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A comparison of delamination models: Modeling, properties, and applications

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Abstract

This contribution presents recent results in the modeling and the analysis of delamination problems. It addresses adhesive contact, brittle, and cohesive zone models both in a quasistatic and a viscous, dynamic setting for the bulk part. Also different evolution laws for the delaminating surface are discussed.

1 Continuum-mechanical modeling approaches to fracture

The creation and growth of cracks in a solid body corresponds to the formation of new (interior) surfaces. From a continuum-mechanical modeling point-of-view, the field variables describing the state of the body may display discontinuities along these crack surfaces. To mathematically model an elastic body with an evolving crack one may formulate the system of elasticity on a moving domain characterized by the propagation of the crack surface, which, in turn, has to be described by a suitable evolution law. Over the decades, in literature, many different criteria have been proposed to determine the unknown direction for the extension of an existing crack, famous among them are the maximum energy release rate criterion and the maximum stress criterion, see e.g. [58, 40] for a discussion. However, to handle the formation of discontinuities in general geometries in a mathematically rigorous way requires the formulation of the problem in adequate function spaces, such as $GSBV$-spaces ($GSBD$-spaces) of functions of generalized special bounded variations [4] (deformations [17]), as used e.g. in the Francort-Marigo model for quasistatic brittle fracture [23]. Instead, many models make regularizing assumptions on the discontinuity set. For this, often an additional internal variable is introduced in the spirit of generalized standard materials [29], with the purpose to describe the changes in the elastic behavior of the material caused by the evolving inelastic process. This is the basis for the phase field approach to fracture, where the phase field variable regularizes the model by replacing the lower dimensional crack *surface* by a damaged *volume* of controllable width, where the displacement field displays no discontinuities as associated with an elastic solid. Starting from the Ambrosio-Tortorelli phase field model, which was shown in [27] to approximate the Francort-Marigo model in the sense of Γ-convergence, phase field models for facture have received much attention both in mathematical analysis and numerical simulation, see e.g. [28, 12, 1, 55, 16, 15, 24]. Since they allow it to capture complicated geometric situations, such as crack initiation and branching, they have become a well-established method in a wide range of engineering applications, cf. e.g. [35, 34, 30, 33, 3, 54, 59].

In case of *compounds* of elastic solids, with applications ranging from laminates to layers of rocks and soil, fracture often occurs in terms of *delamination*, i.e. cracks form and propagate along the interface between two material layers. In such a situation of delamination, one may resort to models, where crack initiation and propagation are confined to a prescribed interface $\Gamma_{\mathrm{C}} \subset \mathbb{R}^{d-1}$ between two parts Ω_+, Ω_- of an elastic solid $\Omega \subset \mathbb{R}^d$. The domain of the internal variable, which then describes the state of the bonding along the interface, is consequently confined to Γ_{C}.

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This setup will be considered in the present contribution, which is devoted to the discussion of recent results in modeling and mathematical analysis of delamination processes. Section 2 explains the mathematical modeling at the example of *adhesive contact* and gives an overview on existence results developed in this field. In Section 3, the modeling and analytical challenges of adhesive contact are compared with those of *brittle delamination* and *cohesive zone delamination*.

2 Modeling of delamination processes via energy and dissipation functionals at the case of adhesive contact

Thermodynamically consistent modeling of adhesive contact via internal variables goes back to [25], cf. also [26]. Therein, the internal delamination variable $z : [0,T] \times \Gamma_{\rm C} \to [0,1]$ describes the state of the bonding along the interface $\Gamma_{\rm C}$ between the two parts $\Omega_+, \Omega_- \subset \mathbb{R}^d$ of the body, monitored during a finite time interval $[0,T]$. In what follows, $z(t,x) = 1$ will indicate that the bonding at the point $x \in \Gamma_{\rm C}$ at time $t \in [0,T]$ is fully intact, whereas $z(t,x) = 0$ means that the bonding is completely broken, and $z(t,x) \in (0,1)$ stands for an intermediate state of degradation. In this way, the set $C(t) := \{x \in \Gamma_{\rm C},\, z(t,x) = 0\}$ defines the crack at time $t \in [0,T]$.

In the isothermal setting at small strains the state of a viscoelastic body and an adhesive is thus described by the pair of state variables (u,z) consisting of the displacement field $u : [0,T] \times \Omega \backslash \Gamma_{\rm C} \to \mathbb{R}^d$ and the delamination variable $z : [0,T] \times \Gamma_{\rm C} \to [0,1]$. Their evolution is governed in terms of a tuple $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathscr{V}, \mathscr{K}, \mathscr{R}, \mathscr{E})$ given by suitable state spaces $\mathbf{V}, \mathbf{W}, \mathbf{Z}$, a stored energy $\mathscr{E}$, the kinetic energy $\mathscr{K}$, a dissipation potential $\mathscr{V}$ accounting for viscosity in the bulk, and a dissipation potential $\mathscr{R}$ accounting for the energy dissipated due to the delamination process.

The stored energy functional $\mathscr{E} := \mathscr{E}_{\rm bulk} + \mathscr{E}_{\rm adh}$ is composed of a bulk term defined on $\Omega \backslash \Gamma_{\rm C} = \Omega_+ \cup \Omega_-$ and a surface term defined on $\Gamma_{\rm C}$. Characteristically for adhesive contact models, which additionally also account for non-penetration of the material along $\Gamma_{\rm C}$, the latter is given as follows:

$$\mathscr{E}_{\rm adh} : \mathbf{V} \times \mathbf{Z} \to [0,\infty],\ \ \mathscr{E}_{\rm adh}(u,z) := \int_{\Gamma_{\rm C}} \big(\tfrac{k}{2} z \big| [\![u]\!] \big|^2 + I_{[0,1]}(z) + I_{[0,\infty)}([\![u]\!]_{\mathbf{n}}) \big) \, {\rm d}\mathscr{H}^{d-1} . \tag{1}$$

Here, $k > 0$ is fixed and $\mathscr{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff-measure. Furthermore, $[\![u]\!] := u_+ - u_-$ denotes the jump of the displacements across $\Gamma_{\rm C}$ calculated from the traces u_+, u_- along $\Gamma_{\rm C}$ of the functions $u|_{\Omega_+}, u|_{\Omega_-}$. In a similar manner, $[\![u]\!]_{\mathbf{n}} := [\![u]\!] \cdot \mathbf{n}$ is the jump of the displacements in normal direction to $\Gamma_{\rm C}$ with the normal defined as the outward normal to the subdomain Ω_+. The constraint $z \in [0,1]$ is ensured by the indicator function $I_{[0,1]} : \mathbb{R} \to \{0,\infty\}$ of the interval $[0,1]$, i.e., $I_{[0,1]}(z) = 0$ if $z \in [0,1]$ and $I_{[0,1]}(z) = \infty$ otherwise. Finally, due to the term $I_{[0,\infty)}([\![u]\!]_{\mathbf{n}})$, a finite surface energy $\mathscr{E}_{\rm adh}(u,z)$ corresponds to $[\![u]\!]_{\mathbf{n}} \geq 0$ a.e. on $\Gamma_{\rm C}$, and thus forbids the penetration of matter along $\Gamma_{\rm C}$.

For a linear elastic material the bulk energy may be considered as

$$\mathscr{E}_{\rm bulk} : [0,T] \times \mathbf{V} \to \mathbb{R},\ \ \mathscr{E}_{\rm bulk}(t,u) := \int_{\Omega \backslash \Gamma_{\rm C}} \big(\tfrac{1}{2} \mathbb{C} e(u) : e(u) - f(t) \cdot u \big) \, {\rm d}x \tag{2}$$

with $\mathbb{C} \in \mathbb{R}^{d \times d \times d \times d}$ a symmetric, positively definite fourth-order tensor and $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$ the small strain tensor. The evolution process is driven here by a time-dependent volume force f. The kinetic energy $\mathscr{K}$ is given by

$$\mathscr{K} : \mathbf{W} \to [0,\infty),\ \ \mathscr{K}(v) := \frac{1}{2} \|v\|_{\mathbf{W}}^2 \text{ with } \|v\|_{\mathbf{W}}^2 := \int_\Omega \rho |v|^2 {\rm d}x \tag{3}$$

for some given function $\rho \in L^\infty(\Omega)$ and a constant $\rho_* > 0$ such that $\rho > \rho_*$ a.e. in Ω. In this way, the space $\mathbf{W}$ is equivalent to $L^2(\Omega;\mathbb{R}^d)$.

The dissipation potential $\mathscr{V} : \mathbf{V} \to [0,\infty)$ accounts for viscosity of the bulk material and is here for simplicity assumed to be quadratic:

$$\mathscr{V} : \mathbf{V} \to [0,\infty), \ \ \mathscr{V}(v) := \int_{\Omega\setminus\Gamma_C} \tfrac{1}{2}\mathbb{D}e(v) : e(v)\,\mathrm{d}x, \tag{4}$$

again, with $\mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}$ a symmetric, positively definite fourth-order tensor. Assuming here for simplicity homogeneous Dirichlet conditions along the outer boundary $\partial\Omega$, in view of (2) and (4), it is $\mathbf{V} := \{v \in H^1(\Omega\setminus\Gamma_C),\, v = 0 \text{ a.e. on } \partial\Omega\}$.

The dissipation potential $\mathscr{R}$ controls how much energy is spent on the progression of delamination. In many materials, such as metals or rocks, fracture is a unidirectional progress, in the sense that cracks can only grow but not heal; this feature may be encoded in the dissipation potential by considering $\mathscr{R} : \mathbf{Z} \to [0,\infty]$,

$$\mathscr{R}(v) := \widetilde{\mathscr{R}}(v) + \alpha \int_{\Gamma_C} I_{(-\infty,0]}(v)\,\mathrm{d}\mathscr{H}^{d-1} \ \text{ with } \widetilde{\mathscr{R}}(v) := \int_{\Gamma_C} \widetilde{R}(v)\,\mathrm{d}\mathscr{H}^{d-1}, \tag{5}$$

where $\widetilde{R} : \mathbb{R} \to [0,\infty)$ is assumed to be convex with $\widetilde{R}(0) = 0$ and where $I_{(-\infty,0]} : \mathbb{R} \to \{0,\infty\}$ denotes the indicator function of $(-\infty,0]$ and α is a non-negative constant. Taking $v = \dot{z}$, in view on the modeling assumptions on z, this captures that a growth of delamination corresponds to a decrease of z in time (from the value 1 at most down to 0), so that, formally, its time-derivative always should be non-increasing.

Formally, the evolution of the state variables (u,z) is then captured by the system

$$\begin{aligned}
\rho\ddot{u} - \operatorname{div}\big(\mathbb{C}e(u) + \mathbb{D}e(\dot{u})\big) &= f \text{ in } [0,T]\times\Omega\setminus\Gamma_C, && \text{(6a)}\\
u &= 0 \text{ on } [0,T]\times\partial\Omega, && \text{(6b)}\\
\big[\!\big[\mathbb{C}e(u) + \mathbb{D}e(\dot{u})\big]\!\big]\cdot\mathbf{n} &= 0 \text{ on } [0,T]\times\Gamma_C, && \text{(6c)}\\
\big(\mathbb{C}e(u) + \mathbb{D}e(\dot{u})\big)\cdot\mathbf{n} + kz\big[\!\big[u\big]\!\big] + \partial_{[\![u]\!]} I_{[0,\infty)}\big(\big[\!\big[u\big]\!\big]_{\mathbf{n}}\big) &\ni 0 \text{ on } [0,T]\times\Gamma_C, && \text{(6d)}\\
\partial_{\dot{z}}\big(\widetilde{R}(\dot{z}) + \alpha I_{(-\infty,0]}(\dot{z})\big) + \tfrac{k}{2}\big|\big[\!\big[u\big]\!\big]\big|^2 + \partial_z I_{[0,1]}(z) &\ni 0 \text{ on } [0,T]\times\Gamma_C, && \text{(6e)}\\
\nabla z\cdot\mathbf{n} &= 0 \text{ on } [0,T]\times\Gamma_C, && \text{(6f)}
\end{aligned}$$

and supplemented by initial conditions for (u,z).

The indicator terms I_K of the convex intervals $K \in \{[0,1],(-\infty,0],[0,\infty)\}$ appearing in the functionals (1) and (5) are highly non-smooth, which is why (6d) and (6e) are given in terms of subdifferential inclusions and display the multivalued subdifferentials ∂I_K of the corresponding indicator function. Here, the subdifferential $\partial g(v)$ of a convex function $g : \mathbb{R} \to [0,\infty]$ at v is given by the set of all subderivatives v^*, i.e., $\partial g(v) := \{v^* \in \mathbb{R},\, g(w) - g(v) \geq v^*(w-v) \text{ for all } w \in \mathbb{R}\}$, which is the set of "slopes" defining tangents to g in the point v. In particular, if a function is classically differentiable in a point v, then the subdifferential is single-valued in v and coincides with the derivative in v. For example, the subdifferential of $I_{[0,1]}$ is given by

$$\partial I_{[0,1]}(z) = \begin{cases} \emptyset & \text{if } z \in (-\infty,0)\cup(1,\infty),\\ (-\infty,0] & \text{if } z = 0,\\ \{0\} & \text{if } z \in (0,1),\\ [0,\infty) & \text{if } z = 1 \end{cases}, \tag{7}$$

i.e., the subdifferential is multivalued and, in general, given by an unbounded set. Exactly this bears the main difficulty in the analysis of system (6) and suitable weak notions of solution in suitable functions spaces as well as refined analytical tools have to be established in order to deduce suitable bounds on each of the occuring terms. To be more precise, a first major difficulty comes with the momentum balance (6a) and the corresponding transmission condition (6d) along Γ_{C}. A weak formulation combines the acceleration term with elements of the unbounded subdifferential of the non-penetration constraint and, in general, bounds needed for the analysis can be deduced only on the sum of the two but not on each of the two terms separately. Many works dealing with adhesive contact and involving the non-penetration constraint thus resort to a physical setting with external loadings varying sufficiently slowly so that acceleration terms can be neglected, cf. e.g. [7, 8, 9, 10], or replace the non-penetration constraint by less restrictive conditions, cf. e.g. [43], where tangential slip is admissible. Without the acceleration term it is possible, in particular, to give a weak formulation of (6a)–(6d) in $L^2(0,T;\mathbf{V})$ and to obtain suitable bounds on all terms in this setting, however, as indicated, at the price of neglecting the acceleration term. Only recently, in a series of papers [11, 53, 51], it was indeed possible to handle dynamics in combination with the non-penetration constraint using a weak formulation in $H^1(0,T;\mathbf{V})$, instead. This opens the door for a number of additional, refined estimates, also using test functions, which incapacitate the element of the unbounded subdifferential, so that a separate estimate on the acceleration term becomes accessible. The approach of resorting to smaller spaces of admissible test functions, with properties tightly tailored to handle unbounded terms arising from non-smooth constraints seems to be the key to deal with rate-dependence and dynamics in combination with non-smooth constraints.

Let us now discuss the difficulties related to the flow rule (6e). In literature, two major cases can be distinguished for the density $\widetilde{R}$ involved in the dissipation potential $\mathscr{R}$ from (5): The case, where $\widetilde{R}$ contains a quadratic term and the case where $\widetilde{R}$ is positively 1-homogeneous. As a first step, let us assume for the moment that $\widetilde{R}$ is quadratic, i.e., $\widetilde{R}(v) := \frac{1}{2}|v|^2$. Then, we may choose the space $\mathbf{Z} := L^2(\Gamma_{\mathrm{C}})$ and the flow rule (6e) can be rewritten as follows:

$$\begin{aligned} \dot{z} + \tfrac{k}{2}\big|[\![u]\!]\big|^2 + \xi &\leq 0, && \text{(8a)}\\ \dot{z}(\dot{z} + \tfrac{k}{2}\big|[\![u]\!]\big|^2 + \xi) &= 0, && \text{(8b)}\\ \dot{z} &\leq 0, && \text{(8c)}\\ \xi &\in \partial_z I_{[0,1]}(z). && \text{(8d)} \end{aligned}$$

For $z = 1$, by (7), it is $\xi \in [0,\infty)$. Thus, as long as $\dot{z} = 0$, by (8), it has to hold $\frac{k}{2}\big|[\![u]\!]\big|^2 = \xi = 0$. Delamination sets in, i.e., $\dot{z} < 0$, if and only if $\xi = -(\dot{z} + \frac{k}{2}\big|[\![u]\!]\big|^2)$. As soon as a weakening of the bonds has taken place, so that $z \in (0,1)$, according to (7), it is $\xi = 0$ and then, again by (8), delamination remains static, i.e., $\dot{z} = 0$ as long as $\frac{k}{2}\big|[\![u]\!]\big|^2 = 0$ as well. In turn, delamination will progress any further, i.e., $\dot{z} < 0$, if and only if $\dot{z} = -\frac{k}{2}\big|[\![u]\!]\big|^2$. Once the final state of rupture $z = 0$ is reached, we see that also $\dot{z} = 0$ is feasible, because $\xi \leq -\frac{k}{2}\big|[\![u]\!]\big|^2$ is admissible for any value of $\frac{k}{2}\big|[\![u]\!]\big|^2$ thanks to $\xi \in (-\infty, 0]$ by (7). In other words, the temporal evolution (considered pointwise in Γ_{C}) can be interpreted to stop once the value $z = 0$ is reached. An evolution of adhesive contact of viscous kind has been treated e.g. in [52, 53] and also in [7, 9, 10], the latter works neglect unidirectionality, i.e., $\alpha = 0$ in (5) & (6e).

For comparison, let us now consider a positively 1-homogeneous density $\widetilde{R}$, i.e. $\widetilde{R}(v) = a|v|$ with a constant $a > 0$ and hence we now set $\mathbf{Z} := L^1(\Gamma_{\mathrm{C}})$. Clearly, this $\widetilde{R}$ is convex but not classically differentiable in $v = 0$ and its subdifferential in $v = 0$ is given by $\partial R(0) = [-a, a]$. In combination with

the unidirectionality constraint with $\alpha = 1$, we have that $R = a|v| + I_{(-\infty,0]}(v)$ in (5). Accordingly, (6e) amounts to the following conditions

$$\text{(8c) \& (8d) combined with} \qquad \tfrac{k}{2}\big|[\![u]\!]\big|^2 + \xi \le a\,, \tag{9a}$$

$$\text{and} \quad \dot z\big(\tfrac{k}{2}\big|[\![u]\!]\big|^2 + \xi - a\big) = 0\,. \tag{9b}$$

For $z = 1$, by (7), $\xi \in [0,\infty)$ is possible. Condition (9a) then imposes a threshold, below which $\dot z = 0$ has to hold in order to fulfill (9b). When the threshold is reached, i.e., $\frac{k}{2}\big|[\![u]\!]\big|^2 + \xi = a$, delamination may set in with $\dot z \le 0$. Once the bonds are weakened, $z \in (0,1)$, it is $\xi = 0$. Then delamination may propagate as soon as the displacement jump reaches the threshold $\frac{k}{2}\big|[\![u]\!]\big|^2 = a$. In the final state of rupture $z = 0$, (7) yields $\xi \in (-\infty, 0]$, so that $\frac{k}{2}\big|[\![u]\!]\big|^2 + \xi < a$ can be satisfied in (9a) for any value of $\frac{k}{2}\big|[\![u]\!]\big|^2$. Thus, $\dot z = 0$ must be assumed to fulfill (9b), and hence the temporal evolution (pointwise in Γ_{C}) can be interpreted to stop once the value $z = 0$ is reached. Comparing the evolution conditions (9) induced by a positively 1-homogeneous dissipation potential with (8) given by a quadratic potential, one observes the following: In (9) the attainment of the threshold is independent of the rate $\dot z$ and also (9b) only distiguishes between $\dot z = 0$ and $\dot z \neq 0$, but the particular values of the rate $\dot z$ do not play any role. In contrast, in the corresponding equations of (8) the rate $\dot z$ takes an explicit influence. In this way, a *quadratic* dissipation potential describes a *rate-dependent* evolution, whereas a *positively* 1*-homogeneous* dissipation potential governs a *rate-independent* evolution of the internal variable.

From a mathematical point-of-view rate-independence entails that formulation (6e), resp. (9), cannot provide any bounds on the rate $\dot z$ in Lebesgue-Bochner spaces. Indeed, one observes that solutions are measure-valued in time, so that, in fact, solutions may jump in time, see e.g. [38, 39, 56] for results on the temporal regularity of rate-independent processes with quasistatic evolution of u and also [47] for the coupled rate-dependent/rate-independent situation as in (6). Due to this lack of regularity in time, in general, formulations (6e) and (9) are not well-defined and, instead, one has to resort to an alternative notion of solution, which does not explicitly involve the time derivative. To motivate this alternative notion of solution, observe that (9b) states a power balance, while (9a) defines a local stability criterion of the form: $\langle \mathrm{D}_z \mathscr{E}(t,u(t),z(t)), v\rangle + \mathscr{R}(v) \ge 0$ for all suitable test functions v. Taking $v = \tilde z - z(t)$ and exploiting the convexity wrt. z of the energy functional $\mathscr{E} = \mathscr{E}_{\mathrm{bulk}} + \mathscr{E}_{\mathrm{adh}}$ with $\mathscr{E}_{\mathrm{adh}}$, one arrives at the *semistability* condition for all $t \in [0,T]$:

$$\text{For all test functions } \tilde z: \quad \mathscr{E}(t,u(t),z(t)) \le \mathscr{E}(t,u(t),\tilde z) + \mathscr{R}(\tilde z - z(t))\,. \tag{10}$$

Here, the notion of *semistability* indicates that (10) is a stability condition only for z, whereas u is determined by a weak formulation of the momentum balance (6a). In addition, formally, we may test the weak formulation of (6a) by $\dot u$. Note then, that the sum of this and the power balance (9b) state a power balance for the full system. Then, formally, we may integrate by parts in time to obtain an energy-dissipation balance for the system:

$$\begin{aligned} \mathscr{E}(t,u(t),z(t)) + \int_0^t \int_{\Omega\setminus\Gamma_{\mathrm{C}}} \mathbb{D}e(\dot u) : e(\dot u)\,\mathrm{d}x\,\mathrm{d}\tau + \mathrm{Var}_{\mathscr{R}}(0,t;z) \\ = \mathscr{E}(0,u(0),z(0)) + \int_0^t \partial_\tau \mathscr{E}(\tau,u(\tau,z(\tau))\,\mathrm{d}\tau\,, \end{aligned} \tag{11}$$

with $\mathrm{Var}_{\mathscr{R}}(0,t;z) := \sup_{\text{partitions of } [0,T]} \sum_i \mathscr{R}(z(t_{i+1}) - z(t_i))$ the total variation induced by $\mathscr{R}$, which defines an extended quasi-distance ('extended' because $\mathscr{R}(v) = \infty$ is possible, 'quasi', because, as in (5), $\mathscr{R}$ need not be symmetric). A pair (u,z) that satisfies a suitable weak formulation of the momentum balance together with semistability (10) and the energy-dissipation balance (11) is called a *semistable energetic solution* to the system defined by the tuple $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathscr{V},\mathscr{K},\mathscr{R},\mathscr{E})$. Results for adhesive contact, combined with further physical phenomena, based on this notion of solution can be found e.g. in [43, 44, 45, 46].

The notion of *semistable energetic solutions* for coupled rate-dependent/rate-independent systems was for the first time studied in [48] and analyzed in [47] in a general, abstract setting, allowing for non-convex energies and non-quadratic dissipation potentials $\mathscr{V}$ being convex and of general superlinear growth. Then, in general, (11) holds only as an inequality '$\le$' with $\int_0^t \int_{\Omega\setminus\Gamma_C} \mathbb{D}e(\dot u) : e(\dot u)\,\mathrm{d}x\,\mathrm{d}\tau$ replaced by $\int_0^t \big(\mathscr{V}(\dot u(\tau)) + \mathscr{V}^*(\rho\ddot u(\tau) - \mathrm{D}_u\mathscr{E}(\tau,u(\tau),z(\tau)))\big)\,\mathrm{d}\tau$ for $\mathscr{V}^*$ the convex conjugate of $\mathscr{V}$. In [48, 47] it is also proven that semistable energetic solutions to a system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathscr{V},\mathscr{K},\mathscr{R},\mathscr{E})$ can be obtained using a time-discrete scheme, alternating between u and z at each time-step. Indeed, also in the fully rate-independent case, i.e., when all rate-dependent terms are elimianted from the momentum balance so that the evolution of u is quasistatic, such an alternate time-discrete scheme, results in a similar notion of solution that combines the semistability inequality (10) and an energy-dissipation inequality with the condition that $u(t)$ be a minimizer for $\mathscr{E}(t,\cdot,z(t))$, see e.g. [50] in the setting of adhesive contact. In fact, it has also been observed in [32, 52] that a vanishing-viscosity-and-inertia limit for a system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathscr{V},\mathscr{K},\mathscr{R},\mathscr{E})$ results in a semistable energetic formulation of the corresponding rate-independent, quasistatic limit system. Instead, if one uses in the rate-independent, quasistatic setting a monolithic time-discrete scheme, which simultaneously minimizes $\mathscr{E}(t_i,\cdot,\cdot)+\mathscr{R}(\cdot - z_{i-1})$ in the pair (u,z), then one arrives at the well-studied notion of *energetic solutions*, which consists of a quasistatic energy-dissipation balance akin to (11) and of a global *stability condition*, i.e., $\mathscr{E}(t,u(t),z(t) \le \mathscr{E}(t,\tilde u,\tilde z) + \mathscr{R}(\tilde z - z(t))$ for all test functions $(\tilde u,\tilde z)$. See e.g. [36, 37] for abstract results on *energetic solutions* and [31] for adhesive contact. Energetic and semistable energetic solutions must not coincide: It is shown e.g. in [50] that energetic solutions tend to jump much earlier than semistable energetic solutions.

3 Adhesive contact, brittle, and cohesive zone-type delamination

The adhesive contact surface energy $\mathscr{E}_{\mathrm{adh}}$ from (1) allows for jumps $[\![u]\!]$ of the displacement field even in points of the interface where the bonds of the adhesive are not completely broken, i.e., $z \in (0,1)$. This may apply to a glue which allows the two parts of the body to slightly detach from each other, but not to a brittle material. In particular, brittle Griffith-type delamination, where the only surface energy is contributed by the energy dissipated due to crack-growth, cannot be modeled via $\mathscr{E}_{\mathrm{adh}}$ as long as the adhesive contact term $\int_{\Gamma_C} \frac{k}{2} z \big|[\![u]\!]\big|^2\,\mathrm{d}\mathscr{H}^{d-1}$ is involved. Delamination in a brittle material can rather be described by a surface energy involving the *brittle constraint* $I_{[z|[\![u]\!]|=0]}([\![u]\!],z)$, being $I_{[z|[\![u]\!]|=0]}([\![u]\!],z)=0$ if $z|[\![u]\!]|=0$ and $I_{[z|[\![u]\!]|=0]}([\![u]\!],z)=\infty$ otherwise. This constraint ensures that the displacements are continuous across Γ_C in points where no rupture has occurred yet, i.e., $z(t,x)\in(0,1]$ and allows the displacements to jump across the crack set $C(t)=\{x\in\Gamma_C, z(t,x)=0\}$. Denoting by $\mathscr{E}_{\mathrm{brittle}}$ the surface energy given as in (1), but with the adhesive contact term $\frac{k}{2}z\big|[\![u]\!]\big|^2$ replaced by the brittle constraint $I_{[z|[\![u]\!]|=0]}([\![u]\!],z)$, it looks intuitive that $\mathscr{E}_{\mathrm{adh}}$ approximates $\mathscr{E}_{\mathrm{brittle}}$ as $k\to\infty$. In this spirit it was shown in [49], in the quasistatic, rate-independent setting that *energetic solutions of an adhesive contact problem* approximate an *energetic solution of a brittle delamination problem* as $k\to\infty$. Analogous approximation results for *semistable energetic solutions* in the *quasistatic, rate-independent setting*, resp. in the *coupled rate-dependent/rate-independent setting* have been obtained in [50], resp. in [45] featuring non-penetration but $\rho=0$ in (6a), and in [46] for $\rho>0$ but neglecting non-penetration. Indeed, in the coupled rate-dependent/rate-independent setting the main challenge lies in the limit passage in the weak formulation of the momentum balance. For $z:[0,T]\times\Gamma_C\to[0,1]$ satisfying semistability (10) the brittle constraint imposes a non-smooth constraint (alike the non-penetration condition), but which addition-

ally *depends on the properties of* z. This is a clear increase of difficulty, because suitable test functions v_k have to be tailored to the properties of z_k, in such a way as to, on the one hand, prevent in the weak formulation of the momentum balance a blow-up of the term $\int_{\Gamma_C} kz_k(t)[\![u_k(t)]\!][\![v_k]\!]\,\mathrm{d}\mathscr{H}^{d-1}$ and, on the other hand, to recover all the information on the crack set of $z(t)$, being the semistable limit function at time $t \in [0,T]$ of the solutions $(z_k(t))_k$ of the adhesive problems. In particular [46] shows that the evolution of rate-independent brittle Griffith-type delamination in a visco-elastic solid with dynamic effects, can be described in terms of a weak form of the momentum balance, semistability (10), and an energy-dissipation balance. Thus, it contributes to the recent developments in the analytical understanding of dynamic fracture [18, 19, 20].

Alike adhesive contact, also cohesive zone models, pioneered e.g. in [22, 6, 42], regard fracture as a gradual phenomenon in which separation takes place across an extended crack 'tip', or cohesive zone. Thus, they also contain finite surface energy terms involving the displacement jump across Γ_C and, possibly an internal variable. Yet, therein, the internal variable ζ rather has the role to keep track of the history of the maximal opening displacement in normal direction. Thus, if ideally $\zeta(t,x) := \sup_{s\in[0,t]} [\![u(t,x)]\!]_{\mathbf{n}}$, it always has to hold $[\![u]\!]_{\mathbf{n}} \leq \zeta$. This can be enforced by the indicator $I_{[0,\zeta(t,x)]}$ of the interval $[0,\zeta(t,x)]$. Inspired by [41], to allow for a different loading and unloading behavior of the surface energy, [57] considers

$$\mathscr{E}_{\mathrm{coh}}(u,\zeta) := \int_{\Gamma_C} \Big(\frac{\phi_{\mathrm{coh}}(\zeta)}{2\zeta^2}\big|\,[\![u]\!]_{\mathbf{n}}\big|^2 + I_{[0,\zeta]}\big([\![u]\!]_{\mathbf{n}}\big) + I_{[0,\zeta^*]}(\zeta) + G(\zeta)\Big)\,\mathrm{d}\mathscr{H}^{d-1} \tag{12}$$

with ϕ_{coh} a typical cohesive energy as proposed in [41] and G a gradient term (quadratic or Sobolev-Slobodeckij-type). Due to its role, ζ monotonously increases in time. The indicator $I_{[0,\zeta^*]}$ then ensures that the model is meaningful as long as the maximum opening displacements do not exceed an utter maximum ζ^*. [57] shows the existence of semistable energetic solutions for this model in the coupled rate-dependent/rate-independent setting (without inertia); to accomodate the increasing nature of ζ, the rate-independent dissipation is of the form $R(v) := a|v| + I_{[0,\infty)}(v)$.

While [41] and consequently [57] consider the cohesive term and the constraint $I_{[0,\zeta(t,x)]}$, keeping track of the opening history, as a part of the surface *energy*, many other analytical works consider them as part of the dissipation potential. In this setup, in [21], existence of energetic solutions in the rate-independent, quasistatic setting was shown without any gradient term G, thus calling for a different notion of convergence to overcome compactness issues. More recently, vanishing viscosity techniques have been applied in cohesive zone models with respect to the internal variable, [14, 2], the first also accounting for history dependence of the crack opening in a Young-measure setting. [13, 5] study cohesive zone delamination for a visco-elastic solid without introducing an internal variable and prove existence of solutions as well as a vanishing viscosity limit.

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