Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Optimal sensor placement: A robust approach

Michael Hintermüller[1,2], Carlos N. Rautenberg[2], Masoumeh Mohammadi[2],

Martin Kanitsar[3]

submitted: 12.08.2016

[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: michael.hintermueller@wias-berlin.de

[2] Department of Mathematics
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin, Germany
E-Mail: carlos.rautenberg@math.hu-berlin.de
hint@math.hu-berlin.de
masoumeh.mohammadi@math.hu-berlin.de

[3] Institute for Mathematics and Scientific Computing
University of Graz
Heinrichstrasse 36
8010 Graz, Austria
E-Mail: martin.kanitsar@uni-graz.at

No. 2287
Berlin 2016

2010 *Mathematics Subject Classification.* 35K86, 47J20, 49J40, 49M15, 65J15, 65K10.

Key words and phrases. Sensor placement, trace class operators, sensitivities, Riccati equation.

This work was carried out in the framework of the DFG under grant no. HI 1466/7-1 “Free Boundary Problems and Level Set Methods” as well as the Research Center MATHEON supported by the Einstein Foundation Berlin within projects OT1, SE5 and SE15.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We address the problem of optimally placing sensor networks for convection-diffusion processes where the convective part is perturbed. The problem is formulated as an optimal control problem where the integral Riccati equation is a constraint and the design variables are sensor locations. The objective functional involves a term associated to the trace of the solution to the Riccati equation and a term given by a constrained optimization problem for the directional derivative of the previous quantity over a set of admissible perturbations. The paper addresses the existence of the derivative with respect to the convective part of the solution to the Riccati equation, the well-posedness of the optimization problem and finalizes with a range of numerical tests.

1 Introduction

The optimal placement of sensors is an important engineering application related to energy efficiency, potable water monitoring, and detection of structure integrity in buildings, among others. Moreover, it raises a number of challenges in, e.g., the applied sciences, but also in mathematics where optimization criteria influence control schemes, state estimation and/or filtering of an underlying time-evolution process. A general feature of this problem is that the evolution process is infinite dimensional and described by a system of partial differential equations (PDEs). The induced sensor output, in contrast, is finite dimensional as sensor locations are considered to be points only, and their measurements are either values of the state process at those points, or they are an integral average on an effective range from the sensor location.

In the sensor placement context, a proper definition of a useful and mathematically sound optimization objective is a problem in its own right. This is partially related to the fact that the concept of *maximal observability* admits no simple rigorous definition for distributed parameters systems. It should further be noted that, even in finite dimensions, an appropriate definition of this criterion is not always clear as simple choices may lead to pathological examples. This and other issues were studied by Khapalov (see [39, 40, 41, 38, 42]) in a robust setting for parabolic and hyperbolic problems.

The first mathematically rigorous approach, in the infinite dimensional stochastic setting, for the optimal sensor placement was taken by Bensoussan (see [6, 7]). In his work, the optimization criterion involves deviations of the state with respect to the Kalman-Bucy filter, as it was also proved by Bensoussan that the filter is well-defined in this setting. For a detailed historical development and further results see Curtain's work in [19, 20]. In a finite dimensional setting, a similar approach was considered by Athans (see [2]), and analogous results to the ones of Bensoussan were obtained through the use of Pontryagin's Maximum Principle.

In recent years, significant contributions were due to Demetriou and contributors, see [21, 22, 23, 27, 24, 26, 25, 31] and references therein. His work combines mathematical and engineering approaches to the optimal sensor and actuator placement, with static and moving networks of controllers and sensors, together with a variety of objectives and engineering applications. Approaches considering sensor placement for optimal control were developed by Burns and King in [11, 13, 14, 12] where the main focus is considered on optimizing gains for optimal feedback controllers. Combined approaches involving sensor/actuator locations have also been developed by Morris and contributors (see [36, 43, 44, 52] and references therein) with diverse applications that include, for example, optimal damping of structure vibrations.

Recently and in the vein of Bensoussan's approach, Burns and collaborators (see [15, 10, 17, 16]) have provided a general framework for determining optimal location and trajectories of sensor networks for optimal filtering. In that work, the optimization setting is based on considering solutions to the Riccati equation on the Schatten p-class and on minimizing a functional involving the trace of that solution. Further, an approximation and gradient descent scheme was developed for the implementation of solution algorithms. This paper builds on the aforementioned research. Indeed, we consider an optimization problem that leads to a robust optimal sensor location: in addition to the functional that penalizes deviations with respect to the Kalman-Bucy filter and in extension to Burns' and collaborator's work, we consider a *worst case scenario* functional involving a further optimization problem for directional sensitivities over a set of admissible perturbations. This endows the objective functional not only with the "good information" criterion but also with the feature that good locations should not be susceptible to perturbations that may render the sensor location subpar.

The infinite dimensional process that requires estimation is of convection-diffusion type where the convective part is generated by a baseline stationary velocity profile $\mathbf{v}$, obtained from solving a Navier-Stokes system. We consider perturbations $\mathbf{v} + \mathbf{h}$ via a family of profiles $\mathbf{h}$ that may be of a different nature as $\mathbf{v}$, e.g, the regularity of $\mathbf{h}$ may be lower than $\mathbf{v}$. Accordingly, a main question to be answered in the paper is how the perturbation $\mathbf{h}$ affects deviations of the state with respect to the output of the Kalman-Bucy filter. Such deviations will determine our measure of robustness.

The paper is organized as follows. In 2 we formulate the optimization problem of interest together with the underlying convection-diffusion PDE and the Riccati equation associated with the problem. Further, in 2.1 we provide some notation and basic results of trace-class operators which are of utmost importance in our setting. In 3, we explicitly describe two families of perturbations $\mathbf{h}$ in 3.1 and 3.2, respectively. Subsequently, in 3.3 we show that the perturbed differential operators generates a C_0-semigroup of contractions $S_{\mathbf{h}}(t)$ over $L^2(\Omega)$, and that, under additional assumptions, the domain of these generators is invariant. In 4 we assess the differentiability of the map that goes from the perturbation space into the solution of the Riccati equation. This is done by first addressing differentiability properties of the semigroup $\mathbf{h} \mapsto S_{\mathbf{h}}(t)$ in 4.1 and other maps involving it. Further, in 5, it is proven that the original optimization problem is well-posed. We end the paper with section 6 by a number of 2D and 3D numerical examples in complex geometries and non-trivial locations for inlets and outlets.

2 Problem Formulation

We consider a connected Lipschitz domain $\Omega \subset \mathbb{R}^\ell$ with $\ell = 2$ or $\ell = 3$ and a stationary velocity profile $\mathbf{v} : \Omega \to \mathbb{R}^\ell$ that solves (weakly) the following Navier-Stokes equations:

$$\begin{cases} -\dfrac{1}{\mathrm{Re}}\Delta \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = 0, & \text{in } \Omega, \\ \mathrm{div}(\mathbf{v}) = 0, & \text{in } \Omega, \\ \mathbf{v} = \mathbf{v}_{\text{in}}, & \text{on } \partial\Omega_{\text{in}}, \\ p\mathbf{n} - \dfrac{1}{\mathrm{Re}}\dfrac{\partial \mathbf{v}}{\partial \mathbf{n}} = \mathbf{0}, & \text{on } \partial\Omega_{\text{out}}, \\ \mathbf{v} = \mathbf{0}, & \text{on } \partial\Omega_{\text{wall}}, \end{cases} \tag{NS}$$

where Re is the Reynolds number, $p : \Omega \to \mathbb{R}$ is the pressure, the inflow profile $\mathbf{v}_{\text{in}} : \partial\Omega_{\text{in}} \to \mathbb{R}^\ell$ is prescribed, and the boundary $\partial\Omega$ is assumed to contain the disjoint sets $\partial\Omega_{\text{in}}$ and $\partial\Omega_{\text{out}}$ of positive $(\ell-1)$-Lebesgue measure, respectively, and where $\partial\Omega_{\text{wall}}$ satisfies $\partial\Omega \equiv \overline{\partial\Omega_{\text{wall}}} \cup \overline{\partial\Omega_{\text{in}}} \cup \overline{\partial\Omega_{\text{out}}}$. The unit normal to $\partial\Omega$ is given by $\mathbf{n}$ and the normal derivative of $\mathbf{v}$ on the boundary is denoted by $\frac{\partial \mathbf{v}}{\partial \mathbf{n}}$. The boundary condition on $\partial\Omega_{\text{in}}$ imply a fixed inflow velocity and the one on $\partial\Omega_{\text{out}}$ translates to free outflow on the outlet. Further, the condition on the remainder of the boundary is a no slip condition for the flow.

The quantity of interest $u : (0,T) \times \Omega \to \mathbb{R}$ is considered to diffuse with constant $(\mathrm{RePr})^{-1}$, where Pr is the Prandtl constant, and to be transported (or convected) due to the velocity profile $\mathbf{v} + \mathbf{h}$. Here, $\mathbf{v}$ solves (NS) and $\mathbf{h} : \Omega \to \mathbb{R}^\ell$ belongs to some admissible set of perturbations of $\mathbf{v}$ which we denote by $\mathbf{H}(\Omega)$. It is assumed that the value of u vanishes at a certain portion of the boundary $\partial\Omega_D := \partial\Omega_{\text{in}} \cup \partial\Omega_{\text{out}}$ and that the normal derivative vanishes at the rest of the boundary $\partial\Omega_N := \partial\Omega_{\text{wall}}$. Furthermore, we consider the dynamics stochastically perturbed. Hence, u satisfies the following stochastic Cauchy problem with mixed boundary conditions:

$$\begin{cases} \partial_t u + (\mathbf{v} + \mathbf{h})\cdot\nabla u - \dfrac{1}{\mathrm{RePr}}\Delta u - \sigma = f, & \text{in } (0,T) \times \Omega \\ u = 0, & \text{on } (0,T) \times \partial\Omega_D \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } (0,T) \times \partial\Omega_N, \end{cases} \tag{A}$$

and $u(0,\cdot) = u_0 + \xi$, on Ω, where σ is an L^2-valued Wiener process, the source $f \in L^2((0,T) \times \Omega))$, $u_0 \in L^2(\Omega)$ is arbitrary and ξ is an L^2-valued Gaussian random process with zero mean and uncorrelated to σ.

The output of the system considers perturbed measurements $\{z_i\}_{i=1}^n$ of u which are due to n sensors at locations $\hat{x} := \{\hat{x}_i\}_{i=1}^n$ and in the n admissible regions $\Gamma^i_{\text{ad}} \subset \overline{\Omega}$, respectively. At time $t \in (0,T)$, these measurements are given by

$$z_i(t) = \int_\Omega K_i(y - \hat{x}_i) u(t,y) \mathrm{d}y + \nu_i, \qquad i = 1, 2, \ldots, n. \tag{B}$$

Here, it is supposed that $K_i \in L^2(\Omega)$ and the support of $y \mapsto K_i(y - \hat{x}_i)$ is restricted to a region around the sensor location $\hat{x}_i$. The perturbations ν_i are Wiener processes uncorrelated with ν_j, for $j \neq i$, ξ and σ. We write $\nu := \{\nu_i\}_{i=1}^n$.

The problem associated with (A) and (B) can be formulated as the following abstract infinite dimensional stochastic evolution system:

$$\begin{aligned} \dot{u} &= -A(\mathbf{h})u + B\eta(t), \\ z &= C_{\hat{x}}u + \nu(t), \end{aligned} \tag{AB}$$

where $u(0) = u_0 + \xi$, and $-A(\mathbf{h}) := \frac{1}{\mathrm{RePr}}\Delta - (\mathbf{v} + \mathbf{h}) \cdot \nabla$, in a sense that is specified later. It is further considered that η is a Wiener process with values in a separable Hilbert space H and $B \in \mathscr{L}(H, L^2(\Omega))$, which implies that $B\eta$ is an L^2-valued Wiener process and $B\eta = \sigma$. By hypotheses, the triple (η, ν, ξ) is assumed to be uncorrelated. Finally, for $t \in (0, T)$, $C_{\hat{x}} \in \mathscr{L}(L^2(\Omega), \mathbb{R}^n)$ is defined by the sensor measurement output (B) as $C_{\hat{x}} := \{C_{\hat{x}_i}\}_{i=1}^n$ with

$$C_{\hat{x}_i}(\varphi) = \int_\Omega K_i(t, y - \hat{x}_i)\varphi(y)\mathrm{d}y, \qquad \forall \varphi \in L^2(\Omega) \tag{1}$$

Since there is no direct access to the variable of interest u but only to the measured output z, the implementation of an appropriate filter is of interest. Provided that $A(\mathbf{h})$, B and $C_{\hat{x}}$ satisfy certain assumptions, the output of a generalized Kalman-Bucy filter determines a stochastic $L^2(\Omega)$-valued process, which we denote by $\tilde{u}(\cdot)$. However, the reliability of such a filtering scheme depends substantially on how "close" $\tilde{u}(\cdot)$ is to the real state $u(\cdot)$. In this vein, the expected value of $|u(t) - \tilde{u}(t)|^2_{L^2(\Omega)}$ with $t \in (0, T)$ is given by the trace of an operator $\Sigma(t)$, i.e.,

$$\mathbb{E}\big\{|u(t) - \tilde{u}(t)|^2_{L^2(\Omega)}\big\} = \mathrm{Tr}\,\Sigma(t), \tag{2}$$

with $\Sigma : (0, T) \to \mathscr{L}(L^2(\Omega))$, the solution (in a sense specified later) to the following Riccati equation:

$$\begin{cases} \dot{\Sigma} = -A(\mathbf{h})\Sigma - \Sigma A^*(\mathbf{h}) + BR_2B^*(t) - \Sigma(C^*_{\hat{x}}R_1^{-1}C_{\hat{x}})(t)\Sigma, \\ \Sigma(0) = \Sigma_0, \end{cases} \tag{3}$$

where the operators $R_1(\cdot)$ and $R_2(\cdot)$ are the incremental covariances of η and ν (see [7]), respectively, and Σ_0 is the covariance operator of ξ.

Since A and C are maps depending on the perturbation $\mathbf{h}$ and on the sensors locations $\hat{x} = \{\hat{x}_i\}_{i=1}^n$, respectively, it follows that the solution to (3) satisfies

$$\Gamma_{\text{ad}} \times \mathbf{H}(\Omega) \ni (\hat{x}, \mathbf{h}) \mapsto \Sigma(\hat{x}, \mathbf{h}), \tag{4}$$

where $\mathbf{H}(\Omega)$ is the set of admissible perturbations to $\mathbf{v}$ and $\Gamma_{\text{ad}} := \prod_{i=1}^n \Gamma^i_{\text{ad}}$ is the region for admissible location of the sensors. The study of the map (4) is one of the major focus points of this work.

It follows from (2) that criteria for the quality of information of the sensors should be related to reducing the value of $\mathrm{Tr}\,\Sigma(t)$ for each t and to not increase it significantly under the family perturbations $\mathbf{H}(\Omega)$. This second criterion is related to the sensitivity of (4) with respect to $\mathbf{h} \in \mathbf{H}(\Omega)$. Hence, the following optimization problem arises naturally from these two quality criteria:

Problem $(\mathbb{P})$ *Let $\lambda \in [0,1]$ and $M > 0$ be given. Consider*

$$\min_{\hat{x}\in\Gamma_{\text{ad}}} J(\hat{x}) := \lambda J_1(\hat{x}) + (1-\lambda) J_2(\hat{x}),$$

where

$$J_1(\hat{x}) := \int_0^T \operatorname{Tr}\left(\Sigma(\hat{x},\mathbf{0})(t)\right)\,\mathrm{d}t \quad \text{and} \quad J_2(\hat{x}) := \sup_{|\mathbf{z}|_{\mathbf{H}(\Omega)}\le M} \int_0^T \operatorname{Tr}\left(W(\hat{x},\mathbf{0})\mathbf{z}(t)\right)\,\mathrm{d}t,$$

with $W(\hat{x},\mathbf{0})\mathbf{z}$ denoting the directional derivative of $\mathbf{h} \mapsto \Sigma(\hat{x},\mathbf{h})$ at zero in direction $\mathbf{z}$, and $W(\hat{x},\mathbf{0})$ is its Fréchet derivative at zero.

A few words concerning problem $(\mathbb{P})$ and its analysis are in order. We consider $-A(\mathbf{h})$ as the infinitesimal generator of a C_0-semigroup $S_{\mathbf{h}}(t)$ over $L^2(\Omega)$ and with domain $\mathscr{D}(-A(\mathbf{h}))$. This is studied in 3. Further, the Riccati equation (3) is considered in its integral form:

$$\Sigma(t) = S_{\mathbf{h}}(t)\Sigma_0 S_{\mathbf{h}}^*(t) + \int_0^t S_{\mathbf{h}}(t-s)\big(BR_2B^* - \Sigma C_{\hat{x}}^* R_1^{-1} C_{\hat{x}}\Sigma\big)(s) S_{\mathbf{h}}^*(t-s)\,\mathrm{d}s, \quad \text{(R)}$$

which makes it amenable to our approach. Additionally, conditions on B and $C_{\hat{x}}$ so that the solution to the integral equation satisfies $\Sigma(\hat{x},\mathbf{h}) \in C([0,T];\mathscr{I}_1)$ are given, where $\mathscr{I}_1$ is the Banach space of trace class operators as described in 2.1 below. Further, $W(\hat{x},\mathbf{0})\mathbf{z}$ refers to the directional derivative of the map $\mathbf{H}(\Omega) \ni \mathbf{h} \mapsto \Sigma(\hat{x},\mathbf{h}) \in C([0,T];\mathscr{I}_1)$ in direction $\mathbf{z} \in \mathbf{H}(\Omega)$. The associated differentiability proof is given in 4.2. As it is a known fact that the trace is a linear bounded functional over $\mathscr{I}_1$ and also $W(\hat{x},\mathbf{0}) \in \mathscr{L}(\mathbf{H}(\Omega), C([0,T];\mathscr{I}_1))$ we can conclude that $\int_0^T \operatorname{Tr}(W(\hat{x},\mathbf{0})(\cdot)(t))\,\mathrm{d}t$ is an element of the dual of $\mathbf{H}(\Omega)$ and $J_2(\hat{x})$ a scaled dual norm of the aforementioned functional.

It is clear that a low value in $J_1(\hat{x})$ implies good quality of information associated to the location of the sensors in $\hat{x}$. On the other hand, $J_2(\hat{x})$ measures the worst case scenario with respect to perturbations $\mathbf{h}$ bounded in energy $|\mathbf{h}|_{\mathbf{H}(\Omega)} \le M$. High values $J_2(\hat{x})$ indicate that the location $\hat{x}$ is susceptible to strong changes of the map

$$\mathbf{h} \mapsto \int_0^T \operatorname{Tr}\left(\Sigma(\hat{x},\mathbf{h})(t)\right)\,\mathrm{d}t, \tag{5}$$

under perturbations of $\mathbf{h}$, as well.

Furthermore, in addition to J_1 and J_2, we are interested in a third map, $\hat{x} \mapsto J_3(\hat{x})$, that is associated to only one given perturbation and defined as

$$J_3(\hat{x}) := \int_0^T \operatorname{Tr}\left(W(\hat{x},\mathbf{0})\mathbf{h}^*(t)\right)\,\mathrm{d}t,$$

for some fixed $\mathbf{h}^* \in \mathbf{H}(\Omega)$. While J_2 measures a worst-case-scenario, J_3 is the directional derivative at zero of the map (5) in direction $\mathbf{h}^*$.

2.1 Trace class operators

Throughout this paper we consider $\mathscr{H} \equiv L^2(\Omega)$ over the field $\mathbb{C}$, so that it is a separable complex Hilbert space. An operator $D \in \mathscr{L}(\mathscr{H})$ is called *non-negative* if $(Dx, x) \geq 0$ for all $x \in \mathscr{H}$ which is denoted by $D \geq 0$. Since $\mathscr{H}$ is a complex Hilbert space it follows that if $D \in \mathscr{L}(\mathscr{H})$ and $D \geq 0$, D is self-adjoint (see [47]), i.e., $D^* = D$.

If $D \geq 0$, then *the trace of* D is defined by $\mathrm{Tr}\,(D) := \sum_{n=1}^{\infty} \langle \phi_n, A\phi_n \rangle$, where $\{\phi_n\}_{n=1}^{\infty}$ is any orthonormal basis of $\mathscr{H}$. In this case, the value of $\mathrm{Tr}\,(D)$ may be $+\infty$ but it is invariant with respect to orthonormal bases. The polar decomposition (see for example [47]) of $D \in \mathscr{L}(\mathscr{H})$ is uniquely given as $D = U|D|$ where $|D| \in \mathscr{L}(\mathscr{H})$ and $|D| \geq 0$ and U is the unique partial isometry such that Ker $U =$ Ker$|D|$. The following definition fixes the Banach space where the trace can be applied and is finite.

Definition 1. *The set of all $D \in \mathscr{L}(\mathscr{H})$ such that $\mathrm{Tr}\,(|D|) < \infty$ is denoted by $\mathscr{I}_1$. If $D \in \mathscr{I}_1$, then the $\mathscr{I}_1$-norm of D is defined as $|D|_{\mathscr{I}_1} := \mathrm{Tr}\,(|D|) < \infty$.*

Endowed with the $\mathscr{I}_1$-norm, the linear space $\mathscr{I}_1$ is a Banach space (see [29], [33] or [48]). If $D \in \mathscr{I}_1$, then D is a compact operator and $|D|_{\mathscr{L}(\mathscr{H})} \leq |D|_{\mathscr{I}_1}$. The class $\mathscr{I}_1$ is called the space of Trace Class (or Nuclear) operators. It is known (see for example [48]), that the finite rank operators are dense (in the $\mathscr{I}_1$-norm) in $\mathscr{I}_1$ and that $\mathscr{I}_1$ is a two-sided $*$-ideal in the ring $\mathscr{L}(\mathscr{H})$, i.e., $\mathscr{I}_1$ is a vector space and:

(1) If $D \in \mathscr{I}_1$ and $E \in \mathscr{L}(\mathscr{H})$, then $DE \in \mathscr{I}_1$ and $ED \in \mathscr{I}_1$. Further, $|DE|_{\mathscr{I}_1} \leq |D|_{\mathscr{I}_1}|E|_{\mathscr{L}(\mathscr{H})}$ and $|ED|_{\mathscr{I}_1} \leq |D|_{\mathscr{I}_1}|E|_{\mathscr{L}(\mathscr{H})}$

(2) If $D \in \mathscr{I}_1$ then $D^* \in \mathscr{I}_1$, and $|D^*|_{\mathscr{I}_1} = |D|_{\mathscr{I}_1}$.

The trace is a continuous linear functional over $\mathscr{I}_1$ (see [29]). Consequently, if $D \in \mathscr{I}_1$, the value $\mathrm{Tr}\,(D) = \sum_{n=1}^{\infty}(\phi_n, D\phi_n)$ does not depend on the choice of the orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$.

3 Semigroup Setting

We now consider problem (A) in a semigroup setting in which $\mathbf{h}$ represents perturbations of the infinitesimal generator of a semigroup. The state space is given by the closure of $E(\Omega) := \{\phi \in C^\infty(\Omega) : \overline{\text{supp}\,\phi} \cap \partial\Omega_D = \emptyset\}$ in the $H^1(\Omega)$-norm, i.e.,

$$H^1_D(\Omega) := \overline{E(\Omega)}^{H^1(\Omega)}. \tag{6}$$

We note that $H^1_D(\Omega)$, when endowed with the inner product

$$((v, w)) := \int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x, \qquad \forall v, w \in H^1_D(\Omega), \tag{7}$$

is a Hilbert space and that the closure (6) can be also taken with respect to the $H^1_0(\Omega)$-norm without changing the outcome. This follows from the fact that since $|\partial\Omega_D| > 0$, the norm defined as $v \mapsto |v|_D := |\nabla v|_{L^2(\Omega)^\ell} + |\int_{\partial\Omega_D} v \,\mathrm{d}S|$ is equivalent to the usual norm on $H^1(\Omega)$. Hence, for $v \in H^1_D(\Omega)$ we have that $|v|_D = |\nabla v|_{L^2(\Omega)^\ell}$ and then $|v|_{H^1(\Omega)} \leq C|v|_{H^1_D(\Omega)}$, for some constant $C > 0$ not depending on v.

Further note that $(H^1_D(\Omega), L^2(\Omega), H^1_D(\Omega)^*)$ forms a Gelfand triple. Indeed, the natural injection $H^1_D(\Omega) \hookrightarrow L^2(\Omega)$ is dense and continuous and we identify the dual of $L^2(\Omega)$ with itself. The previous implies that the injection $L^2(\Omega) \hookrightarrow H^1_D(\Omega)^*$ is also dense and continuous (see [50] or [51], for example).

Let $\alpha := (\mathrm{RePr})^{-1}$, and define the form a as

$$a(u, w) := \alpha \int_\Omega \nabla u \cdot \nabla w \,\mathrm{d}x + \int_\Omega (\mathbf{v} \cdot \nabla u) w \,\mathrm{d}x + \int_\Omega (\mathbf{h} \cdot \nabla u) w \,\mathrm{d}x. \tag{8}$$

The well-posedness of the form, its domain of definition and properties are tied directly to $\mathbf{v}$ and $\mathbf{h}$. In this connection, the regularity of $\mathbf{v}$ and $\mathbf{h}$ is important and it is addressed next. Since $\mathbf{v}$ is assumed to be the weak solution to (NS), then $\mathbf{v} \in H^1(\Omega)^\ell$. In general, no extra global regularity can be expected due to the "do nothing" boundary condition in $\partial\Omega_{\text{out}}$, i.e., we assume that $\mathbf{v} \notin H^2(\Omega)^\ell$. Additionally we have that $\mathrm{div}(\mathbf{v}) = 0$ in Ω, $\mathbf{v} = 0$ in $\partial\Omega_{\text{wall}} \equiv \partial\Omega_N$ and $\mathbf{v} = \mathbf{v}_{\text{in}}$ in $\partial\Omega_{\text{in}} \subset \partial\Omega_D$.

For the admissible space of perturbations, in this section, we consider (at this point) two possibilites: $H(\Omega)$, and $\tilde{H}(\Omega)$. The first choice is a space with the same regularity as $\mathbf{v}$ and the second one results from a particular construction via PDEs that is considered within numerical implementation and possesses lower regularity of its elements. While, the first approach works for $\ell = 2, 3$, the second one, is mathematically rigorous only for $\ell = 2$.

3.1 First Perturbation Approach

Define

$$H(\Omega) := \{\mathbf{h} \in H^1(\Omega)^\ell : \text{ div}\, \mathbf{h} = 0 \text{ in } \Omega, \quad \mathbf{n} \cdot \mathbf{h}|_{\partial\Omega_{\text{wall}}} = 0\}, \tag{9}$$

where $\mathbf{n}$ is the outer unit normal to $\partial\Omega$. As with $H^1_D(\Omega)$, note that $H(\Omega)$ endowed with the inner product in (7) is a Hilbert space. Further, note that $\mathbf{v} \in H(\Omega)$, where $\mathbf{v}$ is the solution to (NS).

Since, at this point, neither $\mathbf{v}$ or $\mathbf{h}$ are necessarily in $L^\infty(\Omega)^\ell$, it is not straightforward to state that the operator associated with $a(\cdot, \cdot)$ generates a C_0-semigroup. Hence, we need to clarify as the necessary properties for the form first. For this reason, we need the following result:

Proposition 2. *Let $\mathbf{h} \in H(\Omega)$ and $\ell = 2, 3$. Then, the form in* (8) *is well-defined as $a : H^1_D(\Omega) \times H^1_D(\Omega) \to \mathbb{R}$, and further, it is bilinear, continuous and coercive.*

Proof. First, note that $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q < \infty$ if $\ell = 2$, for $2 \leq q \leq 6$ if $\ell = 3$. Then, for $\ell = 2, 3$, $z \in H^1(\Omega)$ and $u, w \in H^1_D(\Omega)$, we have

$$\left| \int_\Omega z \frac{\partial u}{\partial x_i} w \,\mathrm{d}x \right| \leq |z|_{L^4(\Omega)} |w|_{L^4(\Omega)} |u|_{H^1_D(\Omega)} \leq C_1 |z|_{H^1(\Omega)} |w|_{H^1_D(\Omega)} |u|_{H^1_D(\Omega)},$$

for some $C_1 > 0$. Hence, the integrals involving $\mathbf{h}$ and $\mathbf{v}$ in the definition of $a(\cdot,\cdot)$, are well defined and bounded. This implies that the form $a(\cdot,\cdot)$ is bilinear. Further,

$$|a(u,w)| \leq C|w|_{H^1_D(\Omega)}|u|_{H^1_D(\Omega)}, \tag{10}$$

for some C depending linearly on α, $|\nabla \mathbf{v}|_{L^2(\Omega)^{\ell\times\ell}}$ and $|\nabla \mathbf{h}|_{L^2(\Omega)^{\ell\times\ell}}$. Hence, $a(\cdot,\cdot)$ is also continuous.

Let $\mathbf{z} \in H^1(\Omega)^\ell$ and $\mathbf{n}\cdot\mathbf{z}|_{\partial\Omega_{\text{wall}}} = 0$ (note that $\partial\Omega_{\text{wall}} \equiv \partial\Omega_N$). Then, for $u \in H^1_D(\Omega)$, we observe by Green's formula (see [45, Chapter 3, Theorem 1.1]) that

$$\begin{aligned}\int_\Omega z_i \frac{\partial u}{\partial x_i} u \,\mathrm{d}x &= -\int_\Omega u \frac{\partial}{\partial x_i}(z_i u)\,\mathrm{d}x + \int_{\partial\Omega} z_i u^2 n_i \,\mathrm{d}S,\\ &= -\left(\int_\Omega u^2 \frac{\partial z_i}{\partial x_i} + u\frac{\partial u}{\partial x_i} z_i \,\mathrm{d}x\right) + \int_{\partial\Omega} z_i u^2 n_i\,\mathrm{d}S,\end{aligned}$$

where we have used that $z_i u \in W^{1,3/2}(\Omega)$ and $u \in W^{1,2}(\Omega)$ for $\ell = 2,3$ and hence of usage the Green's formula is justified. Summation of the above identity over the index i implies

$$\begin{aligned}\int_\Omega (\mathbf{z}\cdot\nabla u)u\,\mathrm{d}x &= -\left(\int_\Omega u^2\,\mathrm{div}(\mathbf{z}) + (\mathbf{z}\cdot\nabla u)u\,\mathrm{d}x\right) + \int_{\partial\Omega}(\mathbf{z}\cdot\mathbf{n})u^2\,\mathrm{d}S\\ &= -\int_\Omega (\mathbf{z}\cdot\nabla u)u\,\mathrm{d}x,\end{aligned}$$

where we have used that $\mathrm{div}(\mathbf{z}) = 0$ in Ω, $\mathbf{z}\cdot\mathbf{n}|_{\partial\Omega_N} = 0$ and $u|_{\partial\Omega_D} = 0$. Therefore,

$$\int_\Omega (\mathbf{z}\cdot\nabla u)u\,\mathrm{d}x = 0, \tag{11}$$

and we obtain

$$a(u,u) = \alpha|u|^2_{H^1_D(\Omega)}, \tag{12}$$

i.e., $a(\cdot,\cdot)$ is coercive. □

3.2 Second Perturbation Approach

For the second approach concerning the perturbations in $\Omega \subset \mathbb{R}^2$ we consider $\partial\Omega$ to be of class C^3 and pursue the following construction. Let $\Gamma_0 \subset \partial\Omega_D$ and Γ_1 denote open disjoint subsets of $\partial\Omega$ consisting of a finite number of connected open subsets. Note that this implies that $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ is equal to a finite number of points. Given $g : \Gamma_0 \to \mathbb{R}$, consider the following mixed boundary value problem:

$$\begin{cases} -\Delta w = 0, & \text{in } \Omega,\\ w = g, & \text{on } \Gamma_0,\\ \dfrac{\partial w}{\partial \mathbf{n}} = 0, & \text{on } \Gamma_1.\end{cases} \tag{M}$$

We define $\tilde{H}(\Omega)$ as

$$\tilde{H}(\Omega) := \\ \{\nabla w : w \text{ weakly solves (M) for some Lipschitz } g \in W^{2-\frac{1}{p},p}(\Gamma_0), 1 < p < 4/3\}.$$

A few words are in order concerning $\tilde{H}(\Omega)$. If $\mathbf{h} \in \tilde{H}(\Omega)$, then this implies that $\mathbf{h} = \nabla w$ for some weak solution to (M) determined by a Lipschitz function $g \in W^{2-\frac{1}{p},p}(\Gamma_0)$ where $1 < p < 4/3$. PDE regularity theory implies $w \in W^{2,p}(\Omega) \cap W^{1,q}(\Omega)$ for all $q < 4$ (see [49, Theorem 7.9, Chapter 3]). By the first and third equations in (M), we have that $\mathrm{div}(\mathbf{h}) = \Delta w = 0$ a.e. in Ω, $\mathbf{h} \in W^{1,p}(\Omega)$ and $\mathbf{h} \cdot \mathbf{n} = 0$ in the sense of $W^{1-1/p,p}(\Gamma_1)$ (see, for example, [34]). We see in the following results that this is enough regularity for the well-posedness of the form $a(\cdot, \cdot)$, its coercivity, continuity and bilinearity.

Proposition 3. *Suppose that* $\mathbf{h} \in \tilde{H}(\Omega)$ *and* $\ell = 2$. *Then, the form in* (8) *is well defined as* $a : H^1_D(\Omega) \times H^1_D(\Omega) \to \mathbb{R}$ *and further, it is bilinear, continuous and coercive.*

Proof. Since $\mathbf{h} \in \tilde{H}(\Omega)$, by definition this implies that $\mathrm{div}(\mathbf{h}) = 0$ a.e. in Ω, $\mathbf{h} \in W^{1,p}(\Omega)^2$ and $\mathbf{h} \cdot \mathbf{n} = 0$ in the sense of $W^{1-1/p,p}(\Gamma_1)$ all for some $1 < p < 4/3$. Then, it follows that the integral involving $\mathbf{h}$ in the definition of $a(\cdot, \cdot)$ in (8) is well-defined: Note that the following continuous embeddings hold true $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $H^1(\Omega) \hookrightarrow L^{q^*}(\Omega)$ with

$$2 < p^* := \frac{2p}{2-p} < 4, \qquad \text{and} \qquad 4 < q^* := \frac{2p^*}{p^*-2} < \infty.$$

Then, if $h \in W^{1,p}(\Omega)$ and $u, w \in H^1_D(\Omega)$, we have that $h\frac{\partial u}{\partial x_i} w$ is integrable and bounded as

$$\left| \int_\Omega h \frac{\partial u}{\partial x_i} w \, \mathrm{d}x \right| \leq |h|_{L^{p^*}(\Omega)} |u|_{H^1_D(\Omega)} |w|_{L^{q^*}(\Omega)} \leq C |h|_{L^{p^*}(\Omega)} |u|_{H^1_D(\Omega)} |w|_{H^1_D(\Omega)},$$

for some $C > 0$. From this we again obtain the boundedness as in (10) (and hence continuity) of the form $a(\cdot, \cdot)$ but where the constant C, now depends linearly on α, $|\nabla \mathbf{v}|_{L^2(\Omega)^{2\times 2}}$ and $|\mathbf{h}|_{W^{1,p}(\Omega)^2}$. Hence, $a(\cdot, \cdot)$ is also continuous.

Now we focus on obtaining (11) for $\mathbf{h} \in \tilde{H}(\Omega)$ and $u \in H^1_D(\Omega)$. For that, we first show that $u^2 \in W^{1,q}(\Omega)$, with $1 \leq q < 2$. By Sobolev embeddings, $u \in L^r(\Omega)$, with $2 \leq r < \infty$, and consequently $u^2 \in L^{r/2}(\Omega)$. On the other hand, $\partial_{x_i} u \in L^2(\Omega)$ so that $\partial_{x_i} u^2 = 2u \, \partial_{x_i} u \in L^{2r/r+2}$. Since $r \geq 2$, we have $r/2 > 2r/(r+2)$, which implies that $u^2 \in W^{1,2r/(r+2)}$. Noting that the range of $[2, +\infty) \ni r \mapsto 2r/(r+2)$ is $[1, 2)$, we conclude $u^2 \in W^{1,q}(\Omega)$ for $1 \leq q < 2$.

Further, since $h_i \in W^{1,p}(\Omega)$, with $1 < p < 4/3$, there exist $q \in [1, 2)$ such that $\frac{1}{p} + \frac{1}{q} \leq \frac{\ell+1}{\ell} = \frac{3}{2}$, and where $1 \leq p, q < \ell = 2$, it follows that Green's formula (see [45, Chapter 3, Theorem 1.1]) is applicable in the following:

$$\int_\Omega h_i \frac{\partial (u^2)}{\partial x_i} \, \mathrm{d}x = -\int_\Omega u^2 \frac{\partial h_i}{\partial x_i} \, \mathrm{d}x + \int_{\partial\Omega} h_i u^2 n_i \, \mathrm{d}S.$$

Summation of the above identity along i, yields

$$\int_\Omega (\mathbf{h}\cdot\nabla u)u \,\mathrm{d}x = -\int_\Omega u^2 \operatorname{div}(\mathbf{h})\,\mathrm{d}x + \int_{\partial\Omega} (\mathbf{h}\cdot\mathbf{n})u^2 \,\mathrm{d}S = 0,$$

where we have used that $\operatorname{div}(\mathbf{h}) = 0$ a.e. in Ω, $\mathbf{h}\cdot\mathbf{n}|_{\partial\Omega_N} = 0$ and $u|_{\partial\Omega_D} = 0$. This implies that (12) also holds in this case, and hence $a(\cdot,\cdot)$ is coercive. □

3.3 Infinitesimal Generator

We are now in position to prove that the operator induced by the form $a(\cdot,\cdot)$ generates a C_0-semigroup on $L^2(\Omega)$, the pivot space in $(H^1_D(\Omega), L^2(\Omega), H^1_D(\Omega)^*)$ for the cases where $\mathbf{h}\in H(\Omega)$ and $\mathbf{h}\in\tilde{H}(\Omega)$. We further prove that the domain of the generator is invariant to the perturbations under an additional assumption. The latter is of utmost importance for determining sensitivity properties of $\mathbf{h}\mapsto\Sigma(\mathbf{h})$.

Proposition 4. *Let $\mathbf{h}\in H(\Omega)$ or $\mathbf{h}\in\tilde{H}(\Omega)$. Then, the associated operator $-A(\mathbf{h})$ to the form $a : H^1_D(\Omega)\times H^1_D(\Omega)\to\mathbb{R}$ generates a contractive holomorphic C_0-semigroup $S_{\mathbf{h}}(t)$ over $L^2(\Omega)$.*

Further, for any two $\mathbf{h}_1,\mathbf{h}_2\in H(\Omega)\cap L^\infty(\Omega)^\ell$ for $\ell = 2,3$ or $\mathbf{h}_1,\mathbf{h}_2\in\tilde{H}(\Omega)\cap L^\infty(\Omega)^2$, we have

$$\mathscr{D} := \mathscr{D}(A(\mathbf{h}_1)) \equiv \mathscr{D}(A(\mathbf{h}_2)), \tag{13}$$

i.e., the operator is domain invariant.

Proof. From the several possible approaches to do so, we follow [1] and drop the $\mathbf{h}$-dependence notation in this paragraph. Define A and its domain $\mathscr{D}(A)$ as follows: Given $w\in H^1_D(\Omega)$, $z\in L^2(\Omega)$, we say that $w\in\mathscr{D}(A)$ and $Aw = z$ if

$$a(w,v) = (z,v), \qquad \forall v\in H^1_D(\Omega).$$

Since the form $a : H^1_D(\Omega)\times H^1_D(\Omega)\to\mathbb{R}$ is bilinear, continuous and coercive, it follows that $A : \mathscr{D}(A)\to L^2(\Omega)$ generates a contractive holomorphic C_0-semigroup $S(t)$ over $L^2(\Omega)$ (see [1, Theorem 4.2]).

We argue that $\mathscr{D}(A(\mathbf{h}))\equiv\mathscr{D}(A(\mathbf{0}))$ if $\mathbf{h}\in H(\Omega)\cap L^\infty(\Omega)^\ell$ for $\ell = 1,2$ or $\mathbf{h}\in\tilde{H}(\Omega)\cap L^\infty(\Omega)^2$. In order to emphasize the $\mathbf{h}$-dependence, we write $a_{\mathbf{h}}(\cdot,\cdot)$ for $a(\cdot,\cdot)$ in (8) and further

$$a_{\mathbf{h}}(w,v) := a_0(w,v) + (\mathbf{h}\cdot\nabla w, v), \qquad \forall w,v\in H^1_D(\Omega).$$

Suppose that $w_0\in\mathscr{D}(A(\mathbf{0}))$. Hence $a_0(w_0,v) = (z_0,v)$ for some $z_0\in L^2(\Omega)$ and for all $v\in H^1_D(\Omega)$. Then, $a_{\mathbf{h}}(w_0,v) = (z_0,v)+(\mathbf{h}\cdot\nabla w_0,v)$ for all $v\in H^1_D(\Omega)$. Since $z_0+\mathbf{h}\cdot\nabla w_0\in L^2(\Omega)$ it follows that $w_0\in\mathscr{D}(A(\mathbf{h}))$, so that $\mathscr{D}(A(\mathbf{0}))\subset\mathscr{D}(A(\mathbf{h}))$. The reverse inclusion is shown analogously. □

Since A changes when $\mathbf{h}$ does, it follows that for each $\mathbf{h}\in H(\Omega)$ (or $\mathbf{h}\in\tilde{H}(\Omega)$), the above procedure provides one associated semigroup $S_{\mathbf{h}}(t)$. The study of the perturbations of $\mathbf{h}\mapsto S_{\mathbf{h}}(t)$ and also $\mathbf{h}\mapsto\Sigma(\mathbf{h})$ is carried out in the following section.

4 Sensitivity Analysis

We now establish sensitivity results for $\mathbf{h} \mapsto \Sigma_{\mathbf{h}}$. For this purpose, in what follows we assume that $\mathscr{D} = \mathscr{D}(A(\mathbf{h}))$ for every chosen $\mathbf{h}$. Hence, it is useful to consider the following:

$$H^\infty(\Omega) := H(\Omega) \cap L^\infty(\Omega)^\ell \quad \ell = 2, 3, \qquad \text{and} \qquad \tilde{H}^\infty(\Omega) := \tilde{H}(\Omega) \cap L^\infty(\Omega)^2,$$

which, endowed with the usual norm for intersection of Banach spaces, are Banach spaces in their own right. Throughout this section we use the following notation: we write $\mathbf{H}(\Omega)$ either for $H^\infty(\Omega)$ or $\tilde{H}^\infty(\Omega)$, $S_{\mathbf{h}}(t)$ denotes the semigroup generated by $A(\mathbf{h})$, $S(t) := S_{\mathbf{0}}(t)$, $A := A(\mathbf{0})$ and $P(\mathbf{h}) := A(\mathbf{h}) - A$.

4.1 Sensitivity of $\mathbf{h} \mapsto S_{\mathbf{h}}(t)$

In this section we provide the preparatory results and lemmas necessary to prove the differentiability of the map $\mathbf{h} \mapsto \Sigma_{\mathbf{h}}$. We provide three results, 5, 6, and 7. The first one endows a stronger regularity characterization to the semigroup $S_{\mathbf{h}}(t)$ based on PDE theory. 6 proves the differentiability of the composition of functions involving $S_{\mathbf{h}}(t)$ and finally 7 provides additional bounds required in 4.2.

In order to provide a better regularity characterization of $S_{\mathbf{h}}(t)$, first note that for $\phi \in L^2(\Omega)$, $t \mapsto u(t, \cdot) := S_{\mathbf{h}}(t)\phi$ is the unique (weak solution) to

$$\begin{cases} \partial_t u + (\mathbf{v} + \mathbf{h}) \cdot \nabla u - \alpha \Delta u = 0, & \text{in } (0,T) \times \Omega; \\ u(0, \cdot) = \phi, & \text{in } \Omega; \\ u = 0, & \text{on } (0,T) \times \partial\Omega_D; \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } (0,T) \times \partial\Omega_N; \end{cases} \tag{14}$$

and hence we have the following result.

Lemma 5. *Let $\phi \in L^2(\Omega)$ and $\mathbf{h} \in \mathbf{H}(\Omega)$. Then, $S_{\mathbf{h}}(t)\phi \in H^1_D(\Omega)$ for $t > 0$, and for any $T > 0$ it holds true that*

$$\int_0^T |S_{\mathbf{h}}(t)\phi|^2_{H^1_D(\Omega)} \,\mathrm{d}t \le \alpha^{-1} |\phi|^2_{L^2(\Omega)}, \tag{15}$$

and

$$\int_0^T |S_{\mathbf{h}}(t)\phi - S(t)\phi|^2_{H^1_D(\Omega)} \,\mathrm{d}t \le C |\mathbf{h}|^2_{\mathbf{H}(\Omega)} |\phi|^2_{L^2(\Omega)}, \tag{16}$$

for some constant $C > 0$.

Proof. Since $a : H^1_D(\Omega) \times H^1_D(\Omega) \to \mathbb{R}$ is bilinear, continuous and coercive when $\mathbf{h} \in \mathbf{H}(\Omega)$ by previous results, the weak formulation of (14) is given by the following problem: Find $u \in L^2(0,T; H^1_D(\Omega))$, with $\partial_t u \in L^2(0,T; H^1_D(\Omega)^*)$, such that $u(0) = \phi$ and for a.e. $t \in (0,T)$

$$\langle \partial_t u, v \rangle_{(H^1_D)^*, H^1_D} + a(u, v) = 0, \qquad \forall v \in H^1_D(\Omega).$$

Note that if $u \in L^2(0,T;H^1_D(\Omega))$, and $\partial_t u \in L^2(0,T;H^1_D(\Omega)^*)$, then it follows that $u \in C([0,T];L^2(\Omega))$ (see [51]) and hence the condition $u(0) = \phi$ is considered in the sense of $L^2(\Omega)$. Further, from standard PDE theory (see [30]), it is straightforward to prove that the problem admits a unique solution.

Using $v = u$ above, and noting that $\langle \partial_t u, u\rangle_{(H^1_D)^*,H^1_D} = \frac{1}{2}\frac{d}{dt}|u(t)|^2_{L^2(\Omega)}$ as well as $a(u,u) = \alpha|u|^2_{H^1_D(\Omega)}$, it follows that

$$|u(t)|^2_{L^2(\Omega)} + 2\alpha\int_0^t |u(s)|^2_{H^1_D(\Omega)}\,\mathrm{d}s = |\phi|^2_{L^2(\Omega)}, \tag{17}$$

by integrating between 0 and $t \leq T$ the previous equality.

Let u_1 and u_0 denote the weak solutions for $\mathbf{h} \neq 0$ and $\mathbf{h} = 0$, respectively. Then, by considering $v = u_1 - u_0$ in the weak formulations on the respective problems and adding both we obtain that

$$\langle \partial_t(u_1 - u_0), u_1 - u_0\rangle + a_0(u_1 - u_0, u_1 - u_0) = -(\mathbf{h}\cdot\nabla u, u_1 - u_0).$$

Integrating between 0 and t yields

$$|(u_1 - u_0)(t)|^2_{L^2(\Omega)} + 2\alpha\int_0^t |(u_1 - u_0)(s)|^2_{H^1_D(\Omega)}\,\mathrm{d}s \leq \\ 2|\mathbf{h}|_{\mathbf{H}(\Omega)}\left(\int_0^t |u_1(s)|^2_{H^1_D(\Omega)}\,\mathrm{d}s\right)^{1/2}\left(\int_0^t |(u_1 - u_0)(s)|^2_{L^2(\Omega)}\,\mathrm{d}s\right)^{1/2}, \tag{18}$$

by using the Hölder inequality. Therefore, by (17), it follows that

$$|(u_1 - u_0)(t)|^2_{L^2(\Omega)} \leq \frac{\sqrt{2}}{\sqrt{\alpha}}|\mathbf{h}|_{\mathbf{H}(\Omega)}|\phi|_{L^2(\Omega)}\left(\int_0^t |(u_1 - u_0)(s)|^2_{L^2(\Omega)}\,\mathrm{d}s\right)^{1/2}.$$

Although Gronwall's inequality is not directly applicable, Bihari's nonlinear generalization is (see [9, 28]) and gives

$$|(u_1 - u_0)(t)|^2_{L^2(\Omega)} \leq \frac{t}{\alpha}|\mathbf{h}|^2_{\mathbf{H}(\Omega)}|\phi|^2_{L^2(\Omega)}, \qquad \forall t \in [0,T],$$

which applied to (18) together with (17) implies

$$\int_0^t |(u_1 - u_0)(s)|^2_{H^1_D(\Omega)}\,\mathrm{d}s \leq C|\mathbf{h}|^2_{\mathbf{H}(\Omega)}|\phi|^2_{L^2(\Omega)}, \tag{19}$$

for some $C > 0$ involving T and α, and this finalizes the proof. □

With the aid of 5, we are now in shape to prove sensitivity results concerning the semigroup $S_{\mathbf{h}}(t)$ in the following setting.

Theorem 6. *Let $\Gamma_1 \in C([0,T];\mathscr{I}_1)$ and $\Gamma_2 \in L^\infty(0,T;\mathscr{I}_1)$ with $\Gamma_1(t)^* = \Gamma_1(t)$ and $\Gamma_2(t)^* = \Gamma_2(t)$ for $t \in [0,T]$. Define*

$$R_1(\mathbf{h})(t) := S_{\mathbf{h}}(t)\Gamma_1(t)S^*_{\mathbf{h}}(t), \quad \text{and} \quad R_2(\mathbf{h})(t) := \int_0^t S_{\mathbf{h}}(t-s)\Gamma_2(s)S_{\mathbf{h}}(t-s)^*\,\mathrm{d}s,$$

for $t \in [0,T]$. Then, R_1 and R_2 are well-defined, they map $\mathbf{H}(\Omega)$ into $C([0,\tau];\mathscr{I}_1)$ and are Fréchet differentiable at zero.

Proof. The fact that $R_1(\mathbf{h}), R_2(\mathbf{h}) \in C([0,\tau];\mathscr{I}_1)$ is given in [17]. Note that the integral that defines R_2 is well-defined as a Bochner integral only because Γ_2 is compact-valued.

We focus first on R_1. Let $\phi \in \mathscr{D} \subset L^2(\Omega)$. We follow an argument in [46, Chapter 3]. Note that $s \mapsto H(s) := S(t-s)S_{\mathbf{h}}(s)\phi$ is differentiable in $(0,t)$ (here, we have that $S_{\mathbf{h}}(s)\phi \in \mathscr{D}$) with derivative $H'(s) = S(t-s)P(\mathbf{h})S_{\mathbf{h}}(s)\phi$ where $P(\mathbf{h}) = A(\mathbf{h}) - A(\mathbf{0})$. Integrating between 0 and t yields

$$S_{\mathbf{h}}(t)\phi = S(t)\phi + \int_0^t S(t-s)P(\mathbf{h})S_{\mathbf{h}}(s)\phi \,\mathrm{d}s, \qquad \forall \phi \in \mathscr{D}.$$

Since $\mathscr{D}$ is dense in $L^2(\Omega)$, and $S_{\mathbf{h}}(t) - S(t)$ is bounded on $L^2(\Omega)$, the above holds also for all $\phi \in L^2(\Omega)$. Then, for $\phi \in L^2(\Omega)$, we observe

$$S_{\mathbf{h}}(t)\phi - S(t)\phi - \int_0^t S(t-s)P(\mathbf{h})S(s)\phi \,\mathrm{d}s = \int_0^t S(t-s)P(\mathbf{h})(S_{\mathbf{h}}(s) - S(s))\phi \,\mathrm{d}s,$$

where we have used that $\int_0^t S(t-s)P(\mathbf{h})S(s)\phi \mathrm{d}s$ is well-defined: note that $S(\cdot)\phi \in L^2(0,T;H_D^1(\Omega))$, so that $P(\mathbf{h})S(\cdot)\phi = (\mathbf{h}\cdot\nabla)S(\cdot)\phi \in L^2(0,T;L^2(\Omega))$ which proves the claim since $\int_0^t S(t-s)f(s)\mathrm{d}s$ is well defined for $f \in L^1(0,T;L^2(\Omega))$. Also note that $t \mapsto \int_0^t S(t-s)P(\mathbf{h})S(s)\phi\mathrm{d}s \in L^2(\Omega)$ is continuous (see [46]).

Further, for $\Gamma_1 \in C([0,T];\mathscr{I}_1)$, the above equality in $L^2(\Omega)$ implies that

$$\begin{aligned} S_{\mathbf{h}}(t)\Gamma_1(t) - S(t)\Gamma_1(t) - \int_0^t S(t-s)P(\mathbf{h})S(s)\Gamma_1(t)\,\mathrm{d}s \\ = \int_0^t S(t-s)P(\mathbf{h})(S_{\mathbf{h}}(s) - S(s))\Gamma_1(t)\,\mathrm{d}s \end{aligned} \tag{20}$$

in the operator sense and where the integrals are strong Bochner integrals (see [35]), i.e., the integrands may fail to be measurable as operator valued functions, but they are measurable as $L^2(\Omega)$-functions when acting on elements in $L^2(\Omega)$. Additionally, since $\Gamma_1(t) \in \mathscr{I}_1$, all the terms in the above equality are also in $\mathscr{I}_1$. Further, since the maps $t \mapsto S_{\mathbf{h}}(t)$, $t \mapsto S(t)$, and $t \mapsto \int_0^t S(t-s)P(\mathbf{h})S(s)(\cdot)\,\mathrm{d}s$ are strongly continuous, all terms in (20) belong to $C([0,T];\mathscr{I}_1)$ (see for example [17]).

Since $|S(t)|_{\mathscr{L}(L^2(\Omega))} \leq 1$, $P(\mathbf{h})y = \mathbf{h}\cdot\nabla y$ for $y \in H_D^1(\Omega)$, and because (16) holds, we have

$$\left| S_{\mathbf{h}}(t)\Gamma_1(t) - S(t)\Gamma_1(t) - \int_0^t S(t-s)P(\mathbf{h})S(s)\Gamma_1(t)\,\mathrm{d}s \right|_{\mathscr{I}_1} \leq C_1|\mathbf{h}|^2_{\mathbf{H}(\Omega)}|\Gamma_1(t)|_{\mathscr{I}_1},$$

for some $C_1 > 0$, as the following inequalities hold true

$$\left|\int_0^t S(t-s)P(\mathbf{h})(S_{\mathbf{h}}(s)-S(s))\Gamma_1(t)\,\mathrm{d}s\right|_{\mathscr{I}_1} \leq \tag{21}$$

$$\leq |\Gamma_1(t)|_{\mathscr{I}_1} \sup_{|\phi|_{L^2(\Omega)}\leq 1}\left(\left|\int_0^t S(t-s)P(\mathbf{h})(S_{\mathbf{h}}(s)-S(s))\phi\,\mathrm{d}s\right|_{L^2(\Omega)}\right)$$
$$\leq |\Gamma_1(t)|_{\mathscr{I}_1} \sup_{|\phi|_{L^2(\Omega)}\leq 1}\left(\int_0^t |P(\mathbf{h})(S_{\mathbf{h}}(s)-S(s))\phi|_{L^2(\Omega)}\,\mathrm{d}s\right)$$
$$\leq |\Gamma_1(t)|_{\mathscr{I}_1}|\mathbf{h}|_{\mathbf{H}(\Omega)} \sup_{|\phi|_{L^2(\Omega)}\leq 1}\left(\int_0^t |(S_{\mathbf{h}}(s)-S(s))\phi|_{H^1_D(\Omega)}\,\mathrm{d}s\right)$$
$$\leq T^{1/2}|\Gamma_1(t)|_{\mathscr{I}_1}|\mathbf{h}|_{\mathbf{H}(\Omega)} \sup_{|\phi|_{L^2(\Omega)}\leq 1}\left(\int_0^T |(S_{\mathbf{h}}(s)-S(s))\phi|^2_{H^1_D(\Omega)}\,\mathrm{d}s\right)^{1/2}$$
$$\leq CT^{1/2}|\Gamma_1(t)|_{\mathscr{I}_1}|\mathbf{h}|^2_{\mathbf{H}(\Omega)},$$

where we have used (16), and that if $R \in \mathscr{L}(L^2(\Omega))$, then $|R\Gamma|_{\mathscr{I}_1} \leq |R|_{\mathscr{L}(L^2(\Omega))}|\Gamma|_{\mathscr{I}_1}$.

Hence, we obtain

$$\sup_{t\in[0,T]}\left|S_{\mathbf{h}}(t)\Gamma_1(t) - S(t)\Gamma_1(t) - \int_0^t S(t-s)P(\mathbf{h})S(s)\Gamma_1(t)\,\mathrm{d}s\right|_{\mathscr{I}_1}$$
$$\leq C_1|\mathbf{h}|^2_{\mathbf{H}(\Omega)} \sup_{t\in[0,T]}|\Gamma(t)|_{\mathscr{I}_1}.$$

Therefore, $\mathbf{H}(\Omega) \ni \mathbf{h} \mapsto S_{\mathbf{h}}(\cdot)\Gamma_1(\cdot) \in C([0,T];\mathscr{I}_1)$ if Fréchet differentiable at zero with directional derivative $(W_1(\Gamma_1)\mathbf{h})(t) := \int_0^t S(t-s)P(\mathbf{h})S(s)\Gamma_1(t)\,\mathrm{d}s$ for $\mathbf{h} \in \mathbf{H}(\Omega)$, $t \in [0,T]$. Since operators in $\mathscr{I}_1$ have the same $\mathscr{I}_1$-norm as their adjoints, and since $\Gamma_1(t)^* = \Gamma_1(t)$ for $t \in [0,T]$, we also observe that

$$\sup_{t\in[0,T]}|\Gamma_1(t)S_{\mathbf{h}}(t)^* - \Gamma_1(t)S(t)^* - W_2(\Gamma_1)\mathbf{h}(t)|_{\mathscr{I}_1} \leq C_1|\mathbf{h}|^2_{\mathbf{H}(\Omega)}\sup_{t\in[0,T]}|\Gamma(t)|_{\mathscr{I}_1},$$

where $(W_2(\Gamma_1)\mathbf{h})(t) := (W_1(\Gamma_1)\mathbf{h})(t)^*$. Finally, since we can write

$$R_1(\mathbf{h}) - R_1(0) = (S_{\mathbf{h}}(t)\Gamma_1(t) - S(t)\Gamma_1(t))S(t)^* + S(t)(\Gamma_1(t)S_{\mathbf{h}}(t) - \Gamma_1(t)S(t))^*$$
$$+ (S_{\mathbf{h}} - S)(t)\Gamma_1(t)(S_{\mathbf{h}} - S)(t)^*,$$

the differentiability of $R_1(\mathbf{h})$ is implied by the differentiability of $\mathbf{H}(\Omega) \ni \mathbf{h} \mapsto S_{\mathbf{h}}(\cdot)\Gamma_1(\cdot) \in C([0,T];\mathscr{I}_1)$ and $\mathbf{H}(\Omega) \ni \mathbf{h} \mapsto \Gamma_1(\cdot)S_{\mathbf{h}}(\cdot)^* \in C([0,T];\mathscr{I}_1)$, and the fact that $\sup_{t\in[0,T]}|(S_{\mathbf{h}} - S)(t)\Gamma_1(t)(S_{\mathbf{h}} - S)(t)^*|_{\mathscr{I}_1} = o(|\mathbf{h}|_{\mathbf{H}(\Omega)})$ (by similar arguments as the ones above).

Now we focus on R_2. Consider the equality in (20). We change t to $t-s$, and Γ_1 for $\Gamma_2(s)S_{\mathbf{h}}(t-s)^*$. Integration with respect to s from 0 to t yields

$$\int_0^t S_{\mathbf{h}}(t-s)\Gamma_2(s)S_{\mathbf{h}}(t-s)^*\,\mathrm{d}s - \int_0^t S(t-s)\Gamma_2(s)S_{\mathbf{h}}(t-s)^*\,\mathrm{d}s$$
$$- \int_0^t\int_0^{t-s} S(t-s-\sigma)P(\mathbf{h})S(\sigma)\Gamma_2(s)S(t-s)^*\,\mathrm{d}\sigma\,\mathrm{d}s = I_1 \tag{22}$$

with

$$I_1(t) := \int_0^t \int_0^{t-s} S(t-s-\sigma)P(\mathbf{h})(S_{\mathbf{h}}-S)(\sigma)\Gamma_2(s)S_{\mathbf{h}}(t-s)^* \,\mathrm{d}\sigma\,\mathrm{d}s$$
$$+ \int_0^t \int_0^{t-s} S(t-s-\sigma)P(\mathbf{h})S(\sigma)\Gamma_2(s)(S_{\mathbf{h}}-S)(t-s)^* \,\mathrm{d}\sigma\,\mathrm{d}s.$$

Note that the first two terms in (22) are Bochner integrals but the remaining ones are strong Bochner integrals.

Analogously as done in the first part of the proof in (21), we bound I_1 by

$$|I_1(t)|_{\mathscr{I}_1} \leq C_2|\mathbf{h}|^2_{\mathbf{H}(\Omega)}|\Gamma_2|_{L^\infty(0,T;\mathscr{I}_1)},$$

for some $C_2 > 0$, and all $t \in [0,T]$. Once more we consider the equality in (20), exchange t to $t-s$, and Γ_1 to $\Gamma_2(s)S(t-s)^*$. Then, integrating with respect to s from 0 to t yields

$$\int_0^t S_{\mathbf{h}}(t-s)\Gamma_2(s)S(t-s)^*\,\mathrm{d}s - \int_0^t S(t-s)\Gamma_2(s)S(t-s)^*\,\mathrm{d}s$$
$$- \int_0^t \int_0^{t-s} S(t-s-\sigma)P(\mathbf{h})S(\sigma)\Gamma_2(s)S(t-s)^*\,\mathrm{d}\sigma\,\mathrm{d}s = I_2(t), \quad (23)$$

where

$$I_2(t) := \int_0^t \int_0^{t-s} S(t-s-\sigma)P(\mathbf{h})(S_{\mathbf{h}}(\sigma)-S(\sigma))\Gamma_2(s)S(t-s)^*\,\mathrm{d}\sigma\,\mathrm{d}s,$$

and analogously as with I_1, we observe

$$|I_2(t)|_{\mathscr{I}_1} \leq C_3|\mathbf{h}|^2_{\mathbf{H}(\Omega)}|\Gamma_2|_{L^\infty(0,T;\mathscr{I}_1)},$$

for some $C_3 > 0$. Finally, taking the adjoint of (23), adding the resulting expression to (22) and noting that $|I_2(t)^*|_{\mathscr{I}_1} = |I_2(t)|_{\mathscr{I}_1}$, we observe that

$$\left|\int_0^t S_{\mathbf{h}}(t-s)\Gamma_2(s)S_{\mathbf{h}}(t-s)^*\,\mathrm{d}s - \int_0^t S(t-s)\Gamma_2(s)S(t-s)^*\,\mathrm{d}s - (W(\Gamma_2)\mathbf{h})(t)\right|_{\mathscr{I}_1}$$
$$\leq C|\mathbf{h}|^2_{\mathbf{H}(\Omega)}|\Gamma_2|_{L^\infty(0,T;\mathscr{I}_1)}, \quad (24)$$

for some $C > 0$ and $W(\Gamma_2)\mathbf{h} \in C([0,T];\mathscr{I}_1)$ which completes the proof. □

Analogous techniques as the ones displayed above are used for the following result which is required in the proof of the sensitivity of the Riccati equation.

Lemma 7. *Let $E_1, E_2 \in L^\infty(0,T;\mathscr{I}_1)$. Then there exist $C_i > 0$ for $i = 1,2,3,4$, such that*

$$\left|\int_0^t (S_{\mathbf{h}}-S)(t-s)(E_1E_2)(s)(S_{\mathbf{h}}-S)(t-s)^*\,\mathrm{d}s\right|_{\mathscr{I}_1} \leq C_1|E_1|_X|E_2|_X|\mathbf{h}|^2_{\mathbf{H}(\Omega)};$$
$$\left|\int_0^t S_{\mathbf{h}}(t-s)(E_1E_2)(s)S_{\mathbf{h}}(t-s)^*\,\mathrm{d}s\right|_{\mathscr{I}_1} \leq C_2|E_1|_X|E_2|_X;$$
$$\left|\int_0^t (S_{\mathbf{h}}-S)(t-s)(E_1E_2)(s)S_{\mathbf{h}}(t-s)^*\,\mathrm{d}s\right|_{\mathscr{I}_1} \leq C_3|E_1|_X|E_2|_X|\mathbf{h}|_{\mathbf{H}(\Omega)};$$
$$\left|\int_0^t S_{\mathbf{h}}(t-s)(E_1E_2)(s)(S_{\mathbf{h}}-S)(t-s)^*\,\mathrm{d}s\right|_{\mathscr{I}_1} \leq C_4|E_1|_X|E_2|_X|\mathbf{h}|_{\mathbf{H}(\Omega)},$$

where $X = L^\infty(0,T;\mathscr{I}_1)$.

Proof. We only prove the first inequality as other follow identically. First note that the integral in the first inequality is a well-defined Bochner integral with values in I_1. Further, by Hölder's inequality

$$\left|\int_0^t (S_{\mathbf{h}} - S)(t-s)(E_1E_2)(s)(S_{\mathbf{h}} - S)(t-s)^* \,\mathrm{d}s\right|_{\mathscr{I}_1} \leq \left(\int_0^t |(S_{\mathbf{h}} - S)(t-s)E_1(s)|^2_{\mathscr{I}_1} \,\mathrm{d}s\right)^{1/2} \left(\int_0^t |(S_{\mathbf{h}} - S)(t-s)E_2^*(s)|^2_{\mathscr{I}_1} \,\mathrm{d}s\right)^{1/2},$$

where we have used that $|L^*|_{\mathscr{I}_1} = |L|_{\mathscr{I}_1}$ for $L \in \mathscr{I}_1$. Further,

$$\begin{aligned}\int_0^t |(S_{\mathbf{h}} - S)(t-s)E_1(s)|^2_{\mathscr{I}_1} \,\mathrm{d}s &\leq |E_1|_X \sup_{|\phi|_{L^2(\Omega)}\leq 1} \int_0^t |(S_{\mathbf{h}} - S)(t-s)\phi|^2_{L^2(\Omega)} \\ &\leq C|E_1|_X |\mathbf{h}|^2_{\mathbf{H}(\Omega)},\end{aligned}$$

where we exploit 5 and that if $R \in \mathscr{L}(L^2(\Omega))$ and $E \in \mathscr{I}_1$, then we have $|RE|_{\mathscr{I}_1} \leq |R|_{\mathscr{L}(L^2(\Omega))}|E|_{\mathscr{I}_1}$. An analogous bound can be obtained for E_1 exchanged by E_2^*. Considering the latter inequality in the first one of the proof, the result follows. □

4.2 Sensitivity of the solution to the Riccati equation

In this section we concentrate on the map $\mathbf{h} \mapsto \Sigma(\mathbf{h})$. The results in the previous section allow us to prove 10, the main result of the section which provides the differentiability result for the aforementioned map. For that matter, we first need the continuity of the map $\mathbf{h} \mapsto \Sigma(\mathbf{h})$ which is given in what follows.

First, for the sake of brevity we adopt the following notation and keep it throughout the rest of the section. Let $\Sigma_0 \in \mathscr{I}_1$ with $\Sigma_0 \geq 0$, and consider $F \in L^1(0,T;\mathscr{I}_1), G \in L^\infty(0,T;\mathscr{L}(L^2(\Omega)))$, with $F(t) \geq 0$ and $G(t) \geq 0$ for $t \in [0,T]$. Then, for $\mathbf{h} \in \mathbf{H}(\Omega)$ and denote $\Sigma_{\mathbf{h}}$ to the unique pointwise non-negative function $\Pi \in C([0,T];\mathscr{I}_1)$ that satisfies

$$\Pi(t) = S_{\mathbf{h}}(t)\Sigma_0 S_{\mathbf{h}}^*(t) + \int_0^t S_{\mathbf{h}}(t-s)\big(F - \Pi G \Pi\big)(s) S_{\mathbf{h}}^*(t-s)\,\mathrm{d}s, \tag{25}$$

and we further denote by Σ to $\Sigma_{\mathbf{h}}$ for $\mathbf{h} = \mathbf{0}$.

We can now prove that the map defined above is continuous with respect to the topology given for the perturbations $\mathbf{h}$.

Lemma 8. *The map*

$$\mathbf{H}(\Omega) \ni \mathbf{h} \mapsto \Sigma_{\mathbf{h}} \in C([0,T];\mathscr{I}_1)$$

is continuous at zero.

Proof. Note first that $|\Sigma_{\mathbf{h}}(t)|_{\mathscr{I}_1}, |\Sigma(t)|_{\mathscr{I}_1} \leq |\Sigma_0|_{\mathscr{I}_1}$ for all $t \in [0,T]$ (see [17]). Secondly, note that by 6, we have that $Z_{\mathbf{h}} \to Z_{\mathbf{0}}$ in $C([0,T];\mathscr{I}_1)$ as $\mathbf{h} \to \mathbf{0}$ in $\mathbf{H}(\Omega)$, where

$$Z_{\mathbf{h}}(t) := S_{\mathbf{h}}(t)\Sigma_0 S_{\mathbf{h}}^*(t) + \int_0^t S_{\mathbf{h}}(t-s)F(s)S_{\mathbf{h}}^*(t-s)\,\mathrm{d}s, \qquad t \in [0,T].$$

Further, by (24) and since $|(\Sigma_{\mathbf{h}} G \Sigma_{\mathbf{h}})(t)|_{\mathscr{I}_1} \leq |\Sigma_0|_{\mathscr{I}_1}^2 |G|_{L^\infty(0,T;\mathscr{L}(L^2(\Omega)))}$, it follows that $Y_{\mathbf{h}} \to 0$ in $C([0,T];\mathscr{I}_1)$ as $\mathbf{h} \to \mathbf{0}$ in $\mathbf{H}(\Omega)$, where

$$Y_{\mathbf{h}}(t) := \int_0^t S_{\mathbf{h}}(t-s)(\Sigma_{\mathbf{h}} G \Sigma_{\mathbf{h}})(s)S_{\mathbf{h}}^*(t-s) - S(t-s)(\Sigma_{\mathbf{h}} G \Sigma_{\mathbf{h}})(s)S^*(t-s)\,\mathrm{d}s,$$

for $t \in [0,T]$. Define $W_{\mathbf{h}} \in C([0,T];\mathscr{I}_1)$ as $W_{\mathbf{h}}(t) := Z_{\mathbf{h}}(t) - Z_{\mathbf{0}}(t) - Y_{\mathbf{h}}(t)$ for $t \in [0,T]$ and note that because of 6, we have that

$$|W_{\mathbf{h}}(t)|_{\mathscr{I}_1} \leq C_1 |\mathbf{h}|_{\mathbf{H}(\Omega)}, \tag{26}$$

for some $C_1 > 0$ when $\mathbf{h}$ is restricted to some bounded set in $\mathbf{H}(\Omega)$.

It follows from (25), that

$$\begin{aligned}(\Sigma_{\mathbf{h}} - \Sigma)(t) &= W_{\mathbf{h}}(t) - \int_0^t S_{\mathbf{h}}(t-s)\big(\Sigma_{\mathbf{h}} G \Sigma_{\mathbf{h}} - \Sigma G \Sigma\big)(s) S_{\mathbf{h}}^*(t-s)\,\mathrm{d}s, \\ &= W_{\mathbf{h}}(t) - \int_0^t S_{\mathbf{h}}(t-s)\big(\Sigma_{\mathbf{h}} G(\Sigma_{\mathbf{h}} - \Sigma) + (\Sigma_{\mathbf{h}} - \Sigma) G \Sigma\big)(s) S_{\mathbf{h}}^*(t-s)\,\mathrm{d}s.\end{aligned}$$

Hence,

$$|(\Sigma_{\mathbf{h}} - \Sigma)(t)|_{\mathscr{I}_1} \leq |W_{\mathbf{h}}|_{C([0,T];\mathscr{I}_1)} + C_2 \int_0^t |(\Sigma_{\mathbf{h}} - \Sigma)(s)|_{\mathscr{I}_1}\,\mathrm{d}s,$$

where $C_2 := 2|\Sigma_0|_{\mathscr{I}_1} |G|_{L^\infty(0,T;\mathscr{L}(L^2(\Omega)))}$. Then, by application of Gronwall's inequality and (26), we observe

$$|(\Sigma_{\mathbf{h}} - \Sigma)(t)|_{\mathscr{I}_1} \leq C_3 |W_{\mathbf{h}}|_{C([0,T];\mathscr{I}_1)} \leq C_3 C_1 |\mathbf{h}|_{\mathbf{H}(\Omega)}, \tag{27}$$

for some $C_3 > 0$, which proves the initial statement. □

Before we prove the differentiability results concerning $\mathbf{h} \mapsto \Sigma_{\mathbf{h}}$, we prove that the sensitivity equation associated with the Riccati equation is well-posed.

Lemma 9. *Let $H, \Sigma \in C([0,T];\mathscr{I}_1)$, and $G \in L^\infty(0,T;\mathscr{L}(L^2(\Omega)))$. There exists a unique solution $\Lambda \in C([0,T];\mathscr{I}_1)$ to the following integral equation*

$$\Lambda(t) = H(t) - \int_0^t S(t-s)(\Lambda G \Sigma + \Sigma G \Lambda)(s) S^*(t-s)\,\mathrm{d}s. \tag{28}$$

Proof. Note first that the integrand is Bochner measurable, the integral is well-defined as a $\mathscr{I}_1$-valued Bochner integral and that the right hand side belongs to $C([0,T];\mathscr{I}_1)$ (see [16]).

The proof follows a classical renormalization technique due to Bielecki [8] also applied similarly in [16]. On the space $C([0,T];\mathscr{I}_1)$ we consider the norm $|F|_\lambda := \sup_{t\in[0,T]} e^{-\lambda t}|F(t)|_1$ for $F \in C([0,T];\mathscr{I}_1)$, which is equivalent to the usual norm on $C([0,T];\mathscr{I}_1)$.

Denote the right hand side of (28) by $L(\Lambda)$. It follows that $L : C([0,T];\mathscr{I}_1) \to C([0,T];\mathscr{I}_1)$ and

$$\begin{aligned}|L(\Lambda_1)(t) - L(\Lambda_2)(t)|_{\mathscr{I}_1} &\leq 2M \int_0^t |\Lambda_1(s) - \Lambda_2(s)|_{\mathscr{I}_1}\,\mathrm{d}s \leq \\ &\leq 2M|\Lambda_1 - \Lambda_2|_\lambda \int_0^t e^{\lambda s}\,\mathrm{d}s \leq \frac{2Me^{\lambda t}}{\lambda}|\Lambda_1 - \Lambda_2|_\lambda,\end{aligned}$$

where $M := |G|_{L^\infty(0,T;\mathscr{L}(L^2(\Omega)))}|\Sigma|_{C([0,T];\mathscr{I}_1)}$. Therefore,

$$|L(\Lambda_1) - L(\Lambda_2)|_\lambda \leq \frac{2M}{\lambda}|\Lambda_1 - \Lambda_2|_\lambda.$$

Hence, for $\lambda > 0$ sufficiently large L is a contraction, and hence there is an unique solution to (28). $\square$

We can now finally state our sensitivity results for the integral Riccati equation.

Theorem 10. *The map*

$$\mathbf{H}(\Omega) \ni \mathbf{h} \mapsto \Sigma_\mathbf{h} \in C([0,T];\mathscr{I}_1)$$

is Fréchet differentiable at zero.

Proof. Note that since F and G are point-wise non-negative, they subsequently are point-wise self-adjoint as we consider $L^2(\Omega)$ as a complex Hilbert space. It follows by Lemma 6 that $\mathbf{H}(\Omega) \ni \mathbf{h} \mapsto R_\mathbf{h} \in C([0,T];\mathscr{I}_1)$ defined as

$$R_\mathbf{h}(t) := S_\mathbf{h}(t)\Sigma_0 S_\mathbf{h}^*(t) + \int_0^t S_\mathbf{h}(t-s)F(s)S_\mathbf{h}^*(t-s)\,\mathrm{d}s,$$

for $t \in [0,T]$, is Fréchet differentiable at zero and define $R := R_\mathbf{0}$.

We prove that the Gâteaux derivative of $\mathbf{h} \mapsto \Sigma_\mathbf{h}$ at zero and in direction $\mathbf{h}$, denoted as $\Lambda\mathbf{h}$, solves

$$(\Lambda\mathbf{h})(t) = (W'(\mathbf{0})\mathbf{h})(t) - \int_0^t S(t-s)((\Lambda\mathbf{h})G\Sigma + \Sigma G(\Lambda\mathbf{h}))(s)S^*(t-s)\,\mathrm{d}s, \tag{29}$$

where Σ solves (25) for $\mathbf{h} = 0$, i.e., it satisfies

$$\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)\big(F - \Sigma G\Sigma\big)(s)S^*(t-s)\,\mathrm{d}s, \tag{30}$$

and $W'(\mathbf{0})$ is the Fréchet derivative at zero of

$$W(\mathbf{h})(t) := R_{\mathbf{h}}(t) - \int_0^T S_{\mathbf{h}}(t-s)(\Sigma G \Sigma)(s) S_{\mathbf{h}}^*(t-s)\,\mathrm{d}s.$$

Further, note that $\Lambda : \mathbf{H}(\Omega) \to C([0,T]; \mathscr{I}_1)$ is linear and bounded, and then it will follow that Λ is the Fréchet derivative of $\mathbf{h} \mapsto \Sigma_{\mathbf{h}}$ at zero.

Note that the following identity holds

$$(\Sigma_{\mathbf{h}} - \Sigma)(t) = (R_{\mathbf{h}} - R)(t) + N_{\mathbf{h}}(t) + Q_{\mathbf{h}}(t) + O_{\mathbf{h}}(t), \tag{31}$$

where

$$\begin{aligned} N_{\mathbf{h}}(t) := -\int_0^T & (S_{\mathbf{h}} - S)(t-s)(\Sigma G \Sigma)(s) S^*(t-s) + \\ & \qquad\qquad S(t-s)(\Sigma G \Sigma)(s)(S_{\mathbf{h}} - S)^*(t-s)\,\mathrm{d}s; \\ Q_{\mathbf{h}}(t) := -\int & S(t-s)((\Sigma_{\mathbf{h}} - \Sigma) G \Sigma + G \Sigma (\Sigma_{\mathbf{h}} - \Sigma))(s) S^*(t-s)\,\mathrm{d}s. \end{aligned}$$

and

$$\begin{aligned} O_{\mathbf{h}}(t) := & \int_0^T (S_{\mathbf{h}} - S)(t-s)(\Sigma G \Sigma)(s)(S_{\mathbf{h}} - S)(t-s)^*(t-s)\,\mathrm{d}s + \\ & \int_0^T (S_{\mathbf{h}} - S)(t-s)(\Sigma G(\Sigma_{\mathbf{h}} - \Sigma) + (\Sigma_{\mathbf{h}} - \Sigma) G \Sigma)(s) S_{\mathbf{h}}^*(t-s)\,\mathrm{d}s + \\ & \int_0^T S(t-s)(\Sigma G(\Sigma_{\mathbf{h}} - \Sigma) + (\Sigma_{\mathbf{h}} - \Sigma) G \Sigma)(s)(S_{\mathbf{h}} - S)^*(t-s)\,\mathrm{d}s + \\ & \int_0^T S_{\mathbf{h}}(t-s)((\Sigma_{\mathbf{h}} - \Sigma) G (\Sigma_{\mathbf{h}} - \Sigma))(s) S_{\mathbf{h}}^*(t-s)\,\mathrm{d}s. \end{aligned}$$

Additionally, note that since (27) and 7 hold true, we observe

$$|O_{\mathbf{h}}(t)|_{\mathscr{I}_1} \le C_1 |\mathbf{h}|^2_{\mathbf{H}(\Omega)}, \tag{32}$$

for some $C_1 > 0$. Furthermore, by 7 and the proof of 6,

$$|(R_{\mathbf{h}} - R)(t) + N_{\mathbf{h}}(t) - (W'(\mathbf{0})\mathbf{h})(t)|_{\mathscr{I}_1} \le C_2 |\mathbf{h}|^2_{\mathbf{H}(\Omega)},$$

for some $C_2 > 0$ and

$$\begin{aligned} \left| Q_{\mathbf{h}}(t) + \int_0^t S(t-s)((\Lambda \mathbf{h}) G \Sigma + \Sigma G (\Lambda \mathbf{h}))(s) S^*(t-s)\,\mathrm{d}s \right|_{\mathscr{I}_1} & \\ \le C_3 |(\Sigma_{\mathbf{h}} - \Sigma - (\Lambda \mathbf{h}))(t)|_{\mathscr{I}_1}, & \end{aligned}$$

for some $C_3 > 0$ and all $t \in [0,T]$.

Therefore, it follows from (31) that

$$|(\Sigma_{\mathbf{h}} - \Sigma - (\Lambda \mathbf{h}))(t)|_{\mathscr{I}_1} \le C_4 |\mathbf{h}|^2_{\mathbf{H}(\Omega)} + C_3 \int_0^t |(\Sigma_{\mathbf{h}} - \Sigma - (\Lambda \mathbf{h}))(s)|_{\mathscr{I}_1}\,\mathrm{d}s,$$

for some $C_4 > 0$. Finally, by Gronwall's inequality, we have

$$\frac{|(\Sigma_{\mathbf{h}} - \Sigma - (\Lambda \mathbf{h}))(t)|_{\mathscr{I}_1}}{|\mathbf{h}|_{\mathbf{H}(\Omega)}} \leq C_5 |\mathbf{h}|_{\mathbf{H}(\Omega)},$$

for some $C_5 > 0$, which prove the initial statement since the map $\mathbf{h} \mapsto \Lambda \mathbf{h}$ is linear and bounded. $\square$

5 The Optimization Problem

We are now in shape to prove the result concerning Problem $(\mathbb{P})$. Recall that $\Gamma^i_{\text{ad}} \subset \Omega$ is closed and bounded for $i = 1, 2, \dots, n$, and that the admissible location set is given by $\Gamma_{\text{ad}} \equiv \prod_{i=1}^n \Gamma^i_{\text{ad}}$. Throughout this section we involve the following assumption.

Assumption 1. *Let the following statements to hold true:*

(a) $\Sigma_0 \in \mathscr{I}_1$ *and* $\Sigma_0 \geq 0$.

(b) $BR_2B^*(\cdot) \in L^\infty(0,T;\mathscr{I}_1)$ *and* $BR_2B^*(t) \geq 0$ *for a.e.* $t \in (0,T)$.

(c) $C^*_{\hat{x}} R_1^{-1} C_{\hat{x}}(\cdot) \in L^\infty(0,T;\mathscr{I}_1)$ *and* $C^*_{\hat{x}} R_1^{-1} C_{\hat{x}}(t) \geq 0$ *for a.e.* $t \in (0,T)$*, and for each* $\hat{x} \in \Gamma_{ad}$*. Further, we assume that the map* $\hat{x} \mapsto C_{\hat{x}}$ *is such that, the map*

$$\Gamma_{ad} \ni \hat{x} \mapsto C^*_{\hat{x}} R_1^{-1} C_{\hat{x}} \in L^\infty(0,T;\mathscr{I}_1), \tag{33}$$

is continuous.

A a few words are in order concerning 1. Since the maps R_2 and R_1^{-1} are $\mathscr{I}_1$-valued by definition (see [7]), the same holds true for BR_2B^* and $C^*_{\hat{x}} R_1^{-1} C_{\hat{x}}$, and since $R_2 \geq 0$ and $R_1^{-1} \geq 0$, the same follows for BR_2B^* and $C^*_{\hat{x}} R_1^{-1} C_{\hat{x}}$. The assumption involving the essential boundedness of both of these maps is largely satisfied for $C^*_{\hat{x}} R_1^{-1} C_{\hat{x}}$ and on the majority of applications for BR_2B^*. For example, when variant spatial intensity of the noise is allowed involving B like a multiplication operator with an $L^\infty(\Omega)$ function. Finally, the continuity of the map (33) is satisfied in the majority of the sensor scenarios (see [16]), and as consequence, it implies that the map

$$\Gamma_{\text{ad}} \ni \hat{x} \mapsto \Sigma_{\hat{x}} \in C([0,T];\mathscr{I}_1), \tag{34}$$

is continuous where $\Sigma_{\hat{x}}$ is the solution to (25) for $F \equiv BR_2B^*$, $G \equiv C^*_{\hat{x}} R_1^{-1} C_{\hat{x}}$ and $\mathbf{h} = \mathbf{0}$ (see [16]). We are now in position to prove the main result in this section.

Theorem 11. *Problem* $(\mathbb{P})$ *admits a solution.*

Proof. Let $\{\hat{x}_k\}$ be an infimizing sequence for Problem $(\mathbb{P})$, and denote also by $\{\hat{x}_k\}$ to the convergent subsequence in Γ_{ad} with limiting point $\hat{x}^* \in \Gamma_{\text{ad}}$. It follows by assumption that

$$J_1(\hat{x}_k) \to J_1(\hat{x}^*), \tag{35}$$

given that $\Sigma_{\hat{x}_k} \to \Sigma_{\hat{x}^*}$ in $C([0,T];\mathscr{I}_1)$.

Define $G_k := C^*_{\hat{x}_k} R_1^{-1} C_{\hat{x}_k}$ and $G := C^*_{\hat{x}^*} R_1^{-1} C_{\hat{x}^*}$. We now prove that the Fréchet derivative at zero of

$$\mathbf{h} \mapsto \int_0^t S_{\mathbf{h}}(t-s)(\Sigma G_k \Sigma)(s) S^*_{\mathbf{h}}(t-s)\,\mathrm{d}s =: L_k(\mathbf{h}),$$

converges to the one of $\mathbf{h} \mapsto \int_0^t S_{\mathbf{h}}(t-s)(\Sigma G \Sigma)(s) S^*_{\mathbf{h}}(t-s)\,\mathrm{d}s =: L(\mathbf{h})$. It follows from the proof of 6 (analogous as the bound for I_1 in (22)) that

$$\left| -\int_0^t \int_0^{t-s} S(t-s-\sigma) P(\mathbf{h}) S(\sigma)(G-G_k)(s) S(t-s)^*\,\mathrm{d}\sigma\,\mathrm{d}s \right|_{\mathscr{I}_1} \\ \leq C_1 |\mathbf{h}|_{H(\Omega)} |G-G_k|_{L^\infty(0,T;\mathscr{I}_1)},$$

for some $C_1 > 0$. Hence, from the proof of 6, the Fréchet derivative of $\mathbf{h} \mapsto L_k(\mathbf{h}) - L(\mathbf{h})$ is bounded by

$$|L_k'(\mathbf{0}) - L'(\mathbf{0})|_{\mathscr{L}(\mathbf{H}(\Omega), C([0,T];\mathscr{I}_1))} \leq \sup_{\mathbf{h} \in \mathbf{H}(\Omega)} \frac{|(L_k'(\mathbf{0}) - L'(\mathbf{0}))\mathbf{h}|_{C([0,T];\mathscr{I}_1)}}{|\mathbf{h}|_{\mathbf{H}(\Omega)}} \\ \leq 2C_1 |G-G_k|_{L^\infty(0,T;\mathscr{I}_1)},$$

which proves the statement at the beginning of the paragraph.

From the previous section we know that the maps $\mathbf{h} \mapsto \Sigma_{\mathbf{h},\hat{x}^*}, \Sigma_{\mathbf{h},\hat{x}_k}$ are differentiable at zero. The difference between both directional derivatives at zero in the $\mathbf{h}$ direction gives:

$$(\Lambda_k \mathbf{h} - \Lambda \mathbf{h})(t) = ((L_k'(\mathbf{0}) - L'(\mathbf{0}))\mathbf{h})(t) \\ - \int_0^t S(t-s)(\Lambda_k \mathbf{h} G_k \Sigma_k + \Sigma_k G_k \Lambda_k \mathbf{h} - \Lambda \mathbf{h} G \Sigma - \Sigma G \Lambda \mathbf{h})(s) S^*(t-s)\,\mathrm{d}s.$$

Additionally, for $X = C([0,T];\mathscr{I}_1)$, we know that $|\Lambda_k \mathbf{h}|_X, |\Lambda \mathbf{h}|_X, |\Sigma|_X, |\Sigma_k|_X$ are uniformly bounded in k for some $C_2 > 0$ and $|\mathbf{h}|_{\mathbf{H}(\Omega)} = 1$, and further that $\nu_k := |G_k \Sigma_k - G\Sigma|_X \to 0$ as $k \to \infty$. Hence, adding and subtracting the terms $S(t-s)(\Lambda G_k \Sigma_k)(s) S^*(t-s)$ and $S(t-s)(\Sigma_k G_k \Lambda)(s) S^*(t-s)$ in the above integral, yields

$$|(\Lambda_k \mathbf{h} - \Lambda \mathbf{h})(t)|_{\mathscr{I}_1} \leq C_3(|G-G_k|_{L^\infty(0,T;\mathscr{I}_1)} + \nu_k) + C_4 \int_0^t |(\Lambda_k \mathbf{h} - \Lambda \mathbf{h})(s)|_{\mathscr{I}_1}\,\mathrm{d}s,$$

for some $C_3, C_4 > 0$ and all $t \in [0,T]$, with $T < +\infty$. Then, by Gronwall's inequality, we obtain

$$|(\Lambda_k \mathbf{h} - \Lambda \mathbf{h})(t)|_{\mathscr{I}_1} \leq C_5(|G-G_k|_{L^\infty(0,T;\mathscr{I}_1)} + \nu_k),$$

which implies

$$|\Lambda_k - \Lambda|_{\mathscr{L}(\mathbf{H}(\Omega), C([0,T];\mathscr{I}_1))} \leq \sup_{|\mathbf{h}|_{\mathbf{H}(\Omega)}=1} \max_{t \in [0,T]} |(\Lambda_k \mathbf{h} - \Lambda \mathbf{h})(t)|_{\mathscr{I}_1} \\ \leq C_5(|G-G_k|_{L^\infty(0,T;\mathscr{I}_1)} + \nu_k).$$

Also, note for J_2 in Problem $(\mathbb{P})$, we have that

$$J_2(\hat{x}_k) = \sup_{|\mathbf{z}|_{\mathbf{H}(\Omega)} \leq M} L(\Lambda_k \mathbf{z}), \qquad \text{and} \qquad J_2(\hat{x}^*) = \sup_{|\mathbf{z}|_{\mathbf{H}(\Omega)} \leq M} L(\Lambda \mathbf{z}),$$

where $L(\cdot) := \int_0^T \text{Tr}\,(\cdot)\,\mathrm{d}t$ is a bounded linear functional on $C([0,T];\mathscr{I}_1)$. It follows that,

$$|J_2(\hat{x}_k) - J_2(\hat{x}^*)| \leq \sup_{|\mathbf{z}|_{\mathbf{H}(\Omega)} \leq M} |L((\Lambda_k - \Lambda)\mathbf{z})| \leq M|L|_Y |\Lambda_k - \Lambda|_{\mathscr{L}(\mathbf{H}(\Omega), C([0,T];\mathscr{I}_1))},$$

where $Y = C([0,T];\mathscr{I}_1)^*$ which implies that

$$J_2(\hat{x}_k) \to J_2(\hat{x}^*). \tag{36}$$

Finally (35) and (36) prove that $\hat{x}^*$ is a minimizer of Problem $(\mathbb{P})$. □

6 Numerical Tests

We consider numerical tests in 2D and 3D in a variety of settings. In all our tests we utilize $A(\mathbf{h}) = -\alpha\Delta + (\mathbf{v} + \mathbf{h}) \cdot \nabla$ induced by the bilinear form $a : H^1_D(\Omega) \times H^1_D(\Omega) \to \mathbb{R}$ defined in (8), the input map $B\eta = 1 \cdot \eta$ and the output one $C_{\hat{x}_i}$ defined as

$$C_{\hat{x}_i}(\varphi) := \int_\Omega e^{-r|y - \hat{x}_i|^2_{\mathbb{R}^n}} \varphi(y)\,\mathrm{d}y, \qquad \forall \varphi \in L^2(\Omega), \tag{37}$$

for some $r > 0$ and we consider $R_1 = R_2 = I$.

It follows that the sensor kernel is smooth enough so that also the derivative of the map $\hat{x} \mapsto \Sigma(\hat{x}, \mathbf{h}) \in C([0,T];\mathscr{I}_1)$ exists (see [16]) which is enough to derive gradient descent methods for the optimal placement of sensors in the case $\lambda = 1$ in $(\mathbb{P})$.

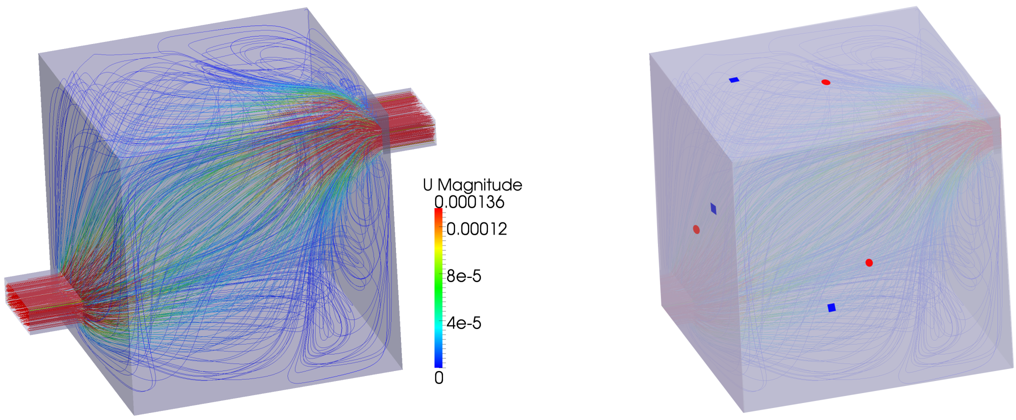
We focus on the following tests associated with Problem $(\mathbb{P})$:

- **a.** Consider $\lambda = 1$ and obtain the full map $\hat{x} \mapsto J_1(\hat{x})$ for the one sensor case.
- **b.** Consider $\lambda = 0$ and obtain the full map $\hat{x} \mapsto J_2(\hat{x})$ for the one sensor case.
- **c.** Study robust locations for a fixed specific perturbation $\mathbf{h}$. Specifically, study if there exist significant differences between the maps $\hat{x} \mapsto J_2(\hat{x})$ and $\hat{x} \mapsto J_3(\hat{x})$. That is, between

$$\sup_{|\mathbf{z}|_{\mathbf{H}(\Omega)} \leq M} \int_0^T \mathrm{Tr}\left(W(\hat{x}, \mathbf{0})\mathbf{z}(t)\right)\,\mathrm{d}t, \quad \text{and} \quad \int_0^T \mathrm{Tr}\left(W(\hat{x}, \mathbf{0})\mathbf{h}^*(t)\right)\,\mathrm{d}t,$$

  for some fixed $\mathbf{h}^*$, where $\hat{x} \in \Omega$
- **d.** Multiple sensor location for 2D and 3D cases for the case $\lambda = 1$.

Since all tasks are computationally very intensive, a variety of model reduction techniques and algorithms are used. Each of them is made explicit in what follows. Also, task **d.** is performed with a projected gradient descent method with an Armijo line search and later specified.



(a) (b)

Figure 1: Streamlines generated from $\mathbf{v}$ in the cube geometry in 1(a). Initial (red circles) and optimal sensor network location (blue squares) in 1(b) where the admissible location region for the sensors Γ^i_{ad} for $i = 1, 2, 3$ are the walls in which they are initialized.

6.1 Discretization

The discrete stationary flow $\mathbf{v}$ is obtained in a different environment as the rest of the discrete variables due to the subtleties of Navier-Stokes solvers. In fact, the stationary velocity $\mathbf{v}$ is computed using Star-CD CCM+ considering the system in (NS). Here, extensions of the inlet and outlet structures are used to stabilize the inflow at the inlet and to prevent back-flow at the outlet. In all cases, the solution to Navier-Stokes is computed with a constant inflow value $\mathbf{v}^0_{\text{in}}$, $(7.5 \cdot 10^{-4}$ in the 3D example and $2 \cdot 10^{-3}$ in the 2D case) and at the real inlet, a parabolic profile is developed (see 1(a) and 2(a)). Hence, an approximate parabolic $\mathbf{v}_{\text{in}}$ is considered on $\partial\Omega_{\text{in}}$. Since the finite volume method is involved a polyhedral mesh with boundary layers is utilized. The resulting meshes approximately consists of 250.000 cells and 1.000.000 grid points. Later, $\mathbf{v}^N$ is exported to be used in the the finite element scheme described below.

Let $H^N = \text{span}\,\{\phi_i^N\}_{i=1}^N \subset H^1_D(\Omega)$ be the finite dimensional subspace approximating the state space $L^2(\Omega)$ where the basis functions $\phi_i^N(\cdot)$, $i = 1, \dots, N$, are continuous piecewise linear splines. We define these functions on a quadrilateral mesh for $\Omega \subset \mathbb{R}^2$, and on a hexahedral mesh when $\Omega \subset \mathbb{R}^3$, while each mesh consists of N nodes. We characterize the basis functions with the properties that ϕ_i^N is unity at node i, and zero at all other nodes. Moreover, ϕ_i^N is nonzero only at those parts of Ω which contain some neighbour node with node i.

Using standard notation we denote the mass $M^N := [(\phi_i^N, \phi_j^N)]_{i,j}$ and stiffness matrix $K^N := [a(\phi_i^N, \phi_j^N)]_{i,j}$ and in our scenario the discrete approximation $A^N(\mathbf{h})$ and B^N to $A(\mathbf{h})$ and B, respectively, are given by

$$-A^N(\mathbf{h}^N) := -(M^N)^{-1}K^N, \qquad \text{and} \qquad B^N := (M^N)^{-1}P^N(1),$$

where $P^N : L^2(\Omega) \to H^N$ is the orthogonal projection of $L^2(\Omega)$ onto H^N and with $\mathbf{h}^N$ the discrete perturbation described 6.2; see the monograph by Banks [3] for a detailed explanation of this choice for the discrete versions of the operators involved. Note that these matrices are not explicitly computed but considered implicitly as $M^N A^N(\mathbf{h}) := -K^N$ and $M^N B^N = P^N(1)$.

Further, in this setting, the approximation $A^N(\mathbf{h}^N)$ satisfies the Trotter-Kato approximation theorem, so that the semigroup $S^N_{\mathbf{h}}(t) := e^{-tA^N(\mathbf{h}^N)}$ satisfies $S^N_{\mathbf{h}}(t)P^N z \to S(t)z$ for each $z \in H^1_D(\Omega)$, uniformly on compact intervals (see [3, Chapter 12]). Further, the discrete version of $C_{\hat{x}_i}$ is then given by

$$C^N_{\hat{x}_i} := \left[C_{\hat{x}_i}(\phi^N_j)\right]_j.$$

6.2 Perturbation Generation and J_2 discretization

The perturbations $\mathbf{h}$ are computed as the gradient of the solution to the mixed boundary value problem (M) where $\Gamma_1 \equiv \partial\Omega_{\text{wall}}$ and $g = 0$ in all but one inlet that we denote by $\Gamma \subset \partial\Omega_{\text{in}}$. For practical reasons of implementation, we consider $g : \Gamma \to \mathbb{R}$ to have zero boundary conditions in the sense of the trace so that $g \in H^1_0(\Gamma)$. Hence, every $\mathbf{h}$ considered is constructed as

$$\mathbf{h} = \nabla Z(g),$$

where Z is the solution mapping to (M) for some $g \in H^1_0(\Gamma)$.

We utilize in our numerical tests

$$J_2(\hat{x}) := \sup_{|g|_{H^1_0(\Gamma)} \leq M} \int_0^T \mathrm{Tr}\left(W(\hat{x}, \mathbf{0})\nabla Z(g)(t)\right) \,\mathrm{d}t,$$

for some $M > 0$. Provided that $g \mapsto \int_0^T \mathrm{Tr}\left(W^N(\hat{x}, \mathbf{0})(t)\nabla Z(g)\right) \,\mathrm{d}t$ is a bounded linear functional over $H^1_0(\Gamma)$ (see the remarks after the definition of Problem $(\mathbb{P})$), there exists $L_{\hat{x}} \in H^{-1}(\Gamma)$, such that

$$J_2(\hat{x}) = \sup_{|g|_{H^1_0(\Gamma)} \leq M} L_{\hat{x}}(g) = M|L_{\hat{x}}|_{H^{-1}(\Gamma)}.$$

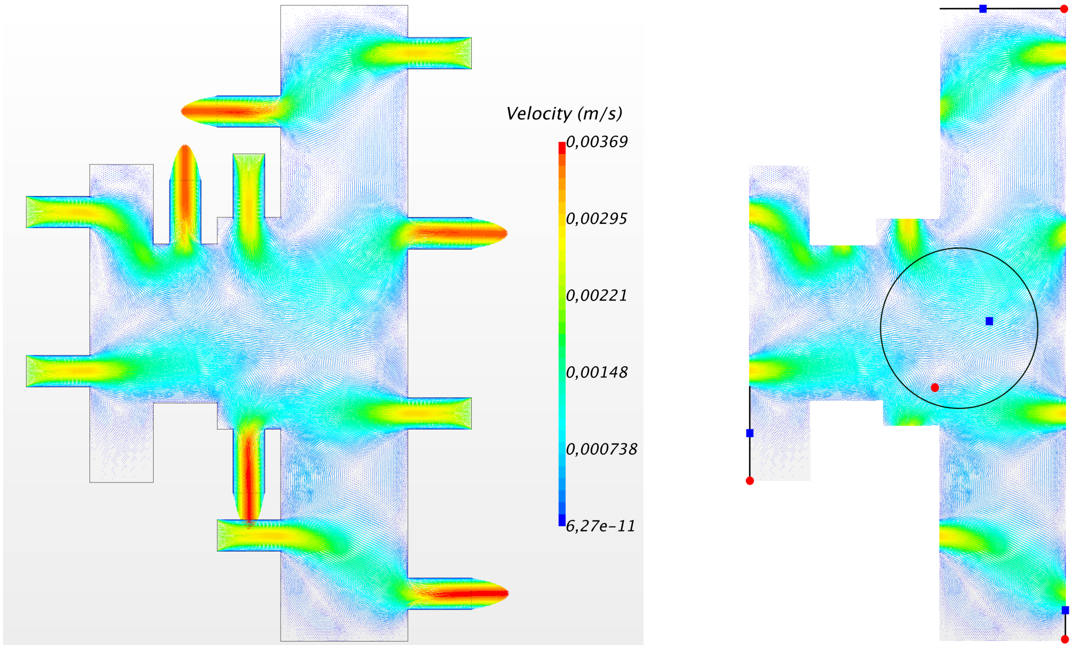
Hence, upon identification of $L_{\hat{x}}$, the value $J_2(\hat{x})$ is the M scaled dual norm of $L_{\hat{x}}$. This entails that once $L_{\hat{x}}$ is given, $|L_{\hat{x}}|_{H^{-1}(\Gamma)}$ is computed as $|\mathcal{R}^{-1}L|_{H^1_0(\Gamma)}$ where $\mathcal{R}$ is the associated Riesz map, i.e., $\mathcal{R}^{-1}L$ solves the problem: Find $y \in H^1_0(\Gamma)$ such that $-\Delta y = L$ in Γ and $y = 0$ on $\partial\Gamma$.

As there is no direct access to $W(\hat{x}, \mathbf{0})$, but only to $W(\hat{x}, \mathbf{0})\mathbf{z}$ for some $\mathbf{z} \in \mathbf{H}(\Omega)$, the construction of $L_{\hat{x}}$ can be done over a basis $\{\psi_j\}$ of $H^1_0(\Gamma)$. Hence, $\langle L_{\hat{x}}, \psi_j\rangle$ is identified, and since $L_{\hat{x}}(\psi) = \sum_j \langle L_{\hat{x}}, \psi_j\rangle (\psi, \psi_j)_{H^1_0(\Gamma)}\psi_j$ the full $L_{\hat{x}}$ can be represented with respect to this basis. In the finite element scheme, the $\{\psi_j\}$ reduce to a basis of linear splines and the described above procedure is used to determine the value of $J_2(\hat{x})$ for a given sensor location $\hat{x}$.

6.3 The Riccati equation solver

The approximation of the Riccati equation can be performed as follows. For the time step $\delta > 0$, and given Σ^N_k we consider Σ^N_{k+1} to be the solution of the following equation

$$\frac{\Sigma^N_{k+1} - \Sigma^N_k}{\delta} = -A^N(h^N)\Sigma^N_{k+1} - \Sigma^N_{k+1}(A^N(h^N))^* + B^N(B^N)^* - \Sigma^N_{k+1}(C^N_{\hat{x}})^*(C^N_{\hat{x}})\Sigma^N_{k+1}, \tag{38}$$



(a) (b)

Figure 2: Velocity profile $\mathbf{v}$ in the airport geometry in 2(a). Initial (red circles) and optimal sensor network location (blue squares) where the admissible location region for the sensors Γ^i_{ad} for $i = 1, 2, 3, 4$ are given by the black segments on the walls and the encircled region on the center of Ω

which reduces to the resolution of an algebraic Riccati equation. As the above equation represents the implicit Euler scheme applied to (3), it provides an approximation to (R). Although the discrete problem is very high dimensional, it can be handled by state of the art model reduction techniques and advanced Kleinman-Newton variations available now from Benner and collaborators ([4, 5]). We utilize an inexact low-rank Newton-ADI method (see [5]) after the model reduction provided in [4]. Convergence results of these discrete approximations towards their infinite dimensional counterparts are not available (and perhaps not attainable). However, we have tested these approximations in contrast to inexact Kleinman-Newton ([18, 32]) with no model reduction on the finest mesh that still allowed storage of the full versions of $A^N, B^N, C^N_{\hat{x}}$, and Σ^N_k and noticed that errors are neglectable: For example, in the 3D tests, this comparison was done with mesh size $1/10$ and cube volume equal to 1. Model reduction techniques allowed us then to treat mesh sizes down to $1/40$ on the same cube. Note that without model reduction, the number of elements the matrix Σ^N_k is in the order of $4.7 \cdot 10^9$ entries, which makes it even challenging to store.

In case $(\Sigma^N_{k+1} - \Sigma^N_k)\delta^{-1} \simeq 0$, then (38) reduces to the computation of an unique algebraic Riccati equation

$$0 = -A^N(\mathbf{h}^N)\Sigma^N - \Sigma^N(A^N(\mathbf{h}^N))^* + B^N(B^N)^* - \Sigma^N(C^N_{\hat{x}})^*(C^N_{\hat{x}})\Sigma^N. \qquad (39)$$

The approximation of the directional derivative Λ^N at zero and in the $\mathbf{h}^N$ direction, in both cases (39) and (38), reduces to the computation of Lyapunov equations (see [37]) of the general form

$$D^N\Lambda^N + \Lambda^N(D^N)^* = E^N,$$

where D^N is a function of $A^N(\mathbf{0})$, $C^N_{\hat{x}}$, Σ^N and E^N is a function of Σ^N, $A^N(\mathbf{h}^N) - A^N(\mathbf{0})$ and possibly the previous time-step value of D^N in the case of (38). It should be noted that, in general, the above equation need not to have a solution (see [37]), and requirements of the matrix components such as that the stabilizability of the pair $(-A^N(\mathbf{h}^N), (C^N_{\hat{x}})^*(C^N_{\hat{x}}))$ are needed.

6.4 2D Tests

The velocity profile is generated with $\mathbf{v}^0_{\text{in}} = 2 \cdot 10^{-3}$ and a Reynolds number given by Re $\simeq 13$. The resulting velocity profile $\mathbf{v}^N$ is shown in 2(a). Also, the inlet Γ used to generate perturbations is the only inlet whose inflow is perpendicular to the x-axis (center of the image).

In all of these examples we consider a diffusion coefficient $\alpha = 0.05$ and the sensor $C_{\hat{x}_i}$ of the form (37) is considered with parameter $r = 50$ in the kernel. The mesh size is given by 0.0062 which results in a number of entries of the non-reduced Riccati equation on the order of $2.7 \cdot 10^8$ elements.

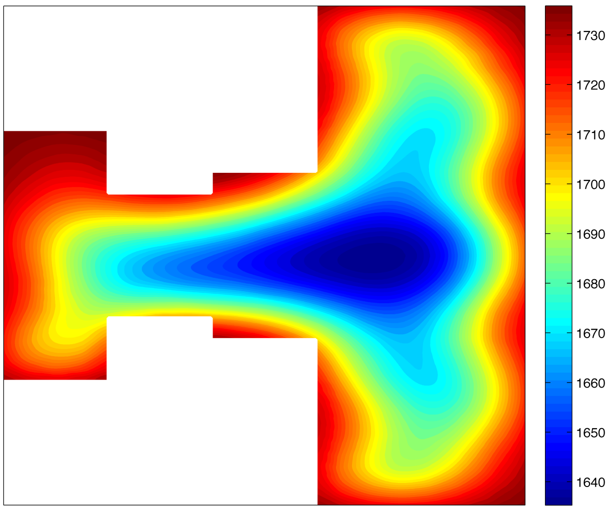


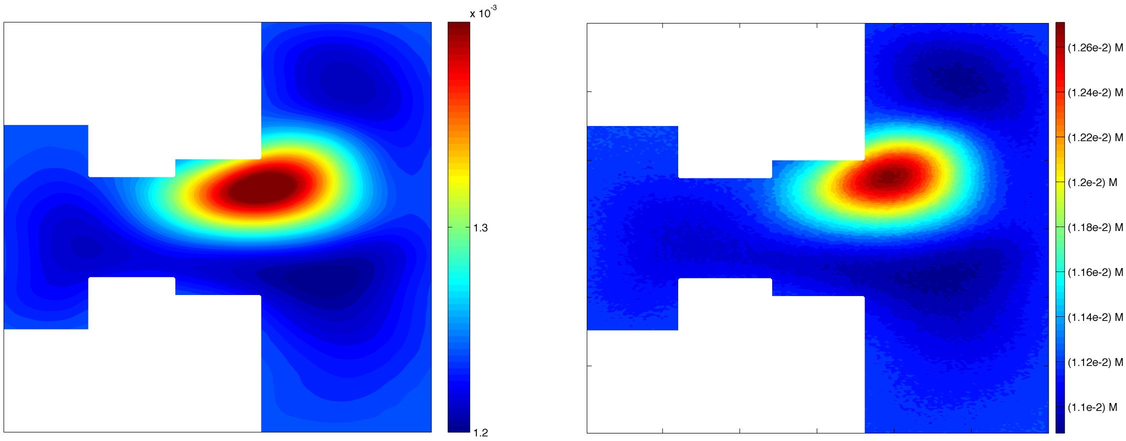
Figure 3: Colormap of $\Omega \ni \hat{x} \mapsto J_1(\hat{x})$ for the one sensor case.

Concerning the test in **a.**, the behaviour of $\hat{x} \mapsto J_1(\hat{x})$ is depicted in 6.4. This shows that the in the case of one sensor, and when no robustness is demanded from the sensor location, the best possible placement (provided that the admissible location for the sensor is the entire domain) is unique. The range of values in 6.4 depends significantly with respect to the diffusion coefficient α and the parameter in the exponential in the kernel r.

We turn the attention to numerical tests **b.** and **c.**: We consider one perturbation $\mathbf{h}$ generated with $g = 1.5811 \cdot 10^{-4}$ in Γ and compute $\hat{x} \mapsto J_3(\hat{x})$ (defined at beginning of 6) in the one sensor example. This should be compared with $\hat{x} \mapsto J_2(\hat{x})$, in which for each sensor position $\hat{x}$ the worst perturbation is considered with bound $|g|_{H_0^1(\Omega)} \leq M$. The discrete versions of the maps $\hat{x} \mapsto J_3(\hat{x})$ and $\hat{x} \mapsto J_2(\hat{x})$ for $\hat{x} \in \Omega$ are given in 4(a) and 4(b), respectively.

Further, it should be noticed that 6.4 and 4(b) provide good evidence that although walls, edges, and corners do not provide optimal places in terms of quality of information, they are relatively robust places. This holds even for corners which are significantly close to locations of perturbation.

Finally, we consider **d.**: In 2(b) we show the behaviour of the optimization procedure ($\lambda = 1$) in the case of 4 sensors: we consider 3 sensors restricted to confined regions on the walls, where these regions are the dark segments on the boundary of Ω, and one sensor on the interior of the domain restricted to the ellipse depicted with black boundary. The red circles show the initial position of the sensors and the blue squares the final position of these sensors at the end of the optimization routine. The solution algorithm is a projected gradient descent over the sensor position with Armijo line search. The reduction of the objective functional from initial points to final points is around 3% which due to the restricted locations of the sensors is also the best possible gain in this scenario.



(a) (b)

Figure 4: Approximations to $\hat{x} \mapsto J_3(\hat{x})$ and $\hat{x} \mapsto J_2(\hat{x})$ are given in 4(a) and 4(b),respectively. Note that the units in the colormap of J_2 are given with respect to M, the bound on $|g|_{H_0^1(\Omega)} \leq M$ in the generation of $\mathbf{h}$ (see 6.2). Note that although J_2 represents the value associated with the worst perturbation, it has a qualitative behaviour similar to the one determined by only one perturbation.

6.5 3D Tests

We utilize a cube geometry with one inlet in a lower position close to a corner and one outlet in a higher position close to the corner that diagonally opposes the first one (see 1(a)). The mesh size is given by $1/40$ with unitary sides.

For the generation of the stationary velocity profile $\mathbf{v}$, we consider $\mathbf{v}_{\text{in}}^0 = 7.5 \cdot 10^{-4}$ and kinematic viscosity $\nu = 1.56 \cdot 10^{-5}$ which determines in our example that the Reynolds number is given by Re $\simeq 9.6$. The streamlines generated from the velocity profile $\mathbf{v}$ that is utilized in the operator $A(\mathbf{h})$ are seen in 1(a). In these examples we utilize a diffusion coefficient $\alpha = 0.005$ and the sensor $C_{\hat{x}_i}$ of the form (37) is considered with parameter $r = 60$ in the kernel.

In 6.5 we solve the numerical task **a.** and observe the behaviour of the iso-surfaces of $\hat{x} \mapsto J_1(\hat{x})$. There seems to be a symmetry associated with the direction of the main flow, but better sensor locations are closer to the outlet than to the inlet. It becomes clear that walls, edges, and corners (in this order) are bad places for location (this is also seen in the 2D example). This is mainly explained by the fact that the operator $C_{\hat{x}_i}$ when restricted to any of those locations, reduce the area of integration and hence less information is attained. Additionally, in 5(f), we observe $\hat{x} \mapsto J_1(\hat{x})$ when restricted to the boundary of our domain. Although hard to notice due to the colormap, in general, regions closer to the outlet are relatively better to locate sensors than regions closer to the inlet.

Associated with task **b.** and **c.**, we observe that the behaviour of $\hat{x} \mapsto J_2(\hat{x})$ is significantly different to the one in the 2D test. The map $\eta \mapsto \{x \in \Omega : J_2(\hat{x}) < \eta\}$, as η decreases, creates an uniform filling from top to bottom of the cube with no significant variation on the x and y direction. Here, it seems that higher locations are more robust, and lower locations are less robust, with no significant features in other coordinate directions. Analogously as done in the 2D case, we consider one perturbation $\mathbf{h}$ generated with $g = 1.1180 \cdot 10^{-4}$ in $\Gamma_{\text{in}} = \Gamma$ and show $\hat{x} \mapsto J_3(\hat{x})$ (defined at beginning of 6) in the one sensor example. The results of this test are shown in 6.5. The results, for this one perturbation, show that locations near the inlet, where the perturbation is gene rated, worsen in terms of quality of information, while location near the outlet would perform better.

Finally, for task **d.**, we observe in 1(b) the initial sensor network location (red circles) comprised of three sensors (each one located at the center of a side of the cube) and the optimized sensor network location (blue squares), and where the admissible location for each one of the sensors corresponds to the wall in which the sensor is initialized. The optimization problem is considered for $\lambda = 1$ and the solution algorithm utilized was projected gradient descent over the sensor position with Armijo line search. The reduction of the objective functional from initial points to final points is around 4%.

7 Conclusion and future work

A theoretical framework for studying an optimization problem associated with a robust and optimal sensor network placement is introduced. The source of complexity in the problem is given by the perturbation of the differential operator of the underlying convection-diffusion process. Since the problem is formulated as an optimization problem where the integral Riccati equation is a constraint, the sensitivity of the Riccati equation with respect to perturbations of the convection-diffusion differential operator is developed. Existence of solutions for the optimization problem is proven and a variety of numerical tests are shown.

It remains an open question and source of further research where the functional $\hat{x} \mapsto J_2(\hat{x})$ is differentiable. If this is the case, then a variety of solutions algorithms could be applied for the solution of Problem $(\mathbb{P})$. Regularity of J_2 added to the complex Riccati framework would make classical schemes applicable.

The consideration of moving sensors which become increasingly common in practical applications is in the scope of future research. However, it should be noted that this adds another level of complexity to the problem. Additionally, as two objectives are considered, one is naturally confronted with computing Pareto optima (Pareto front) which is also within future research endeavours.

References

[1] W. Arendt and A. F. M. Elst. *Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations: 21st International Workshop on Operator Theory and Applications, Berlin, July 2010*, chapter From Forms to Semigroups, pages 47–69. Springer Basel, Basel, 2012.

[2] M. Athans, D. Willner, M. I. O. T. L. L. LAB., and U. S. N. T. I. Service. *Properties of Matched and Mismatched Kalman Filters*. Technical note. Massachusetts Institute of Technology, Lincoln Laboratory, 1976.

[3] H. T. Banks. *A functional analysis framework for modeling, estimation and control in science and engineering*. CRC Press, Boca Raton, FL, 2012.

[4] P. Benner, P. Benner, and J. Saak. A galerkin-newton-adi method for solving large-scale algebraic riccati equations. 2010.

[5] P. Benner, M. Heinkenschloss, J. Saak, and H. K. Weichelt. Inexact low-rank newton-adi method for large-scale algebraic riccati equations. Preprint MPIMD/15-06, Max Planck Institute Magdeburg, May 2015. Available from `http://www.mpi-magdeburg.mpg.de/preprints/`.

[6] A. Bensoussan. Filtrage optimal des systèmes linéaires. 1971.

[7] A. Bensoussan. Optimization of sensors' location in a distributed filtering problem. In *Stability of stochastic dynamical systems*, pages 62–84. Springer, 1972.

[8] A. Bielecki. Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaires. *Bull. Acad. Polon. Sci. Cl. III.*, 4:261–264, 1956.

[9] I. Bihari. A generalization of a lemma of bellman and its application to uniqueness problems of differential equations. *Acta Mathematica Academiae Scientiarum Hungarica*, 7(1):81–94.

[10] J. A. Burns, E. M. Cliff, and C. N. Rautenberg. A distributed parameter control approach to optimal filtering and smoothing with mobile sensor networks. In *Proceedings of the 2009 17th Mediterranean Conference on Control and Automation*, pages 181–186. IEEE Computer Society, 2009.

[11] J. A. Burns and B. B. King. A note on the regularity of solutions of infinite-dimensional Riccati equations. *Appl. Math. Lett.*, 7(6):13–17, 1994.

[12] J. A. Burns and B. B. King. A reduced basis approach to the design of low-order feedback controllers for nonlinear continuous systems. *J. Vib. Control*, 4(3):297–323, 1998.

[13] J. A. Burns, B. B. King, et al. Optimal sensor location for robust control of distributed parameter systems. In *IEEE Conference on Decision and Control*, volume 4, pages 3967–3967. Institute of Electrical Engineers Inc (IEE), 1994.

[14] J. A. Burns, B. B. King, and Y.-R. Ou. Computational approach to sensor/actuator location for feedback control of fluid flow systems, 1995.

[15] J. A. Burns and C. N. Rautenberg. Bochner integrable solutions to riccati partial differential equations and optimal sensor placement. In *American Control Conference (ACC), 2011*, pages 2368–2373. IEEE, 2011.

[16] J. A. Burns and C. N. Rautenberg. The infinite-dimensional optimal filtering problem with mobile and stationary sensor networks. *Numer. Funct. Anal. Optim.*, 36(2):181–224, 2015.

[17] J. A. Burns and C. N. Rautenberg. Solutions and Approximations to the Riccati Integral Equation with Values in a Space of Compact Operators. *SIAM J. Control Optim.*, 53(5):2846–2877, 2015.

[18] J. A. Burns, E. W. Sachs, and L. Zietsman. Mesh independence of Kleinman-Newton iterations for Riccati equations in Hilbert space. *SIAM J. Control Optim.*, 47(5):2663–2692, 2008.

[19] R. Curtain. A survey of infinite-dimensional filtering. *SIAM Rev.*, 17:395–411, 1975.

[20] R. F. Curtain. Infinite-dimensional filtering. *SIAM J. Control*, 13:89–104, 1975.

[21] M. A. Demetriou. Integrated actuator–sensor placement and hybrid controller design of flexible structures under worst case spatiotemporal disturbance variations. *Journal of Intelligent Material Systems and Structures*, 15(12):901–921, 2004.

[22] M. A. Demetriou. Power management of sensor networks for detection of a moving source in 2-d spatial domains. In *2006 American Control Conference*, pages 6–pp. IEEE, 2006.

[23] M. A. Demetriou. Process estimation and moving source detection in 2-d diffusion processes by scheduling of sensor networks. In *2007 American Control Conference*, pages 3432–3437. IEEE, 2007.

[24] M. A. Demetriou. Guidance of mobile actuator-plus-sensor networks for improved control and estimation of distributed parameter systems. *IEEE Trans. Automat. Control*, 55(7):1570–1584, 2010.

[25] M. A. Demetriou. Adaptive control of 2-D PDEs using mobile collocated actuator/sensor pairs with augmented vehicle dynamics. *IEEE Trans. Automat. Control*, 57(12):2979–2993, 2012.

[26] M. A. Demetriou. Enforcing consensus on adaptive parameter estimation of structurally perturbed infinite dimensional systems. *IEEE Trans. Automat. Control*, 57(12):3147–3152, 2012.

[27] M. A. Demetriou and I. I. Hussein. Estimation of spatially distributed processes using mobile spatially distributed sensor network. *SIAM J. Control Optim.*, 48(1):266–291, 2009.

[28] S. S. Dragomir and M. City. Some gronwall type inequalities and applications. *RGMIA Monographs, Victoria University*, 2000.

[29] N. Dunford and J. T. Schwartz. *Linear operators. Part II.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. Spectral theory. Selfadjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication.

[30] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, 1998.

[31] F. Fahroo and M. A. Demetriou. Optimal actuator/sensor location for active noise regulator and tracking control problems. *J. Comput. Appl. Math.*, 114(1):137–158, 2000. Control of partial differential equations (Jacksonville, FL, 1998).

[32] F. Feitzinger, T. Hylla, and E. W. Sachs. Inexact Kleinman-Newton method for Riccati equations. *SIAM J. Matrix Anal. Appl.*, 31(2):272–288, 2009.

[33] I. C. Gohberg and M. G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators*. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.

[34] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[35] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957. rev. ed.

[36] D. Kasinathan and K. Morris. H_∞-optimal actuator location. *IEEE Trans. Automat. Control*, 58(10):2522–2535, 2013.

[37] C. Kenney and G. Hewer. The sensitivity of the algebraic and differential Riccati equations. *SIAM J. Control Optim.*, 28(1):50–69, 1990.

[38] A. Khapalov. L^∞-exact observability of the heat equation with scanning pointwise sensor. *SIAM J. Control Optim.*, 32(4):1037–1051, 1994.

[39] A. Y. Khapalov. Optimal measurement trajectories for distributed parameter systems. *Systems Control Lett.*, 18(6):467–477, 1992.

[40] A. Y. Khapalov. Continuous observability for parabolic system under observations of discrete type. *IEEE Trans. Automat. Control*, 38(9):1388–1391, 1993.

[41] A. Y. Khapalov. Observability of hyperbolic systems with interior moving sensors. In *Analysis and optimization of systems: state and frequency domain approaches for infinite-dimensional systems (Sophia-Antipolis, 1992)*, volume 185 of *Lecture Notes in Control and Inform. Sci.*, pages 489–499. Springer, Berlin, 1993.

[42] A. Y. Khapalov. Exact observability of the time-varying hyperbolic equation with finitely many moving internal observations. *SIAM J. Control Optim.*, 33(4):1256–1269, 1995.

[43] K. Morris. Linear-quadratic optimal actuator location. *IEEE Trans. Automat. Control*, 56(1):113–124, 2011.

[44] K. Morris, M. A. Demetriou, and S. D. Yang. Using $\mathbb{H}_2$-control performance metrics for the optimal actuator location of distributed parameter systems. *IEEE Trans. Automat. Control*, 60(2):450–462, 2015.

[45] J. Nečas. *Direct methods in the theory of elliptic equations*. Springer Monographs in Mathematics. Springer, Heidelberg, 2012. Translated from the 1967 French original by Gerard Tronel and Alois Kufner, Editorial coordination and preface by Šárka Nečasová and a contribution by Christian G. Simader.

[46] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[47] M. Reed and B. Simon. *Methods of modern mathematical physics. I*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.

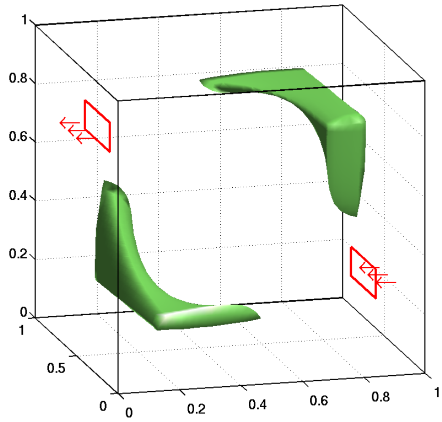
[48] J. R. Ringrose. *Compact non-self-adjoint operators*. Van Nostrand Reinhold Co., London, 1971.

[49] J.-F. Rodrigues. *Obstacle problems in mathematical physics*, volume 134 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987. Notas de Matemática [Mathematical Notes], 114.

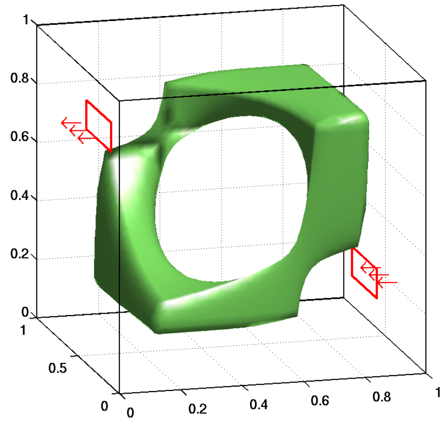
[50] R. E. Showalter. *Hilbert space methods for partial differential equations*. Pitman, London-San Francisco, Calif.-Melbourne, 1977. Monographs and Studies in Mathematics, Vol. 1.

[51] R. E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. American Mathematical Society, 1997.

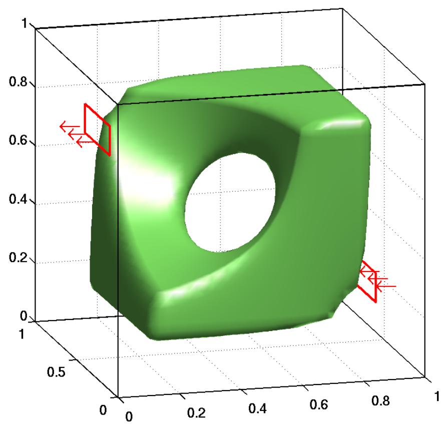
[52] S. Valadkhan, K. Morris, and A. Khajepour. Robust control of smart material-based actuators. In *Recent advances in learning and control*, volume 371 of *Lecture Notes in Control and Inform. Sci.*, pages 249–262. Springer, London, 2008.



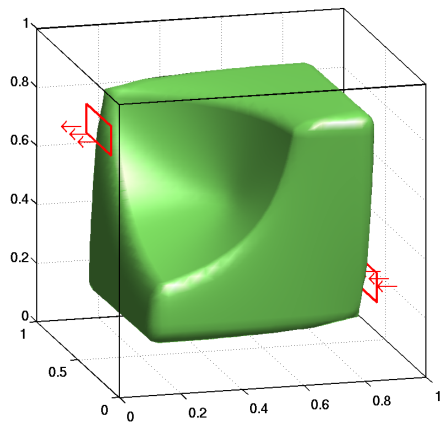
(a) $J_1(\hat{x}) = 4.7 \cdot 10^6$



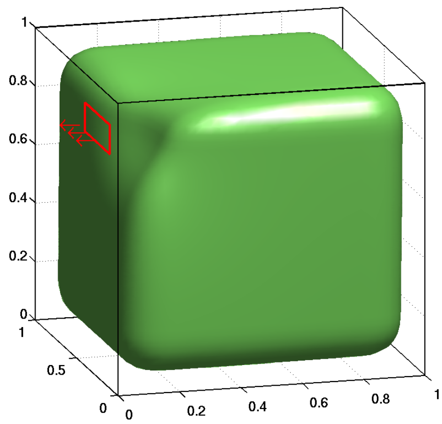
(b) $J_1(\hat{x}) = 4.71 \cdot 10^6$



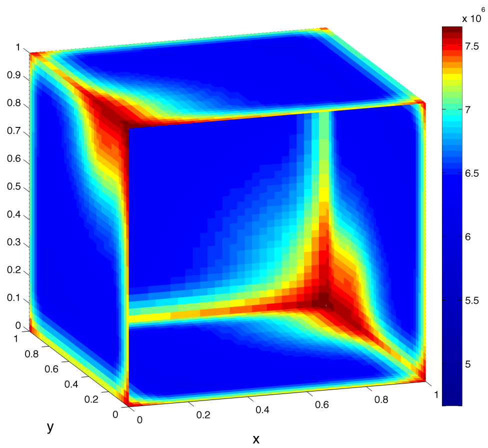
(c) $J_1(\hat{x}) = 4.73 \cdot 10^6$



(d) $J_1(\hat{x}) = 4.75 \cdot 10^6$

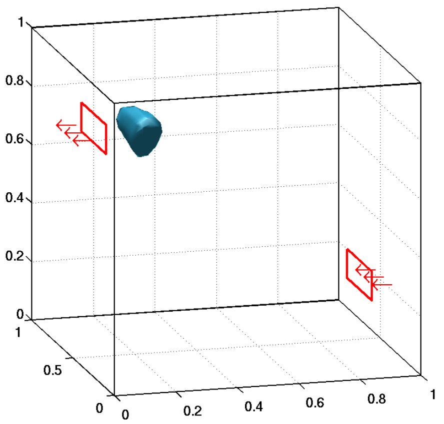


(e) $J_1(\hat{x}) = 5.65 \cdot 10^6$

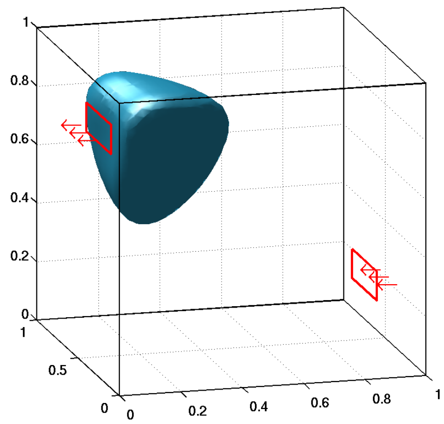


(f) Values of $\partial\Omega \ni \hat{x} \mapsto J_1(\hat{x})$

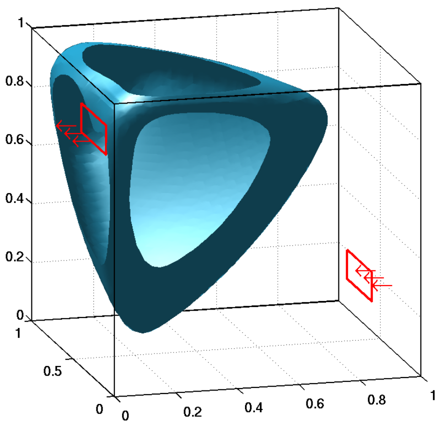
Figure 5: Isosurfaces for $\hat{x} \mapsto J_1(\hat{x})$, when $\hat{x}$ is restricted to the interior of the domain in 5(a)-5(e) and values of $\hat{x} \mapsto J_1(\hat{x})$ when restricted to the boundary in 5(f)



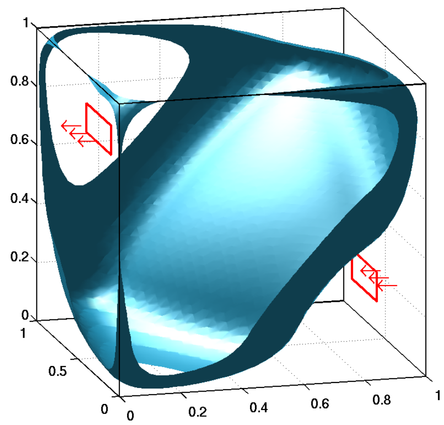
(a) $J_3(\hat{x}) = -366.73$



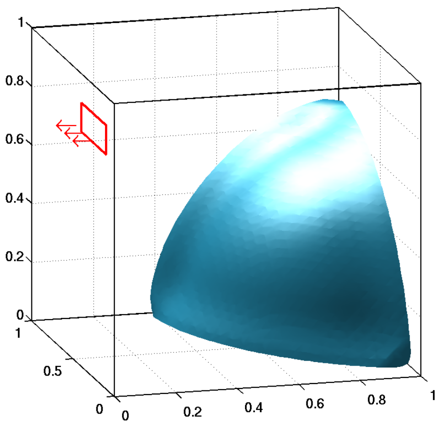
(b) $J_3(\hat{x}) = -182.06$



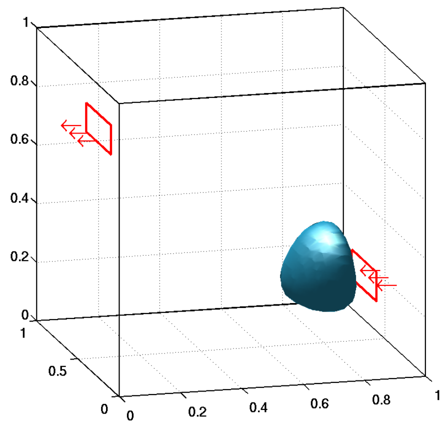
(c) $J_3(\hat{x}) = -1.19$



(d) $J_3(\hat{x}) = 99.71$



(e) $J_3(\hat{x}) = 196.81$



(f) $J_3(\hat{x}) = 465.25$

Figure 6: Isosurfaces for $\hat{x} \mapsto J_3(\hat{x})$, when $\hat{x}$ is restricted to the interior of the domain, are shown in Figures 6(a)-6(f).