = \xi_i$ for $i \in S$, and $P_S(\xi)_i = 0$ otherwise. We define a linear subspace $W_S \subset \mathbb{R}^N$ via

$$W_S := \operatorname{span}\left\{P_S(\gamma^1), \ldots, P_S(\gamma^s),\, P_S(\hat{\gamma}^1), \ldots, P_S(\hat{\gamma}^{\hat{s}^\Gamma})\right\}.$$

The selection S will be called *uncritical* if $\dim(W_S) = |S|$ and *critical* otherwise. For every selection S, we denote S^{c} the complementary selection $\{1, \ldots, N\} \setminus S$. It can easily be shown that the manifold

$$\mathcal{M}_{\text{crit}} := \mathbb{R}^N_+ \cap \bigcup_{S \subset \{1,\ldots,N\},\ S \text{ critical}} W_S \times P_{S^\perp}(\mathbb{R}^N) \tag{22}$$

is the finite union of sub manifolds of dimension at most $N-1$. We say that the *initial compatibility condition* is satisfied if the initial vector of the total partial masses $\bar{\rho}_0 := \int_\Omega \rho^0\, dx \in \mathbb{R}^N_+$ satisfies $\bar{\rho}_0 \notin \mathcal{M}_{\text{crit}}$.

Functional classes: We make use of standard Sobolev spaces. Moreover, the vectorial Orlicz classes $L_\Psi(Q;\, \mathbb{R}^s)$ and $L_{\Psi^*}(Q_T;\, \mathbb{R}^s)$ are then well known. We make use of the notation

$$[D^{\mathsf{R}}]_{L_\Psi(Q;\, \mathbb{R}^s)} := \int_{Q_T} \Psi(D^{\mathsf{R}}(t,\, x))\, dx\, dt\,.$$

For $\hat{\Psi}^\Gamma \in L^\infty(S;\, C^2(\mathbb{R}^{\hat{s}^\Gamma}))$, we define a vectorial Orlicz class $L_{\hat{\Psi}^\Gamma}(S;\, \mathbb{R}^{\hat{s}^\Gamma})$ as the set of all measurable $\hat{D}^{\Gamma,\mathsf{R}} :\, S \to \mathbb{R}^{\hat{s}^\Gamma}$ such that

$$[\hat{D}^{\Gamma,\mathsf{R}}]_{L_{\hat{\Psi}^\Gamma}(S;\, \mathbb{R}^{\hat{s}^\Gamma})} := \int_S \hat{\Psi}^\Gamma(t,\, x,\, \hat{D}^{\Gamma,\mathsf{R}}(t,\, x))\, dS(x)\, dt < +\infty\,.$$

Let us recall (see [DDGG16] for a detailed construction) that there is a non-negative function $\Phi^* \in C([0,T]^2)$, $\Phi^*(t,\, t) = 0$ constructed from the functions Ψ, $\hat{\Psi}^\Gamma$ such that the variable

$$\bar{\rho} := \int_\Omega \rho = \int_\Omega \mathscr{R}(\varrho,\, q)\,, \tag{23}$$

satisfies the estimate $[\bar{\rho}]_{C_{\Phi^*}([0,T];\, \mathbb{R}^N)} := \sup_{t_1,\, t_2 \in [0,T]} \frac{|\bar{\rho}(t_1) - \bar{\rho}(t_2)|}{\Phi^*(t_1,\, t_2)} < +\infty$.

Formulation of the weak problem. Following [DDGG16], [DDGG17a] a solution vector to the initial boundary value problem (1), (2), (3), (7), (8), (9), (10) with initial conditions (=: Problem (P)) is composed of the scalars $\varrho :\, Q \to \mathbb{R}_+$ (total mass density) and $\phi :\, Q \to \mathbb{R}$ (electrical potential) and of the vector fields $q :\, Q \to \mathbb{R}^{N-1}$ (relative chemical potentials), and $v :\, Q \to \mathbb{R}^3$ (barycentric velocity field). Since we want to account for the possibility of vacuum, the productions factors are not everywhere functions of these components only. Thus we also introduce $R\,:\,Q \to \mathbb{R}^s$, $R^\Gamma\,:\,S \to \mathbb{R}^{\hat{s}^\Gamma}$ as variables. For a vector $(\varrho,\,q,\,v,\,\phi,\,R,\,R^\Gamma)$, we recover all variables of the system via

$$\rho = \mathscr{R}(\varrho,\,q) \tag{24a}$$

$$J = -M(\rho)\,D, \qquad D := \nabla\mathcal{E}\,q + \frac{z}{m}\,\nabla\phi \tag{24b}$$

$$r = \sum_{k=1}^{s} \gamma^k\,R_k, \qquad D^{\mathsf{R}}_k := \gamma^k\cdot\mathcal{E}q \text{ for } k = 1,\ldots,s \tag{24c}$$

$$\hat{r} = \sum_{k=1}^{\hat{s}^\Gamma} \hat{\gamma}^k\,R^\Gamma_k, \qquad \hat{D}^{\Gamma,\mathsf{R}}_k := \hat{\gamma}^k\cdot\mathcal{E}q \text{ for } k = 1,\ldots,\hat{s}^\Gamma \tag{24d}$$

$$p = P(\varrho,\,q) \tag{24e}$$

$$n^F = \rho\cdot\frac{z}{m}\,. \tag{24f}$$

For $q \in \mathbb{R}^{N-1}$, we denote $\mathcal{E}\,q := \sum_{i=1}^{N-1} q_i\,\xi^i$, where $\xi^1,\ldots,\xi^{N-1} \in \mathbb{R}^N$ are fixed vectors that are extendable via 1^N to a basis of $\mathbb{R}^N$ (details in Section 4 of [DDGG17b]). The vector fieds $\mathscr{R}$ and the pressure function P are associated with (5), (6) and explicitly constructed in Section 4 of [DDGG17b] (see also [DDGG16]). We next give the main properties of weak solutions.

(1) **Energy conservation:** We say that $(\varrho,\,q,\,v,\,\phi,\,R,\,R^\Gamma)$ satisfies the *(global) energy (in)equality* with free energy function h and mobility matrix M if and only if the associated fields and variables (24) satisfy for almost all $t \in]0,T[$

$$\begin{aligned}
&\int_\Omega \left\{\frac{1}{2}\,\varrho\,v^2 + \frac{1}{2}\,\epsilon_0\,(1+\chi)\,|\nabla\phi|^2 + h(\rho)\right\}(t)\\
&\quad+\int_{Q_t} \left\{\mathbb{S}(\nabla v)\,:\,\nabla v + M\,D\cdot D + (\Psi(D^{\mathsf{R}}) + \Psi^*(-R))\right\}\\
&\quad+\int_{S_t} \{\hat{\Psi}^\Gamma(\cdot,\,\hat{D}^{\Gamma,\mathsf{R}}) + (\hat{\Psi}^\Gamma)^*(\cdot,\,-R^\Gamma)\}\\
&\overset{(\le)}{=} \int_\Omega \left\{\frac{1}{2}\,\varrho_0\,|v^0|^2 + \frac{1}{2}\,\epsilon_0\,(1+\chi)\,|\nabla\phi_0(0)|^2 + h(\rho^0)\right\}\\
&\int_{Q_t} \left\{n^F\,\phi_{0,t} - \epsilon_0\,(1+\chi)\,\nabla\phi\cdot\nabla\phi_{0,t}\right\} - \int_\Omega \left\{n^F\,\phi_0 - \epsilon_0\,(1+\chi)\,\nabla\phi\cdot\nabla\phi_0\right\}\Big|_0^t\\
&\quad+\int_{S_t} ((\hat{r} + J^0)\cdot\frac{z}{m}\,\phi_0 + J^0\cdot\mathcal{E}q)\,.
\end{aligned} \tag{25}$$

(2) **Balance of total partial masses:** We say that $(\varrho,\,q,\,v,\,\phi,\,R,\,R^\Gamma)$ satisfies the *balance of total partial masses* if the vector field (cf. (23)) is subject to

$$\bar{\rho}(t) = \bar{\rho}^0 + \int_0^t \left\{\int_\Omega r + \int_\Gamma (\hat{r} + J^0)\right\}(s)\,ds \quad \text{for all } t \in [0,T]\,. \tag{26}$$

with $\bar{\rho}^0 := \int_\Omega \rho^0\,dx$.

(3) **Natural class:** We say that $(\varrho,\,q,\,v,\,\phi,\,R,\,R^\Gamma)$ belongs to the class $\mathcal{B}(T,\,\Omega,\,\alpha,\operatorname{rk}M,\,\Psi,\,\Psi^\Gamma)$

if and only if the number

$$
\begin{aligned}
&[(\varrho,\, q,\, v,\, \phi,\, R,\, R^{\Gamma})]_{\mathcal{B}(T,\,\Omega,\,\alpha,\,\mathrm{rk}\,M,\,\Psi,\,\Psi^{\Gamma})} := \\
&\quad \|\varrho\|_{L^{\infty,\alpha}(Q)} + \|v\|_{W^{1,0}_2(Q;\mathbb{R}^3)} + \|\sqrt{\varrho}\, v\|_{L^{\infty,2}(Q;\mathbb{R}^3)} + \|\phi\|_{L^\infty(Q)} + \|\nabla\phi\|_{L^{\infty,\beta}(Q)} \\
&\quad + \|\nabla q\|_{W^{1,0}_2(Q;\mathbb{R}^{N-1})} + [D^{\mathrm{R}}]_{L_\Psi(Q;\mathbb{R}^s)} + [\hat{D}^{\Gamma,\mathrm{R}}]_{L_{\hat{\Psi}^\Gamma}(S;\mathbb{R}^{\hat{s}^\Gamma})} \\
&\quad + \|J\|_{L^{2,\frac{2\alpha}{1+\alpha}}(Q;\mathbb{R}^{N\times 3})} + [-R]_{L_{\Psi^*}(Q;\mathbb{R}^s)} + [-R^\Gamma]_{L_{(\hat{\Psi}^\Gamma)^*}(S;\mathbb{R}^{\hat{s}^\Gamma})} + \|p\|_{L^{\min\{1+\frac{1}{\alpha},\,\frac{5}{3}-\frac{1}{\alpha}\}}(Q)} \\
&\quad + [\bar{\rho}]_{C_{\Phi^*}([0,T])}
\end{aligned}
$$

is finite ($\beta := \min\left\{r(\Omega,\,\Gamma),\, \frac{3\alpha}{(3-\alpha)^+}\right\}$).

(4) **Weak solution:** We call a vector $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma) \in \mathcal{B}(T,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma)$ *weak solution* to the Problem (P) if the energy inequality and the balance of partial total masses are valid, and if the quantities ρ, J, r and $\hat{r}$, p and n^F obeying the definitions (24) satisfy the relations

$$
\begin{aligned}
&-\int_Q \rho\cdot\psi_t - \int_Q (\rho_i\, v + J^i)\cdot\nabla\psi_i && (27)\\
&\qquad = \int_\Omega \rho^0\cdot\psi(0) + \int_Q r\cdot\psi + \int_{S_T} (\hat{r} + J^0)\cdot\psi \qquad \forall\,\psi\in C^1_c([0,T[;\, C^1(\overline{\Omega};\,\mathbb{R}^N)) \\
&-\int_Q \varrho\, v\cdot\eta_t - \int_Q \varrho\, v\otimes v\,:\,\nabla\eta - \int_Q p\,\mathrm{div}\,\eta + \int_Q \mathbb{S}(\nabla v)\,:\,\nabla\eta && (28)\\
&\qquad = \int_\Omega \varrho_0\, v^0\cdot\eta(0) - \int_Q n^F\,\nabla\phi\cdot\eta \qquad \forall\,\eta\in C^1_c([0,T[;\, C^1_c(\Omega;\,\mathbb{R}^3)) \\
&\epsilon_0\,(1+\chi)\int_Q \nabla\phi\cdot\nabla\zeta = \int_Q n^F\,\zeta \qquad \forall\,\zeta\in L^1(0,T;\, W^{1,2}_\Gamma(\Omega))\,, && (29)
\end{aligned}
$$

$\phi = \phi_0$ as traces on $]0,T[\times\Gamma$

if the r and $\hat{r}$ obey their representation (7c), (10b) in the vacuum free sets

$$Q^+(\varrho) := \{(t,x)\in Q\,:\, \varrho(t,x) > 0\} \tag{30}$$

$$S^+(\varrho) := \{(t,x)\in S\,:\, \exists\,\mathcal{U} \text{ open },\ (t,x)\in\mathcal{U},\ \lambda_4((\mathcal{U}\cap Q)\setminus Q^+(\varrho)) = 0\}\,. \tag{31}$$

Approximation. The approximation method was exposed in [DDGG17b]. It is involving three positive parameters, say σ, δ and τ. The mobility matrix M is regularised in order to ensure ellipticity and allow a control on $\nabla\mu$

$$M_\sigma(\rho) = M(\rho) + \sigma\,\mathrm{Id}\,.$$

The free energy function is modified in order to stabilise both the vector ρ and the vector μ (see [DDGG17b], Section 5 for details). As a result, there are $\tilde{c}_0,\ \tilde{c}_1 > 0$, and $\tau_0(\alpha,\,\alpha_0) > 0$ such that if $\tau \le \tau_0$

$$h_{\tau,\delta}(\rho) \ge \tilde{c}_0\,(|\rho|^\alpha + \delta\,|\rho|^{\alpha_\delta} + \tau\,\Phi_\omega(\mu)) - \tilde{c}_1$$

for all $\rho\in\mathbb{R}^N_+$ and $\mu\in\mathbb{R}^N$ connected by the identity $\rho = \nabla h^*_{\tau,\delta}(\mu)$. Here we distinguish between the original growth exponent α of the free energy, and the stabilisation exponent $\alpha_\delta > 3$. Moreover, $\Phi_\omega(\mu) = \sum_{i=1}^N \omega'(\mu_i)\,\mu_i - \omega(\mu_i)$ where $\omega\in C^2(\mathbb{R})$ is a convex, nondecreasing function, which is unbounded with sublinear growth at $-\infty$ (Example in [DDGG17b], Section 5).

We say that $(\mu,\, v,\, \phi)$ satisfies the approximate energy (in)equality if and only if the corresponding vector $(\varrho,\, q,\, v,\, \phi,\, R,\, R^{\Gamma})$ satisfies the energy (in)equality (25), with free energy function $h_{\tau,\delta}$ and mobility matrix M_{σ}. For $\delta > 0$, $\sigma > 0$ and $\tau \geq 0$ we call weak solution to the problem $(P_{\tau,\,\sigma,\,\delta})$ a vector $(\mu,\, v,\, \phi) \in \mathcal{B}$ subject to the energy inequality and such that the quantities

$$\begin{aligned}
\rho &= \nabla h^*_{\tau,\delta}(\mu)\\
J &= -M_{\sigma}(\rho)\, D, \quad D := \frac{\nabla\mu}{\theta} + \frac{1}{\theta}\,\frac{z}{m}\,\nabla\phi\\
r &= \sum_{k=1}^{s} \hat{\gamma}^k\, \bar{R}_k(D^{\mathsf{R}}), \quad D^{\mathsf{R}} = (\gamma^1\cdot\mu, \ldots, \gamma^s\cdot\mu)\\
\hat{r} &= \sum_{k=1}^{\hat{s}^{\Gamma}} \hat{\gamma}^k\, \hat{R}^{\Gamma}_k(t, x,\, \hat{D}^{\Gamma,\mathsf{R}}), \quad \hat{D}^{\Gamma,\mathsf{R}} = (\hat{\gamma}^1\cdot\mu, \ldots, \hat{\gamma}^{\hat{s}^{\Gamma}}\cdot\mu)\\
p &= h^*_{\tau,\delta}(\mu)\\
n^F &= \rho\cdot\frac{z}{m}
\end{aligned} \tag{32}$$

satisfy the identities (27), (29), and instead of (28)

$$\begin{aligned}
&-\int_Q \varrho\, v\cdot\eta_t - \int_Q \varrho\, v\otimes v \,:\, \nabla\eta - \int_Q p\,\operatorname{div}\eta + \int_Q \mathbb{S}(\nabla v)\,:\,\nabla\eta \\
&= \int_{\Omega} \varrho_0\, v^0\cdot\eta(0) - \int_Q n^F\,\nabla\phi\cdot\eta - \int_Q (\sum_{i=1}^{N} J^i\cdot\nabla)\eta\cdot v \qquad \forall\eta \in C^1_c([0,T[;\, C^1_c(\Omega;\mathbb{R}^3))\,.
\end{aligned} \tag{33}$$

Since $\sum_{i=1}^N J^i \neq 0$, it is necessary to add this term in the momentum equation (33) in order to preserve the energy identity.

Estimates. We define

$$\begin{aligned}
\mathcal{B}_0 := &\|\rho^0\|_{L^{\alpha}(\Omega)} + \tau\,\|\Phi_{\omega}(\mu^0)\|_{L^1(\Omega)} + \|\sqrt{\varrho_0}\, v^0\|_{L^2(\Omega)} + \|\phi_0\|_{L^{\infty}(Q)}\\
&+ \|\phi_0\|_{L^{\infty}(0,T;\, W^{1,2}(\Omega))} + \|\phi_{0,t}\|_{W^{1,0}_2(Q)} + \|\phi_{0,t}\|_{L^{\alpha'}(Q)} + \|\jmath\|_{L^{\infty}(S;\,\mathbb{R}^{\hat{s}^{\Gamma}})}\,,
\end{aligned} \tag{34}$$

Assume that either (cf. (22))

$$\operatorname{dist}(\bar{\rho}_0,\, \mathcal{M}_{\mathsf{crit}}) > 0 \tag{35}$$

or that

$$T \leq T_0\,, \tag{36}$$

for a certain time T_0 determined by the magnitude of the data. For $\delta \geq 0$, $\sigma \geq 0$ and $\tau \geq 0$ every weak solution to the problem $(P_{\tau,\,\sigma,\,\delta})$ satisfies the bound

$$[(\varrho,\, q,\, v,\, \phi,\, R,\, R^{\Gamma})]_{\mathcal{B}(T,\,\alpha,\,N-1,\,\Psi,\,\Psi^{\Gamma})} \leq C(\mathcal{B}_0)\,. \tag{37}$$

If moreover $\delta > 0$ then

$$[(\varrho,\, q,\, v,\, \phi,\, R,\, R^{\Gamma})]_{\mathcal{B}(T,\,\alpha_{\delta},\,N-1,\,\Psi,\,\Psi^{\Gamma})} \leq C(\delta,\, \mathcal{B}_0)\,. \tag{38}$$

If $\tau > 0$ and $\sigma > 0$, the weak solution takes the form of a reduced vector $(\mu,\, v\,\phi)$ for which the additional bounds

$$\sqrt{\sigma}\,\|\nabla\mu\|_{L^2(Q)} + \min\{\sigma,\,\tau^2\}\,\|\mu\|_{L^{2,3}(Q)} \leq C(\mathcal{B}_0)\,, \tag{39}$$

$$\|\mathbb{1}\cdot J\|_{L^2(Q)} \leq C(\mathcal{B}_0)\,\sqrt{\sigma}\,, \quad \|\tau\,\omega'(\mu)\|_{L^{\infty,\alpha}(Q)} \leq C(\mathcal{B}_0)\,\tau^{1/\alpha'}\,, \tag{40}$$

$$\|((\mathbb{1}\cdot J^{\sigma})\cdot\nabla\ln\varrho_{\sigma})^{+}\|_{L^1(Q)} \leq C_0\,\sqrt{\sigma}\,. \tag{41}$$

are valid. In this case we for simplicity write

$$[(\mu,\, v,\, \phi)]_{\mathcal{B}(T,\,\alpha_\delta,\,N,\,\Psi,\,\Psi^\Gamma)} \leq C(\delta,\,\sigma,\,\tau,\,\mathcal{B}_0)\,. \tag{42}$$

Existence theorems. The theorems were announced in our survey [DDGG17a]. In this paper we shall complete their proofs.

Theorem 2.1. *[Global-in-time existence] Let $\Omega \in C^{0,1}$. Assume that the free energy function h satisfies* (11) *and* (12) *and that the mobility matrix M satisfies* (13) *and* (14)*. Let $\Psi \in C^2(\mathbb{R}^s)$ and $\Psi^\Gamma \in C^2(\mathbb{R}^{s^\Gamma})$ be strictly convex and satisfy* (15)*,* (16)*,* (17)*. Assume that the initial data ρ^0 and v^0, and the boundary data $\jmath^{ext}$, ϕ_0 are non degenerate in the sense of* (20)*, and that one of the following conditions is valid:*

(1) $\alpha \geq 2$*;*

(2) $\frac{9}{5} \leq \alpha < 2$ *and* $r(\Omega,\,\Gamma) > \alpha'$*;*

(3) $\frac{3}{2} < \alpha < \frac{9}{5}$*,* $r(\Omega,\,\Gamma) > \alpha'$ *and the vectors $m \in \mathbb{R}^N_+$ and $V \in \mathbb{R}^N_+$ are parallel.*

Then, for $T > 0$ arbitrary, the problem (P) possesses a weak solution. Moreover the following information on the complete vanishing of species is available:

$$\lambda_1\big(\{t \in [0,T]\,:\, \inf_{i=1,\ldots,N} \bar{\rho}_i(t) = 0\}\big) = 0\,.$$

If one starts with total initial masses on the critical manifold, then it is possible that certain species completely vanish after finite time, and the solution then exists only up to this time. Afterwards, it might be necessary to restart the system with a smaller number of species.

Theorem 2.2. *[Local-in-time existence] Same assumptions as in Theorem 2.1, with $\bar{\rho}_0 \in \mathcal{M}_{crit}$.*

Then, there are a time $0 < T_0$ depending only on the data (cf. (36)*) and a time $T_0 \leq T^* \leq +\infty$ such that there is a weak solution $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma) \in \mathcal{B}(t,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma)$ to (P_t) for all $t < T^*$. Moreover the following alternative concerning T^* is valid:*

(1) Either $T^* = +\infty$*;*

(2) Or there exists $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma) \in \mathcal{B}(T^*,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma)$ which is a weak solution to (P_t) for all $t < T^*$ such that $\inf_{i=1,\ldots,N} \bar{\rho}_i(t) > 0$ for all $t \in [0, T^*[$ and $\lim_{t\to T^*} \inf_{i=1,\ldots,N} \bar{\rho}_i(t) = 0$. Moreover $\liminf_{t\to T^*} \|q(t)\|_{L^1(\Omega;\,\mathbb{R}^{N-1})} = +\infty$.

3 Compactness

Our aim in this section is to derive a general compactness tool in order to pass to the limit with approximate solutions to the problem (P). Since we do not want to specify with which of the approximation parameters δ, σ or τ we pass to the limit, we will consider families indexed by a generic parameter $\epsilon > 0$.

In order to obtain the compactness we shall need the informations on distributional times derivative contained in the system (27), (28) (or (33) instead). For technical reasons it is convenient to express these informations in an older (though elementary) fashion (see [Hop51], Lemma 5.1 for the inspiring precursor of all Aubin–Lions–type techniques). For the sake of brevity, we introduce an auxiliary vector $\mathcal{A}$ associated with the solution vector $(\varrho,\, q,\, v,\, \phi,\, R,\, R^{\Gamma})$ and the auxiliary quantities (24) via

$$\mathcal{A} := (J,\, \varrho\, v,\, r,\, \hat{r},\, \nabla v,\, \varrho\, v \otimes v,\, v \otimes (\mathbb{1}\cdot J),\, p,\, n^F\, \nabla\phi) \in [L^1(Q)]^a\,, \tag{43}$$

where $a = 5N + 34$ is the number of scalar components of the vector $\mathcal{A}$. Due to the structure of the weak formulation, the identities

$$\begin{pmatrix} \int_\Omega \rho^\epsilon(t)\cdot\psi \\ \int_\Omega \varrho_\epsilon(t)\, v^\epsilon(t)\cdot\eta \end{pmatrix} = \begin{pmatrix} \int_\Omega \rho^0\cdot\psi \\ \int_\Omega \varrho_0(t)\, v^0\cdot\eta \end{pmatrix} + \begin{pmatrix} \int_0^t\int_\Omega \sum_{j=0,1} \mathscr{L}^{1,j}(\mathcal{A})\cdot D^j\psi \\ \int_0^t\int_\Omega \sum_{j=0,1} \mathscr{L}^{2,j}(\mathcal{A})\cdot D^j\eta \end{pmatrix}\,, \tag{44}$$
$$\forall\, t\in[0,T],\quad \forall\,(\psi,\,\eta)\in C^1_c(\Omega;\,\mathbb{R}^N)\times C^1_c(\Omega;\,\mathbb{R}^3)\,.$$

are valid. Here $\mathscr{L}^{i,j}(\mathcal{A})$, $i,j = 1,2$ are certain linear combinations with bounded coefficients of the elements of the vector $\mathcal{A}$. The following observation is elementary.

Remark 3.1. *Consider a family $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0}$ which satisfies a uniform bound in the class $\mathcal{B}(T,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma)$. Define and auxiliary quantity $\mathcal{A}^\epsilon$ in the fashion of* (43)*. If the representation* (44) *is valid, then there is a subsequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that for almost all $t\in[0,T]$ the sequences $\{\rho_{\epsilon_n}(t)\}_{\epsilon>0}$ and $\{\varrho_{\epsilon_n}\, v^{\epsilon_n}(t)\}_{\epsilon>0}$ converge as distribution in Ω.*

Proof. Define $w^\epsilon := (\rho^\epsilon,\, \varrho_\epsilon\, v^\epsilon)$, and let Y be the Banach space $C^1_c(\Omega;\,\mathbb{R}^N)\times C^1_c(\Omega;\,\mathbb{R}^3)$. Obviously, the identity (44) implies the bound

$$\|\partial_t w^\epsilon\|_{L^1(0,T;\,Y^*)} \leq C_0 \sup_{\epsilon\in[0,1]} \|\mathcal{A}^\epsilon\|_{[L^1(Q)]^a}\,.$$

Thus, invoking the Lemma 4 and the Theorem 1 of [Sim86], we obtain that w^ϵ is in a compact subset of $L^p(0,T;\,Y^*)$ for all $1\leq p<+\infty$. Standard results allow to find $w\in L^1(0,T;\,Y^*)$ and to extract a susequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that $w^{\epsilon_n}\to w\in L^1(0,T;\,Y^*)$ and $w^{\epsilon_n}(t)\to w(t)$ in Y^* for almost all $t\in]0,T[$. ☐

We consider a 'solution family' $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0}$ which might for example correspond to free energy functions $\{h_\epsilon\}_{\epsilon>0}$ and mobility matrices $\{M_\epsilon\}_{\epsilon>0}$. We assume that the conditions

$$h_\epsilon(\rho) \geq c_0\,|\rho|^\alpha - c_1, \text{ for all } \rho\in\mathbb{R}^N_+$$
$$M_\epsilon\xi\cdot\xi \geq \underline{\lambda}\,|P_{\mathbb{1}^\perp}\xi|^2 \quad \text{ for all } \xi\in\mathbb{R}^N\,.$$

are satisfied uniformly in ϵ. At first we need to extract weakly convergent sub-sequences.

Lemma 3.2. *Consider a family* $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0}$ *which satisfies a uniform bound in the class* $\mathcal{B}(T,\, \Omega,\, \alpha,\, N-1,\, \Psi,\, \Psi^\Gamma)$*. Define auxiliary quantities* ρ^ϵ, J_ϵ, r^ϵ, $\hat{r}^\epsilon$, p_ϵ, n^F_ϵ *and* $\mathcal{A}^\epsilon$ *in the fashion of* (24)*,* (43)*. Assume that* (44) *is valid. Assume that for almost all* $t \in]0,T[$, $\phi_\epsilon(t)$ *satisfies in the weak sense*

$$-\epsilon_0\,(1+\chi)\,\triangle\phi_\epsilon(t) = \tfrac{z}{m}\cdot\rho^\epsilon(t),\ -\nu\cdot\nabla\phi_\epsilon(t) = 0 \text{ on } \Sigma,\ \phi_\epsilon(t) = \phi_0(t) \text{ on } \Gamma\,.$$

Then, there are

$$\begin{aligned}
&\rho \in L^{\infty,\alpha}(Q;\,\mathbb{R}^N_+), \quad J \in L^{2,\frac{2\alpha}{1+\alpha}}(Q;\,\mathbb{R}^{N\times 3})\\
&-R \in L_\Psi(Q;\,\mathbb{R}^s), \quad -R^\Gamma \in L_{\hat{\Psi}^\Gamma}(S;\,\mathbb{R}^{\hat{s}^\Gamma})\\
&v \in W^{1,0}_2(Q;\,\mathbb{R}^3), \quad p \in L^{\infty,1}(Q)\cap L^{\min\{1+\frac{1}{\alpha},\,\frac{5}{3}-\frac{1}{\alpha}\}}(Q)\\
&\phi \in L^\infty(Q)\cap L^\infty(0,T;\,W^{1,\beta}(\Omega))
\end{aligned}$$

and a subsequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ *such that as* $n\to\infty$*:*

$$\begin{aligned}
&\rho^{\epsilon_n} \to \rho \text{ weakly in } L^\alpha(Q;\,\mathbb{R}^N)\\
&\rho^{\epsilon_n}(t) \to \rho(t) \text{ weakly in } L^\alpha(\Omega;\,\mathbb{R}^N) \text{ for almost all } t\in[0,T]\\
&\bar{\rho}^{\epsilon_n} \to \bar{\rho} \text{ strongly in } C([0,T];\,\mathbb{R}^N)\\
&J_{\epsilon_n} \to J \text{ weakly in } L^{2,\frac{2\alpha}{1+\alpha}}(Q;\,\mathbb{R}^{N\times 3})\\
&R^{\epsilon_n} \to R \text{ weakly in } L^1(Q;\,\mathbb{R}^s),\ R^{\Gamma,\epsilon_n} \to R^\Gamma \text{ weakly in } L^1(S;\,\mathbb{R}^{\hat{s}^\Gamma})\\
&v^{\epsilon_n} \to v \text{ weakly in } W^{1,0}_2(Q;\,\mathbb{R}^3)\\
&p_{\epsilon_n} \to p \text{ weakly in } L^{1+\min\{\frac{1}{\alpha},\,\frac{2}{3}-\frac{1}{\alpha}\}}(Q)\\
&\phi_{\epsilon_n} \to \phi \text{ strongly in } W^{1,0}_2(Q)\\
&\frac{z}{m}\cdot\rho^{\epsilon_n}\,\nabla\phi_{\epsilon_n} \to \frac{z}{m}\cdot\rho\,\nabla\phi \text{ weakly in } L^1(Q;\,\mathbb{R}^3)\\
&\varrho_{\epsilon_n}\,v^{\epsilon_n} \to \varrho\,v \text{ weakly in } L^{2,\frac{6\alpha}{6+\alpha}}(Q;\,\mathbb{R}^3)\\
&(\varrho_{\epsilon_n}\,v^{\epsilon_n})(t) \to \varrho(t)\,v(t) \text{ weakly in } L^{\frac{2\alpha}{1+\alpha}}(\Omega;\,\mathbb{R}^3) \text{ for almost all } t\in[0,T]\\
&\varrho_{\epsilon_n}\,v^{\epsilon_n}\otimes v^{\epsilon_n} \to \varrho\,v\otimes v \text{ weakly in } L^{\frac{5\alpha-3}{3\,\alpha}}(Q;\,\mathbb{R}^{3\times 3})\,.
\end{aligned}$$

Proof. At first, using the bounds in the natural class $\mathcal{B}$ we extract a subsequence such that

$$\begin{aligned}
&\rho^{\epsilon_n} \to \rho \text{ weakly in } L^\alpha(Q;\,\mathbb{R}^N), \quad \bar{\rho}^{\epsilon_n} \to \bar{\rho} \text{ strongly in } C([0,T];\,\mathbb{R}^N)\\
&J_{\epsilon_n} \to J \text{ weakly in } L^{2,\frac{2\alpha}{1+\alpha}}(Q;\,\mathbb{R}^{N\times 3})\\
&R^{\epsilon_n} \to R \text{ weakly in } L^1(Q;\,\mathbb{R}^s), \quad R^{\Gamma,\epsilon_n} \to R^\Gamma \text{ weakly in } L^1(S;\,\mathbb{R}^{\hat{s}^\Gamma})\\
&v^{\epsilon_n} \to v \text{ weakly in } W^{1,0}_2(Q;\,\mathbb{R}^3)\\
&p_{\epsilon_n} \to p \text{ weakly in } L^{1+\min\{\frac{1}{\alpha},\,\frac{2}{3}-\frac{1}{\alpha}\}}(Q)\\
&\varrho_{\epsilon_n}\,v^{\epsilon_n} \to \xi \text{ weakly in } L^{2,\frac{6\alpha}{6+\alpha}}(Q;\,\mathbb{R}^3)\\
&\varrho_{\epsilon_n}\,v^{\epsilon_n}\otimes v^{\epsilon_n} \to \tilde{\xi} \text{ weakly in } L^{\frac{5\alpha-3}{3\,\alpha}}(Q;\,\mathbb{R}^{3\times 3})\\
&\phi_{\epsilon_n} \to \phi \text{ weakly } W^{1,0}_2(Q)\\
&n^F_{\epsilon_n}\,\nabla\phi_{\epsilon_n} \to k_L \text{ weakly in } L^1(Q;\,\mathbb{R}^3)\\
&v^{\epsilon_n}\otimes(\mathbb{1}\cdot J_{\epsilon_n}) \to \hat{\xi} \text{ in } (L^\infty)^*(Q;\,\mathbb{R}^{3\times 3})\,.
\end{aligned}$$

We now make use of the identity (44) via Remark 3.1. Thus, for all $t \in [0,T]$, we realise that the entire sequence $\{\rho^{\epsilon_n}(t)\}$ converges as distributions. Since it is uniformly bounded in $L^{\alpha}(\Omega)$, we obtain that $\{\rho^{\epsilon_n}(t)\}$ weakly converges in $L^{\alpha}(\Omega)$. The limit must be identical with $\rho(t)$ for almost all $t \in [0,T]$. Thus, making use of the Remark 3.3 hereafter, $\rho^{\epsilon_n} \to \rho$ strongly in $[W^{1,0}_2(Q)]^*$, and this allows to show that $\varrho^{\epsilon_n}\, v^{\epsilon_n} \to \varrho\, v$ as distributions in Q. Clearly $\xi = \varrho\, v$.

Next we define $\phi(t) \in W^{1,2}(\Omega)$ to be the unique weak solution to the problem $-\epsilon_0\,(1+\chi)\,\triangle\phi(t) = \frac{z}{m}\cdot\rho(t)$ with the boundary conditions $-\nu\cdot\nabla\phi(t) = 0$ on Σ and $\phi(t) = \phi_0(t)$ on Γ. We can verify due to the Remark 3.3 that for almost all $t \in]0,T[$ the convergence $\phi_{\epsilon_n}(t) \to \phi(t)$ strongly in $W^{1,2}(\Omega)$ takes place. Thus, it also follows that $\frac{z}{m}\cdot\rho^{\epsilon_n}\,\nabla\phi_{\epsilon_n} \to \frac{z}{m}\cdot\rho\,\nabla\phi$ weakly in $L^1(Q)$. It follows that $k_L = n^F\,\nabla\phi$. The Remark 3.1 implies that $\varrho_{\epsilon_n}(t)\,v^{\epsilon_n}(t)$ converges as distributions to $\varrho(t)\,v(t)$ for almost all $t \in]0,T[$, and therefore also weakly in $L^{2\alpha/(1+\alpha)}(\Omega)$. Since $2\alpha/(1+\alpha) > 6/5$, it also follows $\varrho_{\epsilon_n}\,v^{\epsilon_n} \to \varrho\,v$ strongly in $[W^{1,0}_2(Q)]^*$. This in turn allows to show that $\varrho_{\epsilon_n}\,v^{\epsilon_n}\otimes v^{\epsilon_n} \to \varrho\,v\otimes v$ as distributions, that means $\tilde{\xi} = \varrho\,v\otimes v$. □

Remark 3.3.

- *Let $1 \leq p \leq +\infty$. Let $\mathcal{K} : L^p(\Omega) \to W^{1,p}(\Omega)$ be a bounded, compact operator. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset L^p(Q)$ is a sequence such that $u_n(t) \to u(t)$ weakly in $L^p(\Omega)$ for almost all $t \in]0,T[$. Then $\mathcal{K}(u_n(t)) \to \mathcal{K}(u(t))$ strongly in $W^{1,p}(\Omega)$ for almost all $t \in]0,T[$.*
- *If $v_n \to v$ weakly in $W^{1,0}_2(Q)$ and $u_n(t) \to u(t)$ strongly in $[W^{1,2}(\Omega)]^*$ for almost all $t \in]0,T[$, then $u_n\,v_n \to u\,v$ weakly in $L^1(Q)$.*

We next can obtain the strong convergence of the velocity field. This result is in principle known (see [Lio98], page 9). A detailed proof for the present situation is to find in [DDGG16].

Corollary 3.4. *Assumptions of Lemma 3.2. Then, there is a subsequence such that $\varrho_{\epsilon_n}\,(v^{\epsilon_n} - v)$ converges to zero strongly in $L^1(Q)$ and pointwise almost everywhere in Q.*

We now can prove the conditional compactness of the family $\{\rho^{\epsilon}\}_{\epsilon>0}$. We will need the following auxiliary statements.

Lemma 3.5. *Consider a map $\mathscr{R} \in C(\mathbb{R}_{0,+}\times\mathbb{R}^{N-1};\,\mathbb{R}^N_+)$. For $x \in \mathbb{R}_+\times\mathbb{R}^{N-1}$, we denote $x = (x_1,\,\bar{x})$. Let $K \subset L^1(\Omega;\,\mathbb{R}^N)$ be a weakly sequentially compact set, and $K^* \subset L^1(\Omega)$ a sequentially compact set. Let $\phi^1,\,\phi^2,\ldots \in C^{\infty}(\overline{\Omega})$ be a countable, dense subset of $C(\overline{\Omega};\,\mathbb{R}^N)$.*

For all $\delta > 0$, there are $C(\delta) > 0$ and $m(\delta) \in \mathbb{N}$ such that

$$\|\mathscr{R}(w^1) - \mathscr{R}(w^2)\|_{L^1(\Omega)} \leq \delta\left(1 + \sum_{i=1,2}\|\bar{w}^i\|_{W^{1,1}(\Omega)}\right) + C(\delta)\sum_{i=1}^{m}\left|\int_{\Omega}(\mathscr{R}(w^1) - \mathscr{R}(w^2))\cdot\phi^i\right|$$

for all $w^1,\,w^2 \in L^1(\Omega;\,\mathbb{R}_+\times\mathbb{R}^{N-1})$ such that

$$\mathscr{R}(w^i) \in K,\quad w^i_1 \in K^*,\quad \bar{w}_i \in W^{1,1}(\Omega;\,\mathbb{R}^{N-1})\quad \text{for } i = 1,2\,.$$

Proof. Clearly, it is sufficient to prove the inequality

$$\|\mathscr{R}(w^1) - \mathscr{R}(w^2)\|_{L^1(\Omega)} \leq \delta\sum_{i=1,2}\|\bar{w}^i\|_{W^{1,1}(\Omega)} + C(\delta)\sum_{i=1}^{m}\left|\int_{\Omega}(\mathscr{R}(w^1) - \mathscr{R}(w^2))\cdot\phi^i\right|$$

for all $w^1,\, w^2 \in L^1(\Omega;\, \mathbb{R}_+ \times \mathbb{R}^{N-1})$ such that $\mathscr{R}(w^i) \in K$, $w^i_1 \in K^*$ and $\bar{w}^i \in W^{1,1}(\Omega;\, \mathbb{R}^{N-1})$ for $i = 1, 2$ and such that

$$\|\mathscr{R}(w^1) - \mathscr{R}(w^2)\|_{L^1(\Omega)} \geq \delta\,.$$

If this is not true, there is $\delta_0 > 0$ such that for all $n \in \mathbb{N}$ and $i = 1, 2$, we can find $w^{i,n} \in L^1(\Omega;\, \mathbb{R}_+ \times \mathbb{R}^{N-1})$ such that $\mathscr{R}(w^{i,n}) \in K$, $w^{i,n}_1 \in K^*$, $\bar{w}^{i,n} \in W^{1,1}(\Omega;\, \mathbb{R}^{N-1})$ ($i = 1, 2$) satisfying moreover the properties

$$\begin{aligned} &\|\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n})\|_{L^1(\Omega)} \\ &\geq \delta_0 \sum_{i=1,2} \|\bar{w}^{i,n}\|_{W^{1,1}(\Omega)} + n \sum_{i=1}^{n} \left| \int_\Omega (\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n})) \cdot \phi^i \right| \qquad (45) \\ &\|\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n})\|_{L^1(\Omega)} \geq \delta_0\,. \qquad (46) \end{aligned}$$

Since we assume that $\mathscr{R}(w^{i,n}) \in K$ for $i = 1, 2$ and since K is a bounded set of $L^1(\Omega)$, we obtain first that $\|\bar{w}^{i,n}\|_{W^{1,1}(\Omega)} \leq C$ for all $n \in \mathbb{N}$. Thus we can extract a subsequence that we not relabel such that for almost all $x \in \Omega$ there exists $\bar{w}^i(x) := \lim_{n\to\infty} \bar{w}^{i,n}(x)$.

Moreover as $w^{i,n}_1 \in K^*$, we can extract a subsequence such that $w^{i,n}_1 \to w^i_1$ strongly in $L^1(\Omega)$ and almost everywhere in Ω. Consequently, we obtain for a subsequence and for $i = 1, 2$ that

$$w^{i,n} \to w^i := (w^i_1,\, \bar{w}^i) \text{ strongly in } L^1(\Omega;\, \mathbb{R}_+ \times \mathbb{R}^{N-1}) \text{ and a. e. in } \Omega\,.$$

Now using that $\mathscr{R}(w^{i,n}) \in K$, we can pass to a subsequence again to see that $\mathscr{R}(w^{i,n}) \to u^i$ weakly in $L^1(\Omega;\, \mathbb{R}^N)$ for $i = 1, 2$. Obviously the continuity of $\mathscr{R}$ and the pointwise convergence yield $u^i = \mathscr{R}(w^i)$. We next use the second implication of (45), that is,

$$\sum_{i=1}^{n} \left| \int_\Omega (\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n})) \cdot \phi^i \right| \leq c\, n^{-1}\,,$$

so that we easily conclude that $\mathscr{R}(w^1) = \mathscr{R}(w^2)$ almost everywhere in Ω. It remains to observe that $\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n}) \to 0$ in $L^1(\Omega)$ to show that the condition (46) is violated. □

In order to apply the Lemma 3.5 in the context of parabolic problems, we recall the following remark.

Remark 3.6. *A family $\{u_\epsilon\}_{\epsilon\in[0,1]}$ of $C([0,T];\, L^1(\Omega))$ is compact in $C([0,T];\, L^1(\Omega))$ if and only if the set $\bigcup_{\epsilon\in[0,1]} \bigcup_{t\in[0,T]} \{u_\epsilon(t)\}$ is sequentially compact in $L^1(\Omega)$.*

We now state and prove our main compactness tool.

Corollary 3.7. *For $n \in \mathbb{N}$, let $w^n :\, [0,T] \to L^1(\Omega;\, \mathbb{R}_+ \times \mathbb{R}^{N-1})$ be continuous. Assume that $\{w^n_1\}$ is compact in $C([0,T];\, L^1(\Omega))$ (see Remark 3.6). Moreover assume that there is C_1 independent on n such that $\|\bar{w}^n\|_{L^1(Q;\mathbb{R}^{N-1})} + \|\nabla \bar{w}^n\|_{L^1(Q;\mathbb{R}^{N-1})} \leq C_1$. Suppose that $\|\mathscr{R}(w^n)\|_{L^{\infty,\alpha}(Q;\mathbb{R}^{N-1})} \leq C_1$, and that the sequence $\{\mathscr{R}(w^n(t))\}_{n\in\mathbb{N}}$ converges as distributions in Ω for almost all t.*

Then, there is a subsequence (no new labels) for which there exists $\rho(t,\, x) := \lim_{n\to\infty} \mathscr{R}(w^n(t,x))$ for almost all $(t,x) \in Q$, and $\mathscr{R}(w^n(t,x)) \to \rho$ strongly in $L^1(Q;\, \mathbb{R}^N)$.

Proof. For $n \in \mathbb{N}$, the assumptions imply that $\mathscr{R}(w^n(t)) \in L^\alpha(\Omega;\, \mathbb{R}^N)$ for all $t \in [0,T]$. We define $K \subset L^1(\Omega;\, \mathbb{R}^N)$ via $K := \bigcup_{n\in\mathbb{N}} \bigcup_{t\in[0,T]} \{\mathscr{R}(w^n(t))\}$ By assumption K is bounded in $L^\alpha(\Omega)$ and thus also weakly sequentially compact in $L^1(\Omega)$.

By assumption again, the set $K^* := \bigcup_{n\in\mathbb{N}} \bigcup_{t\in[0,T]} \{w_1^n(t)\}$ is compact in $L^1(\Omega)$.

For $\delta > 0$, we apply the inequality of Lemma 3.5 with $w^1 = w^n(t)$, $w^2 := w^{n+p}(t)$ ($p \in \mathbb{N}$). For $t \in [0,T]$ it follows that

$$\begin{aligned}
&\|\mathscr{R}(w^n(t)) - \mathscr{R}(w^{n+p}(t))\|_{L^1(\Omega)} \\
&\quad \le \delta\left(1 + \|\bar{w}^n(t)\|_{W^{1,1}(\Omega)} + \|\bar{w}^{n+p}(t)\|_{W^{1,1}(\Omega)}\right) \\
&\qquad + C(\delta,\, K^*) \sum_{i=1}^m \left|\int_\Omega (\mathscr{R}(w^n(t)) - \mathscr{R}(w^{n+p}(t)))\cdot \phi^i\right| . \qquad (47)
\end{aligned}$$

We integrate the relation (47) over the set $]0,T[$ and this yields

$$\begin{aligned}
&\|\mathscr{R}(w^n) - \mathscr{R}(w^{n+p})\|_{L^1(Q)} \\
&\le \delta\,(T + 2\sup_n \|\bar{w}^n\|_{W^{1,0}_1(Q)}) \\
&\quad + C(\delta) \sum_{i=1}^m \int_0^T \left|\int_\Omega (\mathscr{R}(w^n(t)) - \mathscr{R}(w^{n+p}(t)))\cdot \phi^i\right| \\
&\le \delta\,(T + C) + C(\delta) \sum_{i=1}^m \int_0^T \left|\int_\Omega (\mathscr{R}(w^n(t)) - \mathscr{R}(w^{n+p}(t)))\cdot \phi^i\right|
\end{aligned}$$

The vector fields $\mathscr{R}(w^n)$ weakly converges in $L^1(\Omega;\,\mathbb{R}^N)$ for almost all t to some element $\rho \in L^{\infty,\alpha}(Q;\,\mathbb{R}^N)$. Invoking the triangle inequality,

$$\int_0^T \left|\int_\Omega (\mathscr{R}(w^n(t)) - \mathscr{R}(w^{n+p}(t)))\cdot \phi^i\right| \le 2\sup_{k\ge n} \int_0^T \left|\int_\Omega (\mathscr{R}(w^k(t)) - \rho(t))\cdot \phi^i\right| .$$

It follows that

$$\begin{aligned}
\sup_{p\ge 0} \|\mathscr{R}(w^n) - \mathscr{R}(w^{n+p})\|_{L^1(Q)} &\le \delta\,(T + C) \\
&\quad + 2\,C(\delta) \sum_{i=1}^m \sup_{k\ge n} \int_0^T \left|\int_\Omega (\mathscr{R}(w^k(t)) - \rho(t))\cdot \phi^i\right| .
\end{aligned}$$

Invoking the Fatou Lemma and the bounds in $L^{\infty,\alpha}$

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{k\ge n} \int_0^T \left|\int_\Omega (\mathscr{R}(w^k(t)) - \rho(t))\cdot \phi^i\right| \\
&\quad = \limsup_{n\to\infty} \int_0^T \left|\int_\Omega (\mathscr{R}(w^n(t)) - \rho(t))\cdot \phi^i\right| \\
&\quad \le \int_0^T \limsup_{n\to\infty} \left|\int_\Omega (\mathscr{R}(w^n(t)) - \rho(t))\cdot \phi^i\right| .
\end{aligned}$$

The vector fields $\mathscr{R}(w^n(t))$ weakly converges in $L^1(\Omega)$ for almost all t.

Therefore $\limsup_{n\to\infty} \left|\int_\Omega (\mathscr{R}(w^n(t)) - \rho(t))\cdot \phi^i\right| = 0$ for almost all t. Thus

$$\limsup_{n\to\infty} \sup_{k\ge n} \int_0^T \left|\int_\Omega (\mathscr{R}(w^k(t)) - \rho(t))\cdot \phi^i\right| = 0 .$$

It next follows that

$$\limsup_{n\to\infty}\sup_{p\geq 0}\|\mathscr{R}(w^n)-\mathscr{R}(w^{n+p})\|_{L^1(Q)}\leq\delta\,(T+C)\,,$$

δ is arbitrary, $\limsup_{n\to\infty}\sup_{p\geq 0}\|\mathscr{R}(w^n)-\mathscr{R}(w^{n+p})\|_{L^1(Q)}=0$. This means that $\{\mathscr{R}(w^n)\}$ is a Cauchy sequence in $L^1(Q)$. In particular, we can extract a subsequence such that $\lim_{n\to+\infty}\mathscr{R}(w^n)$ exists almost everywhere in Q. ☐

Corollary 3.8. *Assumptions of Lemma 3.2. Assume moreover that the family of the total mass densities $\{\varrho_\epsilon\}_{\epsilon\geq 0}$ is compact in $L^1(\Omega)$ uniformly with respect to time (sense of Remark 3.6). Then*

$$\rho^{\epsilon_n}\to\rho \text{ strongly in } L^1(Q;\,\mathbb{R}^N)$$
$$\exists q(t,\,x):=\lim_{n\to\infty}q^{\epsilon_n}(t,\,x) \text{ for almost every } (t,\,x) \text{ such that } \varrho(t,\,x)>0\,.$$

Consequently, the identity $\rho=\mathscr{R}(\varrho,\,q)$ is valid at almost every point of the set $\{(t,\,x)\,:\,\varrho(t,\,x)>0\}$.

Proof. We at first obtain the convergence properties of Lemma 3.2 for a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$. We define $w^n=(\varrho_{\epsilon_n},\,q^{\epsilon_n})$, and verify easily that all requirements of the Corollary 3.7 are satisfied. We apply the Corollary 3.7, and this at first shows that $\rho^{\epsilon_n}=\mathscr{R}(\varrho_{\epsilon_n},\,q^{\epsilon_n})$ converges strongly in $L^1(Q)$ and pointwise almost everywhere.

Recall from the paper [DDGG17b] that $\mathscr{R}\in C(\mathbb{R}_{0,+}\times\mathbb{R}^{N-1};\,\mathbb{R}^N)\cap C^1(\mathbb{R}_+\times\mathbb{R}^{N-1};\,\mathbb{R}^N)$ satisfies

$$|\partial_\varrho\mathbb{R}(\varrho,\,q)|\leq C\,\varrho^{\frac{\alpha-1}{2}} \text{ for all } (\varrho,\,q)\in\mathbb{R}_+\times\mathbb{R}^{N-1}\,. \tag{48}$$

Next we make use of the (48) to see that for a certain $\lambda\in[0,1]$

$$\begin{aligned}|\mathscr{R}(\varrho_{\epsilon_n},\,q^{\epsilon_n})-\mathscr{R}(\varrho,\,q^{\epsilon_n})|&=\mathscr{R}_s(\lambda\,\varrho_{\epsilon_n}+(1-\lambda)\,\varrho,\,q^{\epsilon_n})|\,|\varrho_{\epsilon_n}-\varrho|\\&\leq C\,\max\{\varrho_{\epsilon_n},\,\varrho\}^{\frac{\alpha-1}{2}}\,|\varrho_{\epsilon_n}-\varrho|\,.\end{aligned}$$

The latter implies that

$$\begin{aligned}&\|\mathscr{R}(\varrho_{\epsilon_n},\,q^{\epsilon_n})-\mathscr{R}(\varrho,\,q^{\epsilon_n})\|_{L^1(Q)}\\&\quad\leq C\,(\sup_n\|\varrho_{\epsilon_n}\|_{L^\alpha(Q)}+\|\varrho\|_{L^\alpha(Q)})\,\|\varrho_{\epsilon_n}-\varrho\|_{L^{\frac{2\alpha}{1+\alpha}}(Q)}\to 0\,.\end{aligned}$$

Thus passing to a subsequence, $\mathscr{R}(\varrho,\,q^{\epsilon_n})$ converges almost everywhere in Q. Recall from the paper [DDGG17b] that for all $\varrho>0$, the map $\mathscr{R}$ is a bijection between $[\varrho,\,+\infty[\times\mathbb{R}^{N-1}$ and $\{X\in\mathbb{R}^N_+\,:\,X\cdot\mathbb{1}\geq\varrho\}$. Thus, from the existence of $\lim_{n\to\infty}\mathscr{R}(\varrho(t,x),\,q^{\epsilon_n}(t,x))$, it follows that

$$\liminf_{n\to\infty}q^{\epsilon_n}(t,x)=\limsup_{n\to\infty}q^{\epsilon_n}(t,x) \text{ for a. a.} (t,x) \text{ such that } \varrho(t,x)>0\,. \tag{49}$$

Next we make use of the estimates available on q^{ϵ_n} and (49) to see that

$$\lim_{n\to\infty}q^{\epsilon_n}(t,x) \text{ exists in } \mathbb{R}^{N-1} \text{ for almost all } (t,x)\in\{(t,x)\,:\,\varrho(t,x)>0\}\,.\,.$$

Recall at last that $\rho^{\epsilon_n}=\mathscr{R}(\varrho_{\epsilon_n},\,q^{\epsilon_n})$, to see that ρ^{ϵ_n} also converges almost everywhere in Q to $\mathscr{R}(\varrho,\,q)$, and the claim follows. ☐

In order to pass to the limit in the boundary reaction terms, we also discuss the strong convergence of the relative chemical potentials on the boundary $]0,T[\times\Gamma$.

Lemma 3.9. *Assumptions of Corollary 3.8. Then*

$$\exists\, q(t,\,x) := \lim_{n\to\infty} q^{\epsilon_n}(t,\,x) \text{ for almost every } (t,\,x)\in S^+(\varrho)\,.$$

Proof. By definition, the surface $S^+(\varrho)$ is relatively open and possesses an open neighbourhood U in Q such that $|U\cap\{(t,x)\,:\,\varrho(t,\,x)=0\}|=0$. Thus, for $(t_0,\,x^0)\in S^+(\varrho)$ arbitrary, there is $R>0$ such that the cube $Q_R(t_0,x^0)$ with radius $R>0$ and centred at $(t_0,\,x^0)$ is contained in U. For all $\epsilon>0$, there is a constant $c=c(\Omega,\,\epsilon)$ such that

$$\|u\|_{L^1(\Gamma_R(x^0))}\leq \epsilon\,\|\nabla u\|_{L^1(\Omega_R(x^0))}+c(\epsilon,\,\Omega)\,\|u\|_{L^1(\Omega_R(x^0))} \text{ for all } u\in W^{1,1}(\Omega)\,.$$

Here Γ_R and Ω_R denote the intersection of Γ and Ω with $Q_R(x^0)$, the three-dimensional cube with radius R centred at x^0. With the help of this inequality, we obtain for almost all $t\in]t_0-R,\,t_0+R[$ that

$$\begin{aligned}\|q^{\epsilon_n}(t)-q(t)\|_{L^1(\Gamma\cap Q_R(x^0))}\leq&\epsilon\,(\|\nabla q^{\epsilon_n}(t)\|_{L^1(\Omega)}+\|\nabla q(t)\|_{L^1(\Omega)})\\&+c(\epsilon,\,\Omega)\,\|q^{\epsilon_n}(t)-q(t)\|_{L^1(\Omega\cap Q_R(x^0))}\,.\end{aligned}$$

We obtain that

$$\int_{t_0-R}^{t_0+R}\|q^{\epsilon_n}(t)-q(t)\|_{L^1(\Gamma_R(x^0))}\,dt\leq C_0\,\epsilon+c(\epsilon,\,\Omega)\int_{t_0-R}^{t_0+R}\|q^{\epsilon_n}(t)-q(t)\|_{L^1(\Omega_R(x^0))}\,dt$$

Now as $(]t_0-R,\,t_0+R[\times\Omega)\cap Q_R(x^0))$ is a subset of U, we obtain with the help of Corollary 3.8 that $\int_{t_0-R}^{t_0+R}\|q^{\epsilon_n}(t)-q(t)\|_{L^1(\Omega_R(x^0))}\,dt\to 0$ for $n\to\infty$, and this yields $\limsup_{n\to\infty}\int_{t_0-R}^{t_0+R}\|q^{\epsilon_n}(t)-q(t)\|_{L^1(\Gamma_R(x^0))}\,dt=0$. The claim follows. □

It remains to enlighten the global convergence property of the variables $\{q^{\epsilon_n}\}$ inclusively of the set where vacuum possibly occurs. The proof is obvious in view of the *a priori* estimates.

Lemma 3.10. *Assumptions of Corollary 3.8. Then, there is a subsequence such that*

$$\begin{aligned}q^{\epsilon_n}&\to q \text{ weakly in } L^2(]0,T[\times[\Omega\cup\Gamma];\,\mathbb{R}^{N-1})\\ \nabla q^{\epsilon_n}&\to\nabla q \text{ weakly in } L^2(Q;\,\mathbb{R}^{(N-1)\times 3})\\ D^{R,\epsilon_n}&\to(\gamma^1\cdot\mathcal{E}q,\ldots,\gamma^s\cdot\mathcal{E}q) \text{ weakly in } L^1(Q;\,\mathbb{R}^s)\\ \hat{D}^{\Gamma,R,\epsilon_n}&\to(\hat{\gamma}^1\cdot\mathcal{E}q,\ldots,\hat{\gamma}^{\hat{s}^\Gamma}\cdot\mathcal{E}q) \text{ weakly in } L^1(S;\,\mathbb{R}^{\hat{s}^\Gamma})\,.\end{aligned}$$

Finally we can identity the remaining limits.

Corollary 3.11. *Assumptions of Corollary 3.8. Let J, p, r and $\hat{r}$ denote the weak limit of J^{ϵ_n}, p_{ϵ_n}, r^{ϵ_n} and $\hat{r}^{\epsilon_n}$ constructed in the Lemma 3.2. Then, for almost all $t\in]0,T[$*

$$\begin{aligned}&J=M(\rho)\,(\nabla\mathcal{E}\,q+\frac{z}{m}\,\nabla\phi)\\&p=P(\varrho,\,q)\\&r=\sum_{k=1}^{s}\gamma^k\,\bar{R}_k(D^R) \text{ with } D^R_k=\gamma^k\cdot\mathcal{E}q \text{ in } Q^+(\varrho)\\&\hat{r}=\sum_{k=1}^{s^\Gamma}\hat{\gamma}^k\,\hat{R}^\Gamma_k(t,x,\,\hat{D}^{\Gamma,R}) \text{ with } \hat{D}^{\Gamma,R}_k=\hat{\gamma}^k\cdot\mathcal{E}q \text{ on } S^+(\varrho)\,.\end{aligned}$$

Proof. Exploiting the convergence properties stated in the Corollary 3.8 and the Lemma 3.2, 3.10 we see that

$$J_{\epsilon_n} = M(\rho^{\epsilon_n})\,(\nabla\mathcal{E}q^{\epsilon_n} + \tfrac{z}{m}\,\nabla\phi_{\epsilon_n}) \to M(\rho)\,(\nabla\mathcal{E}\,q + \tfrac{z}{m}\,\nabla\phi)$$

weakly in $L^{2,\frac{2\alpha}{1+\alpha}}(Q)$. Moreover, $P(\varrho_{\epsilon_n},\, q^{\epsilon_n}) \to P(\varrho,\, q)$ pointwise in $Q^+(\varrho)$, while $|P(\varrho_{\epsilon_n},\, q^{\epsilon_n})| \leq c\,\varrho_{\epsilon_n}^{\alpha} \to 0$ pointwise in $Q \setminus Q^+(\varrho)$. The other claims are proved similarly. □

We now resume the results of the section formulating our main (conditional) compactness statement.

Proposition 3.12. *Consider a family* $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0}$ *which satisfies a uniform bound in the class* $\mathcal{B}(T,\, \Omega,\, \alpha,\, N-1,\, \Psi,\, \Psi^\Gamma)$. *Assume the condition* (44) *on the time derivatives. Assume that the family* $\{\varrho_\epsilon\}$ *is compact in* $C([0,T];\, L^1(\Omega))$.

Then, there is a limiting element $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma) \in \mathcal{B}$ *and subsequence* $\{\epsilon_n\}_{n\in\mathbb{N}}$ *such that*

$\rho^{\epsilon_n} \to \rho$ *strongly in* $L^1(Q;\, \mathbb{R}^N)$

$J^{\epsilon_n} \to J$ *weakly in* $L^{2,\frac{2\alpha}{1+\alpha}}(Q;\, \mathbb{R}^{N\times 3})$

$R^{\epsilon_n} \to R$ *weakly in* $L^1(Q;\, \mathbb{R}^s)$, $\quad R^{\Gamma,\epsilon_n} \to R^\Gamma$ *weakly in* $L^1(S;\, \mathbb{R}^{\hat{s}^\Gamma})$

$v^{\epsilon_n} \to v$ *weakly in* $W^{1,0}_2(Q;\, \mathbb{R}^3)$

$\phi_{\epsilon_n} \to \phi$ *strongly in* $W^{1,0}_2(Q)$

$n^F_{\epsilon_n}\,\nabla\phi_{\epsilon_n} \to n^F\,\nabla\phi$ *weakly in* $L^1(Q;\, \mathbb{R}^3)$

$\varrho_{\epsilon_n}\, v^{\epsilon_n} \to \varrho\, v$ *strongly in* $L^1(Q;\, \mathbb{R}^3)$

$\varrho_{\epsilon_n}\, v^{\epsilon_n} \otimes v^{\epsilon_n} \to \varrho\, v \otimes v$ *weakly in* $L^{\frac{5\alpha-3}{3\,\alpha}}(Q;\, \mathbb{R}^{3\times 3})\,.$

Here the quantities ρ, J, r, $\hat{r}$, p, n^F *obey the natural definitions* (24).

We finally note an important consequence of Proposition 3.12 concerning the lower semi continuity of the energy identity. The proof is rather obvious and we can omit it.

Corollary 3.13. *Assumptions of Proposition 3.12. Let* $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0}$ *satisfy for* $\epsilon > 0$ *the energy inequality with mobility matrix* $M_\epsilon \geq M$, *and a free energy function* h^ϵ *having the property*

$$\rho^\epsilon \to \rho \in \mathbb{R}^N_{0,+} \Longrightarrow \liminf_{\epsilon\to 0} h^\epsilon(\rho^\epsilon) \geq h(\rho)\,.$$

Then the limiting element $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)$ *constructed in Proposition 3.12 satisfies the energy inequality with free energy function* h *and mobility matrix* M.

We showed that boundedness in the energy class together with the existence of weak time derivatives implies the compactness of the solution vector if the condition $\varrho(t) \in K^*$ for all t is satisfied, where K^* is a compact of $L^1(\Omega)$. Using an extension of the method of Lions for the compressible Navier-Stokes operator, we can show that this condition is satisfied for the approximation schemes of interest to us.

Proposition 3.14. *Consider a family* $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0} \subset \mathcal{B}$ *which is uniformly bounded in the natural class* $\mathcal{B}(T,\, \Omega,\, \alpha,\, N-1,\, \Psi,\, \Psi^\Gamma)$ *and satisfies the assumptions of Lemma 3.2. Let*

$\{\bar{J}^\epsilon\}_{\epsilon>0} \subset L^2(Q;\, \mathbb{R}^3)$ be a family of perturbations such that $\bar{J}^\epsilon \to 0$ strongly in $L^2(Q)$ as $\epsilon \to 0$ and such that

$$\begin{cases} \limsup_{\epsilon\to 0} \|(\bar{J}^\epsilon \cdot \nabla \ln \varrho_\epsilon)^+\|_{L^1(Q)} = 0 & \text{if } \alpha > 3 \\ \bar{J}^\epsilon \equiv 0 & \text{if } \frac{3}{2} < \alpha \leq 3\,. \end{cases}$$

Suppose that the identities

$$- \int_Q \varrho_\epsilon\, \psi_t - \int_Q (\varrho_\epsilon\, v^\epsilon + \bar{J}^\epsilon) \cdot \nabla\psi = \int_\Omega \varrho_0\, \psi(0) \tag{50}$$

$$- \int_Q \varrho_\epsilon\, v^\epsilon \cdot \eta_t - \int_Q \varrho_\epsilon\, v^\epsilon \otimes v^\epsilon \,:\, \nabla\eta - \int_Q p_\epsilon \operatorname{div}\eta + \int_Q \mathbb{S}(\nabla v^\epsilon) \cdot \nabla\eta \tag{51}$$
$$= \int_\Omega \varrho_0\, v^0 \cdot \eta(0) + \int_Q (\bar{J}^\epsilon \cdot \nabla)\eta \cdot v^\epsilon - \int_Q n^F_\epsilon\, \nabla\phi_\epsilon \cdot \eta\,.$$

are valid for all $\psi \in C^1_c([0,T[;\, C^1(\overline{\Omega}))$ and all $\eta \in C^1_c([0,T[;\, C^1_c(\Omega;\, \mathbb{R}^3))$. Assume that either $\alpha \geq 9/5$, or that the function P is convex in the first variable and that $\frac{3}{2} < \alpha < 9/5$.

Then for every sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$, the sequence $\{\varrho_{\epsilon_n}\}_{n\in\mathbb{N}}$ is compact in $C([0,T];\, L^1(\Omega))$.

Remark 3.15. *Insiders in mathematical fluid dynamics will directly conclude from the representation of the pressure $p = P(\varrho,\, q)$, with P increasing in ϱ and with ∇q controlled, that the total mass density must be compact. For readers less familiar with the Lions theory, complete proofs with extensive details have been provided in the Section 10 of [DDGG16].*

4 Existence of solutions

Weak solutions to (P) are defined in the spirit of viscosity solutions by passing to the limit $\tau,\, \sigma \to 0$ and then $\delta \to 0$ in the approximation scheme $(P_{\tau,\sigma,\delta})$.

Proposition 4.1. *Assumptions of the Theorems 2.1, 2.2. For $\tau,\, \sigma > 0$ and $\delta > 0$ assume that there is $(\mu^{\tau,\sigma,\delta},\, v^{\tau,\sigma,\delta},\, \phi_{\tau,\sigma,\delta}) \in \mathcal{B}(T,\, \Omega,\, \alpha,\, N,\, \Psi,\, \Psi^\Gamma)$, subject to the energy inequality and to the global conservation of partial masses, that weakly solves $(P_{\tau,\sigma,\delta})$.*

Then, (P) possesses a weak solution (as stated in the Theorems 2.1, 2.2).

Proof. We first show the claim under the assumptions of the Theorem 2.1 (Global existence).

The validity of the mass conservation identity (27) implies that the vector of total partial mass densities $\bar{\rho}^{\tau,\sigma,\delta} \in C_{\Phi^*}([0,T];\, \mathbb{R}^N)$ satisfies

$$\bar{\rho}^{\tau,\sigma,\delta}(t) \in \{\bar{\rho}^0\} \oplus W \text{ for all } t \in [0,T]\,.$$

We apply the results (37), (38), and we obtain that

$$\begin{aligned} [(\varrho_{\tau,\sigma,\delta},\, q^{\tau,\sigma,\delta},\, v^{\tau,\sigma,\delta},\, \phi_{\tau,\sigma,\delta},\, R^{\tau,\sigma,\delta},\, R^{\Gamma,\tau,\sigma,\delta})]_{\mathcal{B}(T,\,\Omega,\,\alpha_\delta,\,N-1,\,\Psi,\,\Psi^\Gamma)} \leq C(\delta, \mathcal{B}_0) \\ [(\varrho_{\tau,\sigma,\delta},\, q^{\tau,\sigma,\delta},\, v^{\tau,\sigma,\delta},\, \phi_{\tau,\sigma,\delta},\, R^{\tau,\sigma,\delta},\, R^{\Gamma,\tau,\sigma,\delta})]_{\mathcal{B}(T,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma)} \leq C(\mathcal{B}_0)\,. \end{aligned} \tag{52}$$

Here we distinguish the regularisation exponent $\alpha_\delta > 3$ and the original growth exponent $3/2 < \alpha < +\infty$ of the free energy function.

Moreover, time integration in (27) and (33) means that (44) is valid.

We fix $\delta > 0$. By construction, the condition $\alpha_\delta > 3$ is valid. We choose $\tau = \tau(\sigma)$ such that $\tau(\sigma) \to 0$ for $\sigma \to 0$, and we abbreviate $\mu^{\sigma,\delta} = \mu^{\tau(\sigma),\sigma,\delta}$ etc. Aiming to apply the Proposition 3.14, we need to verify the condition

$$\|((\mathbb{1} \cdot J^{\sigma,\delta}) \cdot \nabla \ln \varrho_{\sigma,\delta})^+\|_{L^1(Q)} \to 0 \text{ for } \sigma \to 0\,. \tag{53}$$

This is a direct consequenc of (41). The Proposition 3.14 applied with $\bar{J}^\sigma := \mathbb{1} \cdot J^\sigma$ now guaranties that the family $\{\varrho_{\sigma,\delta}\}_{\sigma>0}$ is compact in $C([0,T];\, L^1(\Omega))$. It remains to apply the Proposition 3.12 in order to obtain the convergence to a weak solution $(\varrho_\delta,\, q^\delta,\, v^\delta,\, \phi_\delta,\, R^\delta,\, R^{\Gamma,\delta}) \in \mathcal{B}(T,\, \Omega,\, \alpha_\delta,\, N-1,\, \Psi,\, \Psi^\Gamma)$ to $(P_{\tau=0,\sigma=0,\delta})$.

For the passage to the limit $\delta \to 0$ the reasoning is the same. The second of the bounds (52) is available. Since there is no perturbation $\bar{J}^\delta$ in the mass conservation equation, the Proposition 3.14 guaranties at once the uniform in time compactness in $C([0,T];\, L^1(\Omega))$ of $\{\varrho_\delta\}_{\delta>0}$, and the Proposition 3.12 guaranties the convergence to a weak solution to (P).

It remains to discuss the case of Theorem 2.2 (Local-in-time existence). Due to (37), $[\bar{\rho}^{\sigma,\delta}]_{C_{\Phi^*}([0,T];\,\mathbb{R}^N)} \leq C_0$. We can extract sub-sequences such that $\rho^{\sigma,\delta}$ converges weakly in $L^\alpha(Q)$, and $\bar{\rho}^{\sigma,\delta}$ converges uniformly on $[0,T]$. We define a time $T^*_{\sigma,\delta}$ via

$$T^*_{\sigma,\delta} = \inf\{t \in [0,T[:\, \inf_{i=1,\ldots,N} \bar{\rho}_i^{\sigma,\delta}(t) = 0\}\,.$$

We know that $T^*_{\sigma,\delta} \geq T_0 > 0$ where T_0 is fixed by the data (cf. (36)). At first we can extract a subsequence such that $T^*_{\sigma,\delta} \to T^*$. Due to the continuity of $\bar{\rho}$, we see that $0 = \inf \bar{\rho}^{\sigma,\delta}(T^*_{\sigma,\delta}) \to \inf \bar{\rho}(T^*)$.

Consider now $T' \in [0,\, T^*[$ arbitrary. Then, for all $\sigma \leq \sigma_0(T^* - T')$, and $\delta \leq \delta_0(T^* - T')$, we establish the estimates (52) with T replaced by T'. We then finish the proof as for Theorem 2.1 with T replaced by T'. By definition, we now have

$$\lim_{T' \to T^*} \min_{i=1,\ldots,N} \bar{\rho}_i(T') = 0\,.$$

Thus, there must exist an index i_1 such that $\bar{\rho}_{i_1}(T^*) = 0$. For $t < T^*$, we then consider the function $\hat{q}_{i_1} = \mu_{i_1} - \max_{i=1,\ldots,N} \mu_i \leq 0$. We can introduce constants

$$\bar{a}_0 := \frac{1}{2|\Omega|} \int_\Omega \varrho_0, \quad \bar{b}_0 = \left(\frac{|\Omega|}{2\|\varrho\|_{L^{\infty,\alpha}(Q)}} \int_\Omega \varrho_0\right)^{\alpha'} \tag{54}$$

and show that the set $A_0(t) := \{x \in \Omega \,:\, \varrho(t,x) \geq \bar{a}_0\}$ satisfies $\lambda_3(A_0(t)) \geq \bar{b}_0$ for all $t \in]0,T[$. Since $\lambda_3(\{x \in \Omega \,:\, \varrho(t,x) \geq k\}) \leq \frac{C_0}{k}$, we easily construct a set $A_1(t) := \{x \in \Omega \,:\, \bar{a}_1 \geq \varrho(t,x) \geq \bar{a}_0\}$ such that $\lambda_3(A_1(t)) \geq \frac{\bar{b}_0}{2}$ for all $t \in]0,T[$. Now observe that $x \in A_1(t)$ implies $|F'(\frac{V}{m} \cdot \rho(t,x))| \leq C(F,\, \bar{a}_0,\, \bar{a}_1)$. Thus for $t \in]0,T[$ and $x \in A_1(t)$

$$\tfrac{k_B\,\theta}{\overline{m}} \ln \tfrac{1}{N} - |\tfrac{V}{m}|\, C(F,\, \bar{a}_0,\, \bar{a}_1) - \|c\|_\infty \leq \max_{i=1,\ldots,N} \mu_i \leq |\tfrac{V}{m}|\, C(F,\, \bar{a}_0,\, \bar{a}_1) + \|c\|_\infty\,.$$

For $t \in]0,T[$ and $x \in A_1(t)$ it follows that $\hat{q}_{i_1}(t,x) \leq \frac{k_B\,\theta}{m_{i_1}} \ln \rho_{i_1}(t,x) + \tilde{C}(F,\, \bar{a}_0,\, \bar{a}_1)$ for $t \in]0,T[$ and $x \in A_1(t)$. Due to the Jensen inequality

$$\tfrac{1}{\lambda_3(A_1(t))} \int_{A_1(t)} |\hat{q}_{i_1}| \geq \tfrac{k_B\,\theta}{m_{i_1}} \ln \frac{1}{\frac{1}{\lambda_3(A_1(t))} \int_{A_1(t)} \rho_{i_1}} - \tilde{C}\,.$$

In this way we easily see that $\liminf_{t \to T^*} \|\hat{q}_{i_1}(t)\|_{L^1(\Omega)} = +\infty$.

□

Due to Proposition 4.1 it is sufficient to prove the solvability of the problem $(P_{\tau,\sigma,\delta})$ in order to complete the proof of the existence Theorems. We are going to carry over this last step by means of a Galerkin approximation described hereafter.

5 Galerkin approximation for $(P_{\tau,\sigma,\delta})$

We choose

(1) A countable, linearly independent system $\eta^1,\, \eta^2, \ldots \in W_0^{1,\infty}(\Omega;\, \mathbb{R}^3)$ dense in $W_0^{1,2}(\Omega;\, \mathbb{R}^3)$ in order to approximate the variable v;

(2) A countable, linearly independent system $\zeta^1,\, \zeta^2, \ldots \in W_\Gamma^{1,\infty}(\Omega)$ dense in $W_\Gamma^{1,2}(\Omega)$ in order to approximate the variable ϕ;

In order to approximate the variables μ we need a countable system $\psi^1,\, \psi^2, \ldots$ of the space $W^{1,\infty}(\Omega;\, \mathbb{R}^N)$ dense in $W^{1,2}(\Omega;\, \mathbb{R}^N)$. For technical reasons, we have to require additional properties of this set. For $n \in \mathbb{N}$, and $i, j \in \{1, \ldots, n\}$ such that $i \leq j$, we introduce the functions $\tilde{\eta}^{i,j} = \eta^i \cdot \eta^j$ with $\eta^1, \ldots, \eta^n$ from (1). By means of an obvious renumbering, we denote these functions $\tilde{\eta}^s$ for $s = 1, \ldots, n\,(n+1)/2$. For all $n \in \mathbb{N}$, we assume that there is $p = p(n) > n$ such that the following additional conditions are valid

$$\begin{cases} \mathbb{1} \in \operatorname{span}\{\psi^1, \ldots, \psi^p\} & \\ \tilde{\eta}^s\, \mathbb{1} \in \operatorname{span}\{\psi^1, \ldots, \psi^p\} & \text{for all } s = 1, \ldots, n\,(n+1)/2 \\ \phi_0\, \frac{z}{m},\, \zeta^s\, \frac{z}{m} \in \operatorname{span}\{\psi^1, \ldots, \psi^p\} & \text{for all } s = 1, \ldots, n \end{cases} \tag{55}$$

Obvious corollaries of this property are

$$\begin{cases} v \in \operatorname{span}\{\eta^1, \ldots, \eta^n\} \implies |v|^2 \in \operatorname{span}\{\psi^1, \ldots, \psi^{p(n)}\} \\ \tilde{\phi} \in \operatorname{span}\{\zeta^1, \ldots, \zeta^n\} \implies (\tilde{\phi} + \phi_0)\, \frac{z}{m} \in \operatorname{span}\{\psi^1, \ldots, \psi^{p(n)}\} \end{cases} \tag{56}$$

For $n \in \mathbb{N}$, we are looking for approximate solutions

$$\begin{aligned} \mu^n \in C^1([0,T];\, W^{1,\infty}(\Omega;\, \mathbb{R}^N)) \quad v^n \in C^1([0,T];\, W_0^{1,\infty}(\Omega;\, \mathbb{R}^3)) \\ \phi_n \in C^1([0,T];\, W^{1,\infty}(\Omega)) \end{aligned} \tag{57}$$

following the ansatz

$$\mu^n = \sum_{\ell=1}^{p(n)} a_\ell(t)\, \psi^\ell(x), \quad v^n = \sum_{\ell=1}^{n} b_\ell(t)\, \eta^\ell(x), \quad \phi_n = \phi_0 + \sum_{\ell=1}^{n} c_\ell(t)\, \zeta^\ell(x)\,. \tag{58}$$

where the vector fields $a = a^{(n)} \in C^1([0,T];\, \mathbb{R}^p)$, $b = b^{(n)} \in C^1([0,T];\, \mathbb{R}^n)$ and $c = c^{(n)} \in C^1([0,T];\, \mathbb{R}^n)$ are to determine.

Our approximation scheme is $(P_{\tau,\sigma,\delta})$ as described in the Section 5 of [DDGG17b]. We project this scheme on the Galerkin space. In order to state approximate equations, we need for $i = 1, \ldots, N$ the free energy functions $h_{\tau,\delta}$. In this point we introduce the abbreviation

$$\mathscr{R}^*(\mu) := \nabla h^*_{\tau,\delta}(\mu) = \nabla (h_\delta)^*(\mu) + \tau\, \omega'(\mu)\,. \tag{59}$$

In order to approximate the equations (1), we consider for $s \in \{1, \ldots, p(n)\}$ the equations

$$\begin{aligned}\int_\Omega \partial_t \mathscr{R}^*(\mu^n) \cdot \psi^s = & \int_\Omega ((\mathscr{R}^*(\mu^n)\, v^n + J^n) \cdot \nabla\psi^s + r(\mu^n) \cdot \psi^s) \\ & + \int_\Gamma (\hat{r}(\mu^n) + J^0) \cdot \psi^s \\ J^{n,i} = & e^i \cdot M(\mathscr{R}^*(\mu^n))\, (\nabla \mu^n + \frac{z}{m}\, \nabla\phi_n)\,.\end{aligned} \tag{60}$$

Introduce a Matrix-valued mapping $\mu \mapsto A^1(\mu) = \{a_{i,j}(\mu)\}_{i,j=1,\ldots,p(n)}$ via

$$a_{i,j}(\mu) := \int_\Omega \mathscr{R}^*_{\ell,\mu_s}(\mu)\, \psi^j_\ell\, \psi^i_s = \int_\Omega (D^2_{l,s} h^*_\delta(\mu) + \tau\, \omega''(\mu_s)\, \delta_{s,\ell})\, \psi^j_\ell\, \psi^i_s\,. \tag{61}$$

Owing to the convexity of h^*_δ and of the function ω, we see that $A^1(\mu)$ is symmetric and positive semi definite. Due to the ansatz (58) for μ^n, we can now express (60) in the equivalent form

$$\begin{aligned}& A^1(\mu^n(t))\, a'(t) = F^1(a(t),\, b(t),\, c(t)) \\ & F^1_s := \int_\Omega (\mathscr{R}^*(\mu^n)\, v^n + J^n) \cdot \nabla\psi^s + \int_\Omega r(\mu^n) \cdot \psi^s + \int_\Gamma (\hat{r}(\mu^n) + J^0) \cdot \psi^s\,.\end{aligned}$$

In order to approximate the equations (2), we consider for $s \in \{1, \ldots, n\}$ the equations

$$\begin{aligned}\int_\Omega \mathscr{R}^*(\mu^n) \cdot \mathbb{1}\, \partial_t v^n \cdot \eta^s = & - \int_\Omega \mathscr{R}^*(\mu^n) \cdot \mathbb{1}\, (v^n \cdot \nabla) v^n \cdot \eta^s + \int_\Omega h^*_{\tau,\delta}(\mu^n)\, \operatorname{div} \eta^s \\ & - \int_\Omega \mathbb{S}(\nabla v^n) \cdot \nabla \eta^s - \int_\Omega (\sum_{i=1}^N J^{n,i} \cdot \nabla) v^n \cdot \eta^s - \int_\Omega \frac{z}{m} \cdot \mathscr{R}^*(\mu^n)\, \nabla\phi_n \cdot \eta^s\,.\end{aligned} \tag{62}$$

Introduce a matrix-valued mapping $\mu \mapsto A^2(\mu) = \{a^{(2)}_{i,j}(\mu)\}_{i,j=1,\ldots,n}$

$$a^{(2)}_{i,j}(\mu) := \int_\Omega \mathscr{R}^*(\mu) \cdot \mathbb{1}\, \eta^i \cdot \eta^j = \int_\Omega (\nabla h^*_\delta(\mu) + \tau\, \omega'(\mu^n)) \cdot \mathbb{1}\, \eta^i \cdot \eta^j\ . \tag{63}$$

Owing to the non negativity of ∇h^*_δ and of ω', we see that $A^2(\mu)$ is symmetric and positive semi definite. Due to the ansatz (58) for v^n and μ^n, we can express (62) in the equivalent form

$$\begin{aligned}& A^2(\mu^n(t))\, b'(t) = F^2(a(t),\, b(t),\, c(t)) \\ & F^2_s := - \int_\Omega \mathscr{R}^*(\mu^n) \cdot \mathbb{1}\, (v^n \cdot \nabla) v^n \cdot \eta^s + \int_\Omega h^*_{\tau,\delta}(\mu^n)\, \operatorname{div} \eta^s \\ & - \int_\Omega \mathbb{S}(\nabla v^n) \cdot \nabla\eta^s + \int_\Omega (\sum_{i=1}^N J^{n,i} \cdot \nabla) \eta^s \cdot v^n - \int_\Omega \frac{z}{m} \cdot \mathscr{R}^*(\mu^n)\, \nabla\phi^n \cdot \eta^s\,.\end{aligned}$$

In order to determine ϕ_n, we use the ansatz $\phi_n = \tilde{\phi}_n + \phi_0$ and we consider the projection onto $\operatorname{span}\{\zeta^1, \ldots, \zeta^n\}^*$ of the Poisson equation, that is

$$\epsilon_0\, (1 + \chi) \int_\Omega \nabla\tilde{\phi}_n \cdot \nabla\zeta^i = -\epsilon_0\, (1 + \chi) \int_\Omega \nabla\phi_0 \cdot \nabla\zeta^i + \int_\Omega \frac{z}{m} \cdot \mathscr{R}^*(\mu^n)\, \zeta^i\,. \tag{64}$$

We make use of the ansatz (58) for ϕ_n, and we see that the vector $c_1, \ldots c_n$ can be determined via for a linear system $A\,c = f$ where

$$A_{i,j} := \epsilon_0\,(1+\chi)\int_\Omega \nabla\zeta^i \cdot \nabla\zeta^j \text{ for } i,j = 1,\ldots,n$$
$$f_i := -\epsilon_0\,(1+\chi)\int_\Omega \nabla\phi_0 \cdot \nabla\zeta^i + \int_\Omega \frac{z}{m}\cdot \mathscr{R}^*(\mu^n)\,\zeta^i \text{ for } i = 1,\ldots,n\,.$$

Since the matrix A is by assumption invertible, we obtain that $c = A^{-1}f =: \tilde{f}(a)$.

Overall, the Galerkin approximation (60), (62), (64) has the form

$$\begin{pmatrix} A^1(a(t)) & 0 \\ 0 & A^2(a(t)) \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} F^1(a(t),\, b(t),\, \tilde{f}(a(t))) \\ F^2(a(t),\, b(t),\, \tilde{f}(a(t))) \end{pmatrix} \tag{65}$$

We consider the initial conditions

$$a(0) = a^{0,n} \in \mathbb{R}^p, \quad b(0) = b^{0,n} \in \mathbb{R}^n\,. \tag{66}$$

Here we require for the reason of consistency that

$$\mu^{0,n} := \sum_{\ell=1}^{p(n)} a_\ell^{0,n}\,\psi^\ell \to \mu^0 := \nabla_\rho h_{\tau,\delta}(\rho^0) \text{ in } L^1(\Omega;\,\mathbb{R}^N)$$
$$v^{0,n} = \sum_{\ell=1}^{n} b_\ell^{0,n}\,\eta^\ell \to v^0 \text{ in } L^1(\Omega;\,\mathbb{R}^3) \text{ as } n\to\infty\,.$$

We moreover assume that

$$\|\mu^{0,n}\|_{L^\infty(\Omega)} \le C_0\,,$$

which by definition also yields for $i = 1,\ldots,N$

$$\rho_i^{0,n} := \mathscr{R}_i^*(\mu^{0,n}) \ge c_0 > 0 \text{ everywhere in } \Omega\,. \tag{67}$$

At first we can obtain local existence for the problem (65), (66).

Proposition 5.1. *There is $\epsilon = \epsilon(n,\, a^{0,n},\, b^{0,n})$ such that the problem* (65)*,* (66) *possesses a solution in $C^1([0,\,\epsilon];\,\mathbb{R}^p\times\mathbb{R}^n)$.*

Proof. Recall (67). Consider the matrix $A^1(\mu^0)$ (cf. (61))

$$A^1_{i,j}(\mu^0) = \int_\Omega D^2_{\ell,s}h^*_{\tau,\delta}(\mu^0)\,\psi^j_\ell\,\psi^i_s\,.$$

Owing to the strict convexity of $h^*_{\tau,\delta}$ on compact sets, $A^1(\mu^0)$ is positive definite and therefore invertible, and $\|[A^1(\mu^0)]^{-1}\| \le C(a^0,\,n)$. The matrix $A^2(\mu^0)$ (cf. (63)) is uniformly invertible because $\nabla h^*_{\tau,\delta}$ is strictly positive on compact sets, and $\|[A^2(\mu^0)]^{-1}\| \le C(a^0,\,n)$.

The system matrix A in (65) satisfies $\det A = \det A^1\,\det A^2$. Thus, A is invertible at $a^0,\,b^0$, and standard perturbation arguments yield the claim. □ □

Next we want establish a continuation property for the solution, and we need *a priori* estimates.

Proposition 5.2. *Assume that the approximate system* (65), (66) *possesses a solution* $(a,\, b) \in C^1([0, T^*[;\, \mathbb{R}^p \times \mathbb{R}^n)$ *for a* $T^* > 0$*. Then,* μ^n*,* v^n *and* ϕ_n *satisfy the dissipation inequality with free energy* $h_{\tau,\delta}$ *and mobility matrix* M_σ*.*

Proof. We apply the ideas of Proposition 6.1 in [DDGG17b]. We can multiply (60) with μ^n. Due to the additional property (55) and to (56) on the system $\{\psi^1, \ldots, \psi^p\}$, we can also multiply (60) with $\frac{z}{m}\, \phi_n$.

Second, we multiply (62) with v^n. Due again to the additional property (55) and to (56) we can also choose $|v^n|^2\, \mathbb{1}$ as a test function in (60) to obtain that the perturbation $\mathbb{1} \cdot J$ vanishes. The claim follows. □ □

Next we verify a continuation criterion.

Proposition 5.3. *Assumptions of Proposition 5.2. Then* $\|\mu^n\|_{L^\infty([0,T^*]\times\Omega)} + \|v^n\|_{L^\infty([0,T^*]\times\Omega)} + \|\phi_n\|_{L^\infty([0,T^*]\times\Omega)} \leq C(n)$*.*

Proof. The bound $\|\mathscr{R}^*(\mu^n)\|_{L^{\infty,\alpha}(Q_{T^*})} \leq C_0$ also yields $\|\mathscr{R}^*(\mu^n)\|_{L^\infty(Q_{T^*})} \leq C(n)$. The reason is that the set $M := \mathscr{R}^*(\mathrm{span}\{\psi^1, \ldots, \psi^{p(n)}\})$ as a subset of $L^1(\Omega;\, \mathbb{R}^N)$ is parametrised by a finite dimensional linear space. Thus, there exists a constant c_M such that $\|u\|_{L^1(\Omega)} \geq c_M\, \|u\|_{L^\infty(\Omega)}$ for all $u \in M$.

We want to obtain a L^∞ bound for μ^n. By construction, for $t \in]0, T^*[$ arbitrary,

$$c\, \tau \sum_{i=1}^N \int_\Omega \sqrt{|\mu_i^n(t)|} \leq \tau \int_\Omega \Phi_\omega(\mu^n) \leq C_0\,.$$

Now we prove: There is $c = c(n)$ such that $|x|_{L^\infty}^{1/2} \leq c\, \||x \cdot \psi|^{1/2}\|_{L^1(\Omega)}$ for all $x \in \mathbb{R}^p$. Otherwise there is for each $j \in \mathbb{N}$ a $x^j \in \mathbb{R}^p$ such that $|x^j|_\infty^{1/2} \geq j\, \||x^j \cdot \psi|^{1/2}\|_{L^1(\Omega)}$. Thus $\||\bar{x}^j \cdot \psi|^{1/2}\|_{L^1(\Omega)} \leq j^{-1}$ with $\bar{x}^j = x^j/|x^j|_{L^\infty}$. For a subsequence, $\bar{x}^j \to \bar{x}$ in $\mathbb{R}^p$, $|\bar{x}|_\infty = 1$. But since $\||\bar{x} \cdot \psi|^{1/2}\|_{L^1(\Omega)} = 0$, we obtain that $\bar{x} \cdot \psi = 0$ in Ω, and due to the choice of the system $\{\psi^1, \ldots, \psi^p\}$, it follows that $\bar{x} = 0$, a contradiction.

It follows that

$$\|\mu^n(t)\|_{L^\infty(\Omega)}^{1/2} \leq k(n)\, |a(t)|_\infty^{1/2} \leq k(n)\, c(n)\, \||\mu^n(t)|^{1/2}\|_{L^1(\Omega)} \leq \frac{C(n)}{\tau}\, C_0$$

and this implies that $\|\mu^n\|_{L^\infty([0,T^*]\times\Omega)} \leq C(n)$. The properties of $\mathscr{R}^*$ entail

$$\inf_{i=1,\ldots,N}\, \inf_{[0,T^*]\times\Omega} \mathscr{R}_i^*(\mu^n) \geq c(n) > 0\,.$$

From the bound $\int_\Omega \mathscr{R}^*(\mu^n(t)) \cdot \mathbb{1}\, |v^n(t)|^2 \leq C_0$, we obtain that $\|v^n\|_{L^\infty([0,T^*]\times\Omega)} \leq C_0\, c(n)^{-1}$. Analogously, $\int_\Omega |\nabla \phi_n(t)|^2 \leq C_0$ implies that $\|\nabla \phi_n\|_{L^\infty([0,T^*]\times\Omega)} \leq C(n)$, and since $\phi_n = \phi_0$ on $[0, T^*] \times \Gamma$, the claim follows. □ □

Corollary 5.4. *Let* $T > 0$*. Then, the approximate system* (65), (66) *possesses a solution* $(a,\, b) \in C^1([0, T];\, \mathbb{R}^p \times \mathbb{R}^n)$*.*

Proof. Owing to the Proposition 5.1, there is $T^* > 0$ such that (65), (66) possesses a solution $(a,\, b) \in C^1([0,\, T^*];\, \mathbb{R}^p \times \mathbb{R}^n)$. Since $\|\mu^n\|_{L^\infty([0,T^*]\times\Omega)} \leq C(n)$, it follows from the properties of $\mathscr{R}^*$ that $\inf_{i=1,\ldots,N} \inf_{[0,T^*]\times\Omega} \mathscr{R}_i^*(\mu^n) \geq c(n)$.

The matrix $A^1(\mu^n(t))$ (cp. (61)) is invertible for all $t \in [0,\, T^*]$, and $\|[A^1(\mu^n(t))]^{-1}\| \leq C(n)$. The matrix $A^2(\mu^n(t))$ (cf. (63)) is uniformly invertible, and the norm of the inverse satisfies a uniform bound $\|[A^2(\mu^n(t))]^{-1}\| \leq C(n)$ on $[0, T^*]$. Due to the Proposition 5.3, the functions $\mu^n(T^*)$, $v^n(T^*)$ and $\phi_n(T^*)$ belong to $L^\infty(\Omega)$ and their norm in this space is bounded independently on t.

Thus, the problem (65), with initial data $(a(T^*), b(T^*))$ possesses solution in an interval $[T^*,\, T^* + \epsilon(n)]$, and the claim follows reiterating this argument. □ □

Proposition 5.5. *Let $n \in \mathbb{N}$ and $T > 0$. The Galerkin approximation* (60)*,* (62)*,* (64)*, possesses a solution with the regularity* (57) *such that the dissipation inequality is valid with free energy function $h_{\tau,\delta}$ and mobility matrix M_σ.*

Uniform estimates We define

$$\rho^n := \mathscr{R}^*(\mu^n) = \nabla(h_\delta)^*(\mu^n) + \tau\, \omega'(\mu^n), \quad p_n := h^*_{\tau,\delta}(\mu^n)\,.$$

The approximate vector of total masses $\bar{\rho}^n \in C^1([0,T];\, \mathbb{R}^N)$ defined via $\bar{\rho}^n(t) = \int_\Omega \rho^n(t)$ satisfies by assumption $\bar{\rho}^n(0) \to \rho^0$ for $n \to \infty$. Therefore, for every $\epsilon > 0$ we find $n_0(\epsilon)$ such that for all $n \geq n_0$

$$\bar{\rho}^n(t) \in B_\epsilon(\bar{\rho}^0) \oplus W = B_\epsilon(\bar{\rho}^0) \oplus \operatorname{span}\{\gamma^1, \ldots, \gamma^s,\, \hat{\gamma}^1, \ldots, \hat{\gamma}^{\hat{s}^\Gamma}\}\,.$$

For $\epsilon \leq \frac{1}{2}\, \operatorname{dist}(\bar{\rho}^0,\, \mathcal{M}_{\text{crit}})$, the bounds (37), (38), (39), (40) are therefore valid.

Thus $[(\mu^n,\, v^n,\, \phi_n)]_{\mathcal{B}(T,\Omega,\alpha,N,\Psi,\Psi^\Gamma)} \leq C(\delta,\, \sigma,\, \tau,\, \mathcal{B}_0)$.

Passage to the limit $n \to \infty$ Due to the condition (55), we can multiply the equations (60) with $\psi = v^n \cdot \eta^s\, \mathbb{1}$, $s \in \{1, \ldots, n\}$ arbitrary. We obtain that

$$\int_\Omega \partial_t \varrho_n\, v^n \cdot \eta^s - \int_\Omega \varrho_n\, v^n \cdot \nabla(v^n \cdot \eta^s) = \int_\Omega (\mathbb{1} \cdot J^n) \cdot \nabla(v^n \cdot \eta^s)\,.$$

Thus, it follows that

$$\begin{aligned}\int_\Omega \partial_t(\varrho_n\, v^n) \cdot \eta^s - \int_\Omega \varrho_n\, \partial_t v^n \cdot \eta^s - \int_\Omega \varrho_n\, (v^n \cdot \nabla) v^n \cdot \eta^s \\ - \int_\Omega \varrho_n\, (v^n \otimes v^n) \,:\, \nabla \eta^s = \int_\Omega (\mathbb{1} \cdot J^n) \cdot \nabla(v^n \cdot \eta^s)\,.\end{aligned}$$

Rearranging terms

$$\begin{aligned}\int_\Omega \partial_t(\varrho_n\, v^n) \cdot \eta^s - \int_\Omega \varrho_n\, (v^n \otimes v^n) \,:\, \nabla \eta^s - \int_\Omega (\mathbb{1} \cdot J^n) \cdot \nabla(v^n \cdot \eta^s) \\ = \int_\Omega \varrho_n\, (\partial_t v^n + (v^n \cdot \nabla) v^n) \cdot \eta^s\,.\end{aligned}$$

Making use of the latter identity and of (62)

$$\begin{aligned}\int_\Omega \partial_t(\varrho_n\, v^n) \cdot \eta^s - \int_\Omega \varrho_n\, (v^n \otimes v^n) \cdot \nabla \eta^s = \int_\Omega p_n\, \operatorname{div} \eta^s - \int_\Omega \mathbb{S}(\nabla v^n) \cdot \nabla \eta^s \\ + \int_\Omega (\sum_{i=1}^N J^{n,i} \cdot \nabla) \eta^s \cdot v^n - \int_\Omega \tfrac{z}{m} \cdot \rho^n)\, \nabla \phi_n \cdot \eta^s\,. \qquad (68)\end{aligned}$$

Due to the identities (60) and (68) we obtain for all $t \in [0,T]$ the representation

$$\begin{pmatrix} \int_\Omega \rho^n(t)\cdot\psi \\ \int_\Omega \varrho_n(t)\, v^n(t)\cdot\eta \end{pmatrix} = \begin{pmatrix} \int_\Omega \rho^0\cdot\psi \\ \int_\Omega \varrho_0(t)\, v^0(t)\cdot\eta \end{pmatrix} + \begin{pmatrix} \int_0^t\int_\Omega \sum_{j=0,1} \mathscr{L}^{1,j}(\mathcal{A}^n)\cdot D^j\psi \\ \int_\Omega \sum_{j=0,1} \mathscr{L}^{2,j}(\mathcal{A}^n)\cdot D^j\eta \end{pmatrix}$$

for all $t \in [0,T]$ and for all $(\psi,\,\eta) \in \mathrm{span}\{\psi^1,\ldots,\,\psi^{p(n)}\} \times \mathrm{span}\{\eta^1,\ldots,\eta^n\}$.

Here $\mathscr{L}i,j(\mathcal{A}^n)$ are linear combinations in $\mathcal{A}$ naturally defined by the right-hands of (60) and (62). Since the systems $\mathrm{span}\{\psi^1,\ldots,\,\psi^{p(n)}\}$ and $\mathrm{span}\{\eta^1,\ldots,\eta^n\}$ are dense in C^1 for $n\to\infty$, we easily show that there is a subsequence such that $\rho^n(t)$ and $\varrho_n(t)\,v^n(t)$ converge as distributions for all $t\in]0,T[$. Thus, the conclusions of Lemma 3.2 are valid and we can produce a limit element $(\mu,\,v,\,\phi)\in\mathcal{B}(T,\,\Omega,\,\alpha,\,N,\,\Psi,\,\Psi^\Gamma)$.

In order to obtain the strong convergence of the sequence, we make use of the estimates valid for the class $\mathcal{B}(T,\,\Omega,\,\alpha,\,N,\,\Psi,\,\Psi^\Gamma)$ for fixed $\tau,\,\sigma>0$, and a variant of Corollary 3.7. Recall the abbreviation (59). We show that for all $\epsilon>0$, there are $C_\epsilon>0$ and $m_\epsilon\in\mathbb{N}$ such that for all $w^1,\,w^2\in W^{1,1}(\Omega;\,\mathbb{R}^N)$

$$\begin{aligned}\|\mathscr{R}^*(w^1)-\mathscr{R}^*(w^2)\|_{L^1(\Omega)} \leq & \epsilon\,(1+\sup_{i=1,2}\|w^i\|_{W^{1,1}(\Omega;\,\mathbb{R}^N)}) \\ & + C_\epsilon\sum_{j=1}^{m_\epsilon}\left|\int_\Omega(\mathscr{R}^*(w^1)-\mathscr{R}^*(w^2))\cdot\phi^j\right|.\end{aligned}$$

Here $\phi^1,\,\phi^2,\ldots$ is a dense subset of $C_c(\Omega;\,\mathbb{R}^N)$. Then, we choose $w^1=\mu^n(t)$ and $w^2=\mu^{n+p}(t)$, and integrate over the interval $[0,T]$. After few straightforward steps, the inequality

$$\begin{aligned}&\|\mathscr{R}^*(\mu^n)-\mathscr{R}^*(\mu^{n+p})\|_{L^1([0,T]\times\Omega)} \\ &\leq \epsilon\,(T+C_0)+C_\epsilon\sum_{j=1}^{m_\epsilon}\int_0^T\left|\int_\Omega(\mathscr{R}^*(\mu^n)-\mathscr{R}^*(\mu^{n+p}))\cdot\phi^j\right|\end{aligned}$$

is attained. In view of Lemma 3.2, $\mathscr{R}^*(\mu^n(t))\to\rho(t)$ as distributions for almost all t. This yields

$$\limsup_{n\to\infty}\sup_{p\geq 0}\|\mathscr{R}(\mu^n)-\mathscr{R}(\mu^{n+p})\|_{L^1([0,T]\times\Omega)}\leq\epsilon\,(T+C_0)\,.$$

We conclude as in the proof of Corollary 3.7 that $\{\mathscr{R}(\mu^n)\}$ converges strongly in $L^1(Q;\,\mathbb{R}^N)$. Then, owing to the uniform bound $\|\mu^n\|_{L^2(Q)}\leq C_0$, we obtain that $\mu:=\lim_{n\to\infty}\mu^n$ exists almost everywhere in Q.

It remains to identify $(\mu,\,v,\,\phi)\in\mathcal{B}(T,\,\Omega,\,\alpha,\,N,\,\Psi,\,\Psi^\Gamma)$ as a weak solution to the problem $(P_{\tau,\sigma,\delta})$. The passage to the limit in the energy identity is unproblematic if it is relaxed to an inequality. The passage to the limit in the integral identities is also straightforward up to one instance: The sequence $\mathbb{1}\cdot J^n\otimes v^n$ satisfies a uniform bound only in $L^{1,\frac{3}{2}}(Q;\mathbb{R}^{3\times 3})$. However, recall that $\mathbb{1}\cdot J^n=\sigma\,(\nabla\sum_{i=1}^N\mu_i^n+\sum_{i=1}^N\frac{z_i}{m_i}\nabla\phi_n)$. Thus, for a test function $\zeta\in C_c^2(Q)$ and $k\in\{1,\ldots,N\}$, $\ell=1,2,3$

$$\int_Q\mathbb{1}\cdot J_k^n\,v_\ell^n\cdot\nabla\zeta=-\sigma\int_Q\sum_{i=1}^N\mu_i^n\,\partial_k(v_\ell^n\cdot\nabla\zeta)+\sigma\sum_{i=1}^N\frac{z_i}{m_i}\int_Q\partial_k\phi_n\,v_\ell^n\cdot\nabla\zeta\,.$$

Since $\mu^n \to \mu$ strongly in $L^2(Q)$, we then can show that

$$\int_Q \mathbb{1} \cdot J_k^n \, v_\ell^n \cdot \nabla\zeta \to -\,\sigma \int_Q \sum_{i=1}^N \mu_i \, \partial_k(v_\ell \cdot \nabla\zeta) + \sigma \, \sum_{i=1}^N \tfrac{z_i}{m_i} \int_Q \partial_k\phi \, v_\ell \cdot \nabla\zeta$$
$$= \int_Q \mathbb{1} \cdot J_k \, v_\ell \cdot \nabla\zeta \, .$$

Thus $\mathbb{1} \cdot J^n \otimes v^n \to \mathbb{1} \cdot J \otimes v$ as distributions.

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