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Strong solutions to nonlocal 2D Cahn–Hilliard–Navier–Stokes systems with nonconstant viscosity, degenerate mobility and singular potential

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Abstract

We consider a nonlinear system which consists of the incompressible Navier-Stokes equations coupled with a convective nonlocal Cahn-Hilliard equation. This is a diffuse interface model which describes the motion of an incompressible isothermal mixture of two (partially) immiscible fluids having the same density. We suppose that the viscosity depends smoothly on the order parameter as well as the mobility. Moreover, we assume that the mobility is degenerate at the pure phases and that the potential is singular (e.g. of logarithmic type). This system is endowed with no-slip boundary condition for the (average) velocity and homogeneous Neumann boundary condition for the chemical potential. Thus the total mass is conserved. In the two-dimensional case, this problem was already analyzed in some joint papers of the first three authors. However, in the present general case, only the existence of a global weak solution, the (conditional) weak-strong uniqueness and the existence of the global attractor were proven. Here we are able to establish the existence of a (unique) strong solution through an approximation procedure based on time discretization. As a consequence, we can prove suitable uniform estimates which allow us to show some smoothness of the global attractor. Finally, we discuss the existence of strong solutions for the convective nonlocal Cahn-Hilliard equation, with a given velocity field, in the three dimensional case as well.

1 Introduction

The so-called model **H** (see, for instance, [38] and references therein) has been proposed to describe the motion of a binary mixture of two isothermal, partially immiscible and incompressible fluids. This model is based on the diffuse interface approach and leads to the formulation of a Cahn-Hilliard-Navier-Stokes (CHNS) system for the average velocity $\boldsymbol{u}$ and the order parameter φ (i.e., the relative concentration of one of the fluid components). In the case of matched constant densities, a rather general CHNS system is the following

$$\boldsymbol{u}_t - 2\mathrm{div}\,(\nu(\varphi)D\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}\,, \tag{1.1}$$

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu)\,, \tag{1.2}$$

$$\mu = a\varphi - K*\varphi + F'(\varphi)\,, \tag{1.3}$$

$$\mathrm{div}(\boldsymbol{u}) = 0\,, \tag{1.4}$$

in $\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^d$, $d = 2,3$, is a bounded smooth domain (say, e.g., of class $\mathcal{C}^2$), $T > 0$ is a prescribed final time, ν stands for the fluid viscosity, D denotes the symmetric gradient, that is, $D\boldsymbol{u} := \big(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\big)/2$ and $\boldsymbol{v}$ is a given external force (the density has been taken equal to one). The Cahn-Hilliard (CH) equation with mobility m and potential F is nonlocal (see, e.g., [6]). The interaction kernel $K : \mathbb{R}^d \to \mathbb{R}$ is a (sufficiently) smooth even function and $a(x) := \displaystyle\int_\Omega K(x-y)dy$, $x \in \Omega$.

System (1.1)–(1.4) is subject to no-slip boundary condition for the velocity $\boldsymbol{u}$ and to homogeneous Neumann boundary condition for the chemical potential μ (which ensures the conservation of the total mass), namely,

$$\boldsymbol{u} = \boldsymbol{0}\,, \qquad m(\varphi)\nabla\mu \cdot \boldsymbol{n} = 0\,, \tag{1.5}$$

on $\partial\Omega \times (0,T)$, and to the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0\,, \qquad \varphi(0) = \varphi_0\,, \tag{1.6}$$

in Ω. Here, $\boldsymbol{n}$ stands for the outward normal to the boundary $\partial\Omega$ of Ω, while $\boldsymbol{u}_0$ and φ_0 are given.

Problem (1.1)–(1.6) has been studied so far under various assumptions on ν, m and F (see [12, 21, 22, 23, 24, 25, 26, 27, 28, 37], cf. also [20] for unmatched densities). However, there are very few results in the physically more relevant case, namely, when the viscosity depends on φ, the mobility m degenerate at pure phases (i.e. $\varphi = \pm 1$) and F is a singular potential (say, of logarithmic type). In this case, the existence of weak solutions ($d = 2,3$) has been proven in [25], where, for simplicity, the viscosity ν was assumed to be constant (as far as existence of weak solutions is concerned the case of a ν depending on φ can be dealt without difficulties as well).

It is worth recalling that for CHNS systems where the CH equation is the standard (local) one (see, for instance, [1, 2, 10, 11, 29, 30, 31, 40, 49, 52]), the case of degenerate mobility and singular potential is already difficult in the case of the CH only (cf. [17]). More precisely, the existence of a weak solution is essentially the only available result as far as we know (see [10]).

Going back to our nonlocal system, in the two dimensional case, the existence of the global attractor has been proven in [25]. This result can be also extended also to the case of ν depending on φ. On the other hand, uniqueness of weak solutions and the connectedness of the global attractor have been established in [21] for the case of constant viscosity only. If the viscosity depends on φ then weak-strong uniqueness has been proven in [21] for constant mobility and regular potential (i.e. defined on $\mathbb{R}$ with polynomially controlled growth). In the more general case (m degenerate and F singular), a conditional weak-strong uniqueness in dimension two was also established in [21] by supposing the existence of a strong solution.

The basic open issue in the two dimensional case is therefore the existence of a strong solution under the mentioned assumptions on ν, m and F. This is precisely the goal of the present contribution.

Proving the existence of strong solutions when ν depends on φ is much more difficult with respect to the case of a constant ν (cf. [21], cf. also Remark 8 below). We recall that, in the simplest case (i.e., ν and m constants and F regular), existence of strong solutions in two dimensions was proven in [24].

The existence of a strong solution to (1.1)–(1.6) paves the road for two further results. The first is concerned with uniform in time regularization estimates, which, in particular, provide a regularity property for the global attractor. The second is concerned with the convective nonlocal CH equation, for which we are able to prove existence of strong solutions also for the more challenging case $d = 3$, under quite general regularity assumptions on the given velocity field. In particular, this allows us to deduce some smoothness for the global attractor.

The plan of the paper follows. In the next section, besides some notation and definitions, the known results on existence and uniqueness of weak solutions are recalled. Section 3 is devoted to state the main regularity result of the paper whose proof is given in Section 4. Section 5 contains uniform in time estimates and the related regularity of the global attractor. In the final Section 6, we extend the analysis of the previous sections to the convective nonlocal CH equation with a given velocity field.

2 Weak solutions: what is known

Let us fix some notation first. We set $H := L^2(\Omega)$, $V := H^1(\Omega)$, and we introduce the classical Hilbert spaces for the incompressible Navier-Stokes equations with no-slip boundary conditions (see, e.g., [51]), namely,

$$G_{div} := \overline{\left\{\boldsymbol{u} \in C_0^\infty(\Omega)^2 : \mathrm{div}(\boldsymbol{u}) = 0\right\}}^{\,L^2(\Omega)^2},$$

and

$$V_{div} := \left\{\boldsymbol{u} \in H_0^1(\Omega)^2 : \mathrm{div}(\boldsymbol{u}) = 0\right\}.$$

Denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the scalar product, respectively, on both H and G_{div}, as well as on $L^2(\Omega)^2$ and $L^2(\Omega)^{2\times 2}$ The notation $\langle\cdot,\cdot\rangle_X$ and $\|\cdot\|_X$ will stand for the duality pairing between a Banach space X and its dual X', and for the norm of X, respectively. For every $f \in V'$, we set $\overline{f} := |\Omega|^{-1}\langle f, 1\rangle_V$. Here $|\Omega|$ is the Lebesgue measure of Ω. The Hilbert space V_{div} is endowed with the scalar product

$$(\boldsymbol{u},\boldsymbol{v})_{V_{div}} = (\nabla\boldsymbol{u},\nabla\boldsymbol{v}) = 2(D\boldsymbol{u}, D\boldsymbol{v}), \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in V_{div},$$

Let us also recall the definition of the Stokes operator $S : D(S) \cap G_{div} \to G_{div}$ in the case of no-slip boundary condition $(1.5)_1$, i.e. $S = -P\Delta$ with domain $D(S) = H^2(\Omega)^d \cap V_{div}$, where $P : L^2(\Omega)^d \to G_{div}$ is the Leray projector (see, for instance, [51]). Notice that we have

$$(S\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v})_{V_{div}} = (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}), \qquad \forall \boldsymbol{u} \in D(S), \quad \forall \boldsymbol{v} \in V_{div}.$$

We also recall that $S^{-1} : G_{div} \to G_{div}$ is a self-adjoint compact operator in G_{div} and by the classical spectral theorems there exists a sequence λ_j with $0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\lambda_j \to \infty$, and a family of $\boldsymbol{w}_j \in D(S)$ which is orthonormal in G_{div} and such that $S\boldsymbol{w}_j = \lambda_j \boldsymbol{w}_j$.

We also recall Poincaré's inequality

$$\lambda_1 \|\boldsymbol{u}\|^2 \le \|\nabla\boldsymbol{u}\|^2, \qquad \forall\, \boldsymbol{u} \in V_{div},$$

and two other inequalities, which are valid in two dimensions of space and will be used repeatedly in the course of our analysis. More precisely, the particular case of the Gagliardo-Nirenberg inequality (see, e.g., [8])

$$\|v\|_{L^{2q}(\Omega)} \le \widehat{C}_2 \|v\|^{1/q} \|v\|_V^{1-1/q}, \qquad \forall\, v \in V, \qquad 2 \le q < \infty, \tag{2.1}$$

as well as Agmon's inequality (see [3])

$$\|v\|_{L^\infty(\Omega)} \le \widehat{C}_3 \|v\|^{1/2} \|v\|_{H^2(\Omega)}^{1/2}, \qquad \forall\, v \in H^2(\Omega). \tag{2.2}$$

In these inequalities, the positive constant $\widehat{C}_2$ depends on q and on $\Omega \subset \mathbb{R}^2$, while the positive constant $\widehat{C}_3$ depends on Ω only.

The trilinear form b appearing in the weak formulation of the Navier-Stokes equations is defined as usual, that is,

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w}\, dx, \qquad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V_{div}.$$

The associated bilinear operator $\mathcal{B}$ from $V_{div} \times V_{div}$ into V_{div}' is defined by $\langle \mathcal{B}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w} \rangle := b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V_{div}$. We also set $\mathcal{B}\,\boldsymbol{u} := \mathcal{B}(\boldsymbol{u}, \boldsymbol{u})$, for every $\boldsymbol{u} \in V_{div}$, and we recall that

$$b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = -\, b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \qquad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V_{div}.$$

In addition, in two dimensions, the following estimate holds

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le \widehat{C}_1 \|\boldsymbol{u}\|^{1/2} \|\nabla\boldsymbol{u}\|^{1/2} \|\nabla\boldsymbol{v}\| \|\boldsymbol{w}\|^{1/2} \|\nabla\boldsymbol{w}\|^{1/2}, \qquad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V_{div},$$

with a constant $\widehat{C}_1 > 0$ that only depends on Ω.

If X is a (real) Banach space, we shall denote by $L^p_{tb}(0,\infty;X)$, $1 \le p < \infty$, the space of functions $f \in L^p_{loc}([0,\infty);X)$ that are translation bounded in $L^p_{loc}([0,\infty);X)$, i.e. such that

$$\|f\|^p_{L^p_{tb}(0,\infty;X)} := \sup_{t \ge 0} \int_t^{t+1} \|f(s)\|^p_X ds < \infty.$$

We are now ready to recall the result on the existence of weak solutions proven in [25]. For completeness, we deal with $d = 2$ and $d = 3$. The assumptions on the kernel K, on the viscosity ν are the following

(K) $K(\cdot - x) \in W^{1,1}(\Omega)$ for almost any $x \in \Omega$ and satisfies

$$K(x) = K(-x)\,, \qquad a(x) := \int_\Omega K(x-y)\,dy \geq 0\,, \quad \text{a.e. } x \in \Omega\,,$$

$$a^* := \sup_{x\in\Omega} \int_\Omega |K(x-y)|\,dy < \infty\,, \qquad b := \sup_{x\in\Omega} \int_\Omega |\nabla K(x-y)|\,dy < \infty\,.$$

(V) The viscosity ν is locally Lipschitz on $\mathbb{R}$ and there exist $\hat{\nu}_1, \hat{\nu}_2 > 0$ such that

$$\hat{\nu}_1 \leq \nu(s) \leq \hat{\nu}_2\,, \qquad \forall s \in \mathbb{R}\,.$$

The mobility m is supposed to be degenerate at ± 1 and the double-well potential F is assumed to be singular (e.g. logarithmic like) and defined in $(-1,1)$. More precisely, we assume the condition

(M) The mobility satisfies $m \in C^1([-1,1])$, $m \geq 0$, $m(s) = 0$ if and only if $s = -1$ or $s = 1$. Moreover, there exists $\epsilon_0 > 0$ such that m is non-increasing in $[1-\epsilon_0, 1]$ and non-decreasing in $[-1, -1+\epsilon_0]$.

Furthermore, m and F are supposed to fulfill the condition

(A1) $F \in C^2(-1,1)$ and $\lambda := mF'' \in C([-1,1])$.

Condition **(A1)** is typical in the analysis of the CH equation with degenerate mobility (see [17, 35, 36, 33]).

As far as F is concerned we assume that it can be written in the following form

$$F = F_1 + F_2\,,$$

where the singular component F_1 and the regular component $F_2 \in C^2([-1,1])$ satisfy the following assumptions.

(A2) There exists $\epsilon_0 > 0$ such that F_1'' is non-decreasing in $[1-\epsilon_0, 1)$ and non-increasing in $(-1, -1+\epsilon_0]$.

(A3) There exists $c_0 > 0$ such that

$$F''(s) + a(x) \geq c_0\,, \qquad \forall s \in (-1,1)\,, \qquad \text{a.e. } x \in \Omega\,.$$

(A4) There exists $\rho \in [0,1)$ such that

$$\rho F_1''(s) + F_2''(s) + a(x) \geq 0\,, \qquad \forall s \in (-1,1)\,, \quad \text{a.e. in } \Omega\,.$$

(A5) There exists $\alpha_0 > 0$ such that

$$m(s) F_1''(s) \geq \alpha_0\,, \qquad \forall s \in [-1,1]\,.$$

We denote by ϵ_0 a positive constant the value of which may possibly vary from line to line.

It is worth recalling that a typical situation is $m(s) = k_1(1-s^2)$ and F given by

$$F(s) = -\frac{\theta_c}{2} s^2 + \frac{\theta}{2}\big((1+s)\log(1+s) + (1-s)\log(1-s)\big), \tag{2.3}$$

where $0 < \theta < \theta_c$, θ being the absolute temperature and θ_c a given critical temperature below which the phase separation takes place.

In [25] the viscosity ν was assumed to be constant just to avoid technicalities, but the results therein also hold for a nonconstant viscosity satisfying **(V)**.

As far as the weak formulation is concerned, we point out that, if the mobility degenerates then the gradient of the chemical potential μ is not controlled in some L^p space. For this reason, and also in order to pass to the limit to prove existence of a weak solution, a suitable reformulation of the definition of weak solution should be introduced in such a way that μ does not appear explicitly (cf. [17], see also [25]).

Remark 1. *It is worth observing that all the results mentioned or proven in this paper hold, in particular, when F is strictly convex and $a \equiv 0$ (see [34]-[36], cf. also the discussion in [37]).*

The definition of weak solution given in [25] is

Definition 1. *Let $\boldsymbol{u}_0 \in G_{div}$, $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$, $\boldsymbol{v} \in L^2(0,T;V_{div}')$ and $0 < T < +\infty$ be given. A couple $[\boldsymbol{u},\varphi]$ is a weak solution to* (1.1)-(1.6) *on $[0,T]$ if*

- *$\boldsymbol{u}$, φ satisfy*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;G_{div}) \cap L^2(0,T;V_{div})\,,\\
&\boldsymbol{u}_t \in L^{4/3}(0,T;V_{div}')\,, \qquad \text{if} \quad d=3\,,\\
&\boldsymbol{u}_t \in L^2(0,T;V_{div}')\,, \qquad \text{if} \quad d=2\,,\\
&\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V)\,,\\
&\varphi_t \in L^2(0,T;V')\,,
\end{aligned}$$

  and

$$\varphi \in L^\infty(Q_T)\,, \qquad |\varphi(x,t)| \leq 1 \quad \text{a.e. } (x,t) \in Q_T := \Omega \times (0,T)\,;$$

- *for every* $\boldsymbol{w} \in V_{div}$*, every* $\psi \in V$ *and for almost any* $t \in (0,T)$ *we have*

$$\langle \boldsymbol{u}_t, \boldsymbol{w}\rangle_{V_{div}} + 2\left(\nu\left(\varphi\right) D\boldsymbol{u}, D\boldsymbol{w}\right) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) = \big((a\varphi - K*\varphi)\nabla\varphi, \boldsymbol{w}\big) + \langle \boldsymbol{v}, \boldsymbol{w}\rangle\,,$$
$$\langle \varphi_t, \psi\rangle_V + \int_\Omega m(\varphi)F''(\varphi)\nabla\varphi\cdot\nabla\psi + \int_\Omega m(\varphi)a\nabla\varphi\cdot\nabla\psi$$
$$+ \int_\Omega m(\varphi)(\varphi\nabla a - \nabla K*\varphi)\cdot\nabla\psi = (\boldsymbol{u}\varphi, \nabla\psi)\,;$$

- *the initial conditions* $\boldsymbol{u}(0) = \boldsymbol{u}_0$*,* $\varphi(0) = \varphi_0$ *hold.*

Recall also that from the regularity properties of the weak solution we have

$$\boldsymbol{u} \in C_w([0,T]; G_{div})\,, \qquad \varphi \in C_w([0,T]; H)\,.$$

Therefore, the initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\varphi(0) = \varphi_0$ make sense.

The results on existence of weak solutions and, in the case of constant viscosity, of their uniqueness, proven in [25, Theorem 2] and in [21, Theorem 4] (cf. also Remark 2), are summarized in the following

Theorem 1. *Assume that **(K)**, **(V)**, **(M)** and **(A1)–(A3)** are satisfied. Let* $\boldsymbol{u}_0 \in G_{div}$ *and* $\varphi_0 \in L^\infty(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$ *and* $M(\varphi_0) \in L^1(\Omega)$*, where* $M \in C^2(-1,1)$ *is defined by* $m(s)M''(s) = 1$ *for all* $s \in (-1,1)$ *and* $M(0) = M'(0) = 0$*. Let also* $\boldsymbol{v} \in L^2_{loc}([0,\infty); V'_{div})$*. Then, for every* $T > 0$ *system* (1.1)–(1.6) *admits a weak solution* $[\boldsymbol{u},\varphi]$ *on* $[0,T]$ *such that* $\overline{\varphi}(t) = \overline{\varphi}_0$ *for all* $t \in [0,T]$*. In addition, if* $d = 2$ *then the weak solution* $[\boldsymbol{u},\varphi]$ *satisfies the energy equation*

$$\frac{1}{2}\frac{d}{dt}\big(\|\boldsymbol{u}\|^2 + \|\varphi\|^2\big) + 2\,\|\sqrt{\nu(\varphi)}D\boldsymbol{u}\|^2 + \int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2 + \int_\Omega a\, m(\varphi)|\nabla\varphi|^2$$
$$= \int_\Omega m(\varphi)(\nabla K*\varphi - \varphi\nabla a)\cdot\nabla\varphi + \int_\Omega (a\varphi - K*\varphi)\,\boldsymbol{u}\cdot\nabla\varphi + \langle \boldsymbol{v},\boldsymbol{u}\rangle\,, \tag{2.4}$$

for almost any $t > 0$*, while if* $d = 3$ *then* $[\boldsymbol{u},\varphi]$ *satisfies the following energy inequality*

$$\frac{1}{2}\big(\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2\big) + 2\int_0^t \|\sqrt{\nu(\varphi)}D\boldsymbol{u}\|^2 + + \int_0^t\int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2$$
$$+ \int_0^t\int_\Omega a\, m(\varphi)|\nabla\varphi|^2 \le \frac{1}{2}\big(\|\boldsymbol{u}_0\|^2 + \|\varphi_0\|^2\big) + \int_0^t\int_\Omega m(\varphi)\big(\nabla K*\varphi - \varphi\nabla a\big)\cdot\nabla\varphi$$
$$+ \int_0^t\int_\Omega \big(a\varphi - K*\varphi\big)\boldsymbol{u}\cdot\nabla\varphi + \int_0^t \langle \boldsymbol{v},\boldsymbol{u}\rangle\,, \qquad \forall t > 0\,. \tag{2.5}$$

Let $d = 2$ *and let* ν *be constant. In addition, suppose that assumptions **(A4)** and **(A5)** are satisfied. Then the weak solution to system* (1.1)-(1.6) *is also unique. Moreover, let* $[\boldsymbol{u}_i,\varphi_i]$ *be two weak solutions corresponding to two initial data* $[\boldsymbol{u}_{0i},\varphi_{0i}]$ *and external force densities* $\boldsymbol{v}_i$*,* *with $\boldsymbol{u}_{0i} \in G_{div}$, $\varphi_{0i} \in L^\infty(\Omega)$ such that $F(\varphi_{0i}) \in L^1(\Omega)$, $M(\varphi_{0i}) \in L^1(\Omega)$ and $\boldsymbol{v}_i \in L^2_{loc}([0,\infty); V'_{div})$, $i = 1,2$. Then, setting $\boldsymbol{u} := \boldsymbol{u}_2 - \boldsymbol{u}_1$, $\varphi := \varphi_2 - \varphi_1$ and $\boldsymbol{v} := \boldsymbol{v}_2 - \boldsymbol{v}_1$, the following continuous dependence estimate holds*

$$\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2_{V'} + \|\boldsymbol{u}\|^2_{L^2(0,t;V_{div})} + \|\varphi\|^2_{L^2(0,t;H)} \leq \left(\|\boldsymbol{u}(0)\|^2 + \|\varphi(0)\|^2_{V'}\right)\Lambda_0(t) + |\overline{\varphi}(0)|^2\,\Lambda_1(t) + \|\boldsymbol{v}\|^2_{L^2(0,T;V'_{div})}\Lambda_2(t)\,, \tag{2.6}$$

for all $t \in [0,T]$, where Λ_0, Λ_1 and Λ_2 are continuous functions which depend on the norms of the two solutions. The functions Λ_i also depend on F, K, ν and Ω.

Remark 2. We observe that in [25, Theorem 2] this kind of additional condition was assumed: there exists $\kappa > 4(a^* - a_* - b_*)$, where $a_* := \inf_{x\in\Omega}\int_\Omega K(x-y)dy$, $b_* := \min_{[-1,1]} F_2''$, and there exists $\epsilon_0 > 0$ such that

$$F_1''(s) \geq \kappa, \qquad \forall s \in (-1,-1+\epsilon_0] \cup [1-\epsilon_0, 1)\,. \tag{2.7}$$

This assumption was helpful in the proof to deduce the equicoercivity $F_\epsilon(s) \geq \delta_1 s^2 - \delta_2$, for all $s \in \mathbb{R}$ (with $\delta_1 > 0$, and $\delta_2 \in \mathbb{R}$ both independent of ϵ), for the family of $\epsilon-$regularizations F_ϵ of F. However, we now show that (2.7) is superfluous. Indeed, it can be removed by employing a variant of the Elliot-Garcke type of approximation (see [25, Proof of Theorem 2]). More precisely, the following approximations $F_{1\epsilon}$ and $F_{2\epsilon}$ for F_1 and F_2, respectively, can be considered (see also [20])

$$F_{1\epsilon}(s) = \begin{cases} F_1(1-\epsilon) + F_1'(1-\epsilon)\big(s-(1-\epsilon)\big) + \frac{1}{2}F_1''(1-\epsilon)\big(s-(1-\epsilon)\big)^2 \\ +\big(s-(1-\epsilon)\big)^3, \qquad s \geq 1-\epsilon\,, \\ F_1(s)\,, \qquad |s| \leq 1-\epsilon\,, \\ F_1(-1+\epsilon) + F_1'(-1+\epsilon)\big(s-(-1+\epsilon)\big) + \frac{1}{2}F_1''(-1+\epsilon)\big(s-(-1+\epsilon)\big)^2 \\ +\big|s-(-1+\epsilon)\big|^3, \qquad s \leq -1+\epsilon\,, \end{cases}$$

$$F_{2\epsilon}(s) = \begin{cases} F_2(1-\epsilon) + F_2'(1-\epsilon)\big(s-(1-\epsilon)\big) + \frac{1}{2}F_2''(1-\epsilon)\big(s-(1-\epsilon)\big)^2\,, \\ \qquad s \geq 1-\epsilon\,, \\ F_2(s)\,, \qquad |s| \leq 1-\epsilon\,, \\ F_2(-1+\epsilon) + F_2'(-1+\epsilon)\big(s-(-1+\epsilon)\big) + \frac{1}{2}F_2''(-1+\epsilon)\big(s-(-1+\epsilon)\big)^2\,, \\ \qquad s \leq -1+\epsilon\,. \end{cases}$$

It is easy to check that $F_\epsilon \in C^{2,1}_{loc}(\mathbb{R})$ and that, due to the lower bound $F''(s) \geq -k$, for all $s \in (-1,1)$, where $k = \|a\|_{L^\infty(\Omega)} - c_0$ (cf. **(A3)**), there exist two constants $k_1 > 0$ and $k_2 \in \mathbb{R}$, which do not depend on ϵ, such that

$$F_\epsilon(s) \geq k_1|s|^3 - k_2\,, \qquad \forall s \in \mathbb{R}\,. \tag{2.8}$$

Moreover, as a consequence of **(A3)**, we still have

$$F_\epsilon''(s) + a(x) \geq c_0 \,, \qquad \forall s \in \mathbb{R} \,, \quad \text{a.e. } x \in \Omega \,. \tag{2.9}$$

and **(A2)** implies that there exists $\epsilon_0 > 0$ such that

$$F_{1\epsilon}(s) \leq F_1(s) + \epsilon^3 \,, \qquad \forall s \in (-1,1) \,, \quad \forall \epsilon \in (0, \epsilon_0] \,. \tag{2.10}$$

Thanks to the bounds (2.8)–(2.10), the argument of [25, Proof of Theorem 2], to which we refer for the details, can still be reproduced, and the same basic estimates for the sequence $[\boldsymbol{u}_\epsilon, \varphi_\epsilon]$ of approximate solutions can be recovered. Moreover, the argument to prove that $|\varphi| \leq 1$ almost everywhere in Q_T remains unchanged. There only remains to show that we can still pass to the limit, as $\epsilon \to 0$, in the term $\int_\Omega m_\epsilon(\varphi_\epsilon) F_\epsilon''(\varphi_\epsilon) \nabla \varphi_\epsilon \cdot \nabla \psi$ (for all $\psi \in V$), which appears in the variational formulation of the approximate problem, in order to prove that the limit couple $[\boldsymbol{u}, \varphi]$ is a weak solution. To this aim, notice that, due to **(A1)** and to the convergence $\varphi_\epsilon \to \varphi$, pointwise almost everywhere in Q_T, it is easy to see that we still have

$$m_\epsilon(\varphi_\epsilon) F_\epsilon''(\varphi_\epsilon) \to m(\varphi) F''(\varphi) \,, \qquad \text{a.e. in } Q_T \,. \tag{2.11}$$

Moreover, there holds

$$\begin{aligned} |m_\epsilon(s) F_\epsilon''(s)| \leq \lambda_\infty &+ 6\, m(1-\epsilon)\big(s - (1-\epsilon)\big) \chi_{[1-\epsilon, +\infty)}(s) \\ &+ 6\, m(-1+\epsilon) \big| s - (-1+\epsilon) \big| \chi_{(-\infty, -1+\epsilon]}(s) \,, \end{aligned} \tag{2.12}$$

where $\lambda_\infty := \|\lambda\|_{L^\infty(-1,1)}$, and χ_E denotes the characteristic function of a set $E \subset \mathbb{R}$. Since φ_ϵ is bounded in $L^r(Q_T)$, where $r = 10/3$ if $d = 2$, and $r = 4$ if $d = 2$, then, by Lebesgue's theorem, (2.11) and (2.12) entail

$$m_\epsilon(\varphi_\epsilon) F_\epsilon''(\varphi_\epsilon) \to m(\varphi) F''(\varphi) \,, \qquad \text{strongly in } L^r(Q_T) \,.$$

This strong convergence, together with the weak convergence $\varphi_\epsilon \rightharpoonup \varphi$ in $L^2(0,T;V)$, allow to pass to the limit in the term above.

Remark 3. It is worth pointing out that, to prove the existence of a weak solution (in the sense of Definition 1) we do not need that the potential F has some singular behavior at the endpoints $s = \pm 1$ (cf. **(A1)–(A3)**). Instead, the key role is played by the degenerate mobility, i.e., by condition **(M)**, with F being also $C^2([-1,1])$. This is enough to ensure the crucial bound $|\varphi| \leq 1$ almost everywhere in Q_T. However, concerning uniqueness and regularity results (see the following sections), assumption **(A5)** implies that F must have some singular behavior at the endpoints, in the sense that, at least, $F''(s) \to \infty$, as $s \to \pm 1$.

Remark 4. By combining **(A1)** with the definition of the function M, we can see that F and M are not independent. Actually, in the statement of Theorem 1, $F(\varphi_0) \in L^1(\Omega)$ is a consequence of $M(\varphi_0) \in L^1(\Omega)$. Moreover, if **(A5)** holds then the two conditions are equivalent (see [25]).

3 Strong solutions in two dimensions

Here we state and prove our main result: the existence of strong solutions to (1.1)–(1.6).

Let us introduce some preliminaries that we shall need in the proof. First of all we observe that equations (1.2)–(1.3) can formally be rewritten as follows

$$\varphi_t + \boldsymbol{u} \cdot \nabla\varphi = \Delta B(\cdot,\varphi) + \mathrm{div}\big(\mathcal{N}(\varphi)\nabla a\big) - \mathrm{div}\big(m(\varphi)(\nabla K * \varphi)\big)\,, \tag{3.1}$$

where we have set

$$B(x,s) = \int_0^s \beta(x,\sigma)d\sigma\,, \qquad \beta(x,s) = m(s)\big(a(x) + F''(s)\big)\,, \tag{3.2}$$

$$\mathcal{N}(s) = sm(s) - \mathcal{M}(s)\,, \qquad \mathcal{M}(s) = \int_0^s m(\sigma)d\sigma\,, \tag{3.3}$$

for all $s \in [-1,1]$ and for a.e. $x \in \Omega$. Notice that we have

$$\nabla B(\cdot,\varphi) = \mathcal{M}(\varphi)\nabla a + \beta(\cdot,\varphi)\nabla\varphi\,. \tag{3.4}$$

Hence the boundary condition $m(\varphi)\nabla\mu \cdot \boldsymbol{n} = 0$ can be rewritten as

$$\big[\nabla B(\cdot,\varphi) + \mathcal{N}(\varphi)\nabla a - m(\varphi)(\nabla K * \varphi)\big] \cdot \boldsymbol{n} = 0. \tag{3.5}$$

Thus the equivalent weak formulation of equations (1.2)–(1.3) is

$$\langle\varphi_t,\psi\rangle_V + \int_\Omega \nabla B(\cdot,\varphi)\cdot\nabla\psi + \int_\Omega \mathcal{N}(\varphi)\nabla a\cdot\nabla\psi - \int_\Omega m(\varphi)(\nabla K*\varphi)\cdot\nabla\psi = (\boldsymbol{u}\varphi,\nabla\psi)\,,$$

for every $\psi \in V$ and for almost any $t \in (0,T)$.

On account of this formulation we can give our definition of strong solution if $d = 2$.

Definition 2. *Let $\boldsymbol{u}_0 \in V_{div}$, $\varphi_0 \in V \cap C^\beta(\bar\Omega)$ for some $\beta \in (0,1)$ and $\boldsymbol{v} \in L^2(0,T;V'_{div})$ and $0 < T < +\infty$ be given. A weak solution $[\boldsymbol{u},\varphi]$ to* (1.1)-(1.6) *on $[0,T]$ corresponding to $[\boldsymbol{u}_0,\varphi_0]$ is called strong solution if*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;V_{div}) \cap L^2\big(0,T;H^2(\Omega)^2\big)\,, \qquad \boldsymbol{u}_t \in L^2(0,T;G_{div})\,,\\
&\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))\,, \qquad \varphi_t \in L^2(0,T;H)\,,\\
&\boldsymbol{u}_t - 2\mathrm{div}(\nu(\varphi)D\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}\,,\\
&\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \Delta B(\cdot,\varphi) + \mathrm{div}\big(\mathcal{N}(\varphi)\nabla a\big) - \mathrm{div}\big(m(\varphi)(\nabla K*\varphi)\big)\,,\\
&\mathrm{div}(\boldsymbol{u}) = 0\,,
\end{aligned}$$

almost everywhere in $\Omega \times (0,T)$ with

$$\boldsymbol{u} = \boldsymbol{0}\,, \quad \big[\nabla B(\cdot,\varphi) + \mathcal{N}(\varphi)\nabla a - m(\varphi)(\nabla K*\varphi)\big]\cdot\boldsymbol{n} = 0\,,$$

almost everywhere on $\partial\Omega \times (0,T)$ and (1.6).

Remark 5. *It is worth noting that, for a strong solution, the nonlocal CH equation can also be written*

$$\varphi_t + \boldsymbol{u} \cdot \nabla\varphi = \mathit{div}\left(m(\varphi)F''(\varphi)\nabla\varphi + m(\varphi)(a\nabla\varphi + \varphi\nabla a - \nabla K * \varphi)\right),$$

almost everywhere in $\Omega \times (0,T)$*, while the boundary condition becomes*

$$\left[(m(\varphi)F''(\varphi)\nabla\varphi + m(\varphi)(a\nabla\varphi + \varphi\nabla a - \nabla K * \varphi)\right] \cdot \boldsymbol{n} = 0,$$

almost everywhere on $\partial\Omega \times (0,T)$*.*

Then we shall use the following lemma to handle the boundary condition (3.5).

Lemma 1. *Let* $\varphi, \psi \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$*. Then* $\varphi\psi \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ *and*

$$\|\varphi\psi\|_{H^{1/2}(\partial\Omega)} \leq \|\varphi\|_{L^\infty(\partial\Omega)}\|\psi\|_{H^{1/2}(\partial\Omega)} + \|\psi\|_{L^\infty(\partial\Omega)}\|\varphi\|_{H^{1/2}(\partial\Omega)}.$$

Proof. The proof is an immediate consequence of the definition of the space $H^{1/2}(\partial\Omega)$ with seminorm given by

$$|\varphi|^2_{H^{1/2}(\partial\Omega)} = \int_{\partial\Omega}\int_{\partial\Omega} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^2} d\Gamma(x) d\Gamma(y), \tag{3.6}$$

where $d\Gamma(\cdot)$ is the surface measure on $\partial\Omega$ (see, e.g., [13, Chapter IX, Section 18]). □

To establish the regularity of solutions we shall also need the kernel K to be more regular than $W^{1,1}_{loc}$. A possible assumption is that $K \in W^{2,1}_{loc}(\mathbb{R}^2)$. However, this assumption excludes physically relevant classes of kernels like, e.g., Newtonian and Bessel kernels. This class can be included by assuming that K is admissible, according to the following definition (see [7, Definition 1]).

Definition 3. *A kernel* $K \in W^{1,1}_{loc}(\mathbb{R}^2)$ *is admissible if the following conditions are satisfied:*

(K1) $K \in C^3(\mathbb{R}^d \setminus \{0\})$*;*

(K2) K *is radially symmetric,* $K(x) = \tilde{K}(|x|)$ *and* $\tilde{K}$ *is non-increasing;*

(K3) $\tilde{K}''(r)$ *and* $\tilde{K}'(r)/r$ *are monotone on* $(0, r_0)$ *for some* $r_0 > 0$*;*

(K4) $|D^3K(x)| \leq C_d|x|^{-3}$ *for some* $C_* > 0$*.*

The advantage of working with admissible kernels is due to the following lemma (cf. [7, Lemma 2]).

Lemma 2. *Let K be admissible. Then, for every $p \in (1,\infty)$, there exists $C_p > 0$ such that*

$$\|\nabla v\|_{L^p(\Omega)^{2\times 2}} \leq C_p \|\psi\|_{L^p(\Omega)}\,, \qquad \forall \psi \in L^p(\Omega)\,,$$

where $v = \nabla K * \psi$.

Notice that, as a consequence of assumption **(K)**, we have $a \in W^{1,\infty}(\Omega)$. If, in addition, K is admissible, then, as a consequence of Lemma 2 (taking $\psi = 1$, and hence $v = \nabla a$), we immediately have that $a \in W^{2,p}(\Omega)$, for all $p \in (1,\infty)$. Hence, the trace of ∇a on the boundary $\partial\Omega$ is well defined, and, in particular, we have $\nabla a \cdot \boldsymbol{n} \in W^{1-1/p,p}(\partial\Omega)$.

Before stating our result we need to replace **(A1)** with the following slightly stronger assumption

(A1)$_1$ $F \in C^3(-1,1)$ and $\lambda := mF'' \in C^1([-1,1])$.

Our main theorem is

Theorem 2. *Let assumptions **(K)**, **(V)**, **(M)**, **(A1)**$_1$, **(A4)–(A5)** hold and suppose that $K \in W^{2,1}_{loc}(\mathbb{R}^2)$ or that K is admissible. Let $\boldsymbol{u}_0 \in G_{div}$, $\varphi_0 \in V \cap L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$, where M is defined as in Theorem 1. Let also $\boldsymbol{v} \in L^2(0,T;G_{div})$. Then, for every $T > 0$, problem* (1.2)–(1.6) *admits a weak solution $[\boldsymbol{u},\varphi]$ on $[0,T]$ such that*

$$\boldsymbol{u} \in L^\infty(0,T;G_{div}) \cap L^2(0,T;V_{div})\,, \qquad \boldsymbol{u}_t \in L^2(0,T;V'_{div})\,, \tag{3.7}$$

$$\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))\,, \qquad \varphi_t \in L^2(0,T;H)\,. \tag{3.8}$$

Assume in addition that $\boldsymbol{u}_0 \in V_{div}$ and that $\varphi_0 \in V \cap C^\beta(\bar{\Omega})$ for some $\beta \in (0,1)$. Then, problem (1.2)–(1.6) *admits a (unique) strong solution satisfying* (3.8) *and*

$$\boldsymbol{u} \in L^\infty(0,T;V_{div}) \cap L^2\left(0,T;H^2(\Omega)^2\right)\,, \qquad \boldsymbol{u}_t \in L^2(0,T;G_{div})\,. \tag{3.9}$$

Finally, suppose that $\varphi_0 \in H^2(\Omega)$ and the following compatibility condition holds

$$\frac{\partial B(\cdot,\varphi_0)}{\partial \boldsymbol{n}} = m(\varphi_0)(\nabla K * \varphi_0)\cdot \boldsymbol{n} - \mathcal{N}(\varphi_0)(\nabla a \cdot \boldsymbol{n})\,, \quad \text{a.e. on } \partial\Omega\,. \tag{3.10}$$

Then, the strong solution also satisfies

$$\varphi \in L^\infty(0,T;H^2(\Omega))\,, \qquad \varphi_t \in L^\infty(0,T;H) \cap L^2(0,T;V)\,. \tag{3.11}$$

Remark 6. *We observe that uniqueness was already proven in [21, Theorem 7]. Actually, a conditional weak-strong uniqueness was established by supposing the existence of a strong solution. That result is no longer a conditional one.*

4 Proof of Theorem 2

The proof is divided into three steps.

Step 1. We first establish the $L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ regularity for φ. For this purpose, we need to carefully deduce higher order estimates on the nonlocal CH in such a way that the only regularity which is exploited for $\boldsymbol{u}$ is the weak one, i.e., $\boldsymbol{u} \in L^\infty(0,T;G_{div}) \cap L^2(0,T;V_{div})$. Indeed, if the viscosity is nonconstant, we cannot directly apply the classical regularity result [51, Theorem 3.10] for the incompressible Navier-Stokes system in 2D (which also requires a regularity assumption on the initial velocity $\boldsymbol{u}_0 \in V_{div}$) and adapt to our situation the argument of [24].

The (formal) idea is to test (3.1) by $B(\cdot,\varphi)_t = \beta(\cdot,\varphi)\varphi_t$. In order to make the argument rigorous, let us develop a suitable approximation scheme. We first approximate problem (3.1), (3.5) with the following

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \Delta B_\epsilon(\cdot,\varphi) + \mathrm{div}\big(\mathcal{N}_\epsilon(\varphi)\nabla a\big) - \mathrm{div}\big(m_\epsilon(\varphi)(\nabla K * Q(\varphi))\big)\,, \tag{4.1}$$

$$\big[\nabla B_\epsilon(\cdot,\varphi) + \mathcal{N}_\epsilon(\varphi)\nabla a - m_\epsilon(\varphi)(\nabla K * Q(\varphi))\big]\cdot\boldsymbol{n} = 0\,, \tag{4.2}$$

where we have set

$$B_\epsilon(x,s) = \int_0^s \beta_\epsilon(x,\sigma)d\sigma\,, \qquad \beta_\epsilon(x,s) = m_\epsilon(s)\big(a(x) + F_\epsilon''(s)\big)\,, \quad \forall s\in\mathbb{R}\,, \quad \text{a.e } x\in\Omega\,,$$

$$\mathcal{N}_\epsilon(s) = s m_\epsilon(s) - \mathcal{M}_\epsilon(s)\,, \qquad \mathcal{M}_\epsilon(s) = \int_0^s m_\epsilon(\sigma)d\sigma\,, \qquad \forall s\in\mathbb{R}\,.$$

Here we the singular potential F is replaced by the regular potential F_ϵ $F_\epsilon = F_{1\epsilon} + F_{2\epsilon}$, with $F_{1\epsilon}$ and $F_{2\epsilon}$ defined by (see [17])

$$F_{1\epsilon}''(s) = \begin{cases} F_1''(1-\epsilon)\,, & s\geq 1-\epsilon\,, \\ F_1''(s)\,, & |s|\leq 1-\epsilon\,, \\ F_1''(-1+\epsilon)\,, & s\leq -1+\epsilon\,, \end{cases} \tag{4.3}$$

$$F_{2\epsilon}''(s) = \begin{cases} F_2''(1-\epsilon)\,, & s\geq 1-\epsilon\,, \\ F_2''(s)\,, & |s|\leq 1-\epsilon\,, \\ F_2''(-1+\epsilon)\,, & s\leq -1+\epsilon\,, \end{cases} \tag{4.4}$$

with $F_{1\epsilon}(0) = F_1(0)$, $F_{1\epsilon}'(0) = F_1'(0)$, $F_{2\epsilon}(0) = F_2(0)$, $F_{2\epsilon}'(0) = F_2'(0)$. Moreover, the degenerate mobility m is replaced by

$$m_\epsilon(s) = \begin{cases} m(1-\epsilon)\,, & s\geq 1-\epsilon\,, \\ m(s)\,, & |s|\leq 1-\epsilon\,, \\ m(-1+\epsilon)\,, & s\leq -1+\epsilon\,. \end{cases} \tag{4.5}$$

In the last term of (4.1), $Q : \mathbb{R} \to \mathbb{R}$ is the truncation function defined as

$$Q(s) = \max\{-1, \min\{1, s\}\}, \qquad \forall s \in \mathbb{R}.$$

Notice that, thanks to condition **(A1)**, we have the bound $|m_\epsilon(s)F''_\epsilon(s)| \leq \lambda_\infty$, for all $s \in \mathbb{R}$ and for all $\epsilon \in (0,1)$, where $\lambda_\infty := \|\lambda\|_{L^\infty(-1,1)}$. On account also of conditions **(A4)** and **(A5)**, there holds

$$0 < \alpha_0(1-\rho) \leq \beta_\epsilon(x,s) \leq k^*, \qquad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \tag{4.6}$$

where $k^* := m_\infty a_\infty + \lambda_\infty$, $m_\infty := \|m\|_{L^\infty(-1,1)}$, $a_\infty := \|a\|_{L^\infty(\Omega)}$ do not depend on ϵ. Moreover, notice that the functions m_ϵ, $\mathcal{M}_\epsilon$ and $\mathcal{N}_\epsilon$ satisfy the following properties

$$0 < m(1-\epsilon) \leq m_\epsilon(s) \leq m_\infty, \quad |\mathcal{M}_\epsilon(s)| \leq m_\infty |s|, \quad |\mathcal{N}_\epsilon(s)| \leq N_\infty, \qquad \forall s \in \mathbb{R}, \tag{4.7}$$

$$|m_\epsilon(s_2) - m_\epsilon(s_1)| \leq m'_\infty |s_2 - s_1|, \quad |\mathcal{M}_\epsilon(s_2) - \mathcal{M}_\epsilon(s_1)| \leq m_\infty |s_2 - s_1|, \quad \forall s_1, s_2 \in \mathbb{R}, \tag{4.8}$$

$$|\mathcal{N}_\epsilon(s_2) - \mathcal{N}_\epsilon(s_1)| \leq N'_\infty |s_2 - s_1|, \qquad \forall s_1, s_2 \in \mathbb{R}, \tag{4.9}$$

where $N_\infty := \|\mathcal{N}\|_{L^\infty(-1,1)}$, $N'_\infty := \|\mathcal{N}'\|_{L^\infty(-1,1)}$ and $m'_\infty := \|m'\|_{L^\infty(-1,1)}$ are independent of ϵ. Indeed, regarding the last bound in (4.7), it is easy to check that, for all $s \geq 1-\epsilon$, we have $\mathcal{N}_\epsilon(s) = \mathcal{N}(1-\epsilon)$ (a similar expression holds for $s \leq -1+\epsilon$). Finally, due to condition **(A1)**$_1$, we have

$$|\beta(x,s_2) - \beta(x,s_1)| \leq \beta'_\infty |s_2 - s_1|, \qquad \forall s_1, s_2 \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \tag{4.10}$$

where $\beta'_\infty := m'_\infty a_\infty + \lambda'_\infty$, and $\lambda'_\infty := \|\lambda'\|_{L^\infty(-1,1)}$.

We now prove that problem (4.1), (4.2), for every fixed $\epsilon > 0$, admits a solution $\varphi \in L^\infty(V) \cap L^2(H^2(\Omega))$, with $\varphi_t \in L^2(H)$. In order to prove this regularity, the choice of the approximation argument is crucial. Indeed, we point out that the use of the Faedo-Galerkin (FG) method is problematic. The reason is that testing the projected (4.1) by $\partial_t B(\cdot, \varphi_n)$ (here φ_n denotes a FG approximate solution) is not allowed, since $B(\cdot, \varphi_n)$ does not belong, in general, to the subspace spanned by the first n elements of the FG basis. The problem is the nonconstant mobility. On the other hand, testing by $\partial_t \varphi_n$ also leads to technical difficulties.

We shall therefore employ a different approximation approach; in particular, the proof will be carried out by means of a time-discretization argument. For simplicity of notation, for the moment we drop the indication of the approximation parameter ϵ. We fix $N \in \mathbb{N}$ and set $\tau = T/N$. We first introduce the following incremental-step problem: for $k = 0, \ldots, N-1$, given $\varphi_k \in V$, find $\varphi_{k+1} \in V$ that solves

$$
\begin{aligned}
-\tau\Delta B(\cdot,\varphi_{k+1})+\varphi_{k+1}=\varphi_k-\tau\boldsymbol{U}_k\cdot\nabla\varphi_{k+1}+\tau\,\mathrm{div}\big(\mathcal{N}(\varphi_k)\nabla a\big)\\
-\tau\,\mathrm{div}\big(m(\varphi_k)(\nabla K*Q(\varphi_k))\big)\,,
\end{aligned}
\tag{4.11}
$$

$$
\frac{\partial B(\cdot,\varphi_{k+1})}{\partial\boldsymbol{n}}=m(\varphi_k)(\nabla K*Q(\varphi_k))\cdot\boldsymbol{n}-\mathcal{N}(\varphi_k)(\nabla a\cdot\boldsymbol{n})\,,\qquad \text{a.e. on } \partial\Omega\,, \tag{4.12}
$$

where $\boldsymbol{U}_k$ are given by

$$
\boldsymbol{U}_k:=\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}\boldsymbol{u}(s)\,ds\,,\qquad k=0,\dots,N-1\,.
$$

We now claim that (4.11), (4.12), for every $\varphi_0\in V$, admit a solution $(\varphi_1,\dots,\varphi_N)\in H^2(\Omega)^N$. Indeed, introducing, for every $k=0,\dots,N-1$, the nonlinear operator $A_k:V\to V'$ defined by

$$
\langle A_k\varphi,\psi\rangle_V:=\tau\big(\nabla B(\cdot,\varphi),\nabla\psi\big)+(\varphi,\psi)-\tau(\boldsymbol{U}_k\varphi,\nabla\psi)\,,\qquad\forall\varphi,\psi\in V\,, \tag{4.13}
$$

and $g_k\in V'$ given by

$$
\langle g_k,\psi\rangle_V:=(\varphi_k,\psi)-\tau\big(\mathcal{N}(\varphi_k)\nabla a,\nabla\psi\big)+\tau\big(m(\varphi_k)(\nabla K*Q(\varphi_k)),\nabla\psi\big)\,,\qquad\forall\psi\in V\,,
$$

then problem (4.11)–(4.12) can be written as

$$
A_k\varphi_{k+1}=g_k,\quad \text{in } V'. \tag{4.14}
$$

We now observe that A_k is pseudomonotone and coercive on V. Indeed, writing the first term on the right-hand side of (4.13) as $\tau(\beta(\cdot,\varphi)\nabla\varphi,\nabla\psi)+\tau(\mathcal{M}(\varphi)\nabla a,\nabla\psi)$, then it is straightforward to check that A_k satisfy all the assumptions of the general results given by [47, Lemma 2.31 and Lemma 2.32] (for pseudomonotonicity) and by [47, Lemma 2.35] (for coercivity). This can be seen by taking $a(x,r,s):=\tau\beta(x,r)s+\tau\mathcal{M}(r)\nabla a(x)-\tau\boldsymbol{U}_kr$, $b(x,r):=0$, and $c(x,r,s):=r$ in [47, Lemma 2.31, Lemma 2.32 and Lemma 2.35]. Therefore (4.14) admits a solution $\varphi_{k+1}\in V$ (see [9], cf. also [47, Theorem 2.6]).

Using a bootstrap argument we find that $\varphi_{k+1}\in H^2(\Omega)$, for $k=0,\dots,N-1$. Indeed, owing to (4.7)–(4.9), from (4.11) and (4.12) we deduce that $\Delta B(\cdot,\varphi_{k+1})\in L^{2-\gamma}(\Omega)$, for all $0<\gamma\leq 1$, and $\partial B(\cdot,\varphi_{k+1})/\partial\boldsymbol{n}\in H^{1/2}(\partial\Omega)$. From elliptic regularity theory, we then infer that $B(\cdot,\varphi_{k+1})\in W^{2,2-\gamma}(\Omega)$. Hence we have $\nabla B(\cdot,\varphi_{k+1})\in W^{1,2-\gamma}(\Omega)^2$, for all $0<\gamma\leq 1$. This, by comparison in (3.4), implies that $\nabla\varphi_{k+1}\in L^4(\Omega)^2$. Therefore, the right-hand side of (4.11) is in $L^2(\Omega)$ and by applying elliptic regularity theory again we get that $B(\cdot,\varphi_{k+1})\in H^2(\Omega)$. Hence, $\nabla B(\cdot,\varphi_{k+1})\in H^1(\Omega)^2$ and, thanks to (4.6) and (4.10), it is easy to check that we also have $\nabla\beta(\cdot,\varphi_{k+1})\in L^4(\Omega)^2$. Then, again by comparison in (3.4), we deduce that $\nabla\varphi_{k+1}\in H^1(\Omega)^2$, whence $\varphi_{k+1}\in H^2(\Omega)$. Moreover, the following identity, which will be useful later, holds

$$
\partial^2_{ij}\varphi_{k+1}=\frac{1}{\beta(\cdot,\varphi_{k+1})}\partial^2_{ij}B(\cdot,\varphi_{k+1})-\frac{1}{\beta^2(\cdot,\varphi_{k+1})}\partial_i\beta(\cdot,\varphi_{k+1})\partial_jB(\cdot,\varphi_{k+1})
$$

$$- \frac{\mathcal{M}(\varphi_{k+1})}{\beta(\cdot,\varphi_{k+1})}\partial_i\left(\partial_j a\right) - \frac{m(\varphi_{k+1})}{\beta(\cdot,\varphi_{k+1})}\partial_i\varphi_{k+1}\partial_j a + \frac{\mathcal{M}(\varphi_{k+1})}{\beta^2(\cdot,\varphi_{k+1})}\partial_i\beta(\cdot,\varphi_{k+1})\partial_j a\,. \tag{4.15}$$

Let us now begin to establish the basic discrete estimates. We first test (4.11) by φ_{k+1} and sum over k from $k=0$ to $k=n$, where $n<N$. By using the following elementary identity

$$\sum_{k=0}^{n}(\varphi_{k+1}-\varphi_k,\varphi_{k+1}) = \frac{1}{2}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 + \frac{1}{2}\|\varphi_{n+1}\|^2 - \frac{1}{2}\|\varphi_0\|^2, \tag{4.16}$$

and (3.4), we get

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 + \frac{1}{2}\|\varphi_{n+1}\|^2 + \tau\sum_{k=0}^{n}\big(\beta(\cdot,\varphi_{k+1})\nabla\varphi_{k+1},\nabla\varphi_{k+1}\big)\\
&= \frac{1}{2}\|\varphi_0\|^2 + \tau\sum_{k=0}^{n}\big(m(\varphi_k)(\nabla K * Q(\varphi_k)) - \mathcal{N}(\varphi_k)\nabla a,\nabla\varphi_{k+1}\big)\\
&- \tau\sum_{k=0}^{n}\big(\mathcal{M}(\varphi_{k+1})\nabla a,\nabla\varphi_{k+1}\big).
\end{aligned} \tag{4.17}$$

Observe that

$$\tau\Big|\sum_{k=0}^{n}\big(\mathcal{M}(\varphi_{k+1})\nabla a,\nabla\varphi_{k+1}\big)\Big| \le \tau\delta\sum_{k=0}^{n}\|\nabla\varphi_{k+1}\|^2 + \tau C_{\delta,m,K}\sum_{k=0}^{n}\|\varphi_{k+1}\|^2, \tag{4.18}$$

$$\tau\Big|\sum_{k=0}^{n}\big(\mathcal{N}(\varphi_k)\nabla a,\nabla\varphi_{k+1}\big)\Big| \le \tau\delta\sum_{k=0}^{n}\|\nabla\varphi_{k+1}\|^2 + C_{\delta,m,K}\,T, \tag{4.19}$$

$$\tau\Big|\sum_{k=0}^{n}\big(m(\varphi_k)(\nabla K * Q(\varphi_k)),\nabla\varphi_{k+1}\big)\Big| \le \tau\delta\sum_{k=0}^{n}\|\nabla\varphi_{k+1}\|^2 + C_{\delta,m,K}\,T. \tag{4.20}$$

Therefore, inserting estimates (4.18)–(4.20) into (4.17), using the lower bound in (4.6), and choosing $\delta>0$ small enough (i.e., $\delta\le\alpha_0(1-\rho)/6$), we obtain the discrete inequality

$$\begin{aligned}
&\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 + \|\varphi_{n+1}\|^2 + \tau\alpha_0(1-\rho)\sum_{k=0}^{n}\|\nabla\varphi_{k+1}\|^2\\
&\le \|\varphi_0\|^2 + C_{m,K}\,T + \tau C_{m,K}\sum_{k=0}^{n}\|\varphi_{k+1}\|^2\\
&= \|\varphi_0\|^2 + C_{m,K}\,T + \tau C_{m,K}\|\varphi_{n+1}\|^2 + \tau C_{m,K}\sum_{k=0}^{n-1}\|\varphi_{k+1}\|^2\,.
\end{aligned}$$

Choosing $\tau>0$ small enough (such that, e.g., $\tau C_{m,K}\le 1/2$), by means of the discrete Gronwall Lemma we hence obtain the estimate

$$\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 + \|\varphi_{n+1}\|^2 + \tau\alpha_0(1-\rho)\sum_{k=0}^{n}\|\nabla\varphi_{k+1}\|^2 \le C_T(1+\|\varphi_0\|^2)\,, \tag{4.21}$$

for $n = 0, \dots, N-1$. The next step now consists in testing (4.11) by $B(\cdot, \varphi_{k+1}) - B(\cdot, \varphi_k)$. We employ for $B(\cdot, \varphi_k)$ the analogue of the elementary (4.16), the lower bound in (4.6), the following discrete integration by parts formula

$$\tau \sum_{k=0}^{n} \big(\mathcal{N}(\varphi_k)\nabla a, \nabla(B(\cdot, \varphi_{k+1}) - B(\cdot, \varphi_k))\big) = \tau\big(\mathcal{N}(\varphi_{n+1})\nabla a, \nabla B(\cdot, \varphi_{n+1})\big)$$
$$- \tau \sum_{k=0}^{n} \big((\mathcal{N}(\varphi_{k+1}) - \mathcal{N}(\varphi_k))\nabla a, \nabla B(\cdot, \varphi_{k+1})\big) - \tau\big(\mathcal{N}(\varphi_0)\nabla a, \nabla B(\cdot, \varphi_0)\big)\,,$$

and a similar formula for the term in the convolution operator, to get

$$\begin{aligned}
&\alpha_0(1-\rho)\frac{1}{\tau}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 + \frac{1}{2}\|\nabla B(\cdot,\varphi_{n+1})\|^2 + \frac{1}{2}\sum_{k=0}^{n}\|\nabla(B(\cdot,\varphi_{k+1}) - B(\cdot,\varphi_k))\|^2 \\
&\le \frac{1}{2}\|\nabla B(\cdot,\varphi_0)\|^2 - \big(\mathcal{N}(\varphi_{n+1})\nabla a, \nabla B(\cdot,\varphi_{n+1})\big) \\
&+ \sum_{k=0}^{n}\big((\mathcal{N}(\varphi_{k+1}) - \mathcal{N}(\varphi_k)\big)\nabla a, \nabla B(\cdot,\varphi_{k+1})\big) \\
&+ \big(\mathcal{N}(\varphi_0)\nabla a, \nabla B(\cdot,\varphi_0)\big) + \big(m(\varphi_{n+1})(\nabla K * Q(\varphi_{n+1})), \nabla B(\cdot,\varphi_{n+1})\big) \\
&- \sum_{k=0}^{n}\big(m(\varphi_{k+1})(\nabla K * Q(\varphi_{k+1})) - m(\varphi_k)(\nabla K * Q(\varphi_k)), \nabla B(\cdot,\varphi_{k+1})\big) \\
&- \big(m(\varphi_0)(\nabla K * Q(\varphi_0)), \nabla B(\cdot,\varphi_0)\big) - \sum_{k=0}^{n}\big(\boldsymbol{U}_k\cdot\nabla\varphi_{k+1}, B(\cdot,\varphi_{k+1}) - B(\cdot,\varphi_k)\big).
\end{aligned} \tag{4.22}$$

Let us now estimate individually the terms on the right-hand side of (4.22). We begin with those terms which are easier to be estimated. We have

$$\begin{aligned}
\big|\big(\mathcal{N}(\varphi_{n+1})\nabla a, \nabla B(\cdot,\varphi_{n+1})\big)\big| &\le N_\infty\|\nabla a\|_\infty|\Omega|^{1/2}\|\nabla B(\cdot,\varphi_{n+1})\| \\
&\le \frac{1}{8}\|\nabla B(\cdot,\varphi_{n+1})\|^2 + C_{m,K,\Omega}\,,
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
\Big|\sum_{k=0}^{n}\big((\mathcal{N}(\varphi_{k+1}) - \mathcal{N}(\varphi_k)\big)\nabla a, \nabla B(\cdot,\varphi_{k+1})\big)\Big| & \\
&\le \sum_{k=0}^{n} N'_\infty\|\nabla a\|_\infty\|\varphi_{k+1}-\varphi_k\|\|\nabla B(\cdot,\varphi_{n+1})\| \\
&\le \frac{\alpha_0(1-\rho)}{4\tau}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 \\
&+ C_{m,K,\alpha_0,\rho}\,\tau\sum_{k=0}^{n}\|\nabla B(\cdot,\varphi_{n+1})\|^2\,,
\end{aligned} \tag{4.24}$$

$$\left|\big(m(\varphi_{n+1})(\nabla K * Q(\varphi_{n+1})), \nabla B(\cdot,\varphi_{n+1})\big)\right| \le m_\infty\, b\, |\Omega|^{1/2} \|\nabla B(\cdot,\varphi_{n+1})\|$$
$$\le \frac{1}{8}\|\nabla B(\cdot,\varphi_{n+1})\|^2 + C_{m,K,\Omega}\,, \tag{4.25}$$

$$\Big|\sum_{k=0}^{n}\big(m(\varphi_{k+1})(\nabla K * Q(\varphi_{k+1})) - m(\varphi_k)(\nabla K * Q(\varphi_k)), \nabla B(\cdot,\varphi_{k+1})\big)\Big|$$
$$\le (m'_\infty|\Omega|^{1/2} + m_\infty)\, b \sum_{k=0}^{n} \|\varphi_{k+1}-\varphi_k\|\|\nabla B(\cdot,\varphi_{k+1})\|$$
$$\le \frac{\alpha_0(1-\rho)}{4\tau}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 + C_{m,K,\Omega,\alpha_0,\rho}\,\tau\sum_{k=0}^{n}\|\nabla B(\cdot,\varphi_{k+1})\|^2\,, \tag{4.26}$$

where $\|\nabla a\|_\infty := \|\nabla a\|_{L^\infty(\Omega)^2}$. The estimate for the last term on the right-hand side of (4.22) is more delicate. We first observe that, by means of a direct computation, the following bounds can be deduced

$$\|\boldsymbol{U}_n\| \le \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})}\,, \qquad \tau\sum_{k=0}^{n}\|\nabla \boldsymbol{U}_k\|^2 \le \|\boldsymbol{u}\|^2_{L^2(0,T;V_{div})}\,, \qquad n = 0,\dots,N-1. \tag{4.27}$$

Then we observe that

$$\Big|\sum_{k=0}^{n}\big(\boldsymbol{U}_k\cdot\nabla\varphi_{k+1}, B(\cdot,\varphi_{k+1}) - B(\cdot,\varphi_k)\big)\Big| \le \frac{\alpha_0(1-\rho)}{4\tau}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2$$
$$+ Ck^{*2}\tau\sum_{k=0}^{n}\|\boldsymbol{U}_k\cdot\nabla\varphi_{k+1}\|^2\,. \tag{4.28}$$

On the other hand, we have

$$Ck^{*2}\tau\sum_{k=0}^{n}\|\boldsymbol{U}_k\cdot\nabla\varphi_{k+1}\|^2 = Ck^{*2}\tau\sum_{k=0}^{n}\Big\|\boldsymbol{U}_k\cdot\frac{1}{\beta(\cdot,\varphi_{k+1})}\big(\nabla B(\cdot,\varphi_{k+1}) - \mathcal{M}(\varphi_{k+1})\nabla a\big)\Big\|^2$$
$$\le \frac{Ck^{*2}\tau}{\alpha_0^2(1-\rho)^2}\sum_{k=0}^{n}\|\boldsymbol{U}_k\|^2_{L^4(\Omega)^2}\|\nabla B(\cdot,\varphi_{k+1})\|^2_{L^4(\Omega)^2}$$
$$+ \frac{Ck^{*2}\tau}{\alpha_0^2(1-\rho)^2}\sum_{k=0}^{n}m_\infty^2\|\nabla a\|_\infty^2\|\boldsymbol{U}_k\|^2_{L^4(\Omega)^2}\|\varphi_{k+1}\|^2_{L^4(\Omega)}$$
$$\le C\tau\sum_{k=0}^{n}\|\boldsymbol{U}_k\|\|\nabla\boldsymbol{U}_k\|\|\nabla B(\cdot,\varphi_{k+1})\|\|B(\cdot,\varphi_{k+1})\|_{H^2(\Omega)}$$
$$+ C\tau\sum_{k=0}^{n}\|\boldsymbol{U}_k\|^2\|\nabla\boldsymbol{U}_k\|^2 + C\tau\sum_{k=0}^{n}\|\varphi_{k+1}\|^2\|\varphi_{k+1}\|_V^2$$
$$\le \delta\tau\sum_{k=0}^{n}\|B(\cdot,\varphi_{k+1})\|^2_{H^2(\Omega)} + C_\delta\tau\sum_{k=0}^{n}\|\boldsymbol{U}_k\|^2\|\nabla\boldsymbol{U}_k\|^2\|\nabla B(\cdot,\varphi_{k+1})\|^2$$

$$+ C\|\boldsymbol{u}\|^2_{L^\infty(0,T;G_{div})}\|\boldsymbol{u}\|^2_{L^2(0,T;V_{div})} + C_T(1+\|\varphi_0\|^2)\,. \tag{4.29}$$

We proceed to estimate the term in the H^2-norm of $B(\cdot,\varphi_{k+1})$. By means of a classical elliptic regularity estimate and by using (4.11), we find

$$\begin{aligned}
&\delta\tau\sum_{k=0}^{n}\|B(\cdot,\varphi_{k+1})\|^2_{H^2(\Omega)}\\
&\leq C\delta\tau\sum_{k=0}^{n}\Big(\|\Delta B(\cdot,\varphi_{k+1})\|^2+\|B(\cdot,\varphi_{k+1})\|^2_V+\Big\|\frac{\partial B(\cdot,\varphi_{k+1})}{\partial\boldsymbol{n}}\Big\|^2_{H^{1/2}(\partial\Omega)}\Big)\\
&\leq\frac{C\delta}{\tau}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2+C\delta\tau\sum_{k=0}^{n}\|\boldsymbol{U}_k\cdot\nabla\varphi_{k+1}\|^2+C\delta\tau\sum_{k=0}^{n}\|\mathrm{div}\big(\mathcal{N}(\varphi_k)\nabla a\big)\|^2\\
&+C\delta\tau\sum_{k=0}^{n}\|\mathrm{div}\big(m(\varphi_k)(\nabla K*Q(\varphi_k))\big)\|^2+C\delta\tau\sum_{k=0}^{n}\|B(\cdot,\varphi_{k+1})\|^2_V\\
&+C\delta\tau\sum_{k=0}^{n}\Big\|\frac{\partial B(\cdot,\varphi_{k+1})}{\partial\boldsymbol{n}}\Big\|^2_{H^{1/2}(\partial\Omega)}\,.
\end{aligned}\tag{4.30}$$

As far as the boundary term in (4.30) is concerned, on account of (4.12) we have

$$\begin{aligned}
&C\delta\tau\sum_{k=0}^{n}\Big\|\frac{\partial B(\cdot,\varphi_{k+1})}{\partial\boldsymbol{n}}\Big\|^2_{H^{1/2}(\partial\Omega)}\leq C\delta\tau\sum_{k=0}^{n}\big(\|m(\varphi_k)(\nabla K*Q(\varphi_k))\cdot\boldsymbol{n}\|^2_{H^{1/2}(\partial\Omega)}\\
&+\|\mathcal{N}(\varphi_k)\nabla a\cdot\boldsymbol{n}\|^2_{H^{1/2}(\partial\Omega)}\big)\leq C\delta\tau\sum_{k=0}^{n}\big(\|m(\varphi_k)\|^2_{L^\infty(\partial\Omega)}\|(\nabla K*Q(\varphi_k))\cdot\boldsymbol{n}\|^2_{H^{1/2}(\partial\Omega)}\\
&+\|m(\varphi_k)\|^2_{H^{1/2}(\partial\Omega)}\|(\nabla K*Q(\varphi_k))\cdot\boldsymbol{n}\|^2_{L^\infty(\partial\Omega)}+\|\mathcal{N}(\varphi_k)\|^2_{L^\infty(\partial\Omega)}\|\nabla a\cdot\boldsymbol{n}\|^2_{H^{1/2}(\partial\Omega)}\\
&+\|\mathcal{N}(\varphi_k)\|^2_{H^{1/2}(\partial\Omega)}\|\nabla a\cdot\boldsymbol{n}\|^2_{L^\infty(\partial\Omega)}\big)\leq C\delta\tau\sum_{k=0}^{n}m_\infty^2\|K*Q(\varphi_k)\|^2_{H^2(\Omega)}\\
&+C\delta\tau\sum_{k=0}^{n}(2\,{m'_\infty}^2\|\varphi_k\|^2_{H^{1/2}(\partial\Omega)}+2\,m_0^2|\partial\Omega|_1)\,b^2+C\delta\tau\sum_{k=0}^{n}N_\infty^2\|a\|^2_{H^2(\Omega)}\\
&+C\delta\tau\sum_{k=0}^{n}{N'_\infty}^2\|\varphi_k\|^2_{H^{1/2}(\partial\Omega)}\,b^2\leq C\delta T+C\delta\tau\sum_{k=0}^{n}\|\varphi_k\|^2_V\leq C_T\,\delta(1+\|\varphi_0\|^2_V),
\end{aligned}\tag{4.31}$$

where $m_0:=m(0)$. In the chains of estimates (4.31) we have employed Lemma 1, the classical trace theorem, the definition of the space $H^{1/2}(\partial\Omega)$ to estimate the term $\|m(\varphi_k)\|_{H^{1/2}(\partial\Omega)}$ (cf. (3.6)), Lemma 2 to estimate the terms in the H^2-norms, the fact that Q is bounded, and inequality (4.21) in the last estimate. Furthermore, we have used the fact that if $\varphi\in H^1(\Omega)$ and $|\varphi|\leq\zeta$ a.e. in Ω for some positive constant ζ (with Ω smooth enough), then the trace $\gamma_0\varphi:=\varphi|_{\partial\Omega}\in H^{1/2}(\partial\Omega)$ of φ on the boundary $\partial\Omega$ satisfies $|\gamma_0\varphi|\leq\zeta$ a.e. on $\partial\Omega$ and, moreover, if $L\in C^1(\mathbb{R})$, then $L(\varphi)\in H^1(\Omega)$ and $\gamma_0 L(\varphi)=L(\gamma_0\varphi)$. We point out that the truncation function Q allows to control the $L^\infty(\partial\Omega)$–norm of $\nabla K*Q(\varphi_k)\cdot\boldsymbol{n}$ by avoiding the control of the $L^\infty(\Omega)$–norm of φ_k. This is the reason for the introduction of Q in (4.1).

The third, fourth and fifth term on the right-hand side of (4.30) can be estimated as follows

$$C\delta\tau\sum_{k=0}^{n}\|\mathrm{div}\big(\mathcal{N}(\varphi_k)\nabla a\big)\|^2 \le C\delta\tau\sum_{k=0}^{n}\big(2N_\infty^2\|a\|_{H^2(\Omega)}^2 + 2{N'_\infty}^2\|\nabla a\|_\infty^2\|\nabla\varphi_k\|^2\big)$$
$$\le C_T\,\delta(1+\|\varphi_0\|_V^2)\,, \tag{4.32}$$

$$C\delta\tau\sum_{k=0}^{n}\|\mathrm{div}\big(m(\varphi_k)(\nabla K * Q(\varphi_k))\big)\|^2$$
$$\le C\delta\tau\sum_{k=0}^{n}\big(2m_\infty^2\|K*Q(\varphi_k)\|_{H^2(\Omega)}^2 + 2{m'_\infty}^2\,b^2\,\|\nabla\varphi_k\|^2\big)$$
$$\le C_T\,\delta(1+\|\varphi_0\|_V^2)\,, \tag{4.33}$$

$$C\delta\tau\sum_{k=0}^{n}\|B(\cdot,\varphi_{k+1})\|_V^2 \le C\,\delta\tau\sum_{k=0}^{n}\big(k^{*2}+2m_\infty^2\|\nabla a\|_\infty^2\big)\|\varphi_{k+1}\|^2$$
$$+ C\,\delta\tau\sum_{k=0}^{n}2k^{*2}\|\nabla\varphi_{k+1}\|^2 \le C_T\,\delta(1+\|\varphi_0\|^2)\,, \tag{4.34}$$

where we have used again Lemma 2 and (3.4), (4.7)–(4.9), (4.21).

We now insert (4.31)–(4.34) into (4.30) and then we insert the resulting inequality into (4.29). By fixing $\delta > 0$ small enough, we obtain

$$Ck^{*2}\tau\sum_{k=0}^{n}\|\boldsymbol{U}_k\cdot\nabla\varphi_{k+1}\|^2 \le \frac{\alpha_0(1-\rho)}{8\tau}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2$$
$$+ C\tau\sum_{k=0}^{n}\|\boldsymbol{U}_k\|^2\|\nabla\boldsymbol{U}_k\|^2\|\nabla B(\cdot,\varphi_{k+1})\|^2$$
$$+ C\|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})}^2\|\boldsymbol{u}\|_{L^2(0,T;V_{div})}^2 + C_T(1+\|\varphi_0\|_V^2)\,. \tag{4.35}$$

By employing (4.35), (4.23)–(4.26) and (4.28), from (4.22) we get

$$\frac{1}{\tau}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 + \|\nabla B(\cdot,\varphi_{n+1})\|^2 + \sum_{k=0}^{n}\|\nabla\big(B(\cdot,\varphi_{k+1})-B(\cdot,\varphi_k)\big)\|^2$$
$$\le C_T(1+\|\varphi_0\|_V^2) + C\|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})}^2\|\boldsymbol{u}\|_{L^2(0,T;V_{div})}^2$$
$$+ C\sum_{k=0}^{n}(\tau+\tau\|\boldsymbol{U}_k\|^2\|\nabla\boldsymbol{U}_k\|^2)\|\nabla B(\cdot,\varphi_{k+1})\|^2$$
$$\le C_T(1+\|\varphi_0\|_V^2) + C\|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})}^2\|\boldsymbol{u}\|_{L^2(0,T;V_{div})}^2$$
$$+ C(\tau+\|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})}^2\tau\|\nabla\boldsymbol{U}_n\|^2)\|\nabla B(\cdot,\varphi_{n+1})\|^2$$

$$+ C\sum_{k=0}^{n-1}(\tau+\tau\|\boldsymbol{U}_k\|^2\|\nabla\boldsymbol{U}_k\|^2)\|\nabla B(\cdot,\varphi_{k+1})\|^2 \,. \tag{4.36}$$

Observe that we have

$$\tau\|\nabla\boldsymbol{U}_n\|^2 \leq \int_{n\tau}^{(n+1)\tau}\|\nabla\boldsymbol{u}(s)\|^2\,ds\,.$$

Hence, for every $\eta > 0$, there exists $\tau_\eta > 0$, which only depends on η (and on $\boldsymbol{u}$), such that $\tau\|\nabla\boldsymbol{U}_n\|^2 < \eta$ for all $0 < \tau < \tau_\eta$ and for all $n < N$. By using this fact, we can take τ small enough in such a way that the third term on the right-hand side of the last inequality (4.36) can be absorbed into the term $\|\nabla B(\cdot,\varphi_{n+1})\|^2$ on the left-hand side. Therefore, on account of (4.27), by applying the discrete Gronwall Lemma to the ensuing discrete inequality, from (4.36) we obtain

$$\begin{aligned}&\frac{1}{\tau}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 + \|\nabla B(\cdot,\varphi_{n+1})\|^2 + \sum_{k=0}^{n}\|\nabla\big(B(\cdot,\varphi_{k+1})-B(\cdot,\varphi_k)\big)\|^2\\ &\leq \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)\,, \qquad n=0,\dots,N-1\,.\end{aligned} \tag{4.37}$$

We can now proceed to prove the $L^2(H^2)$–regularity of φ. Let us first notice that (4.30), combined with (4.31)–(4.35), (4.27) and (4.37), implies that

$$\tau\sum_{k=0}^{n}\|B(\cdot,\varphi_{k+1})\|^2_{H^2(\Omega)} \leq \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)\,, \quad n=0,\dots,N-1\,. \tag{4.38}$$

This estimate yields, in particular, a control on the gradient of $B(\cdot,\varphi_{k+1})$ in L^p, for $2 < p < \infty$. Indeed, from (4.38) we have

$$\tau\sum_{k=0}^{n}\|\nabla B(\cdot,\varphi_{k+1})\|^2_V \leq \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)\,.$$

This, by (2.1) and (4.37), implies that

$$\tau\sum_{k=0}^{n}\|\nabla B(\cdot,\varphi_{k+1})\|^{2p/(p-2)}_{L^p(\Omega)} \leq \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)\,. \tag{4.39}$$

Thanks to (3.4) and to the bound

$$\|\nabla\beta(\cdot,\varphi_{k+1})\|_{L^p(\Omega)^2} \leq m_\infty\|\nabla a\|_\infty|\Omega|^{1/p} + (a_\infty m'_\infty + \lambda'_\infty)\|\nabla\varphi_{k+1}\|_{L^p(\Omega)^2}\,,$$

from (4.39) we also have

$$\tau\sum_{k=0}^{n}\|\nabla\varphi_{k+1}\|^{2p/(p-2)}_{L^p(\Omega)^2} + \tau\sum_{k=0}^{n}\|\nabla\beta(\cdot,\varphi_{k+1})\|^{2p/(p-2)}_{L^p(\Omega)^2} \leq \mathbb{Q}\,, \tag{4.40}$$

where $\mathbb{Q} = \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)$. Thus, using (4.39), (4.40) (written for $p = 4$), and (4.38), from (4.15) we find the desired bound

$$\tau \sum_{k=0}^{n} \|\varphi_{k+1}\|^2_{H^2(\Omega)} \leq \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)\,, \qquad n = 0, \ldots, N-1\,. \tag{4.41}$$

We now need to introduce the functions $\widehat{\varphi}_N$, $\overline{\varphi}_N$, and $\widetilde{\varphi}_N$ which interpolate the values φ_n piecewise linearly, backward, and forward constantly, respectively, on the partition. Namely,

$$\begin{aligned}
&\widehat{\varphi}_N(t) := \gamma_n(t)\varphi_n + (1-\gamma_n(t))\varphi_{n+1}\,, \qquad \gamma_n(t) := n + 1 - (t/\tau)\,,\\
&\overline{\varphi}_N(t) := \varphi_{n+1}\,,\\
&\widetilde{\varphi}_N(t) := \varphi_n\,,
\end{aligned}$$

for $n\tau < t < (n+1)\tau$, $n = 0, \ldots, N-1$. As a consequence of estimates (4.21), (4.37) and (4.41), we have

$$\begin{aligned}
&\|\widehat{\varphi}_N'\|^2_{L^2(0,T;H)} + \|\widehat{\varphi}_N\|^2_{L^\infty(0,T;V)} + \|\overline{\varphi}_N\|^2_{L^\infty(0,T;V)} + \|\widetilde{\varphi}_N\|^2_{L^\infty(0,T;V)} + \|\overline{\varphi}_N\|^2_{L^2(0,T;H^2(\Omega))}\\
&+ \frac{3}{\tau}\|\widehat{\varphi}_N - \overline{\varphi}_N\|^2_{L^2(0,T;H)} + \frac{3}{\tau}\|\widehat{\varphi}_N - \widetilde{\varphi}_N\|^2_{L^2(0,T;H)} \leq \mathbb{Q}\,,
\end{aligned} \tag{4.42}$$

where $\mathbb{Q} = \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)$. Moreover, (4.21) and (4.37) also yield

$$\|B(\cdot, \overline{\varphi}_N)\|_{L^\infty(0,T;V)} \leq \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)\,. \tag{4.43}$$

Problem (4.11)–(4.12) can be rewritten in terms of the interpolating functions $\widehat{\varphi}_N$, $\overline{\varphi}_N$, $\widetilde{\varphi}_N$ as follows

$$\widehat{\varphi}_N' = \Delta B(\cdot, \overline{\varphi}_N) + \boldsymbol{u}_N \cdot \nabla\overline{\varphi}_N + \mathrm{div}\big(\mathcal{N}(\widetilde{\varphi}_N)\nabla a\big) - \mathrm{div}\big(m(\widetilde{\varphi}_N)(\nabla K * Q(\widetilde{\varphi}_N))\big)\,, \tag{4.44}$$

$$\frac{\partial B(\cdot, \overline{\varphi}_N)}{\partial \boldsymbol{n}} = m(\widetilde{\varphi}_N)(\nabla K * Q(\widetilde{\varphi}_N)) \cdot \boldsymbol{n} - \mathcal{N}(\widetilde{\varphi}_N)(\nabla a \cdot \boldsymbol{n}) \qquad \text{a.e. on } \partial\Omega, \tag{4.45}$$

where $\boldsymbol{u}_N$ are defined by $\boldsymbol{u}_N(t) := \boldsymbol{U}_n$, for $n\tau < t < (n+1)\tau$, $n = 0, \ldots, N-1$. The variational formulation of (4.44)–(4.45) reads

$$\begin{aligned}
&\langle \widehat{\varphi}_N', \psi\rangle_V + (\nabla B(\cdot, \overline{\varphi}_N), \nabla\psi) = -(\boldsymbol{u}_N \overline{\varphi}_N, \nabla\psi)\\
&+ \big(m(\widetilde{\varphi}_N)(\nabla K * Q(\widetilde{\varphi}_N)), \nabla\psi\big) - \big(\mathcal{N}(\widetilde{\varphi}_N)\nabla a, \nabla\psi\big)\,, \qquad \forall\psi \in V.
\end{aligned} \tag{4.46}$$

Owing to (4.42) and employing classical compactness results, we deduce that there exists $\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ with $\varphi_t \in L^2(0,T;H)$, such that, up to a subsequence, we have

$$\widehat{\varphi}_N \rightharpoonup \varphi\,, \qquad \text{weakly}^* \text{ in } L^\infty(0,T;V)\,, \tag{4.47}$$

$$\widehat{\varphi}_N' \rightharpoonup \varphi_t\,, \qquad \text{weakly in } L^2(0,T;H)\,, \tag{4.48}$$

$$\widehat{\varphi}_N \to \varphi\,, \qquad \text{strongly in } C^0([0,T];L^q(\Omega))\,, \quad 2 \le q < \infty\,, \tag{4.49}$$

$$\overline{\varphi}_N \rightharpoonup \varphi\,, \qquad \text{weakly}^* \text{ in } L^\infty(0,T;V)\,, \quad \text{weakly in } L^2(0,T;H^2(\Omega))\,, \tag{4.50}$$

$$\widetilde{\varphi}_N \rightharpoonup \varphi\,, \qquad \text{weakly}^* \text{ in } L^\infty(0,T;V)\,, \tag{4.51}$$

$$\overline{\varphi}_N \to \varphi\,, \qquad \text{strongly in } L^2(0,T;H)\,, \tag{4.52}$$

$$\widetilde{\varphi}_N \to \varphi\,, \qquad \text{strongly in } L^2(0,T;H)\,, \tag{4.53}$$

$$B(\cdot,\overline{\varphi}_N) \rightharpoonup B(\cdot,\varphi)\,, \qquad \text{weakly}^* \text{ in } L^\infty(0,T;V)\,, \quad \text{weakly in } L^2(0,T;H^2(\Omega))\,. \tag{4.54}$$

Since $\widetilde{\varphi}_N \to \varphi$ pointwise almost everywhere in $\Omega \times (0,T)$, by virtue of the boundedness of the functions m, $\mathcal{N}$ and Q, and by Lebesgue's theorem, we also have

$$m(\widetilde{\varphi}_N) \to m(\varphi)\,, \quad Q(\widetilde{\varphi}_N) \to Q(\varphi)\,, \quad \mathcal{N}(\widetilde{\varphi}_N) \to \mathcal{N}(\varphi)\,, \quad \text{strongly in } L^q(\Omega)\,, \tag{4.55}$$

for all $q \in [2,\infty)$. Moreover, we have

$$\boldsymbol{u}_N \to \boldsymbol{u}\,, \qquad \text{strongly in } L^2(0,T;V_{div})\,. \tag{4.56}$$

Indeed, it easy to check that $\boldsymbol{u}_N = P_N\boldsymbol{u}$, where P_N is the projector in $L^2(V_{div})$ onto the subspace $\mathcal{S}_N := \{\boldsymbol{v} \in L^2(0,T;V_{div}) \,:\, \boldsymbol{v}|_{(n\tau,(n+1)\tau)} = \boldsymbol{v}_n,\, \boldsymbol{v}_n \in V_{div},\, n = 0,\dots,N-1\}$. Since $\cup_{N\ge1}\mathcal{S}_N$ is dense in $L^2(V_{div})$, then (4.56) follows.

By means of the weak and strong convergences (4.47)–(4.56), we can now pass to the limit in (4.46) in a standard fashion, and recover the weak formulation of problem (4.1)–(4.2). Notice that we can also pass to the limit directly in (4.44)–(4.45) and prove that (4.1)–(4.2) are satisfied also strongly almost everywhere in $\Omega \times (0,T)$ and on $\partial\Omega \times (0,T)$, respectively.

We have thus proven that, for every $\epsilon > 0$, problem (4.1)–(4.2) admits a solution $\varphi_\epsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$. We can also see, by passing to the liminf in (4.42), that the sequence of φ_ϵ is uniformly bounded with respect to ϵ in these spaces (just recall that all constants in (4.6)–(4.10) are independent of ϵ). Therefore, there exists a limit function, which we still denote by $\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$, such that, up to a subsequence, the same convergences as (4.47)–(4.55) hold for the sequence of φ_ϵ to φ. These convergences allow to pass to the limit in the variational formulation of problem (4.1)–(4.2) and recover the variational formulation of the following problem

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \Delta B(\cdot,\varphi) + \operatorname{div}\big(\mathcal{N}(\varphi)\nabla a\big) - \operatorname{div}\big(m(\varphi)(\nabla K * Q(\varphi))\big)\,, \tag{4.57}$$

$$\big[\nabla B(\cdot,\varphi) + \mathcal{N}(\varphi)\nabla a - m(\varphi)(\nabla K * Q(\varphi))\big]\cdot\boldsymbol{n} = 0\,, \qquad \text{on } \partial\Omega\times(0,T)\,. \tag{4.58}$$

We now show that φ satisfies the bound $|\varphi| \le 1$, a.e. in $\Omega \times (0,T)$. This allows to remove the function Q in problem (4.57), (4.58) and hence to conclude Step 1, proving that φ solves problem (3.1) and (3.5). To this purpose, we know that φ_ϵ also satisfies the weak formulation (cf. Definition 1) of problem

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}(m_\epsilon(\varphi)\nabla\mu)\,, \tag{4.59}$$

$$\mu = a\varphi - K * Q(\varphi) + F_\epsilon'(\varphi)\,, \tag{4.60}$$

$$m_\epsilon(\varphi)\nabla\mu\cdot\boldsymbol{n} = 0\,, \quad \text{on } \partial\Omega\times(0,T)\,. \tag{4.61}$$

We can therefore argue as in [25, Proof of Theorem 2]. More precisely, we introduce the C^2 function M_ϵ defined by $m_\epsilon(s)M_\epsilon''(s) = 1$, for all $s\in\mathbb{R}$, $M_\epsilon(0) = M_\epsilon'(0) = 0$, and we test (4.59) by $M_\epsilon'(\varphi_\epsilon)$. This gives the estimate

$$\frac{d}{dt}\int_\Omega M_\epsilon(\varphi_\epsilon) + \frac{c_0}{2}\|\nabla\varphi_\epsilon\|^2 \le \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)\,,$$

where $c_0 = (1-\rho)\alpha_0/m_\infty$. Then, on account of the fact that for ϵ small enough, we have $M_\epsilon(s)\le M(s)$ for all $s\in(-1,1)$ (cf. assumption **(M)**). Thus, recalling that $M(\varphi_0)\in L^1(\Omega)$, we deduce the bound

$$\|M_\epsilon(\varphi_\epsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\big)\,.$$

We can now follow the same lines of [25, Proof of Theorem 2], which rely on an argument devised in [17, Proof of Theorem 1] (see also [10, Proof of Theorem 2.3]), and get the desired claim. This concludes the proof of the first part of the theorem. Namely, there exists a weak solution such that φ is smoother (see (3.8)).

Step 2. We now establish the $L^\infty(0,T;V_{div})\cap L^2(0,T;H^2(\Omega)^2)$ regularity for $\boldsymbol{u}$, assuming that $\boldsymbol{u}_0\in V_{div}$ and $\varphi_0\in V\cap C^\beta(\overline{\Omega})$ for some $\beta\in(0,1)$. The argument, which (formally) consists in testing the Navier-Stokes equations (1.1) by $\boldsymbol{u}_t$, follows exactly the lines of [21, Proof of Theorem 5, Step 2]. The key tool is a regularity result for the inhomogeneous Stokes system in non-divergence form, namely,

$$\begin{cases} -\omega(x)\,\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}(x)\,, & \text{in }\Omega\,,\\ \mathrm{div}\,(\boldsymbol{u}) = 0\,, & \text{in }\Omega\,,\\ u = 0\,, & \text{on }\partial\Omega\,. \end{cases} \tag{4.62}$$

We report the result for the reader's convenience:

Proposition 1. *[50, Proposition 2.1] Let $\boldsymbol{f}\in L^2(\Omega)^2$ and $\omega\in C^\delta\left(\overline{\Omega}\right)$, for some $\delta\in(0,1)$, such that $0<\lambda_0\le\omega(x)\le\lambda_1<\infty$ for all $x\in\overline{\Omega}$. Then any solution $[\boldsymbol{u},\pi]\in H^2(\Omega)^2\times H^1(\Omega)$ of (4.62) satisfies the estimate*

$$\|\boldsymbol{u}\|_{H^2(\Omega)^2} + \|\pi\|_{H^1(\Omega)} \le C\left(\|\boldsymbol{f}\|_{L^2(\Omega)^2} + \|\pi\|_{L^2(\Omega)}\right),$$

for some constant $C = C(\lambda_0,\lambda_1,\Omega,\|\omega\|_{C^\delta(\overline{\Omega})}) > 0$.

This result is applied to the Navier-Stokes system (1.1) after writing it in the following form

$$-\nu(\varphi)\Delta\boldsymbol{u}+\nabla\hat{\pi}=\boldsymbol{f}, \tag{4.63}$$

where

$$\boldsymbol{f}:=(a\varphi-K*\varphi)\nabla\varphi+\boldsymbol{v}-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\boldsymbol{u}_t+2\nu'(\varphi)D\boldsymbol{u}\,\nabla\varphi\,,\qquad \hat{\pi}:=\pi-F(\varphi)\,, \tag{4.64}$$

and allows to bound the H^2-norm of $\boldsymbol{u}$ in terms of the L^2-norm of $\boldsymbol{u}_t$. The only thing to establish is the Hölder regularity for φ (this in turn implies Hölder regularity for $\nu(\varphi)$, which is required in order to apply Proposition 1. We therefore need to suitably extend the argument of [21, Lemma 2] where the Hölder regularity a bounded weak solution to the convective nonlocal CH equation with constant mobility and regular potential was proven. This can be done thanks to assumptions **(A1)**, **(A4)** and **(A5)**. More precisely, we can prove the following

Lemma 3. *Assume $d=2$ and **(A1), (A4), (A5)**. Let $\boldsymbol{u}\in L^\infty(T',T;G_{div})\cap L^2(T',T;V_{div})$, for some $T>T'\geq 0$ and let φ be a bounded weak solution to* (1.2)*,* (1.3)*,* (1.5)$_2$*. Then there exists constants $C>0$ and $\alpha\in(0,1)$ depending on $\|\varphi\|_{L^\infty(Q_{T',T})}$ and on $\|\boldsymbol{u}\|_{L^4(Q_{T',T})}$, respectively, such that*

$$|\varphi(x,t)-\varphi(y,s)|\leq C\left(|x-y|^\alpha+|t-s|^{\alpha/2}\right), \tag{4.65}$$

for every $(x,t),(y,s)\in\bar{Q}_{T',T}:=\bar{\Omega}\times[T',T]$.

Proof. Following the lines of [21, Proof of Lemma 2] (cf. also [43]), let $k\in\mathbb{R}$ and $\eta=\eta\,(x,t)\in[0,1]$ be a continuous piecewise-smooth function which is supported on the space-time cylinders $Q_{t_0,t_0+\tau}\,(r):=B_r\,(x_0)\times(t_0,t_0+\tau)$, where $B_r\,(x_0)$ denotes the (open) ball centered at x_0 of radius $r>0$. As usual for the interior Hölder regularity, one takes $x_0\in\Omega$, while $x_0\in\partial\Omega$ for the corresponding boundary estimate and then exploits a standard compactness argument, in which $\overline{\Omega}$ may be covered by a finite number of such balls. We thus multiply (1.2), (1.3), which can be written as

$$\varphi_t+\boldsymbol{u}\cdot\nabla\varphi=\text{div}\,(\beta(\cdot,\varphi)\nabla\varphi+\boldsymbol{\kappa})\,,\qquad \boldsymbol{\kappa}(x,t):=m(\varphi)\,(\varphi\nabla a-\nabla K*\varphi)\,,$$

by $\eta^2\varphi_k^+$, where $\varphi_k^+:=\max\{0,\varphi-k\}$, integrate the resulting identity over $Q_{t_0,t}:=\Omega\times(t_0,t)$, where $T'\leq t_0<t<t_0+\tau\leq T$, to deduce that

$$\begin{aligned}&\int_{Q_{t_0,t}}\varphi_t\eta^2\varphi_k^+dxdt+\int_{Q_{t_0,t}}\beta(\cdot,\varphi)\nabla\varphi_k^+\cdot\nabla\left(\eta^2\varphi_k^+\right)dxdt\\&=\int_{Q_{t_0,t}}\boldsymbol{u}\varphi\cdot\nabla\left(\eta^2\varphi_k^+\right)dxdt+\int_{Q_{t_0,t}}\boldsymbol{\kappa}\,(x,t)\cdot\nabla\left(\eta^2\varphi_k^+\right)dxdt\,,\end{aligned} \tag{4.66}$$

Since we have $\nabla\varphi_k^+ \cdot \nabla\left(\eta^2\varphi_k^+\right) = \left|\nabla\left(\eta\varphi_k^+\right)\right|^2 - |\nabla\eta|^2\left(\varphi_k^+\right)^2$, we obtain from (4.66) and the assumptions **(A4)** and **(A5)** (cf. (5.17)) that

$$\begin{aligned}
&\frac{1}{2}\sup_{s\in(t_0,t)}\int_\Omega\left(\eta\varphi_k^+\right)^2(s)\,dx + \alpha_0(1-\rho)\int_{Q_{t_0,t}}\left|\nabla\left(\eta\varphi_k^+\right)\right|^2 dxdt \\
&\quad\le \frac{1}{2}\int_\Omega\left(\eta\varphi_k^+\right)^2(t_0)\,dx + \int_{Q_{t_0,t}}\left(\varphi_k^+\right)^2|\eta\eta_t|\,dxdt \\
&\quad + k^*\int_{Q_{t_0,t}}\left(\varphi_k^+\right)^2|\nabla\eta|^2\,dxdt + \int_{Q_{t_0,t}}\boldsymbol{u}\varphi\cdot\nabla\left(\eta^2\varphi_k^+\right)dxdt \\
&\quad + \int_{Q_{t_0,t}}\boldsymbol{\kappa}(x,t)\cdot\nabla\left(\eta^2\varphi_k^+\right)dxdt\,, \qquad (4.67)
\end{aligned}$$

where the constant k^* is the same as in (4.6). The fourth term on the right-hand side of (4.66) can still be estimated in the same fashion as in [50, Proof of Lemma 3.2], using the fact that $\boldsymbol{u}\in L^4\left(Q_{T',T}\right)$ is also divergence free and arguing by elementary Hölder's and Young's inequalities, to find that

$$\begin{aligned}
&\left|\int_{Q_{t_0,t}}\boldsymbol{u}\varphi\cdot\nabla\left(\eta^2\varphi_k^+\right)dxdt\right| \\
&\le \frac{1}{4}\left\|\eta\varphi_k^+\right\|^2_{L^\infty(t_0,t;H)} + \frac{1}{4}\alpha_0(1-\rho)\left\|\nabla\left(\eta\varphi_k^+\right)\right\|^2_{L^2(Q_{t_0,t})} + C_0\left\|\nabla\eta\varphi_k^+\right\|^2_{L^2(Q_{t_0,t})}, \quad (4.68)
\end{aligned}$$

where $C_0>0$ depends on α_0,ρ and the $L^4\left(Q_{T',T}\right)-$norm of $\boldsymbol{u}$ only. For the final term on the right-hand side of (4.67), we employ Hölder's and Young's inequalities again to deduce that

$$\begin{aligned}
\left|\int_{Q_{t_0,t}}\boldsymbol{\kappa}(x,t)\cdot\nabla\left(\eta^2\varphi_k^+\right)dxdt\right| &= \left|\int_{Q_{t_0,t}}\left(\boldsymbol{\kappa}(x,t)\cdot\varphi_k^+\eta\nabla\eta + \eta\boldsymbol{\kappa}(x,t)\cdot\nabla\left(\eta\varphi_k^+\right)\right)dxdt\right| \\
&\le C_1\int_{Q_{t_0,t}}|\eta|^2\,dxdt + \frac{1}{2}\int_{Q_{t_0,t}}\left(\varphi_k^+\right)^2|\nabla\eta|^2\,dxdt \\
&\quad + \frac{1}{4}\alpha_0(1-\rho)\int_{Q_{t_0,t}}\left|\nabla\left(\eta\varphi_k^+\right)\right|^2dxdt\,, \qquad (4.69)
\end{aligned}$$

where $C_1>0$ depends only on α_0,ρ and the $L^\infty\left(Q_{T',T}\right)$-norm of $\boldsymbol{\kappa}$. Inserting the estimates (4.68) and (4.69) into the right-hand side of (4.67), we infer the existence of a constant $C_2 = C_2\left(C_0,C_1,k^*\right)>0$ such that

$$\begin{aligned}
&\frac{1}{2}\sup_{s\in(t_0,t)}\int_\Omega\left(\eta\varphi_k^+\right)^2(s)\,dx + \alpha_0(1-\rho)\int_{Q_{t_0,t}}\left|\nabla\left(\eta\varphi_k^+\right)\right|^2dxdt \le \int_\Omega\left(\eta\varphi_k^+\right)^2(t_0)\,dx \\
&\quad + C_2\left(\int_{Q_{t_0,t}}\left(\varphi_k^+\right)^2|\eta\eta_t|\,dxdt + \int_{Q_{t_0,t}}\left(\varphi_k^+\right)^2|\nabla\eta|^2\,dxdt + \int_{Q_{t_0,t}}|\eta|^2\,dxdt\right). \qquad (4.70)
\end{aligned}$$

Arguing in a similar fashion, inequality (4.70) also holds with φ replaced by $-\varphi$. In particular, these inequalities imply that φ is an element of the class $\mathcal{B}_2\left(Q_{T',T}, 1, \gamma, 4, 1, 1\right)$ in the sense of [43, Chapter II, Section 7], for some $\gamma = \gamma\left(C_2\right)$, cf. inequality (7.5) of [43, Chapter II, Section 7, Remark 7.2]. Therefore, on account of [43, Chapter II, Section 7, Theorem 7.1], the Hölder continuity (4.65) of φ follows. This ends the proof. □

The approximation argument that can be employed to show that $\boldsymbol{u} \in L^\infty(V_{div}) \cap L^2(H^2(\Omega)^2)$ is the same as the one of Step 3 of [21, Proof of Theorem 5], to which we refer for the details. We just recall the main points: 1) φ is suitably mollified in the viscosity term of the Navier-Stokes equation only, namely, the following problem is considered:

$$\boldsymbol{u}_t - 2\text{div}\left(\nu(\varphi_\delta) D\boldsymbol{u}\right) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = (a\varphi - K * \varphi)\nabla\varphi + \boldsymbol{v}\,, \tag{4.71}$$

$$\text{div}(\boldsymbol{u}_\delta) = 0\,, \tag{4.72}$$

with initial condition $\boldsymbol{u}_\delta(0) = \boldsymbol{u}_0$ and no-slip boundary condition; 2) [1, Theorem 8] is applied to get a strong local in time solution $\boldsymbol{u}_\delta$ to (4.71)–(4.72), satisfying

$$\boldsymbol{u}_\delta \in H^1(0, T_\delta; G_{div}) \cap L^2(0, T_\delta; H^2(\Omega)^2) \cap L^\infty(0, T_\delta; V_{div})\,,$$

for some $T_\delta \leq T$; 3) thanks to Lemma 3, we have $\nu(\varphi_\delta) \in C^{\gamma,\gamma/2}(\bar{\Omega} \times [0,T])$, for some $0 < \gamma \leq \min\{\alpha, \beta\}$, and this allows us to apply Proposition 1 to (4.63)–(4.64) (written with $\boldsymbol{u}_\delta$ and φ_δ in place of $\boldsymbol{u}$ and φ, respectively). Arguing as in [21, Proof of Theorem 5, Step 2], we test the Navier-Stokes equations (4.71) by $\partial_t \boldsymbol{u}_\delta$. It is then easy to deduce a differential inequality of the form

$$\begin{aligned}\frac{d}{dt}\int_\Omega \nu(\varphi_\delta)|D\boldsymbol{u}_\delta|^2 + \frac{1}{8}\|\partial_t \boldsymbol{u}_\delta\|^2 &\leq C\left(\|\boldsymbol{l}_\delta\|^2 + \|\boldsymbol{v}\|^2 + \|\nabla\varphi_\delta\|^2\right)\\ &+ C(\|\boldsymbol{u}_\delta\|^2\|\nabla \boldsymbol{u}_\delta\|^2 + \|\nabla\varphi_\delta\|_{L^4}^4 + \|\partial_t\varphi_\delta\|^2)\|D\boldsymbol{u}_\delta\|^2,\end{aligned} \tag{4.73}$$

where

$$\boldsymbol{l}_\delta := -\frac{\varphi_\delta^2}{2}\nabla a - (J * \varphi_\delta)\nabla\varphi_\delta + \boldsymbol{v}.$$

From (4.73), on account of **(V)**, of the improved regularity for φ obtained in Step 1 and of the fact that we have $\varphi \in L^4(0,T; W^{1,4}(\Omega))$, (these regularities yield that $\partial_t\varphi_\delta$ is bounded in $L^2(0,T;H)$ and that φ_δ is bounded in $L^4(0,T;W^{1,4}(\Omega))$, uniformly w.r.t. δ), on account of the uniform w.r.t. δ bound of $\boldsymbol{u}_\delta$ in $L^\infty(0,T;G_{div}) \cap L^2(0,T;V_{div})$ (which stems from the energy identity obtained by testing (4.71) by $\boldsymbol{u}_\delta$ in G_{div}), and also of the condition on the initial velocity field $\boldsymbol{u}_0 \in V_{div}$, by means of Gronwall's lemma and of Proposition 1 once again, we can prove that $\boldsymbol{u}_\delta$ is bounded in $L^\infty(0,T_\delta;V_{div}) \cap H^1(0,T_\delta;G_{div})$ uniformly w.r.t. δ, and, by comparison in (4.71), that $\boldsymbol{u}_\delta$ is uniformly bounded in $L^2(0,T_\delta;H^2(\Omega)^2)$. These estimates entail, in particular, that $\boldsymbol{u}_\delta$ can be extended to any interval $(0,T)$, for all $T > 0$; 4) the passage to the limit in (4.71), (4.72), as $\delta \to 0$, is performed, by employing compactness arguments and the strong convergence $\varphi_\delta(t) \to \varphi$ in V, for almost ant $t \in (0,T)$. This gives a strong solution $\tilde{\boldsymbol{u}}$ to the same problem solved by the weak solution $\boldsymbol{u}$. Finally, 4) the limit velocity field $\tilde{\boldsymbol{u}} = \boldsymbol{u}$, on account of the uniqueness for Navier-Stokes equation with a given (nonconstant) viscosity. Therefore, existence of a strong solution satisfying (3.8) and (3.9) is proven. The uniqueness of this strong solution follows from [21, Theorem 7]. This concludes the proof of the second part of Theorem 2.

Step 3. In order to prove the last part, the idea is to differentiate (3.1) in time and test the resulting equation by φ_t. To make the argument rigorous, we employ the same time-discretization scheme of Step 1, taking the improved regularity for $\boldsymbol{u}$ (cf. Step 2) into account. Therefore, for $k = 1,\dots,N-1$, we consider problem (4.11)–(4.12) (where, in (4.11), the discrete time derivative $(\varphi_{k+1}-\varphi_k)/\tau$ is made explicit) at step k and at step $k-1$. Taking the difference between the two equations (4.11) written for these steps, testing the resulting identity by $(\varphi_{k+1}-\varphi_k)/\tau$, and summing over $k=1,\dots,n$, with $n \le N-1$, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{n}\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau} - \frac{\varphi_k-\varphi_{k-1}}{\tau}, \frac{\varphi_{k+1}-\varphi_k}{\tau}\Big) \\
&= -\sum_{k=1}^{n}\Big(\nabla\big(B(\cdot,\varphi_{k+1}) - B(\cdot,\varphi_k)\big), \nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big) \\
&- \sum_{k=1}^{n}\Big(\boldsymbol{U}_k\cdot\nabla\varphi_{k+1} - \boldsymbol{U}_{k-1}\cdot\nabla\varphi_k\,, \frac{\varphi_{k+1}-\varphi_k}{\tau}\Big) \\
&- \sum_{k=1}^{n}\Big(\big(\mathcal{N}(\varphi_k) - \mathcal{N}(\varphi_{k-1})\big)\nabla a, \nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big) \\
&+ \sum_{k=1}^{n}\Big(m(\varphi_k)\big(\nabla K * Q(\varphi_k)\big) - m(\varphi_{k-1})\big(\nabla K * Q(\varphi_{k-1})\big)\,, \nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big)\,,
\end{aligned}
\tag{4.74}
$$

where, again, for simplicity of notation, the explicit indication of the parameter ϵ is omitted.

Let us now estimate the terms on the right-hand side of (4.74). As far as the first term is concerned, we have

$$
\begin{aligned}
&\sum_{k=1}^{n}\Big(\nabla\big(B(\cdot,\varphi_{k+1}) - B(\cdot,\varphi_k)\big), \nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big) \\
&= \tau\sum_{k=1}^{n}\Big(\beta(\cdot,\varphi_{k+1})\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\,, \nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big) \\
&+ \sum_{k=1}^{n}\Big(\big(\beta(\cdot,\varphi_{k+1}) - \beta(\cdot,\varphi_k)\big)\nabla\varphi_k\,, \nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big)
\end{aligned}
$$

$$+\sum_{k=1}^{n}\Big(\big(\mathcal{M}(\varphi_{k+1})-\mathcal{M}(\varphi_k)\big)\nabla a\,,\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big)$$

$$\geq \frac{1}{2}\alpha_0(1-\rho)\tau\sum_{k=1}^{n}\Big\|\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big\|^2-\frac{\tau}{\alpha_0(1-\rho)}\sum_{k=1}^{n}\Big\|\frac{\beta(\cdot,\varphi_{k+1})-\beta(\cdot,\varphi_k)}{\tau}\nabla\varphi_k\Big\|^2$$

$$-\frac{\tau}{\alpha_0(1-\rho)}\sum_{k=1}^{n}\Big\|\frac{\mathcal{M}(\varphi_{k+1})-\mathcal{M}(\varphi_k)}{\tau}\nabla a\Big\|^2. \tag{4.75}$$

On the other hand, in light of (4.10), (4.37) and (3.4), we have

$$\Big\|\frac{\beta(\cdot,\varphi_{k+1})-\beta(\cdot,\varphi_k)}{\tau}\nabla\varphi_k\Big\|^2$$

$$\leq {\beta'_\infty}^2\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|^2_{L^4(\Omega)}\|\nabla\varphi_k\|^2_{L^4(\Omega)^2}$$

$$\leq C\Big(\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|^2+\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|\Big\|\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big\|\Big)\Big\|\frac{\nabla B(\cdot,\varphi_k)-\mathcal{M}(\varphi_k)\nabla a}{\beta(\cdot,\varphi_k)}\Big\|^2_{L^4(\Omega)^2}$$

$$\leq \frac{1}{4}\alpha_0(1-\rho)\Big\|\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big\|^2+\mathbb{Q}\,\|B(\cdot,\varphi_k)\|^2_{H^2(\Omega)}\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|^2+\mathbb{Q}\,\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|^2. \tag{4.76}$$

Therefore, we get

$$\sum_{k=1}^{n}\Big(\nabla\big(B(\cdot,\varphi_{k+1})-B(\cdot,\varphi_k)\big),\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big)\geq\frac{1}{4}\alpha_0(1-\rho)\tau\sum_{k=1}^{n}\Big\|\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big\|^2$$

$$-\mathbb{Q}\,\tau\sum_{k=1}^{n}\|B(\cdot,\varphi_k)\|^2_{H^2(\Omega)}\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|^2-\mathbb{Q}\,\tau\sum_{k=1}^{n}\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|^2. \tag{4.77}$$

Regarding the second term on the right-hand side of (4.74), we have

$$\sum_{k=1}^{n}\Big(\boldsymbol{U}_k\cdot\nabla\varphi_{k+1}-\boldsymbol{U}_{k-1}\cdot\nabla\varphi_k\,,\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)=\tau\sum_{k=1}^{n}\Big(\frac{\boldsymbol{U}_k-\boldsymbol{U}_{k-1}}{\tau}\cdot\nabla\varphi_k\,,\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)$$

$$\leq\tau\sum_{k=1}^{n}\Big\|\frac{\boldsymbol{U}_k-\boldsymbol{U}_{k-1}}{\tau}\Big\|\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|_{L^4(\Omega)}\Big\|\frac{\nabla B(\cdot,\varphi_k)-\mathcal{M}(\varphi_k)\nabla a}{\beta(\cdot,\varphi_k)}\Big\|_{L^4(\Omega)^2}$$

$$\leq\tau\,\mathbb{Q}\sum_{k=1}^{n}\Big\|\frac{\boldsymbol{U}_k-\boldsymbol{U}_{k-1}}{\tau}\Big\|\Big(\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|+\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|^{1/2}\Big\|\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big\|^{1/2}\Big)$$

$$\cdot\big(\|B(\cdot,\varphi_k)\|^{1/2}_{H^2(\Omega)}+1\big)\leq\delta\tau\sum_{k=1}^{n}\Big\|\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big\|^2+C_\delta\,\tau\sum_{k=1}^{n}\Big\|\frac{\boldsymbol{U}_k-\boldsymbol{U}_{k-1}}{\tau}\Big\|^2$$

$$+\mathbb{Q}_\delta\,\tau\sum_{k=1}^{n}\|B(\cdot,\varphi_k)\|^2_{H^2(\Omega)}\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|^2+\mathbb{Q}_\delta\,\tau\sum_{k=1}^{n}\Big\|\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big\|^2, \tag{4.78}$$

where $\delta>0$ will be fixed later. Finally, the last two terms on the right-hand side of (4.74) are estimated as follows

$$\sum_{k=1}^{n}\Big(\big(\mathcal{N}(\varphi_k)-\mathcal{N}(\varphi_{k-1})\big)\nabla a,\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big)$$

$$\leq C\,\tau \sum_{k=1}^{n} \Big\| \frac{\varphi_k - \varphi_{k-1}}{\tau} \Big\| \Big\| \nabla\Big(\frac{\varphi_{k+1} - \varphi_k}{\tau} \Big) \Big\|$$

$$\leq \delta\tau \sum_{k=1}^{n} \Big\| \nabla\Big(\frac{\varphi_{k+1} - \varphi_k}{\tau} \Big) \Big\|^2 + C_\delta\,\tau \sum_{k=1}^{n} \Big\| \frac{\varphi_k - \varphi_{k-1}}{\tau} \Big\|^2 , \tag{4.79}$$

$$\sum_{k=1}^{n} \Big(m(\varphi_k)\big(\nabla K * Q(\varphi_k)\big) - m(\varphi_{k-1})\big(\nabla K * Q(\varphi_{k-1})\big)\,, \nabla\Big(\frac{\varphi_{k+1} - \varphi_k}{\tau} \Big) \Big)$$

$$\leq (m_\infty + m'_\infty)\, b\,\tau \sum_{k=1}^{n} \Big\| \frac{\varphi_k - \varphi_{k-1}}{\tau} \Big\| \Big\| \nabla\Big(\frac{\varphi_{k+1} - \varphi_k}{\tau} \Big) \Big\|$$

$$\leq \delta\tau \sum_{k=1}^{n} \Big\| \nabla\Big(\frac{\varphi_{k+1} - \varphi_k}{\tau} \Big) \Big\|^2 + C_\delta\,\tau \sum_{k=1}^{n} \Big\| \frac{\varphi_k - \varphi_{k-1}}{\tau} \Big\|^2 . \tag{4.80}$$

By applying (4.16) to the left-hand side of (4.74), inserting estimates (4.77)–(4.80) into the right-hand side, choosing δ small enough and taking (4.37) into account, we obtain

$$\frac{1}{2} \Big\| \frac{\varphi_{n+1} - \varphi_n}{\tau} \Big\|^2 + \frac{1}{2} \sum_{k=1}^{n} \Big\| \frac{\varphi_{k+1} - \varphi_k}{\tau} - \frac{\varphi_k - \varphi_{k-1}}{\tau} \Big\|^2$$
$$+ \frac{1}{8}\alpha_0(1-\rho)\tau \sum_{k=1}^{n} \Big\| \nabla\Big(\frac{\varphi_{k+1} - \varphi_k}{\tau} \Big) \Big\|^2 \leq \frac{1}{2} \Big\| \frac{\varphi_1 - \varphi_0}{\tau} \Big\|^2$$
$$+ \mathbb{Q}\,\tau \sum_{k=1}^{n} \|B(\cdot,\varphi_k)\|^2_{H^2(\Omega)} \Big\| \frac{\varphi_{k+1} - \varphi_k}{\tau} \Big\|^2 + C\,\tau \sum_{k=1}^{n} \Big\| \frac{\boldsymbol{U}_k - \boldsymbol{U}_{k-1}}{\tau} \Big\|^2 + \mathbb{Q} . \tag{4.81}$$

Observe now that we have (cf. (4.30)–(4.35) and (4.37))

$$\tau \|B(\cdot,\varphi_k)\|^2_{H^2(\Omega)} \leq \frac{C}{\tau} \|\varphi_k - \varphi_{k-1}\|^2 + C\tau\,,$$

where here the constant C depends on the norm of $\boldsymbol{u}$ in $L^\infty(0,T;G_{div})$ and in $L^2(0,T;V_{div})$. Therefore, from (4.81) we obtain

$$\frac{1}{2} \Big\| \frac{\varphi_{n+1} - \varphi_n}{\tau} \Big\|^2 + \frac{1}{2} \sum_{k=1}^{n} \Big\| \frac{\varphi_{k+1} - \varphi_k}{\tau} - \frac{\varphi_k - \varphi_{k-1}}{\tau} \Big\|^2$$
$$+ \frac{1}{8}\alpha_0(1-\rho)\tau \sum_{k=1}^{n} \Big\| \nabla\Big(\frac{\varphi_{k+1} - \varphi_k}{\tau} \Big) \Big\|^2 \leq \frac{1}{2} \Big\| \frac{\varphi_1 - \varphi_0}{\tau} \Big\|^2$$
$$+ \mathbb{Q} \sum_{k=0}^{n-1} \frac{1}{\tau} \|\varphi_{k+2} - \varphi_{k+1}\|^2 \Big\| \frac{\varphi_{k+1} - \varphi_k}{\tau} \Big\|^2 + \frac{\mathbb{Q}}{\tau} \sum_{k=1}^{n} \|\varphi_{k+1} - \varphi_k\|^2$$
$$+ C\,\tau \sum_{k=1}^{n} \Big\| \frac{\boldsymbol{U}_k - \boldsymbol{U}_{k-1}}{\tau} \Big\|^2 + \mathbb{Q} . \tag{4.82}$$

The delicate point is now the control of the L^2-norm of the quotient $(\varphi_1-\varphi_0)/\tau$ on the right-hand side. To this goal, let us first point out a remarkable consequence we have from the improved regularity of the velocity field obtained in Step 2, which concerns the solvability of the incremental-step problem (4.11)–(4.12). Indeed, for a given $\varphi_k\in V$, $k=0,\dots N-1$, let us introduce the nonlinear operator $\mathcal{B}_k : D(\mathcal{B}_k)\subset H\to H$, defined by

$$\mathcal{B}_k\varphi := -\Delta B(\cdot,\varphi)+\boldsymbol{U}_k\cdot\nabla\varphi-\mathrm{div}\big(\mathcal{N}(\varphi_k)\nabla a\big)+\mathrm{div}\big(m(\varphi_k)(\nabla K * Q(\varphi_k))\big)\,,$$
$$D(\mathcal{B}_k):=\Big\{\varphi\in H^2(\Omega)\ :\ \frac{\partial B(\cdot,\varphi)}{\partial \boldsymbol{n}}=m(\varphi_k)(\nabla K*Q(\varphi_k))\cdot\boldsymbol{n}-\mathcal{N}(\varphi_k)(\nabla a\cdot\boldsymbol{n})\,,\ \text{a.e. on } \partial\Omega\Big\}\,.$$

We prove that there exists $\tau_0=\tau_0(\boldsymbol{u})>0$ such that we have

$$\|(\varphi_2-\varphi_1)+\tau(\mathcal{B}_k\varphi_2-\mathcal{B}_k\varphi_1)\|\geq\frac{\alpha_0(1-\rho)}{2k^*}\|\varphi_2-\varphi_1\|\,,\qquad\forall\varphi_1,\varphi_2\in D(\mathcal{B}_k)\,,\tag{4.83}$$

and for all $0<\tau\leq\tau_0$. This, in particular, implies that the solution to each incremental-step problem (4.11)–(4.12), for $k=0,\dots,N-1$, is unique.

In order to prove (4.83), we first observe that, for all $\varphi_1,\varphi_2\in D(\mathcal{B}_k)$, we have

$$\begin{aligned}&\big(\varphi_2-\varphi_1+\tau(\mathcal{B}_k\varphi_2-\mathcal{B}_k\varphi_1),B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\geq\alpha_0(1-\rho)\|\varphi_2-\varphi_1\|^2\\&+\tau\|\nabla\big(B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\|^2-\tau\big(\boldsymbol{U}_k\cdot(\varphi_2-\varphi_1),\nabla\big(B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\big).\end{aligned}\tag{4.84}$$

Thanks to the improved regularity (3.9), we have

$$\|\boldsymbol{U}_k\|_{V_{div}}\leq\|\boldsymbol{u}\|_{L^\infty(0,T;V_{div})}\,,\qquad\|\boldsymbol{U}_k\|_{H^2(\Omega)^2}\leq\frac{1}{\sqrt{\tau}}\|\boldsymbol{u}\|_{L^2(0,T;H^2(\Omega)^2)}\,.\tag{4.85}$$

Hence, by means of $(4.85)_2$ and by Agmon's inequality (2.2), the last term on the right-hand side of (4.84) can be estimated as follows

$$\begin{aligned}&\tau\big|\big(\boldsymbol{U}_k\cdot(\varphi_2-\varphi_1),\nabla\big(B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\big)\big|\\&\leq\tau\|\boldsymbol{U}_k\|_{L^\infty(\Omega)^2}\|\varphi_2-\varphi_1\|\|\nabla\big(B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\|\\&\leq\frac{\tau}{2}\|\nabla\big(B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\|^2+\frac{\tau}{2}\,\hat{C}_3^2\,\|\boldsymbol{U}_k\|\|\boldsymbol{U}_k\|_{H^2(\Omega)^2}\|\varphi_2-\varphi_1\|^2\\&\leq\frac{\tau}{2}\|\nabla\big(B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\|^2+\frac{\sqrt{\tau}}{2}\,\hat{C}_3^2\,\|\boldsymbol{u}\|_{L^\infty(0,T;G_{div})}\|\boldsymbol{u}\|_{L^2(0,T;H^2(\Omega)^2)}\|\varphi_2-\varphi_1\|^2\,.\end{aligned}\tag{4.86}$$

Therefore, by taking $0<\tau\leq\tau_0$, with τ_0 given by

$$\tau_0:=\frac{\alpha_0^2(1-\rho)^2}{\hat{C}_3^4\|\boldsymbol{u}\|^2_{L^\infty(0,T;G_{div})}\|\boldsymbol{u}\|^2_{L^2(0,T;H^2(\Omega)^2)}}\,,$$

the right-hand side of (4.84) can be estimated from below by

$$\frac{\alpha_0(1-\rho)}{2}\|\varphi_2-\varphi_1\|^2+\frac{\tau}{2}\|\nabla\big(B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\|^2\,. \tag{4.87}$$

On the other hand, due to (4.6), we have

$$\begin{aligned}&\big(\varphi_2-\varphi_1+\tau(\mathcal{B}_k\varphi_2-\mathcal{B}_k\varphi_1),B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\\&\le k^*\|\varphi_2-\varphi_1+\tau(\mathcal{B}_k\varphi_2-\mathcal{B}_k\varphi_1)\|\|\varphi_2-\varphi_1\|\\&\le\frac{\alpha_0(1-\rho)}{4}\|\varphi_2-\varphi_1\|^2+\frac{k^{*2}}{\alpha_0(1-\rho)}\|\varphi_2-\varphi_1+\tau(\mathcal{B}_k\varphi_2-\mathcal{B}_k\varphi_1)\|^2\,.\end{aligned} \tag{4.88}$$

Hence, from (4.84), (4.87) and (4.88) we get

$$\begin{aligned}\frac{k^{*2}}{\alpha_0(1-\rho)}\|\varphi_2-\varphi_1+\tau(\mathcal{B}_k\varphi_2-\mathcal{B}_k\varphi_1)\|^2\ge{}&\frac{\alpha_0(1-\rho)}{4}\|\varphi_2-\varphi_1\|^2\\&+\frac{\tau}{2}\|\nabla\big(B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\big)\|^2\,,\end{aligned}$$

and this proves the desired claim (4.83). Therefore, for $0<\tau\le\tau_0$, and for every $k=0,\dots,N-1$, the resolvent operator $J_{k,\tau}:=(I+\tau\mathcal{B}_k)^{-1}$ is single-valued and Lipschitz continuous from H to H. Indeed we have

$$\|J_{k,\tau}\psi_2-J_{k,\tau}\psi_1\|\le\frac{2k^*}{\alpha_0(1-\rho)}\|\psi_2-\psi_1\|\,,\qquad\forall\psi_1,\psi_2\in H\,,\quad 0<\tau\le\tau_0\,. \tag{4.89}$$

Notice that, if the first term φ_k on the right-hand side of (4.11) is assumed in H, the solvability of problem (4.11)–(4.12) still holds, arguing as at the beginning of Step 1. Indeed, the nonlinear operator A_k is the same and we still have $g_k\in V'$.

Let us now go back to the problem of controlling the L^2-norm of the quotient $(\varphi_1-\varphi_0)/\tau$. By employing (4.89) for $k=0$, using the assumption on φ_0 which yields that $\varphi_0\in D(\mathcal{B}_0)$, and assuming that $0<\tau\le\tau_0$, we find

$$\begin{aligned}\Big\|\frac{\varphi_1-\varphi_0}{\tau}\Big\|&=\Big\|\frac{J_{0,\tau}\varphi_0-J_{0,\tau}(I+\tau\mathcal{B}_0)\varphi_0}{\tau}\Big\|\le\frac{2k^*}{\alpha_0(1-\rho)}\|\mathcal{B}_0\varphi_0\|\\&\le C\big(\|\Delta B(\cdot,\varphi_0)\|+\|\boldsymbol{u}\|_{L^\infty(0,T;V_{div})}\|\varphi_0\|_{H^2(\Omega)}+\|\varphi_0\|_V+1\big)\,,\end{aligned} \tag{4.90}$$

where we have also used $(4.85)_1$.

Finally, there remains to bound the last sum on the right-hand side of (4.81). To this aim, we can first easily see that the following estimate holds

$$\|\boldsymbol{u}(k\tau)-\boldsymbol{U}_k\|^2\le\frac{\tau}{3}\int_{k\tau}^{(k+1)\tau}\|\boldsymbol{u}_t(s)\|^2\,ds\,.$$

By employing this estimate, a simple computation yields

$$\frac{1}{\tau}\sum_{k=1}^{n}\|\boldsymbol{U}_k-\boldsymbol{U}_{k-1}\|^2\le c\|\boldsymbol{u}_t\|^2_{L^2(0,T;G_{div})},\tag{4.91}$$

where the constant c can be given by $c=10/3$.

We can now apply the discrete Gronwall Lemma to (4.81), taking (4.37), (4.90) and (4.91) into account, to obtain

$$\Big\|\frac{\varphi_{n+1}-\varphi_n}{\tau}\Big\|^2+\tau\sum_{k=1}^{n}\Big\|\nabla\Big(\frac{\varphi_{k+1}-\varphi_k}{\tau}\Big)\Big\|^2\le\mathbb{Q}\big(\|\varphi_0\|_{H^2(\Omega)},\|\boldsymbol{u}\|_{L^\infty(0,T;V_{div})\cap H^1(0,T;G_{div})}\big)\,.\tag{4.92}$$

From this discrete estimate we get the following new bound for the approximate solutions $\widehat{\varphi}_N$, $\overline{\varphi}_N$ introduced in Step 1

$$\|\widehat{\varphi}_N'\|^2_{L^\infty(0,T;H)}+\|\widehat{\varphi}_N'\|^2_{L^2(0,T;V)}\le\mathbb{Q}\big(\|\varphi_0\|_{H^2(\Omega)},\|\boldsymbol{u}\|_{L^\infty(0,T;V_{div})\cap H^1(0,T;G_{div})}\big)\,.$$

Therefore, in addition to (4.47)–(4.54), we also have, up to a subsequence,

$$\widehat{\varphi}_N'\rightharpoonup\varphi_t\,,\qquad\text{weakly* in } L^\infty(0,T;H)\,,\qquad\text{weakly in } L^2(0,T;V)\,,$$

and this proves $(3.11)_2$. Moreover, since we have (cf. (4.30)–(4.35) and (4.37))

$$\|B(\cdot,\varphi_{n+1})\|^2_{H^2(\Omega)}\le C\Big\|\frac{\varphi_{n+1}-\varphi_n}{\tau}\Big\|^2+C\,,$$

then, thanks to (4.92), we get the bound

$$\|B(\cdot,\overline{\varphi}_N)\|_{L^\infty(0,T;H^2(\Omega))}\le\mathbb{Q}\big(\|\varphi_0\|_{H^2(\Omega)},\|\boldsymbol{u}\|_{L^\infty(0,T;V_{div})\cap H^1(0,T;G_{div})}\big)\,.$$

This obviously implies

$$\begin{aligned}&\|\overline{\varphi}_N\|_{L^\infty(0,T;W^{1,p}(\Omega))}+\|B(\cdot,\overline{\varphi}_N)\|_{L^\infty(0,T;W^{1,p}(\Omega))}\\&+\|\beta(\cdot,\overline{\varphi}_N)\|_{L^\infty(0,T;W^{1,p}(\Omega))}\le\mathbb{Q}\big(\|\varphi_0\|_{H^2(\Omega)},\|\boldsymbol{u}\|_{L^\infty(0,T;V_{div})\cap H^1(0,T;G_{div})}\big)\,.\end{aligned}\tag{4.93}$$

Hence, recalling (4.15) (written in terms of the approximate solutions $\overline{\varphi}_N$) and using (4.93), we infer

$$\|\overline{\varphi}_N\|_{L^\infty(0,T;H^2(\Omega))}\le\mathbb{Q}\big(\|\varphi_0\|_{H^2(\Omega)},\|\boldsymbol{u}\|_{L^\infty(0,T;V_{div})\cap H^1(0,T;G_{div})}\big)\,.\tag{4.94}$$

Therefore, up to a subsequence, we have

$$\overline{\varphi}_N\rightharpoonup\varphi\,,\qquad\text{weakly* in } L^\infty(0,T;H^2(\Omega))\,,$$

whence we get $(3.11)_1$. The argument to pass to the limit in (4.44)–(4.45), and also to prove the pointwise bound $|\varphi|\le 1$, is the same as in Step 1 (here we can also rely on even stronger convergence results). The proof of Theorem 2 is finished.

Remark 7. It is not known whether a strong solution according to Definition 2 also satisfies equations (1.2)–(1.3) and the related boundary condition in a strong sense. This occurs if we can guarantee the validity of a strict separation property, namely, the fact that φ stay uniformly away from the pure phases (see, e.g., [41, 42] for a slightly different version of nonlocal CH equation). An intermediate situation holds if $F'(\varphi_0) \in H$ (see [25, Theorem 3]). In this case the weak formulation where $\mu \in L^2(0,T;V)$ appears explicitly can be recovered (cf. [25, Definition 1]).

Remark 8. If ν is constant then we can prove the existence of strong solutions to (1.1)–(1.5) by using a different argument which exploits the classical regularity result [51, Theorem 3.10] for the two dimensional incompressible Navier-Stokes system. This was the strategy followed in [24, Proof of Theorem 2]. Indeed, notice that (1.1) can be rewritten in the form

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\hat{\pi} = (a\varphi - K*\varphi)\,\nabla\varphi + \boldsymbol{v}, \tag{4.95}$$

where the modified pressure $\hat{\pi} := \pi - F(\varphi)$ has been introduced. Thanks to the regularity properties of the weak solution (cf., in particular, the bound $|\varphi| \leq 1$) and to the assumption on $\boldsymbol{v}$, we see that the right-hand side of (4.95) belongs to $L^2\left(0,T;L^2\left(\Omega\right)^2\right)$. Hence, under the assumption that $\boldsymbol{u}_0 \in V_{div}$, the regularity (3.7) for the velocity field $\boldsymbol{u}$ immediately follows from applying [51, Theorem 3.10] to (4.95). Once (3.7) is available, we can devise an easier argument in Step 1, by using (4.28) and $(4.85)_2$ to estimate the last term on the right-hand side of (4.22) simply as follows

$$\begin{aligned}\Big|\sum_{k=0}^{n}\big(\boldsymbol{U}_k\cdot\nabla\varphi_{k+1}, B(\cdot,\varphi_{k+1}) - B(\cdot,\varphi_k)\big)\Big| &\leq \frac{\alpha_0(1-\rho)}{4\tau}\sum_{k=0}^{n}\|\varphi_{k+1}-\varphi_k\|^2 \\ &\quad + C{k^*}^2\|\boldsymbol{u}\|^2_{L^2(0,T;H^2(\Omega)^2)}\sum_{k=0}^{n}\|\nabla\varphi_{k+1}\|^2 .\end{aligned} \tag{4.96}$$

This estimate, together with (4.23)–(4.26), still yield a discrete Gronwall's inequality from (4.22) (cf. (4.36)) and thus allows to obtain the regularity $\varphi \in L^\infty(0,T;V)$, $\varphi_t \in L^2(0,T;H)$. Notice that the assumption that $K \in W^{2,1}_{loc}$ or that K is admissible is not required in this argument (only **(K)** is enough). This regularity assumption on the kernel is needed only in Step 3, in order to prove that $\varphi \in L^2(0,T;H^2(\Omega))$ and, provided $\varphi_0 \in H^2(\Omega)$ satisfies (3.10), that (3.11) holds.

Remark 9. Assume that $\boldsymbol{u}_0 \in V_{div}$ and $\varphi_0 \in H^2(\Omega)$ satisfies (3.10). By integrating (4.73) in time and by passing to the liminf in (4.42), (4.94)*, we can also prove that there exists a continuous and nondecreasing function* $\mathbb{Q}_1 : [0,\infty) \to [0,+\infty)$ *which only depends on the data* F*,* m*,* K*,* ν*,* Ω*,* T*,* $\boldsymbol{u}_0$ *and* φ_0*, such that*

$$\begin{aligned}&\|\boldsymbol{u}\|_{L^\infty([0,T];V_{div})\cap L^2(0,T;H^2(\Omega)^2)} + \|\boldsymbol{u}_t\|_{L^2([0,T];G_{div})} \\ &+ \|\varphi\|_{L^\infty([0,T];H^2(\Omega))} + \|\varphi_t\|_{L^\infty([0,T];H)\cap L^2(0,T;V)} \\ &\leq \mathbb{Q}_1\left(\|\boldsymbol{v}\|_{L^2(0,T;G_{div})}\right).\end{aligned} \tag{4.97}$$

Remark 10. We point out that the estimates in the proof of Theorem 2 rely essentially on:

(i) the boundedness and Lipschitz continuity properties of the nonlinear functions β, m, $\mathcal{M}$, $\mathcal{N}$, given by (4.6)–(4.10);

(ii) the fact that φ is bounded (cf. the control of the boundary term in (4.31)).

Therefore, the argument of Theorem 2 also works for other classes of mobilities and double-well potentials, provided they ensure the validity of (i) and (ii). An example is given by a nondegenerate mobility and a regular potential defined on the whole real line and satisfying the assumptions of, e.g., [24, Theorem 2]. The boundedness of φ follows by simply adapting the Alikakos iteration argument (see [6, Theorem 2.1]). More precisely, in this case, the uniform bound in $L^\infty(\Omega)$ of φ_{k+1} (cf. Step I of the proof of Theorem 2) will be proven below (cf. proof of Theorem 4).

5 Uniform estimates

In this section we establish some uniform in time regularization estimates. To this aim we shall first formally deduce the same kind of higher order bounds which were derived rigorously in the context of the time-discretization scheme in the proof of Theorem 2. These will be the basis for constructing uniform in time estimates. As a consequence, we establish a regularity property for the global attractor of the dynamical system generated by (1.1)–(1.6), the existence of which was proven in [25]. We point out that the argument of Proposition 2 below can be made rigorous by means of time discretization combined with a discrete variant of the uniform Gronwall lemma (see [48, Lemma 3]). Thus, we proceed formally just for the sake of brevity.

Proposition 2. *Suppose that assumptions **(K), (V), (M), (A1)**$_1$, **(A4)–(A5)** are satisfied and suppose that $K \in W^{2,1}_{loc}(\mathbb{R}^2)$ or that K is admissible. Let $\boldsymbol{u}_0 \in G_{div}$, $\varphi_0 \in V \cap L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$, where M is defined as in Theorem 1. Let also $\boldsymbol{v} \in L^2_{tb}(0,\infty;G_{div})$. Then there exists a weak solution $[\boldsymbol{u},\varphi]$ to system* (1.2)–(1.6) *such that*

$$\boldsymbol{u} \in L^\infty\left(0,\infty;G_{div}\right) \cap L^2_{tb}\left(0,\infty;V_{div}\right), \qquad \boldsymbol{u}_t \in L^2_{tb}\left(0,\infty;V_{div}'\right), \tag{5.1}$$

$$\varphi \in L^\infty(0,\infty;V) \cap L^2_{tb}(0,\infty;H^2(\Omega)), \qquad \varphi_t \in L^2_{tb}(0,\infty;H). \tag{5.2}$$

If, in addition, $\boldsymbol{u}_0 \in V_{div}$ and $\varphi_0 \in V \cap C^\beta(\bar\Omega)$ for some $\beta \in (0,1)$. Then, the (unique) strong solution given by Theorem 2 satisfies (5.2) *and*

$$\boldsymbol{u} \in L^\infty\left(0,\infty;V_{div}\right) \cap L^2_{tb}\left(0,\infty;H^2\left(\Omega\right)^2\right), \qquad \boldsymbol{u}_t \in L^2_{tb}\left(0,\infty;G_{div}\right). \tag{5.3}$$

Finally, suppose that $\varphi_0 \in H^2(\Omega)$ satisfies (3.10)*. Then, the strong solution also enjoys the following properties*

$$\varphi \in L^\infty(0,\infty;H^2(\Omega)), \qquad \varphi_t \in L^\infty(0,\infty;H) \cap L^2_{tb}(0,T;V). \tag{5.4}$$

Moreover, there exists a constant $\Gamma = \Gamma(\kappa)$*, depending on* $\kappa \in [0,1]$*, on* $\|\boldsymbol{v}\|_{L^2_{tb}(0,\infty;G_{div})}$ *(and on* F*,* m*,* K*,* ν*,* Ω*), such that, for every initial data* $[\boldsymbol{u}_0, \varphi_0] \in V_{div} \times H^2(\Omega)$*, with* φ_0 *satisfying* (3.10)*,* $F(\varphi_0), M(\varphi_0) \in L^1(\Omega)$ *(hence* $|\varphi_0| \leq 1$ *almost everywhere in* Ω*), and* $|\overline{\varphi}_0| \leq \kappa$*, there exists a time* $t_1 = t_1\big(E(\boldsymbol{u}_0, \varphi_0)\big) \geq 0$*, where* $E(\boldsymbol{u}_0, \varphi_0)$ *is given by* (5.9)*, such that the strong solution corresponding to* $[\boldsymbol{u}_0, \varphi_0]$ *satisfies the following dissipative estimate*

$$\|\boldsymbol{u}(t)\|^2_{V_{div}} + \int_t^{t+1} \|\boldsymbol{u}(s)\|^2_{H^2(\Omega)^2}\, ds + \|\varphi(t)\|^2_{H^2(\Omega)} \leq \Gamma(k)\,, \qquad \forall t \geq t_1\,. \tag{5.5}$$

Proof. Firs we observe that, by arguing as in [25, Proof of Proposition 2], from (2.4) we deduce the following differential inequality

$$\frac{d}{dt}\big(\|\boldsymbol{u}\|^2 + \|\varphi\,\|^2\big) + (1-\rho)\alpha_0\,\|\nabla\varphi\|^2 + \nu_1\|\nabla\boldsymbol{u}\|^2 \leq \hat{C} + \frac{1}{\nu_1\lambda_1}\|\boldsymbol{v}\|^2\,. \tag{5.6}$$

Moreover, again by arguing as in [25, Proof of Proposition 2] (see also [12, Proof of Corollary 2]), from (5.6) we infer the following dissipative estimate

$$\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2 \leq \big(\|\boldsymbol{u}_0\|^2 + \|\varphi_0\|^2\big)e^{-\ell\, t} + L\,, \qquad \forall t \geq 0\,, \tag{5.7}$$

where the positive constant L depends on $\overline{\varphi}_0$ and on $\|\boldsymbol{v}\|_{L^2_{tb}(0;\infty;G_{div})}$. This, in particular, entails that $\boldsymbol{u} \in L^\infty(0,\infty;G_{div})$. Let us now integrate (5.6) between t and $t+1$. We get

$$\begin{aligned}\|\boldsymbol{u}(t+1)\|^2 + \|\varphi(t+1)\,\|^2 + \alpha_0(1-\rho)\int_t^{t+1}\|\nabla\varphi(s)\|^2 ds + \nu_1\int_t^{t+1}\|\nabla\boldsymbol{u}(s)\|^2 ds \\ \leq \|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\,\|^2 + \hat{C} + \frac{1}{\nu_1\lambda_1}\int_t^{t+1}\|\boldsymbol{v}(s)\|^2 ds\,, \qquad \forall t \geq 0\,.\end{aligned} \tag{5.8}$$

Hence, (5.7) and (5.8) yield

$$\frac{\alpha_0(1-\rho)}{2}\int_t^{t+1}\|\nabla\varphi(s)\|^2 ds + \frac{\nu_1}{2}\int_t^{t+1}\|\nabla\boldsymbol{u}(s)\|^2 ds \leq E(\boldsymbol{u}_0,\varphi_0)e^{-\ell\, t} + \Gamma_0\,, \quad \forall t \geq 0\,,$$

where we have set

$$E(\boldsymbol{u}_0,\varphi_0) := \frac{1}{2}\big(\|\boldsymbol{u}_0\|^2 + \|\varphi_0\|^2\big)\,, \tag{5.9}$$

and where $\Gamma_0 = \mathbb{Q}(\kappa, \|\boldsymbol{v}\|_{L^2_{tb}(0,\infty;G_{div})})$, with $\kappa \in [0,1]$ such that $|\overline{\varphi}_0| \leq \kappa$. In particular, this gives

$$\boldsymbol{u} \in L^2_{tb}(0,\infty;V_{div})\,, \qquad \varphi \in L^2_{tb}(0,\infty;V)\,.$$

Moreover, there exists a time $t_0 = t_0\big(E(\boldsymbol{u}_0,\varphi_0)\big) > 0$, which can be given by

$$t_0 = \frac{1}{\ell}\log E(\boldsymbol{u}_0,\varphi_0)$$

such that

$$\frac{\alpha_0(1-\rho)}{2}\int_t^{t+1}\|\nabla\varphi(s)\|^2ds+\frac{\nu_1}{2}\int_t^{t+1}\|\nabla\boldsymbol{u}(s)\|^2ds\leq\Gamma_0+1\,,\qquad\forall t\geq t_0\,. \tag{5.10}$$

Let us now begin with the higher order estimates. We test (3.1) by $B(\cdot,\varphi)_t=\beta(\cdot,\varphi)\varphi_t$. On account of (3.5), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla B(\cdot,\varphi)\|^2+\int_\Omega\beta(\cdot,\varphi)\varphi_t^2+\big(\boldsymbol{u}\cdot\nabla\varphi,\beta(\cdot,\varphi)\varphi_t\big)\\&=-\big(\mathcal{N}(\varphi)\nabla a,\nabla B(\cdot,\varphi)_t\big)+\big(m(\varphi)(\nabla K*\varphi),\nabla B(\cdot,\varphi)_t\big)\,.\end{aligned} \tag{5.11}$$

Observe that

$$\nabla B(\cdot,\varphi)_t=\beta(\cdot,\varphi)\nabla\varphi_t+(m(\varphi)\nabla a+(m'(\varphi)a+\lambda'(\varphi))\nabla\varphi)\varphi_t\,. \tag{5.12}$$

Hence, the two terms on the right-hand side of (5.11) can be written as follows, respectively,

$$\begin{aligned}&-\big(\mathcal{N}(\varphi)\nabla a,\nabla B(\cdot,\varphi)_t\big)=-\frac{d}{dt}\big(\mathcal{N}(\varphi)\nabla a,\beta(\cdot,\varphi)\nabla\varphi\big)+\big(\mathcal{N}'(\varphi)\varphi_t\nabla a,\beta(\cdot,\varphi)\nabla\varphi\big)\\&+\big(\mathcal{N}(\varphi)\nabla a,(m'(\varphi)a+\lambda'(\varphi))\varphi_t\nabla\varphi\big)\\&-\big(\mathcal{N}(\varphi)\nabla a,(m(\varphi)\nabla a+(m'(\varphi)a+\lambda'(\varphi))\nabla\varphi)\varphi_t\big)=-\frac{d}{dt}\big(\mathcal{N}(\varphi)\nabla a,\beta(\cdot,\varphi)\nabla\varphi\big)\\&+\big(\varphi m'(\varphi)\varphi_t\nabla a,\beta(\cdot,\varphi)\nabla\varphi\big)-\big(\mathcal{N}(\varphi)\nabla a,m(\varphi)\varphi_t\nabla a\big)\,,\end{aligned} \tag{5.13}$$

$$\begin{aligned}&\big(m(\varphi)(\nabla K*\varphi),\nabla B(\cdot,\varphi)_t\big)=\frac{d}{dt}\big(m(\varphi)(\nabla K*\varphi),\beta(\cdot,\varphi)\nabla\varphi\big)\\&-\big(m'(\varphi)\varphi_t(\nabla K*\varphi),\beta(\cdot,\varphi)\nabla\varphi\big)-\big(m(\varphi)(\nabla K*\varphi_t),\beta(\cdot,\varphi)\nabla\varphi\big)\\&-\big(m(\varphi)(\nabla K*\varphi),(m'(\varphi)a+\lambda'(\varphi))\varphi_t\nabla\varphi\big)\\&+\big(m(\varphi)(\nabla K*\varphi),(m(\varphi)\nabla a+(m'(\varphi)a+\lambda'(\varphi))\nabla\varphi)\varphi_t\big)\\&=\frac{d}{dt}\big(m(\varphi)(\nabla K*\varphi),\beta(\cdot,\varphi)\nabla\varphi\big)-\big(m'(\varphi)\varphi_t(\nabla K*\varphi),\beta(\cdot,\varphi)\nabla\varphi\big)\\&-\big(m(\varphi)(\nabla K*\varphi_t),\beta(\cdot,\varphi)\nabla\varphi\big)+\big(m(\varphi)(\nabla K*\varphi),m(\varphi)\varphi_t\nabla a\big)\,.\end{aligned} \tag{5.14}$$

Therefore, plugging (5.13) and (5.14) into the differential identity (5.11), we get

$$\begin{aligned}&\frac{1}{2}\frac{d\Phi}{dt}+\int_\Omega\beta(\cdot,\varphi)\varphi_t^2+\big(\boldsymbol{u}\cdot\nabla\varphi,\beta(\cdot,\varphi)\varphi_t\big)\\&=\big(\varphi\, m'(\varphi)\varphi_t\nabla a,\beta(\cdot,\varphi)\nabla\varphi\big)-\big(\mathcal{N}(\varphi)\nabla a,m(\varphi)\varphi_t\nabla a\big)\\&-\big(m'(\varphi)\varphi_t(\nabla K*\varphi),\beta(\cdot,\varphi)\nabla\varphi\big)\\&-\big(m(\varphi)(\nabla K*\varphi_t),\beta(\cdot,\varphi)\nabla\varphi\big)+\big(m(\varphi)(\nabla K*\varphi),m(\varphi)\varphi_t\nabla a\big)\,,\end{aligned} \tag{5.15}$$

where the functional Φ is given by

$$\Phi:=\|\nabla B(\cdot,\varphi)\|^2+2\big(\mathcal{N}(\varphi)\nabla a,\beta(\cdot,\varphi)\nabla\varphi\big)-2\big(m(\varphi)(\nabla K*\varphi),\beta(\cdot,\varphi)\nabla\varphi\big). \tag{5.16}$$

On account of assumptions **(A1)**, **(A4)** and **(A5)**, which ensure that

$$(1-\rho)\alpha_0 \le \beta(x,s) \le k^*\,, \qquad \forall s\in[-1,1]\,, \quad \text{a.e. } x\in\Omega\,, \tag{5.17}$$

it is immediate to estimate the terms on the right-hand side of (5.15). Indeed, the first, third and fourth term can be controlled by

$$\frac{1}{12}(1-\rho)\alpha_0\|\varphi_t\|^2 + C_{m,\lambda,K}\|\nabla\varphi\|^2,$$

while the second and fifth term can be controlled by

$$\frac{1}{12}(1-\rho)\alpha_0\|\varphi_t\|^2 + C_{m,K}\,.$$

As far as the last term on the left-hand side of (5.15) is concerned, taking (3.4) into account, we have

$$\begin{aligned}
|\big(\boldsymbol{u}\cdot\nabla\varphi,\beta(\cdot,\varphi)\varphi_t\big)| &= |(\boldsymbol{u}\cdot(\nabla B(\cdot,\varphi)-\mathcal{M}(\varphi)\nabla a)\,,\varphi_t)|\\
&\le \|\boldsymbol{u}\|_{L^4(\Omega)^2}\|\nabla B\|_{L^4(\Omega)^2}\|\varphi_t\| + m_\infty\|\nabla a\|_\infty\|\boldsymbol{u}\|\|\varphi_t\|\\
&\le C\|\boldsymbol{u}\|^{1/2}\|\nabla\boldsymbol{u}\|^{1/2}\|\nabla B\|^{1/2}\|B\|^{1/2}_{H^2(\Omega)}\|\varphi_t\| + m_\infty\|\nabla a\|_\infty\|\boldsymbol{u}\|\|\varphi_t\|\,.
\end{aligned} \tag{5.18}$$

Let us now control the H^2-norm of $B(\cdot,\varphi)$ in terms of the L^2-norm of φ_t. To this end, we first employ elliptic regularity, namely

$$\|B(\cdot,\varphi)\|_{H^2(\Omega)} \le C\left(\|\Delta B(\cdot,\varphi)\| + \|B(\cdot,\varphi)\|_V + \left\|\frac{\partial B(\cdot,\varphi)}{\partial\boldsymbol{n}}\right\|_{H^{1/2}(\partial\Omega)}\right)\,. \tag{5.19}$$

Then we estimate the boundary term on the right-hand side by taking (3.5) into account. Arguing in a similar way as in the time discrete version (4.31), we find

$$\begin{aligned}
&\left\|\frac{\partial B(\cdot,\varphi)}{\partial\boldsymbol{n}}\right\|_{H^{1/2}(\partial\Omega)} \le \|m(\varphi)(\nabla K*\varphi)\cdot\boldsymbol{n}\|_{H^{1/2}(\partial\Omega)} + \|\mathcal{N}(\varphi)\nabla a\cdot\boldsymbol{n}\|_{H^{1/2}(\partial\Omega)}\\
&\le \|m(\varphi)\|_{L^\infty(\partial\Omega)}\|(\nabla K*\varphi)\cdot\boldsymbol{n}\|_{H^{1/2}(\partial\Omega)} + \|(\nabla K*\varphi)\cdot\boldsymbol{n}\|_{L^\infty(\partial\Omega)}\|m(\varphi)\|_{H^{1/2}(\partial\Omega)}\\
&+ \|\mathcal{N}(\varphi)\|_{L^\infty(\partial\Omega)}\|\nabla a\cdot\boldsymbol{n}\|_{H^{1/2}(\partial\Omega)} + \|\nabla a\cdot\boldsymbol{n}\|_{L^\infty(\partial\Omega)}\|\mathcal{N}(\varphi)\|_{H^{1/2}(\partial\Omega)}\\
&\le m_\infty\,\|K*\varphi\,\|_{H^2(\Omega)} + 3\,b\,m'_\infty\|\varphi\|_{H^{1/2}(\partial\Omega)} + 2\,b\,m_0|\Gamma|_1^{1/2} + N_\infty\,\|a\|_{H^2(\Omega)}\\
&\le (m_\infty+N_\infty)\,C_K + 3\,b\,m'_\infty C_\Omega\,\|\varphi\,\|_V + 2\,b\,m_0|\Gamma|_1^{1/2} \le C_{m,K,\Omega}\,(\|\nabla B(\cdot,\varphi)\|+1)\,.
\end{aligned} \tag{5.20}$$

Notice that, here, the control of the $L^\infty(\partial\Omega)$-norm of the term $\nabla K*\varphi\cdot\boldsymbol{n}$ is automatically provided by the bound $|\varphi|\le 1$, which we are assuming to be available in the framework of these formal estimates (hence, we do not need to introduce a truncation, as done for handling the same control in the time discretization scheme).

Therefore, on account of (3.1), (3.4) and (5.20), from (5.19) we obtain

$$\begin{aligned}
\|B(\cdot,\varphi)\|_{H^2(\Omega)} &\leq C\left(\|\Delta B(\cdot,\varphi)\| + \|\nabla B(\cdot,\varphi)\| + 1\right) \\
&\leq C\left(\|\varphi_t\| + \|\boldsymbol{u}\cdot\nabla\varphi\| + \|\mathrm{div}\,(\mathcal{N}(\varphi)\nabla a)\| + \|\mathrm{div}\,(m(\varphi)(\nabla K * \varphi))\| + \|\nabla B\| + 1\right) \\
&\leq C\left(\|\varphi_t\| + \left\|\boldsymbol{u}\cdot\left(\frac{1}{\beta}\nabla B(\cdot,\varphi) - \frac{1}{\beta}\mathcal{M}(\varphi)\nabla a\right)\right\| + \|\nabla B(\cdot,\varphi)\| + 1\right) \\
&\leq C\left(\|\varphi_t\| + \|\boldsymbol{u}\|_{L^4(\Omega)^2}\|\nabla B(\cdot,\varphi)\|_{L^4(\Omega)^2} + \|\boldsymbol{u}\| + \|\nabla B(\cdot,\varphi)\| + 1\right) \\
&\leq C\left(\|\varphi_t\| + \|\boldsymbol{u}\|^{1/2}\|\nabla\boldsymbol{u}\|^{1/2}\|\nabla B\|^{1/2}\|B\|^{1/2}_{H^2(\Omega)} + \|\boldsymbol{u}\| + \|\nabla B\| + 1\right), \qquad (5.21)
\end{aligned}$$

which, thanks to Young's inequality, entails the desired estimate

$$\|B(\cdot,\varphi)\|_{H^2(\Omega)} \leq C\left(\|\varphi_t\| + \|\boldsymbol{u}\|\,\|\nabla\boldsymbol{u}\|\,\|\nabla B(\cdot,\varphi)\| + \|\boldsymbol{u}\| + \|\nabla B(\cdot,\varphi)\| + 1\right). \qquad (5.22)$$

Estimating the term in the H^2-norm of B in (5.18) by means of (5.22), we get

$$|\big(\boldsymbol{u}\cdot\nabla\varphi, \beta(\cdot,\varphi)\varphi_t\big)| \leq \frac{1}{12}(1-\rho)\alpha_0\|\varphi_t\|^2 + C\left(\|\boldsymbol{u}\|^2\|\nabla\boldsymbol{u}\|^2\|\nabla B(\cdot,\varphi)\|^2 + \|\boldsymbol{u}\|^2 + 1\right). \qquad (5.23)$$

Therefore, by estimating the term coming from convection in (5.15) through (5.23), the other terms as done above, and employing (3.4) once more, we are led to the following differential inequality

$$\frac{d\Phi}{dt} + (1-\rho)\alpha_0\|\varphi_t\|^2 \leq C_{m,\lambda,K}\left(1 + \|\boldsymbol{u}\|^2\|\nabla\boldsymbol{u}\|^2\right)\left(1 + \|\nabla\varphi\|^2\right). \qquad (5.24)$$

On the other hand, it is easy to see that there are two constants $K_1, K_2 > 0$, depending on m, λ and K, such that

$$K_1\left(\|\nabla\varphi(t)\|^2 - 1\right) \leq \Phi(t) \leq K_2\left(\|\nabla\varphi(t)\|^2 + 1\right). \qquad (5.25)$$

Therefore, on account of (5.10) and of the fact that $\boldsymbol{u} \in L^\infty(0,\infty;G_{div})$, by applying the uniform Gronwall Lemma, from (5.24) and (5.25), we can find a time $t_1\big(E(\boldsymbol{u}_0,\varphi_0)\big) := t_0 + 1$ such that

$$\|\varphi(t)\|_V^2 \leq \Gamma_1(\kappa), \qquad \forall t \geq t_1. \qquad (5.26)$$

Moreover, by integrating (5.24) between t and $t+1$, for all $t \geq t_1$, we also get

$$\alpha_0(1-\rho)\int_t^{t+1}\|\varphi_t(s)\|^2 ds \leq \Gamma_2(\kappa), \qquad \forall t \geq t_1. \qquad (5.27)$$

Summing up, we have

$$\varphi \in L^\infty(0,\infty;V), \qquad \varphi_t \in L^2_{tb}(0,\infty;H). \qquad (5.28)$$

We now prove that $\varphi \in L^2_{tb}(0,\infty;H^2(\Omega))$. First, from (5.10), (5.27), (5.26) and (5.22) we infer that we have

$$\int_t^{t+1} \|B(\cdot,\varphi(s))\|^2_{H^2(\Omega)}\,ds \le \Gamma_3(\kappa)\,, \qquad \forall t \ge t_1\,, \tag{5.29}$$

and hence $B(\cdot,\varphi) \in L^2_{tb}(0,\infty;H^2(\Omega))$. This, by Gagliardo-Nirenberg inequality (2.1) and (5.26), implies that (cf. (4.39))

$$\begin{aligned}&\int_t^{t+1} \|\nabla\varphi(s)\|^{2p/(p-2)}_{L^p(\Omega)^2} ds + \int_t^{t+1} \|\nabla B(\cdot,\varphi(s))\|^{2p/(p-2)}_{L^p(\Omega)^2} ds \\ &+ \int_t^{t+1} \|\nabla\beta(\cdot,\varphi(s))\|^{2p/(p-2)}_{L^p(\Omega)^2} ds \le \Gamma_4(\kappa)\,, \qquad 2<p<\infty\,,\end{aligned} \tag{5.30}$$

for all $t \ge t_1$. Thus we have $\varphi, B(\cdot,\varphi), \beta(\cdot,\varphi) \in L^{2p/(p-2)}_{tb}(0,\infty;W^{1,p}(\Omega))$. Notice that we have used the identity $\nabla\beta(\cdot,\varphi) = m(\varphi)\nabla a + (m'(\varphi)\,a + \lambda'(\varphi))\,\nabla\varphi$. As far as the second spatial derivatives $\partial^2_{ij}\varphi$ are concerned, recall that we have the following identity (cf. (4.15))

$$\partial^2_{ij}\varphi = \frac{1}{\beta}\partial^2_{ij}B - \frac{1}{\beta^2}\partial_i\beta\partial_j B - \frac{\mathcal{M}(\varphi)}{\beta}\partial_i\left(\partial_j a\right) - \frac{m(\varphi)}{\beta}\partial_i\varphi\partial_j a + \frac{\mathcal{M}(\varphi)}{\beta^2}\partial_i\beta\partial_j a\,. \tag{5.31}$$

Combining now (4.15) with (5.29) and (5.30) (with $p=4$), we obtain

$$\int_t^{t+1} \|\varphi(s)\|^2_{H^2(\Omega)} ds \le \Gamma_5(\kappa)\,, \qquad \forall t \ge t_1\,,$$

so that $\varphi \in L^2_{tb}(0,\infty;H^2(\Omega))$. This concludes the proof of the first part of the theorem.

Let us now assume that $\boldsymbol{u}_0 \in V_{div}$ and that $\varphi_0 \in V \cap C^\beta(\overline{\Omega})$. On account of (5.10), assumption **(V)**, (5.27) and (5.30) (with $p=4$), by applying the uniform Gronwall Lemma to (4.73) we immediately deduce that

$$\|\boldsymbol{u}(t)\|_{V_{div}} \le \Gamma_6(\kappa)\,, \qquad \forall t \ge t_1\,, \tag{5.32}$$

this yields $\boldsymbol{u} \in L^\infty(0,\infty;V_{div})$. By integrating (4.73) between t and $t+1$, and using Proposition 1, (4.63), (4.64), it is not difficult to obtain

$$\int_t^{t+1} \|\boldsymbol{u}_t(s)\|^2 ds + \int_t^{t+1} \|\boldsymbol{u}(s)\|^2_{H^2(\Omega)^2} ds \le \Gamma_7(\kappa)\,, \qquad \forall t \ge t_1\,. \tag{5.33}$$

Thus we have $\boldsymbol{u}_t \in L^2_{tb}(0,\infty;G_{div})$ and $\boldsymbol{u} \in L^2_{tb}(0,\infty;H^2(\Omega)^2)$.

In order to prove (5.4), we take the time derivative of (3.1) and test the resulting equation by φ_t. By using the boundary condition (3.5), we obtain the following identity:

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 + (\nabla B(\cdot,\varphi)_t, \nabla\varphi_t) = -\left(\boldsymbol{u}_t\cdot\nabla\varphi, \varphi_t\right) - (\varphi\, m'(\varphi)\varphi_t\nabla a, \nabla\varphi_t)$$

$$+ \left(m'(\varphi)\varphi_t \left(\nabla K * \varphi\right), \nabla\varphi_t\right) + \left(m(\varphi)\left(\nabla K * \varphi_t\right), \nabla\varphi_t\right) . \tag{5.34}$$

Owing to (3.4) and (4.6), we have

$$\begin{aligned}(\nabla B(\cdot,\varphi)_t, \nabla\varphi_t) &\geq \alpha_0(1-\rho)\|\nabla\varphi_t\|^2 + \big((m(\varphi)\nabla a + (m'(\varphi)a + \lambda'(\varphi))\nabla\varphi)\,\varphi_t, \nabla\varphi_t\big)\\ &\geq \frac{1}{2}\alpha_0(1-\rho)\|\nabla\varphi_t\|^2 - \frac{m_\infty^2}{\alpha_0(1-\rho)}\|\nabla a\|_\infty^2\|\varphi_t\|^2 - \frac{{\beta'_\infty}^2}{\alpha_0(1-\rho)}\|\varphi_t\nabla\varphi\|^2 ,\end{aligned} \tag{5.35}$$

where the constant β'_∞ is defined as in (4.10). As far as the last term in (5.35) is concerned, on account of (3.4), we have that

$$\begin{aligned}\|\varphi_t\nabla\varphi\|^2 &\leq \|\varphi_t\|_{L^4(\Omega)}^2\|\nabla\varphi\|_{L^4(\Omega)^2}^2\\ &\leq C\left(\|\varphi_t\|^2 + \|\varphi_t\|\|\nabla\varphi_t\|\right)\left\|\frac{1}{\beta}\nabla B(\cdot,\varphi) - \frac{1}{\beta}\mathcal{M}(\varphi)\nabla a\right\|_{L^4(\Omega)^2}^2\\ &\leq C\left(\|\varphi_t\|^2 + \|\varphi_t\|\|\nabla\varphi_t\|\right)\big(\|\nabla B(\cdot,\varphi)\|\|B(\cdot,\varphi)\|_{H^2(\Omega)} + 1\big)\\ &\leq 2\delta\|\nabla\varphi_t\|^2 + C_\delta\|\varphi_t\|^2\|B(\cdot,\varphi)\|_{H^2(\Omega)}^2 + C_\delta\|\varphi_t\|^2,\end{aligned}$$

for all $\delta > 0$, where the first of (5.28) and (3.4) have been taken into account, which yield that $B(\cdot,\varphi) \in L^\infty(0,\infty;V)$. Hence, combining this last estimate with (5.35) and choosing $\delta > 0$ small enough, we obtain the estimate

$$(\nabla B(\cdot,\varphi)_t, \nabla\varphi_t) \geq \frac{1}{4}\alpha_0(1-\rho)\|\nabla\varphi_t\|^2 - C\|\varphi_t\|^2\|B(\cdot,\varphi)\|_{H^2(\Omega)}^2 - C\|\varphi_t\|^2 . \tag{5.36}$$

The H^2-norm of $B(\cdot,\varphi)$ by the L^2-norm of φ_t can be obtained by arguing as above (cf. (5.19)–(5.21)), i.e., by first using elliptic regularity theory and then by estimating the boundary term, to get (5.22). From (5.22), on account of the improved regularity $(5.28)_1$ and (5.32), we get

$$\|B(\cdot,\varphi)\|_{H^2(\Omega)} \leq C\left(\|\varphi_t\| + 1\right) . \tag{5.37}$$

Let us now estimate the terms on the right-hand side of (5.34). For the first term, on account of (3.4), $(5.28)_1$ and (5.37), we have

$$\begin{aligned}&|-(\boldsymbol{u}_t\cdot\nabla\varphi, \varphi_t)|\\ &\quad\leq C\|\boldsymbol{u}_t\|\|\varphi_t\|_{L^4(\Omega)}\left\|\frac{1}{\beta(\cdot,\varphi)}\nabla B(\cdot,\varphi) - \frac{1}{\beta(\cdot,\varphi)}\mathcal{M}(\varphi)\nabla a\right\|_{L^4(\Omega)^2}\\ &\quad\leq C\|\boldsymbol{u}_t\|\left(\|\varphi_t\| + \|\varphi_t\|^{1/2}\|\nabla\varphi_t\|^{1/2}\right)\left(\|\nabla B(\cdot,\varphi)\|^{1/2}\|B(\cdot,\varphi)\|_{H^2(\Omega)}^{1/2} + 1\right)\\ &\quad\leq C\|\boldsymbol{u}_t\|\left(\|\varphi_t\| + \|\varphi_t\|^{1/2}\|\nabla\varphi_t\|^{1/2}\right)\left(\|\varphi_t\|^{1/2} + 1\right)\\ &\quad\leq 3\delta\|\nabla\varphi_t\|^2 + C_\delta\left(\|\varphi_t\|^4 + \|\boldsymbol{u}_t\|^2 + 1\right) .\end{aligned} \tag{5.38}$$

As far as the remaining terms on the right-hand side of (5.34) are concerned, they can simply be controlled by

$$\delta\|\nabla\varphi_t\|^2 + C_\delta\|\varphi_t\|^2 \,. \tag{5.39}$$

Therefore, by taking the estimates (5.36)–(5.39) into account, from (5.34) we can deduce the differential inequality

$$\frac{d}{dt}\|\varphi_t\|^2 + \frac{1}{4}\alpha_0(1-\rho)\|\nabla\varphi_t\|^2 \leq C\left(\|\varphi_t\|^4 + \|\varphi_t\|^2 + \|\boldsymbol{u}_t\|^2 + 1\right) \,. \tag{5.40}$$

Then, using (5.27), (5.33) and the uniform Gronwall Lemma we obtain

$$\|\varphi_t(t)\|^2 \leq \Gamma_8(\kappa)\,, \qquad \forall t \geq t_1\,, \tag{5.41}$$

whence we have $\varphi_t \in L^\infty(0,\infty;H)$. By integrating (5.40) between t and $t+1$, for $t \geq t_1$, we also get

$$\int_t^{t+1} \|\nabla\varphi_t(s)\|^2 ds \leq \Gamma_9(\kappa)\,, \qquad \forall t \geq t_1\,,$$

so that $\varphi_t \in L^2_{tb}(0,\infty;V)$. Finally, we prove that $\varphi \in L^\infty(0,\infty;H^2(\Omega))$. First, notice that (5.37) and (5.41) entail that $\|B(\cdot,\varphi(t))\|_{H^2(\Omega)} \leq \Gamma_{10}(\kappa)$, for all $t \geq t_1$. Then, we have

$$\|\varphi(t)\|_{W^{1,p}(\Omega)} + \|B(\cdot,\varphi(t))\|_{W^{1,p}(\Omega)} + \|\beta(\cdot,\varphi(t))\|_{W^{1,p}(\Omega)} \leq \Gamma_{11}(\kappa)\,, \qquad \forall t \geq t_1\,, \tag{5.42}$$

with $2 < p < \infty$, whence $\varphi, B(\cdot,\varphi), \beta(\cdot,\varphi) \in L^\infty\left(0,\infty;W^{1,p}\left(\Omega\right)\right)$. Therefore, recalling (5.31) and employing (5.42), we deduce

$$\|\varphi(t)\|_{H^2(\Omega)} \leq \Gamma_{12}(\kappa)\,, \qquad \forall t \geq t_1\,,$$

which is the final desired claim. The proof is complete. □

Remark 11. Assume that $\boldsymbol{u}_0 \in V_{div}$, $\varphi_0 \in H^2(\Omega)$ and that the compatibility condition (3.10) is satisfied. Moreover, assume also that

(M)$_1$ The mobility satisfies **(M)** and also $m \in C^2\left([-1,1]\right)$.

(A1)$_2$ $F \in C^4(-1,1)$ and $\lambda := mF'' \in C^2\left([-1,1]\right)$.

Then, the following time continuity properties for the strong solution of Theorem 2 hold

$$\boldsymbol{u} \in C^0([0,T];V_{div})\,, \qquad \varphi \in C^0\big([0,T];H^2(\Omega)\big) \cap C^1([0,T];H)\,. \tag{5.43}$$

Let us sketch the argument for proving (5.43), omitting some details.

The time continuity of the velocity field $(5.43)_1$ is a consequence of the fact that $\boldsymbol{u} \in C_w([0,T];V_{div})$ and of the differential identity

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|^2 - \big(\nu(\varphi)\Delta\boldsymbol{u}, S\boldsymbol{u}\big) - 2\big(\nu'(\varphi)\nabla\varphi\cdot D\boldsymbol{u}, S\boldsymbol{u}\big) + (\mathcal{B}\boldsymbol{u}, S\boldsymbol{u})$$
$$= \big((a\varphi - K*\varphi)\nabla\varphi, S\boldsymbol{u}\big) + (\boldsymbol{v}, S\boldsymbol{u}),$$

which is deduced by testing (4.63) and (4.64) by $S\boldsymbol{u}$ (recall that $S := -P\Delta$ is the Stokes operator, cf. Section 2).

In order to show $(5.43)_2$, we first observe that from (5.34) and from the regularity properties (3.9), (3.8), it is not difficult to see that $\|\varphi_t(\cdot)\|^2 \in C^0([0,T])$. Moreover, (3.11) implies that $\varphi \in C^0([0,T];V)$. From this we infer that $B(\cdot,\varphi) \in C^0([0,T];V)$. Since $\varphi,\ B(\cdot,\varphi) \in L^\infty(0,T;H^2(\Omega))$, we then have $\varphi,\ B(\cdot,\varphi) \in C_w([0,T];H^2(\Omega))$. Also, recalling that $\boldsymbol{u} \in C^0([0,T];L^4(\Omega))$ and $\nabla\varphi \in C_w([0,T];L^4(\Omega))$, we have $\boldsymbol{u}\cdot\nabla\varphi \in C_w([0,T];H)$. It is also easy to see that div$(\mathcal{N}(\varphi)\nabla a)$, div$(m(\varphi)(\nabla K*\varphi)) \in C^0([0,T];H)$. Hence (3.1) yields $\varphi_t \in C_w([0,T];H)$. This weak in time continuity, together with the L^2-norm continuity for φ_t, implies that $\varphi_t \in C^0([0,T];H)$. On the other hand, we also have $\varphi \in C^0([0,T];H^s(\Omega))$, for $1 \le s < 2$, and this entails that $\nabla\varphi \in C^0([0,T];L^4(\Omega))$. Hence, $\boldsymbol{u}\cdot\nabla\varphi \in C^0([0,T];H)$, and from (3.1) again, we infer that $\Delta B(\cdot,\varphi) \in C^0([0,T];H)$. We now employ the following estimate (see [23])

$$\|\varphi_2-\varphi_1\|_{H^2(\Omega)} + \|B(\cdot,\varphi_2)-B(\cdot,\varphi_1)\|_{H^2(\Omega)} \le C\|\Delta(B(\cdot,\varphi_2)-B(\cdot,\varphi_1))\| + C\|\varphi_1-\varphi_2\|_V\,, \tag{5.44}$$

which requires slightly stronger assumptions than **(M)** and **(A1)**, that is, **(M)**$_1$ and **(A1)**$_2$ above. By means of (5.44), we eventually get that $\varphi,\ B(\cdot,\varphi) \in C^0([0,T];H^2(\Omega))$.

Let us now assume that $\boldsymbol{v}$ is time independent, i.e., $\boldsymbol{v} \in G_{div}$. Following [25, Section 5], for $\kappa \in [0,1]$ fixed, we introduce the metric space $\mathcal{X}_\kappa$ defined by

$$\mathcal{X}_\kappa := G_{div} \times \mathcal{Y}_\kappa\,,$$

with $\mathcal{Y}_\kappa$ given by

$$\mathcal{Y}_\kappa := \big\{\varphi \in L^\infty(\Omega) : |\varphi| \le 1 \ \text{ a.e. in } \Omega\,,\quad F(\varphi), M(\varphi) \in L^1(\Omega),\quad |\overline{\varphi}| \le \kappa\big\}\,. \tag{5.45}$$

The metric on $\mathcal{X}_\kappa$ is

$$\boldsymbol{d}_{\mathcal{X}_\kappa}(\boldsymbol{z}_2,\boldsymbol{z}_1) := \|\boldsymbol{u}_2-\boldsymbol{u}_1\| + \|\varphi_2-\varphi_1\|\,,$$

for every $\boldsymbol{z}_1 := [\boldsymbol{u}_1,\varphi_1]$ and $\boldsymbol{z}_2 := [\boldsymbol{u}_2,\varphi_2]$ in $\mathcal{X}_\kappa$.

Suppose that **(K)**, **(V)**, **(M)**, **(A1)**–**(A5)** are satisfied. Then we know that the set $\mathcal{G}_\kappa$ of all weak solutions to (1.1)–(1.6) from $[0,\infty)$ to $\mathcal{X}_k$ (cf. Definition 1 and Theorem 1), corresponding to all initial data $\boldsymbol{z}_0 = [\boldsymbol{u}_0, \varphi_0] \in \mathcal{X}_\kappa$, is a generalized semiflow on $\mathcal{X}_\kappa$ (in the sense of [5]) which possesses a (unique) global attractor $\mathcal{A}_\kappa$ (see [25, Section 5]). Notice that in [25, Section 5] the viscosity ν was assumed to be constant, for simplicity. However, the arguments therein can be easily adapted also to the case of nonconstant viscosity satisfying **(V)**. We also remark that uniqueness of weak solutions is not know in general. However, if ν is constant then, thanks to the uniqueness result of [21, Theorem 4] (cf. (2.6)), the generalized semiflow becomes a semigroup of closed operator on $\mathcal{X}_\kappa$ and the global attractor is connected.

Assume now that the assumptions of Proposition 2 are satisfied. Take $\boldsymbol{z}_0 \in \mathcal{X}_\kappa$ and consider a weak solution $\boldsymbol{z} := [\boldsymbol{u}, \varphi] \in C^0([0,\infty); \mathcal{X}_\kappa)$ corresponding to $\boldsymbol{z}_0$. By integrating (5.6) in time between 0 and $\tau > 0$, we can deduce that, for every $\tau > 0$, there exists $t_\tau \in (0, \tau]$ such that $\boldsymbol{z}(t_\tau) \in V_{div} \times V$. We now consider (5.24) in $[t_\tau, \infty)$. By integrating this differential inequality between t_τ and $t > t_\tau$, we can see that there exists $s_\tau \in (t_\tau, t]$ such that $\varphi_t(s_\tau) \in H$. This, assuming also that $\boldsymbol{u}(s_\tau) \in V_{div}$ and $\varphi(s_\tau) \in V$, owing to (5.22) and (5.31), implies that $\varphi(s_\tau) \in H^2(\Omega)$. Moreover, since the boundary condition (3.5) holds almost everywhere on $\partial\Omega \times (0, T)$, we can suppose that (3.10) holds in s_τ (i.e., with φ_0 replaced by $\varphi(s_\tau)$). Therefore we can apply the last statement of Theorem 2 with initial time s_τ. Let us then consider the metric space

$$\mathcal{W}_\kappa := V_{div} \times \mathcal{Z}_k,$$

where

$$\mathcal{Z}_\kappa := \Big\{ \varphi \in H^2(\Omega) \, : \, \frac{\partial B(\cdot, \varphi)}{\partial \boldsymbol{n}} = m(\varphi)(\nabla K * \varphi) \cdot \boldsymbol{n} - \mathcal{N}(\varphi)(\nabla a \cdot \boldsymbol{n}) \,, \quad \text{a.e. on } \partial\Omega \,, \\ |\varphi| \le 1 \quad \text{a.e. in } \Omega \,, \quad F(\varphi), M(\varphi) \in L^1(\Omega) \,, \quad |\overline{\varphi}| \le \kappa \Big\} \,, \tag{5.46}$$

endowed with the metric

$$\boldsymbol{d}_{\mathcal{W}_\kappa}(\boldsymbol{z}_2, \boldsymbol{z}_1) := \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_{V_{div}} + \|\varphi_2 - \varphi_1\|_{H^2(\Omega)} \,, \qquad \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathcal{W}_\kappa \,,$$

then, for every $\tau > 0$, there exists $s_\tau \in (0, \tau]$ such that $\boldsymbol{z}(s_\tau) \in \mathcal{W}_\kappa$, and starting from the time s_τ, the weak solution corresponding to $\boldsymbol{z}_0$ becomes a (unique) strong solution $\boldsymbol{z} \in C^0([s_\tau, \infty); \mathcal{W}_\kappa)$ (cf. Remark 11). Furthermore, from s_τ on, this solution satisfies the dissipative estimate (5.5), namely, there exists a time $\widetilde{t}_1 = \widetilde{t}_1(E(\boldsymbol{z}_0)) \ge s_\tau$ such that $\boldsymbol{z}$ satisfies (5.5) for all $t \ge \widetilde{t}_1$.

Let us now consider a subset $\mathscr{B} \subset \mathcal{X}_k$, bounded in the metric of $\mathcal{X}_k$. We can choose $\tau = 1$ for every $\boldsymbol{z}_0 \in \mathscr{B}$, and then infer that every weak solution starting from $\boldsymbol{z}_0$ becomes (at some time $s_1 \in (0, 1]$, which depends on $\boldsymbol{z}_0$ and on the weak solution considered from $\boldsymbol{z}_0$) a strong solution satisfying (5.5) for all $t \ge t_1^*$, with $t_1^* = t_1^*(R) \ge 1$, where $R > 0$ is such that $\boldsymbol{d}_{\mathcal{X}_\kappa}(\boldsymbol{w}, \boldsymbol{0}) \le R$, for all $\boldsymbol{w} \in \mathscr{B}$. Therefore, we deduce that there exists a time $t_1^*(\mathscr{B}) \ge 1$, such that

$$\boldsymbol{z}(t) \in \mathcal{B}_{\mathcal{W}_\kappa}\big(\Lambda(k)\big) \,, \qquad \forall t \ge t_1^* \,,$$

where $\Lambda(k) := \Gamma^{1/2}(\kappa)$, and $\mathcal{B}_{\mathcal{W}_\kappa}\big(\Lambda(k)\big)$ is the closed ball in $\mathcal{W}_\kappa$ given by

$$\mathcal{B}_{\mathcal{W}_\kappa}\big(\Lambda(k)\big) := \{\boldsymbol{w} \in \mathcal{W}_\kappa \, : \, \boldsymbol{d}_{\mathcal{W}_\kappa}(\boldsymbol{w}, \boldsymbol{0}) \le \Lambda(k)\} \, .$$

Thanks to the full invariance property of the global attractor $\mathcal{A}_\kappa$, we immediately deduce that $\mathcal{A}_\kappa \subset \mathcal{B}_{\mathcal{W}_\kappa}\big(\Lambda(k)\big)$. In conclusion, we have proven the following regularity result for the global attractor.

Theorem 3. *Let **(K)**, **(V)**, **(M)**, **(A1)**$_1$, **(A4)–(A5)** be satisfied, assume that $K \in W^{2,1}_{loc}(\mathbb{R}^2)$ or that K is admissible, and that $\boldsymbol{v} \in G_{div}$ is independent of time. Then, the global attractor $\mathcal{A}_k$ of the generalized semiflow $\mathcal{G}_k$ associated to system* (1.1)–(1.6) *is such that*

$$\mathcal{A}_\kappa \subset \mathcal{B}_{\mathcal{W}_\kappa}\big(\Lambda(k)\big) \, .$$

Remark 12 (Corrigendum for [24])**.** Similarly to (3.10) of Theorem 2, also in [24, Theorem 2 and Proposition] a compatibility condition, associated with the assumption $\varphi_0 \in H^2(\Omega)$ must be required. More precisely, setting $\mu_0 := a\varphi_0 - J * \varphi_0 + F'(\varphi_0)$ (in [24] J stands for the convolution kernel), the missing condition is $\partial_{\boldsymbol{n}}\mu_0 = 0$ almost everywhere on $\partial\Omega$. Consequently, the metric space $\mathcal{Y}^1_m$, for $m \ge 0$ fixed, introduced before the result on existence of the global attractor (see [24, Theorem 3]) must be defined as follows

$$\mathcal{Y}^1_m := \big\{\varphi \in H^2(\Omega) \, : \, \partial_{\boldsymbol{n}}\mu = 0 \text{ a.e. on } \Omega \, , \; \mu = \varphi - J * \varphi + F'(\varphi) \, , \; |(\varphi, m)| \le m\big\} \, .$$

This observation also applies to [21, Theorem 5], to the definition of the space $\mathcal{K}_\eta$ in [21, Theorem 10]), and to [27, Theorem 2.3].

6 The convective nonlocal CH equation

The results of the previous sections can essentially be established for the nonlocal CH equation with degenerate mobility and with a prescribed (and not necessarily divergence-free) velocity field $\boldsymbol{u}$. We shall consider $d = 2, 3$. However, if $d = 3$ the results are poorer than in the case $d = 2$ (cf. Remark 14).

Theorem 4. *Suppose that assumptions **(K)**, **(M)**, **(A1)**$_1$, **(A4)–(A5)** are satisfied and suppose that $K \in W^{2,1}_{loc}(\mathbb{R}^2)$ or that K is admissible. Let $\varphi_0 \in V \cap L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$, where M is defined as in Theorem 1. Assume also that $\boldsymbol{u}$ is given and*

$$\boldsymbol{u} \in L^{2r/(r-d)}(0, T; L^r(\Omega)^d) \, , \qquad d < r \le \infty \, . \tag{6.1}$$

Then, for every $T > 0$, problem (1.2), (1.3), (1.5)$_2$, (1.6)$_2$ *admits a strong solution φ on $[0, T]$ such that*

$$\varphi \in L^\infty(0, T; V) \cap H^1(0, T; H) \, , \tag{6.2}$$

$$\varphi \in L^2(0,T;H^2(\Omega))\,. \tag{6.3}$$

This solution is also unique, provided $r = \infty$ *when* $d = 3$.

If $d = 2$, $\boldsymbol{u}$ *satisfies the additional regularity*

$$\boldsymbol{u} \in L^s(0,T;L^\infty(\Omega)^2) \cap L^\infty(0,T;L^\sigma(\Omega)^2)\,, \quad s\,,\sigma > 2\,, \qquad \boldsymbol{u}_t \in L^2(0,T;G_{div})\,, \tag{6.4}$$

and $\varphi_0 \in H^2(\Omega)$ *satisfies* (3.10)*, then, the (unique) strong solution also satisfies*

$$\varphi \in L^\infty(0,T;H^2(\Omega))\,, \qquad \varphi_t \in L^\infty(0,T;H) \cap L^2(0,T;V)\,. \tag{6.5}$$

Proof. Since the argument follows the same lines of the time-discretization scheme of Step 1 and of Step 3 in the proof of Theorem 2, we just highlight the main points. The approximate problem (4.1)–(4.2) is considered, and, by applying time-discretization, we are led to formulate the incremental-step problem (4.11)–(4.12).

In view of (6.1), the bootstrap argument to prove that, for $\varphi_0 \in V$, the solution to this problem satisfies $(\varphi_1, \dots, \varphi_N) \in H^2(\Omega)^N$, is now a bit more delicate. Let us sketch this argument only for the case $d = 3$. By comparison in (4.11)–(4.12), we first see that we have $\Delta B(\cdot, \varphi_{k+1}) \in L^{p_1}(\Omega)$, where $p_1 = 2r/(r+2)$, and $\partial B(\cdot, \varphi_{k+1})/\partial \boldsymbol{n} \in H^{1/2}(\partial\Omega)$. From elliptic regularity theory, we then infer that $B(\cdot, \varphi_{k+1}) \in W^{2,p_1}(\Omega)$. Hence, on account also of (3.4), we have $\nabla B(\cdot, \varphi_{k+1}), \nabla\varphi_{k+1} \in W^{1,p_1}(\Omega)$. Thus by Sobolev embedding we get an improved regularity for the convective term $\boldsymbol{U}_k \cdot \nabla\varphi_{k+1}$, which, by means of elliptic regularity again, implies that $B(\cdot, \varphi_{k+1}) \in W^{2,p_2}(\Omega)$, with $1/p_2 = 1/p_1 - 1/3 + 1/r$. By repeating this argument n times, we get $B(\cdot, \varphi_{k+1}) \in W^{2,p_n}(\Omega)$, where $1/p_{n+1} = 1/p_n - 1/3 + 1/r$. This recursive relation can be made explicit and gives

$$p_n = \frac{p_1}{1 - (n-1)\,\sigma p_1}\,, \qquad \sigma := \frac{1}{3} - \frac{1}{r}\,.$$

Therefore, after n steps with n big enough, we have $p_n \geq 2$. The bootstrap argument then leads to $B(\cdot, \varphi_{k+1}) \in H^2(\Omega)$, and, by (4.15), we also have $\varphi_{k+1} \in H^2(\Omega)$ (actually, one could also push the regularity for φ_{k+1} further; however the H^2-regularity is enough for our purposes).

Let us now consider the discrete estimates that can be derived from the incremental-step problem (4.11)–(4.12). The basic estimate (4.21) still holds. As far as estimates (4.22)–(4.26) and (4.28) are concerned, these can be repeated. However, the contribution coming from the convective term $\boldsymbol{U}_k \cdot \nabla\varphi_{k+1}$ in (4.28), instead of being estimated as in (4.29), is now controlled as follows (let us consider just the case $d = 3$, and estimate only the main part of this contribution, recalling (4.6))

$$\tau \sum_{k=0}^{n} \|\boldsymbol{U}_k \cdot \nabla B(\cdot, \varphi_{k+1})\|^2 \leq \tau \sum_{k=0}^{n} \|\boldsymbol{U}_k\|^2_{L^r(\Omega)^3} \|\nabla B(\cdot, \varphi_{k+1})\|^2_{L^{2r/(r-2)}(\Omega)^3}$$

$$\leq \tau \sum_{k=0}^{n} \|\boldsymbol{U}_k\|^2_{L^r(\Omega)^3} \|\nabla B(\cdot,\varphi_{k+1})\|^{2-\frac{6}{r}} \|\nabla B(\cdot,\varphi_{k+1})\|_V^{\frac{6}{r}}$$

$$\leq \delta\tau \sum_{k=0}^{n} \|B(\cdot,\varphi_{k+1})\|^2_{H^2(\Omega)} + C_\delta\, \tau \sum_{k=0}^{n} \|\boldsymbol{U}_k\|^{\frac{2r}{r-3}}_{L^r(\Omega)^3} \|\nabla B(\cdot,\varphi_{k+1})\|^2 \,, \tag{6.6}$$

where $\delta > 0$ is to be fixed later. Here the Gagliardo-Nirenberg inequality has been used. It is easy to see that we have

$$\tau \sum_{k=0}^{n} \|\boldsymbol{U}_k\|^{\frac{2r}{r-3}}_{L^r(\Omega)^3} \leq \|\boldsymbol{u}\|^{\frac{2r}{r-3}}_{L^{\frac{2r}{r-3}}(0,T;L^r(\Omega)^3)} \,. \tag{6.7}$$

Therefore,taking estimates (4.30)–(4.34) into account, from the discrete Gronwall Lemma and from (6.1), (6.7), we can recover estimate (4.37) (the constant $\mathbb{Q}$ now depends on the norm of $\boldsymbol{u}$ on the right-hand side of (6.7)). This allows us to deduce (6.2).

Next, as far as the regularity (6.3) is concerned, let us consider the two cases $d = 2, 3$ separately. In the case $d = 2$, we can argue exactly as in Step 1 in the proof of Theorem 2, by using estimate (4.38), which can now be written into the form

$$\tau \sum_{k=0}^{n} \|B(\cdot,\varphi_{k+1})\|^2_{H^2(\Omega)} \leq \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^{2r/(r-d)}(0,T;L^r(\Omega)^d)}\big) \,, \tag{6.8}$$

and which is derived from (4.30), combined with (4.31)–(4.34), (6.1), (6.6), (6.7), and (4.37). If $d = 3$, the argument requires some care. The first step is to prove a bound in $L^4(0,T;L^4(\Omega)^3)$ for the sequence of $\nabla B(\cdot,\bar{\varphi}_N)$, namely

$$\tau \sum_{k=0}^{n} \|\nabla B(\cdot,\varphi_{k+1})\|^4_{L^4(\Omega)^3} \leq \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^{2r/(r-3)}(0,T;L^r(\Omega)^3)}\big) \,, \qquad n = 0,\dots,N-1 \,. \tag{6.9}$$

This bound is a consequence of (6.8) and of the following Gagliardo-Nirenberg inequality (which holds for every dimension d, see, e.g., [18, 19, 45])

$$\|\nabla B(\cdot,\varphi_{k+1})\|_{L^4(\Omega)^3} \leq C \|B(\cdot,\varphi_{k+1})\|^{1/2}_{L^\infty(\Omega)} \|B(\cdot,\varphi_{k+1})\|^{1/2}_{H^2(\Omega)} \,, \tag{6.10}$$

provided that we prove a uniform bound in $L^\infty(\Omega)$ for the time discrete solutions φ_{k+1} to the incremental-step problem (4.11)–(4.12), namely

$$\sup_{0\leq k\leq n} \|\varphi_{k+1}\|_{L^\infty(\Omega)} \leq C\big(\|\varphi_0\|_{L^\infty(\Omega)}\big) \,, \qquad n = 0,\dots,N-1 \,. \tag{6.11}$$

Once we have (6.9), we also find a bound for $\nabla\bar{\varphi}_N$ and for $\nabla\beta(\cdot,\bar{\varphi}_N)$ in $L^4(0,T;L^4(\Omega)^3)$. Moreover, since we know that $\varphi_{k+1} \in H^2(\Omega)$, then (4.15) holds. From this identity we deduce the bound for $\bar{\varphi}_N$ in $L^2(0,T;H^2(\Omega))$ which yields (6.3). Therefore, we need to prove the uniform $L^\infty(\Omega)$ bound (6.11). This will now be achieved through a Moser-Alikakos iteration argument performed on (4.11)–(4.12).

Let us begin with an elementary identity that can be obtained from $2(a-b)a = a^2-b^2+(a-b)^2$, by multiplying it by a^2, then by multiplying the resulting identity by a^4, and iterating this procedure $m \geq 1$ times. We obtain

$$(a-b)a^{2^m-1} = \frac{1}{2^m}a^{2^m} - \frac{1}{2^m}b^{2^m} + A_m(a,b)\,, \tag{6.12}$$

where $A_m(a,b) \geq 0$ is some polynomial function of order 2^m which we do not write explicitly, since it is not essential.

We now set $p_m := 2^m$, multiply (4.11) by $\varphi_{k+1}^{p_m-1}$, integrate over Ω (taking the boundary condition (4.12) and the incompressibility condition for $\boldsymbol{U}_k$ into account), and sum the resulting identity over k, for $k = 0,\ldots,n$, with $0 \leq n \leq N-1$. By means of (6.12) we easily get the following estimate

$$\begin{aligned}
&\frac{1}{p_m}\int_\Omega \varphi_{n+1}^{p_m} + \frac{4\alpha_0(1-\rho)}{p_m\,p_m'}\,\tau\sum_{k=0}^n\int_\Omega \big|\nabla\big(\varphi_{k+1}^{p_m/2}\big)\big|^2 \leq \frac{1}{p_m}\int_\Omega \varphi_0^{p_m}\\
&- \tau\sum_{k=0}^n \big(\mathcal{M}(\varphi_{k+1})\nabla a, \nabla(\varphi_{k+1}^{p_m-1})\big) - \tau\sum_{k=0}^n \big(\mathcal{N}(\varphi_k)\nabla a, \nabla(\varphi_{k+1}^{p_m-1})\big)\\
&+ \tau\sum_{k=0}^n \big(m(\varphi_k)(\nabla K * Q(\varphi_k)), \nabla(\varphi_{k+1}^{p_m-1})\big)\,,
\end{aligned} \tag{6.13}$$

where p_m' is the conjugate exponent to p_m. Let us estimate the last three terms on the right-hand side of (6.13). We have

$$\begin{aligned}
&\tau\Big|\sum_{k=0}^n \big(\mathcal{M}(\varphi_{k+1})\nabla a, \nabla(\varphi_{k+1}^{p_m-1})\big)\Big| \leq \frac{2}{p_m'}m_\infty\|\nabla a\|_\infty\tau\sum_{k=0}^n\int_\Omega |\varphi_{k+1}^{p_m/2}\nabla\big(\varphi_{k+1}^{p_m/2}\big)|\\
&\leq \frac{\alpha_0(1-\rho)}{p_m p_m'}\,\tau\sum_{k=0}^n\int_\Omega|\nabla(\varphi_{k+1}^{p_m/2})|^2 + \frac{m_\infty^2\|\nabla a\|_\infty^2}{\alpha_0(1-\rho)p_m'}\,p_m\tau\sum_{k=0}^n\int_\Omega|\varphi_{k+1}^{p_m/2}|^2\,,
\end{aligned} \tag{6.14}$$

$$\begin{aligned}
&\tau\Big|\sum_{k=0}^n \big(\mathcal{N}(\varphi_k)\nabla a, \nabla(\varphi_{k+1}^{p_m-1})\big)\Big| \leq \frac{2}{p_m'}N_\infty\|\nabla a\|_\infty\tau\sum_{k=0}^n\int_\Omega |\varphi_{k+1}^{(p_m-2)/2}\nabla(\varphi_{k+1}^{p_m/2})|\\
&\leq \frac{\alpha_0(1-\rho)}{p_m p_m'}\,\tau\sum_{k=0}^n\int_\Omega|\nabla(\varphi_{k+1}^{p_m/2})|^2 + \frac{N_\infty^2\|\nabla a\|_\infty^2}{\alpha_0(1-\rho)p_m'}\,p_m\tau\sum_{k=0}^n\int_\Omega\Big(\frac{1}{p_{m-1}'}|\varphi_{k+1}^{p_m/2}|^2 + \frac{2}{p_m}\Big)\,,
\end{aligned} \tag{6.15}$$

and a similar estimate as (6.15) holds for the last term. By means of these estimates, and setting $\psi_k^{(m)} := \varphi_k^{p_m/2}$, (6.13) yields

$$\int_\Omega|\psi_{n+1}^{(m)}|^2 + \frac{\alpha_0(1-\rho)}{p_m'}\,\tau\sum_{k=0}^n\int_\Omega\big|\nabla\psi_{k+1}^{(m)}\big|^2 \leq \int_\Omega|\psi_0^{(m)}|^2 + C_1p_m^2\tau\sum_{k=0}^n\int_\Omega|\psi_{k+1}^{(m)}|^2 + C_2p_m\,, \tag{6.16}$$

where C_i, $i = 1, 2, \ldots$, shall henceforth denote some positive constants which may depend on m, K, α_0, ρ, Ω and T, but are independent of m and N.

Usinf the following Gagliardo-Nirenberg inequality in three dimensions

$$\|\psi_{k+1}^{(m)}\|^2 \le C\big(\|\psi_{k+1}^{(m)}\|_{L^1(\Omega)}^{4/5}\|\nabla\psi_{k+1}^{(m)}\|^{6/5} + \|\psi_{k+1}^{(m)}\|_{L^1(\Omega)}^2\big)\,, \tag{6.17}$$

and Young's inequality in (6.16), we obtain

$$\int_\Omega |\psi_{n+1}^{(m)}|^2 + \frac{\alpha_0(1-\rho)}{2p'_m}\,\tau\sum_{k=0}^{n}\int_\Omega \big|\nabla\psi_{k+1}^{(m)}\big|^2 \le \int_\Omega |\psi_0^{(m)}|^2 + C_3 p_m^5\tau\sum_{k=0}^{n}\|\psi_{k+1}^{(m)}\|_{L^1(\Omega)}^2\,.$$

The last inequality implies that

$$\begin{aligned}
\int_\Omega \varphi_{n+1}^{p_m} &\le \int_\Omega \varphi_0^{p_m} + C_3 p_m^5\tau\sum_{k=0}^{n}\Big(\int_\Omega |\varphi_{k+1}|^{p_{m-1}}\Big)^2\\
&\le \int_\Omega \varphi_0^{p_m} + C_3 p_m^5 T\max_{0\le k\le n}\Big(\int_\Omega |\varphi_{k+1}|^{p_{m-1}}\Big)^2\\
&\le C_4 p_m^5\Big(\max_{0\le k\le N-1}\Big\{1, \int_\Omega |\varphi_{k+1}|^{p_{m-1}}\Big\}\Big)^2\,,
\end{aligned} \tag{6.18}$$

where we have used the fact that $\|\varphi_0\|_{L^\infty(\Omega)} \le 1$, and the constant C_4 depends on $\|\varphi_0\|_{L^\infty(\Omega)}$. Setting

$$E_m := \max_{0\le k\le N-1}\Big\{1, \int_\Omega |\varphi_{k+1}|^{p_m}\Big\}\,, \qquad \forall m \ge 0\,,$$

from (6.18) we obtain the recursive relation

$$E_m \le C_4\, p_m^5 E_{m-1}^2\,, \qquad m \ge 1\,,$$

so that

$$E_m \le C_4^{\sum_{j=0}^{m-1} 2^j}\prod_{j=0}^{m-1} p_{m-j}^{5\cdot 2^j}\, E_0^{2^m}\,.$$

Hence, we get

$$\begin{aligned}
\max_{0\le k\le N-1}\|\varphi_{k+1}\|_{L^{p_m}(\Omega)} \le C_4 2^{5\sum_{\ell=1}^{m}\frac{\ell}{2^\ell}}\,E_0 &\le C_5\max_{0\le k\le N-1}\Big\{1, \int_\Omega |\varphi_{k+1}|\Big\}\\
&\le C_6\big(\|\varphi_0\|_{L^\infty(\Omega)}\big)\,,
\end{aligned} \tag{6.19}$$

where (4.21) has been taken into account in the last estimate. Letting $m \to \infty$, and using the fact that the constant C_6 does not depend neither on m nor on N, from (6.19) we get (6.11).

We now prove uniqueness of the strong solution satisfying (6.2)–(6.3). Let us start with the case $d = 2$. We take the difference of (3.1) and (3.5) written for two solutions and multiply the resulting identity by $\varphi := \varphi_2 - \varphi_1$ in H. We get

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + \big(\nabla(B(\cdot,\varphi_2) - B(\cdot,\varphi_1)), \nabla\varphi\big) = \big((m(\varphi_2) - m(\varphi_1))(\nabla K * \varphi_2), \nabla\varphi\big)$$
$$+ \big(m(\varphi_1)(\nabla K * \varphi), \nabla\varphi\big) - \big((\mathcal{N}(\varphi_2) - \mathcal{N}(\varphi_1))\nabla a, \nabla\varphi\big)\,. \tag{6.20}$$

Thanks to **(A4)** and **(A5)**, we deduce

$$\big(\nabla(B(\cdot,\varphi_2) - B(\cdot,\varphi_1)), \nabla\varphi\big) \geq \alpha_0(1-\rho)\|\nabla\varphi\|^2 + \big((\beta(\cdot,\varphi_2) - \beta(\cdot,\varphi_1))\nabla\varphi_2, \nabla\varphi\big)$$
$$+ \big((\mathcal{M}(\varphi_2) - \mathcal{M}(\varphi_1))\nabla a, \nabla\varphi\big)\,, \tag{6.21}$$

and, due to (6.2) for φ_2, we have

$$\big|\big((\beta(\cdot,\varphi_2) - \beta(\cdot,\varphi_1))\nabla\varphi_2, \nabla\varphi\big)\big| \leq C(\|\varphi\| + \|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2})\|\nabla\varphi_2\|^{1/2}\|\varphi_2\|^{1/2}_{H^2(\Omega)}\|\nabla\varphi\|$$
$$\leq \frac{1}{4}\alpha_0(1-\rho)\|\nabla\varphi\|^2 + C(1 + \|\varphi_2\|^2_{H^2(\Omega)})\|\varphi\|^2\,.$$

The estimates of the three terms on the right-hand side of (6.20) and of the last term in (6.21) being straightforward, we are led to

$$\frac{d}{dt}\|\varphi\|^2 + \alpha_0(1-\rho)\|\nabla\varphi\|^2 \leq C(1 + \|\varphi_2\|^2_{H^2(\Omega)})\|\varphi\|^2\,.$$

Uniqueness (and also a continuous dependence estimate) then follows by Gronwall's Lemma, on account of (6.3) for φ_2.

For $d = 3$, the test by φ does not work for uniqueness (the difficulty lies in the estimate of the term $\big((\beta(\cdot,\varphi_2) - \beta(\cdot,\varphi_1))\nabla\varphi_2, \nabla\varphi\big)$). The test by $(-\Delta_N)^{-1}\varphi$ works ($-\Delta_N$ being the Laplace operator with homogeneous Neumann boundary condition), provided that $\boldsymbol{u} \in L^2(0,T;L^\infty(\Omega)^3)$. Uniqueness then follows by arguing as in [25, Proposition 4].

Let us now prove the last part of the theorem. If $d = 2$ then we can argue as in Step 3 of the proof of Theorem 2. Identity (4.74) and estimates (4.75)–(4.80) can be rewritten in such a way that the discrete inequality (4.82) holds, where the constant $\mathbb{Q}$ now depends on the norm of $\boldsymbol{u}$ on the right-hand side of (6.7). Also the argument for the control of $(\varphi_1 - \varphi_0)/\tau$ in L^2 still works, with only one difference. More precisely, instead of (4.85), we now have, as a consequence of $(6.4)_1$,

$$\|\boldsymbol{U}_k\|_{L^\infty(\Omega)^2} \leq \frac{1}{\tau^{1/s}}\|\boldsymbol{u}\|_{L^s(0,T;L^\infty(\Omega)^2)}\,.$$

Hence, instead of using Agmon's inequality in (4.86), we can deduce

$$\tau\big|\big(\boldsymbol{U}_k \cdot (\varphi_2 - \varphi_1), \nabla\big(B(\cdot,\varphi_2) - B(\cdot,\varphi_1)\big)\big)\big|$$

$$\leq \tau \|\boldsymbol{U}_k\|_{L^\infty(\Omega)^2} \|\varphi_2 - \varphi_1\| \|\nabla\big(B(\cdot,\varphi_2) - B(\cdot,\varphi_1)\big)\|$$
$$\leq \frac{\tau}{2} \|\nabla\big(B(\cdot,\varphi_2) - B(\cdot,\varphi_1)\big)\|^2 + \frac{1}{2}\tau^{1-\frac{2}{s}} \|\boldsymbol{u}\|^2_{L^s(0,T;L^\infty(\Omega)^2)} \|\varphi_2 - \varphi_1\|^2 .$$

Since $s > 2$, we can choose $0 < \tau \leq \tau_1$, with τ_1 small enough (and depending on the norm of $\boldsymbol{u}$ on the right-hand side of (6.7)), and still obtain (4.89), yielding the desired control for the quotient $(\varphi_1 - \varphi_0)/\tau$. Owing to this control and to $(6.4)_2$ and (4.91), from (4.82) we still get (4.92), which allows to obtain $(6.5)_2$.

Finally, in order to deduce $(6.5)_1$, we can argue as in Step 1 of the proof of Theorem 2, estimating first the H^2-norm of $B(\cdot,\varphi_{k+1})$ by elliptic regularity, and then using (4.11) (cf. (4.30)). The L^2-norm of the convective term, which essentially amount to control $\boldsymbol{U}_k \cdot \nabla B(\cdot,\varphi_{k+1})$, on account of $(6.4)_1$ can now be estimated as

$$\|\boldsymbol{U}_k \cdot \nabla B(\cdot,\varphi_{k+1})\| \leq \|\boldsymbol{U}_k\|_{L^\sigma(\Omega)^2} \|\nabla B(\cdot,\varphi_{k+1})\|_{L^{2\sigma/(\sigma-2)}(\Omega)^2}$$
$$\leq C \|\boldsymbol{u}\|_{L^\infty(0,T;L^\sigma(\Omega)^2)} \|\nabla B(\cdot,\varphi_{k+1})\|^{1-2/\sigma} \|B(\cdot,\varphi_{k+1})\|^{2/\sigma}_{H^2(\Omega)}$$
$$\leq \delta \|B(\cdot,\varphi_{k+1})\|_{H^2(\Omega)} + \mathbb{Q}_\delta\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;L^\sigma(\Omega)^2)}, \|\boldsymbol{u}\|_{L^{2r/(r-2)}(0,T;L^r(\Omega)^2)}\big) .$$

Therefore, choosing $\delta > 0$ small enough, we get

$$\|B(\cdot,\bar{\varphi}_N)\|_{H^2(\Omega)} \leq \mathbb{Q}\big(\|\varphi_0\|_V, \|\boldsymbol{u}\|_{L^\infty(0,T;L^\sigma(\Omega)^2)\cap L^{2r/(r-2)}(0,T;L^r(\Omega)^2)}\big)\big(\|\widehat{\varphi}_N'\| + 1\big) ,$$

which, owing to the bound for $\widehat{\varphi}_N'$ in $L^\infty(0,T;H)$, yields a bound for $B(\cdot,\bar{\varphi}_N)$ in $L^\infty(H^2(\Omega))$, and hence on $\nabla B(\cdot,\bar{\varphi}_N), \nabla\bar{\varphi}_N, \nabla\beta(\cdot,\bar{\varphi}_N)$ in $L^\infty(0,T;L^p(\Omega)^2)$, for all $p < \infty$. Thus, on account of (4.15), we find the desired bound for $\bar{\varphi}_N$ in $L^\infty(0,T;H^2(\Omega))$. Hence, $(6.5)_1$ is proven and the proof is finished. □

Remark 13. The bound (6.11) obviously also holds for $d = 2$. Therefore, the argument relying on (6.10) can be employed, both in Theorem 2 and in Theorem 4 , to deduce the $L^2(0,T;H^2(\Omega))$ regularity for φ in two dimensions as well. However, we point out that, in the case $d = 2$, this regularity can be established without using (6.11).

Remark 14. If $d = 3$ the regularity (6.5) is open, unless we suppose $\lambda := mF_1''$ constant and $a(x) + F_2'' = 0$ almost everywhere in Ω (namely, β is constant; in this case (6.4) is still required). It is worth observing that these assumptions are basically the ones considered in [33]) whose regularity was discussed in [42]. Moreover, if β is constant then uniqueness of the strong solution satisfying (6.2)–(6.3) holds for $d = 3$, also under the more general condition (6.1) (without the need to assume $r = \infty$). Indeed, the second term on the right-hand side of (6.21) vanishes.

Similarly to Proposition 2, by employing the uniform Gronwall Lemma (or, more precisely, its discrete variant, see [48, Lemma 3]), uniform in time regularity estimates can also be established for the convective nonlocal CH equation with a prescribed velocity. We can therefore deduce from Theorem 4 another result obtained by working with translation bounded functions and providing also a dissipative estimate for φ (cf. (5.5)). We omit the statement of this theorem and its proof, since they can be deduced in a straightforward way. Moreover (cf. Remark 14), if $d=3$ and

$$mF_1''=\lambda_0\,,\qquad F_2''(s)+a(x)=0\,,\quad \text{a.e. } x\in\Omega\,, \tag{6.22}$$

where λ_0 is a positive constant, then we can prove that $\varphi\in L^\infty(0,\infty;H^2(\Omega))$ and that $\varphi_t\in L^\infty(0,\infty;H)\cap L^2_{tb}(0,T;V)$, provided $\varphi_0\in H^2(\Omega)$ satisfies (3.10) and $\boldsymbol{u}$ satisfies (6.4) in the corresponding translation bounded spaces.

As far as the time continuity property $(5.43)_2$ is concerned, assume that all the conditions of Theorem 4 and, in addition, suppose that **(M1)**$_1$, **(A1)**$_2$ are fulfilled. By arguing as in the second part of Remark 11, we can easily see that $(5.43)_2$ still holds, under the further regularity $\boldsymbol{u}\in C^0([0,T];L^\sigma(\Omega)^d)$, for some $\sigma>d$, and, if $d=3$, provided that (6.22) holds.

Suppose now that assumptions **(K)**, **(M)**, **(A1)–(A5)** are satisfied and that $\boldsymbol{u}\in L^\infty(\Omega)^d$ is independent of time. Then, from [25, Section 6] we know that (1.2), (1.3), $(1.5)_2$ and $(1.6)_2$ generates a semigroup of closed operators $\{S_\kappa(t)\}_{t\geq0}$, with $\kappa\in[0,1]$ fixed, on the phase space $\mathcal{Y}_\kappa$ defined as in (5.45) and endowed with the metric induced by the L^2-norm, namely $\varphi\in C^0([0,\infty),\mathcal{Y}_\kappa)$ given by $\varphi(t):=S_\kappa(t)\varphi_0$, for all $t\geq0$, is the (unique) weak solution to (1.2), (1.3), $(1.5)_2$ and $(1.6)_2$ corresponding to $\varphi_0\in\mathcal{Y}_\kappa$. According to [25, Theorem 5], this semigroup possesses a connected global attractor $\widetilde{\mathcal{A}}_\kappa$.

Assume now, in addition, that the **(M1)**$_1$ and **(A1)**$_2$ are fulfilled, and, for $d=3$, that (6.22) holds. It is then easy to check that the argument devised at the end of Section 5 to prove the regularity of the global attractor for (1.2)–(1.6), can be adapted to the present situation. This yields

Theorem 5. *Suppose that assumptions **(K)**, **(M)**$_1$, **(A1)**$_2$, **(A4)–(A5)** are satisfied, that $K\in W^{2,1}_{loc}(\mathbb{R}^2)$ or that K is admissible, and that $\boldsymbol{u}\in L^\infty(\Omega)^d$, $d=2,3$, is independent of time. Moreover, if $d=3$, assume that* (6.22) *holds. Then, the global attractor $\widetilde{\mathcal{A}}_k$ of the dynamical system $\big(\mathcal{Y}_k,\{S_\kappa(t)\}_{t\geq0}\big)$ generated by* (1.2), (1.3), $(1.5)_2$, $(1.6)_2$ *is such that*

$$\widetilde{\mathcal{A}}_\kappa\subset\mathcal{B}_{\mathcal{Z}_\kappa}\big(\Lambda(k)\big)\,,$$

where $\mathcal{B}_{\mathcal{Z}_\kappa}\big(\Lambda(k)\big)$ is the closed ball in the metric space $\mathcal{Z}_k$ (cf. (5.46)*), endowed with the metric induced by the H^2-norm, having radius $\Lambda(k)$, for some $\Lambda(k)>0$.*

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References

[1] H. Abels, *On a diffusive interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Ration. Mech. Anal. **194** (2009), 463-506.

[2] H. Abels, *Longtime behavior of solutions of a Navier-Stokes/Cahn-Hilliard system*, Proceedings of the Conference “Nonlocal and Abstract Parabolic Equations and their Applications”, Bedlewo, Banach Center Publ. **86** (2009), 9-19.

[3] S. Agmon, Lectures on elliptic boundary value problems, Mathematical Studies, Van Nostrand, New York, 1965.

[4] W. Arendt, R. Chill, *Global existence for quasilinear diffusion equations in isotropic nondivergence form*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9** (2010), 523-539.

[5] J.M. Ball, *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equation*, J. Nonlinear Sci. **7** (1997), 475-502 (Erratum, J. Nonlinear Sci. **8** (1998), 233).

[6] P.W. Bates, J. Han, *The Neumann boundary problem for a nonlocal Cahn-Hilliard equation*, J. Differential Equations **212** (2005), 235-277.

[7] J. Bedrossian, N. Rodríguez, A. Bertozzi, *Local and global well-posedness for an aggregation equation and Patlak-Keller-Segel models with degenerate diffusion*, Nonlinearity **24** (2011), 1683-1714.

[8] O.V. Besov, V.P. Il’in, S.M. Nikol’skiĭ, Integral representations of functions and embedding theorems. Vol. II, Scripta Series in Mathematics. Edited by M.H. Taibleson. V. H. Winston & Sons, Washington, D.C.; Halsted Press [John Wiley & Sons], New York-Toronto, Ont.-London, 1979.

[9] H. Brézis, *Équations et inéquations non-linéaires dans les espaces vectoriel en dualité*, Ann. Inst. Fourier **18** (1968), 115-176.

[10] F. Boyer, *Mathematical study of multi-phase flow under shear through order parameter formulation*, Asymptot. Anal. **20** (1999), 175-212.

[11] C. Cao, C.G. Gal, *Global solutions for the 2D NS-CH model for a two-phase flow of viscous, incompressible fluids with mixed partial viscosity and mobility*, Nonlinearity **25** (2012), 3211-3234.

[12] P. Colli, S. Frigeri, M. Grasselli, *Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Math. Anal. Appl. **386** (2012), 428-444.

[13] E. DiBenedetto, Real Analysis, Birkhäuser, Boston, Advanced Text Series, 2002.

[14] D.E. Edmunds, W.D. Evans, Spectral theory and differential operators, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1987.

[15] M. Efendiev, A. Miranville, S. Zelik, *Exponential attractors for a nonlinear reaction-diffusion system in* $\mathbb{R}^3$, C. R. Acad. Sci. Paris, Ser. I **330** (2000), 713-718.

[16] M. Efendiev, S. Zelik, *Finite-dimensional attractors and exponential attractors for degenerate doubly nonlinear equations*, Math. Methods Appl. Sci. **32** (2009), 1638-1668.

[17] C.M. Elliott, H. Garcke, *On the Cahn-Hilliard equation with degenerate mobility*, SIAM J. Math. Anal. **27** (1996), 404-423.

[18] A. Friedman, *Partial differential equations*, Holt, Rinehart and Winston, New York, 1969.

[19] E. Gagliardo, *Ulteriori proprietà di alcune classi di funzioni in più variabili*, Ricerche Mat. **8** (1959), 24-51.

[20] S. Frigeri, *Global existence of weak solutions for a nonlocal model for two-phase flows of incompressible fluids with unmatched densities*, Math. Models Methods Appl. Sci. **26** (2016), 1955-1993.

[21] S. Frigeri, C.G. Gal, M. Grasselli *On nonlocal Cahn-Hilliard-Navier-Stokes systems in two dimensions*, J. Nonlinear Sci. **26** (2016), 847-893.

[22] S. Frigeri, M. Grasselli, *Global and trajectories attractors for a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Dynam. Differential Equations **24** (2012), 827-856.

[23] S. Frigeri, M. Grasselli, *Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potentials*, Dyn. Partial Differ. Equ. **9** (2012), 273-304.

[24] S. Frigeri, M. Grasselli, P. Krejčí, *Strong solutions for two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems*, J. Differential Equations **255** (2013), 2597-2614.

[25] S. Frigeri, M. Grasselli, E. Rocca, *A diffuse interface model for two-phase incompressible flows with nonlocal interactions and nonconstant mobility*, Nonlinearity **28** (2015), 1257-1293.

[26] S. Frigeri, M. Grasselli, J Sprekels, *Optimal distributed control of nonlocal Cahn-Hilliard/Navier-Stokes systems in 2D with singular potential and degenerate mobility*, forthcoming.

[27] S. Frigeri, E. Rocca, J. Sprekels, *Optimal distributed control of a nonlocal Cahn-Hilliard/Navier-Stokes system in 2D*, SIAM J. Control Optim. **54** (2016), 221-250.

[28] C.G. Gal, *On an inviscid model for incompressible two-phase flows with nonlocal interaction*, J. Math. Fluid Mechanics (2016), DOI 10.1007/s00021-016-0252-y

[29] C.G. Gal, M. Grasselli, *Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 401-436.

[30] C.G. Gal, M. Grasselli, *Trajectory attractors for binary fluid mixtures in 3D*, Chinese Ann. Math. Ser. B **31** (2010), 655-678.

[31] C.G. Gal, M. Grasselli, *Instability of two-phase flows: a lower bound on the dimension of the global attractor of the Cahn-Hilliard-Navier-Stokes system,* Phys. D **240** (2011), 629-635.

[32] C.G. Gal, M. Grasselli, *Longtime behavior of nonlocal Cahn-Hilliard equations*, Discrete Contin. Dyn. Syst. Ser. A **34** (2014), 145-179.

[33] H. Gajewski, K. Zacharias, *On a nonlocal phase separation model*, J. Math. Anal. Appl. **286** (2003), 11-31.

[34] G. Giacomin, J.L. Lebowitz, *Exact macroscopic description of phase segregation in model alloys with long range interactions*, Phys. Rev. Lett. **76** (1996), 1094-1097.

[35] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits*, J. Statist. Phys. **87** (1997), 37-61.

[36] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. II. Phase motion*, SIAM J. Appl. Math. **58** (1998), 1707-1729.

[37] C.G. Gal, A. Giorgini, M. Grasselli, *The nonlocal Cahn-Hilliard equation with singular potential: well-posedness, regularity and strict separation property*, submitted.

[38] J.S. Kim, *Phase-field models for multi-component fluid flows*, Commun. Comput. Phys. **12** (2012), 613-661.

[39] J.-L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod Gauthier-Villars, Paris, 1969.

[40] C. Liu, J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier spectral method*, Phys. D **179** (2003), 211-228.

[41] S.-O. Londen, H. Petzeltová, *Convergence of solutions of a non-local phase-field system*, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), 653-670.

[42] S.-O. Londen, H. Petzeltová, *Regularity and separation from potential barriers for a non-local phase-field system*, J. Math. Anal. Appl. **379** (2011), 724-735.

[43] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and quasilinear equations of parabolic type, AMS Transl. Monographs **23**, AMS, Providence, R.I. 1968.

[44] A.I. Nazarov, N.N. Ural'tseva, *The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients*, St. Petersburg Math. J. **23** (2012), 93-115.

[45] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa **13** (1959), 115-162.

[46] V. Pata, S. Zelik, *A result on the existence of global attractors for semigroups of closed operators*, Commun. Pure Appl. Anal. **6** (2007), 481-486.

[47] T. Roubíček, Nonlinear Partial Differential Equations with Applications, Birkhäuser, 2005.

[48] J. Shen, *Long time stabilities and convergences for the fully discrete nonliner Galerkin method*, Appl. Anal. **38** (1990), 201-229.

[49] V.N. Starovoitov, *The dynamics of a two-component fluid in the presence of capillary forces*, Math. Notes **62** (1997), 244-254.

[50] Y. Sun, Z. Zhang, *Global regularity for the initial-boundary value problem of the 2-D Boussinesq system with variable viscosity and thermal diffusivity*, J. Differential Equations **255** (2013), 1069-1085.

[51] R. Temam, Navier-Stokes equations and nonlinear functional analysis, Second edition, CBMS-NSF Reg. Conf. Ser. Appl. Math. **66**, SIAM, Philadelphia, PA, 1995.

[52] L. Zhao, H. Wu, H. Huang, *Convergence to equilibrium for a phase-field model for the mixture of two viscous incompressible fluids*, Commun. Math. Sci. **7** (2009), 939-962.