Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Adiabatic theory of champion solitons

Sabrina Pickartz, Uwe Bandelow, Shalva Amiranashvili

submitted: July 08, 2016

Weierstraß-Institut
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: Sabrina.Pickartz@wias-berlin.de
Uwe.Bandelow@wias-berlin.de
Shalva.Amiranashvili@wias-berlin.de

No. 2276
Berlin 2016

2010 *Mathematics Subject Classification.* 78A60, 35Q60, 70H11.

2010 *Physics and Astronomy Classification Scheme.* 42.65.-k, 42.81.Dp, 05.45.Yv, 03.65.Nk.

Key words and phrases. Champion solitons, All-optical switching, Extreme events, Soliton perturbation theory, Event horizons.

Acknowledgments. The authors thank D. Turaev, A. Vladimirov, and M. Wolfrum for discussions on soliton perturbation theory and T. Koprucki for discussions on accelerated frames in Quantum Mechanics. U.B. and Sh.A. acknowledge support of Einstein Foundation Berlin and Research Center MATHEON under Project D-OT2.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider scattering of small-amplitude dispersive waves at an intense optical soliton which constitutes a nonlinear perturbation of the refractive index. Specifically, we consider a single-mode optical fiber and a group velocity matched pair: an optical soliton and a nearly perfectly reflected dispersive wave, a fiber-optical analogue of the event horizon. By combining (i) an adiabatic approach that is used in soliton perturbation theory and (ii) scattering theory from Quantum Mechanics, we give a quantitative account for the evolution of all soliton parameters. In particular, we quantify the increase in the soliton peak power that may result in spontaneous appearance of an extremely large, so-called champion soliton. The presented adiabatic theory agrees well with the numerical solutions of the pulse propagation equation. Moreover, for the first time we predict the full frequency band of the scattered dispersive waves and explain an emerging caustic structure in the space-time domain.

1 Introduction

Solitary solutions of integrable nonlinear wave equations, like the Nonlinear Schrödinger Equation (NLSE) survive after binary collisions and retain their energy and momentum [51, 47]. If integrability is destroyed by a small modification of the NLSE, solitons behave differently: they exchange energy, and the larger solitons grow at the expense of the smaller ones. A large number of collisions contributes to formation of a single huge or champion soliton [50, 18, 36]. In fiber optics, champion solitons, as well as breathers [4, 28] and giant dispersive waves [20] (DW), are natural examples of extreme nonlinear phenomena, e.g., in optical supercontinuum [43, 17].

Extremely large solitary waves are also observed in systems where only a few solitons are present and their binary collisions and exchange of energy are not important [10]. The key process here is interaction of solitons with DWs. It is well known that a DW packet can be trapped by a soliton [22, 23], it can bounce at an accelerating soliton [21] or between two solitons [48]. Scattering of DWs at a nonlinear perturbation created by an optical soliton was studied in Refs. [49, 19, 41, 35] and resulted in accurate predictions of the new-appearing DW frequencies. Moreover, it was found that, under favorable conditions, the scattering leads to a significant increase in the soliton peak power and can yield champion solitons [15, 12].

The purpose of the present paper is to analyze the scattering process and to provide a quantitative description of the soliton amplification. This was achieved by a combination of the so-called *soliton perturbation theory* and the scattering theory from Quantum Mechanics (QM). Incidentally, we show how a low-power DW that is several orders of magnitude weaker than the soliton, can give rise to a significant spontaneous increase in the soliton peak power. Informally speaking, such a champion soliton appears from nowhere, a property of rogue waves that is usually attributed to modulation instability and breathers [2].

2 Background

Scattering of light at a stationary inhomogeneity in an otherwise uniform transparent medium is a well-known topic in optics [7]. In the case of a moving inhomogeneity, the scattering is followed by a frequency conversion. Full analysis of such interactions is challenging [6]. To get a rough idea one can equate the Doppler shifted income and outcome frequencies, ω_i and ω_o

$$\omega_i - \mathbf{k}_i\mathbf{V} = \omega_o - \mathbf{k}_o\mathbf{V},$$

where the incoming and the scattered waves are given by $e^{i(\mathbf{k}_{i,o}\mathbf{r}-\omega_{i,o}t)}$ and $\mathbf{V}$ refers to the velocity of the inhomogeneity. For instance, in a one-dimensional setting with all waves propagating along the z-axis and with the dispersion relation $k_z = \pm\beta(\omega)$ for the forward and backward waves respectively, the possible outcome frequencies of the scattered wave are determined by the algebraic equation

$$\omega_i - \beta(\omega_i)V = \omega_o \mp \beta(\omega_o)V. \tag{1}$$

In a uniform dielectric with $\beta(\omega) = n\omega/c$, where n is the index of refraction and c is the speed of light, we have two standard options for the transmitted and the reflected waves [6, 33]

$$\omega_o = \omega_i \quad \text{and} \quad \omega_o = \frac{1 - nV/c}{1 + nV/c}\omega_i,$$

where for $V > 0$ the inhomogeneity propagates forward with the incoming wave. The reflected wave is the backward one. In a dispersive medium with $n = n(\omega)$, an incoming wave can produce several scattered waves that propagate in different directions with different frequencies [38, 39].

The next level of complexity is introduced by nonlinearity, e.g., the inhomogeneity is given by a robust nonlinear perturbation with its own dynamics. An important example is given by a soliton that propagates together with a small-amplitude DW [35, 40, 45]. In the context of optics, interaction of solitons and DWs is an integral part of supercontinuum generation [16, 42]. The DWs are directly generated by optical solitons due to Cherenkov mechanism [44, 3], thereafter they are scattered at solitons and significantly contribute to the spectral broadening [27, 42]. In the simplest scenario, the forward incoming wave $e^{i\beta(\omega_i)z-i\omega_i t}$ is scattered by a small perturbation of the refractive index yielded by an optical soliton with the carrier frequency ω_s such that the outcome wave $e^{i\beta(\omega_o)z-i\omega_o t}$ does not change its direction of propagation. One can approximate V in Eq. (1) by the group velocity at the soliton frequency $1/\beta'(\omega_s)$ and derive that

$$\beta'(\omega_s) = \frac{\beta(\omega_o) - \beta(\omega_i)}{\omega_o - \omega_i}, \tag{2}$$

where, as illustrated in Fig. 1, the possible values of ω_o are yielded by $\omega_{i,s}$. The trivial $\omega_o = \omega_i$ solution corresponds to the transmitted wave, for the scattered wave $\omega_o \neq \omega_i$. Cherenkov radiation emitted by the soliton as such corresponds to the special case $\omega_i = \omega_s$.

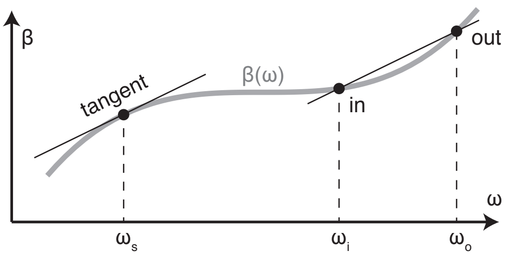

Figure 1: A graphic illustration of Eq. (2). The input DW frequency ω_i and the soliton carrier frequency ω_s yield the output frequency ω_o of the scattered DW.

Solitons in fiber optics are usually described by the Generalized Nonlinear Schrödinger Equation [1, 8] (GNLSE) centered at soliton carrier frequency ω_s and with a dispersion function $\mathfrak{D}(\varpi)$. The latter is defined for the frequency offset ϖ such that

$$\beta(\omega_s + \varpi) = \beta(\omega_s) + \beta'(\omega_s)\varpi + \mathfrak{D}(\varpi). \tag{3}$$

By definition (3), Eq. (2) takes the well-known form $\mathfrak{D}(\omega_o - \omega_s) = \mathfrak{D}(\omega_i - \omega_s)$ for the scattered wave, and the form $\mathfrak{D}(\omega_o - \omega_s) = 0$ for the Cherenkov output frequency [42]. A more accurate analysis of nonlinear mixing of solitons and DWs in the framework of four-wave mixing (FWM) yields radiation at several new frequencies that add to the predictions of Eq. (2) if the corresponding phase-matching conditions are met [49, 19, 41].

In the above studies of DWs scattered at a soliton, the latter is assumed to be an unchanged solitary solution of the NLSE. This approximation is used despite of the fact that NLSE solitons may have different amplitudes and velocities, moreover these parameters can significantly change in a slow adiabatic manner due to an external action as quantified by the soliton perturbation theory [29, 24]. It is then reasonable to ask whether an optical soliton can be manipulated/enhanced by applying a suitable DW. This is indeed the case [15], however, the manipulation is effective only if the following empirical rules are met.

1 Carrier frequencies of the soliton and the DW packet are on either side of the zero dispersion frequency (Fig. 2a). The group velocity dispersion $\beta''(\omega)$ (GVD) is then negative for the soliton and positive for the DW [15].

2 Group velocities of the soliton and the DW packet should be reasonably close (Fig. 2b). If they are too different the pulses go through each other unchanged, if they are too close the pulses may never meet [15].

3 In a favorable situation the DW is almost perfectly scattered at the soliton (Fig. 3a). That is, the intensity of the transmitted wave is very low as compared to the intensity of the scattered wave [15, 9]. The situation is often described as *a group velocity event horizon*, see Ref. [35].

4 The GVD profile $\beta''(\omega)$ should be steep for the soliton and slightly sloping for the DW (Fig. 2a). Otherwise the DW is still scattered but the soliton remains nearly unchanged [10].

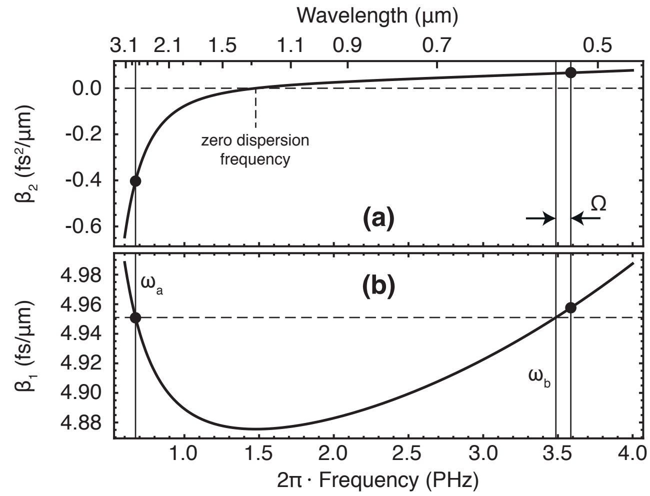

Figure 2: The frequency dependence of (a) GVD $\beta_2 = \beta''(\omega)$ and (b) group delay $\beta_1 = \beta'(\omega)$ for bulk fused silica. The initial soliton frequency ω_a and the DW frequency $\omega_b + \Omega$ are taken from Eq. (4) and labelled by the black points. ω_b is the reference frequency such that $\beta'(\omega_a) = \beta'(\omega_b)$.

5 An extended low-amplitude DW packet works better than a short DW packet with the same energy [14]. In the latter case the initial delay between the DW pulse and the soliton should be large enough to ensure pulse spreading due to dispersion (Fig. 3a).

The above empirical rules have been found by trial and error during numerical investigations of DWs scattering and all-optical switching of solitons [15, 10, 14, 12, 13, 11]. A possible choice of carrier frequencies (wavelengths) of the soliton and the DW packet for bulk fused silica dispersion is shown in Fig. 2(a,b)

$$\begin{aligned} \omega_s|_{z=0} &= \omega_a, & \lambda_s &= 2.8\,\mu\text{m}, \\ \omega_{\text{DW}}|_{z=0} &= \omega_b + \Omega, & \lambda_{\text{DW}} &= 0.525\,\mu\text{m}. \end{aligned} \tag{4}$$

With this choice, we plotted in Fig. 3a how a $\cosh^{-2}(t/t_0)$ DW packet with $t_0 = 100$ fs is scattered on a fundamental soliton with $t_0 = 30$ fs. The initial delay is 600 fs, the initial peak power of the DW packet is 30% that of the soliton. The latter yields a nonlinear correction $\delta n = 2 \cdot 10^{-4}$ to the refractive index. The soliton peak power at $z = 10$ cm is 3.1 times larger than that at $z = 0$.

Figure 3b shows the evolution of the soliton carrier frequency along the fiber. Note that increase in $\omega_s(z)$ leads to a decrease in the GVD $\beta''(\omega_s)$ (Fig. 2a) and to soliton compression (Fig. 3a). Figure 3c shows the spectral window for the DW packet. Scattering at the soliton occurs for $z \gtrsim 4$ cm and yields DWs at a new frequency. In fact, a bunch of frequencies is generated due to the accompanied increase in $\omega_s(z)$.

We develop a quantitative description of interactions like the one shown in Fig. 3. Whenever possible, we would like to specify the above mentioned empirical rules, e.g., refine rule 2 by calculating the optimal DW frequency offset Ω in Fig 2. Construction of such a theory is a difficult

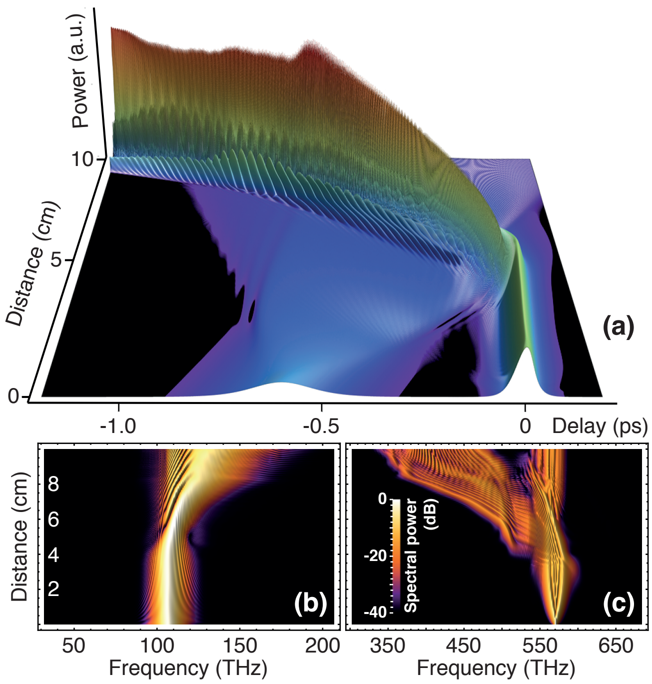

Figure 3: (a) an exemplary scattering of the DW (left pulse) at a soliton (right pulse) in the space-time domain. (b) spectral density for the soliton. (c) spectral density for the DW. See the text following Eq. (4) for parameters.

task and we make several simplifying assumptions to address it. Motivated by Fig. 3, we assume that the soliton and the DW are separated in the frequency domain (although this is not always the case, see Refs. [14, 13]). We then consider two coupled propagation equations, one for the soliton and one for the DW. Each propagation equation applies only to a part of the frequency domain and can be simplified, as opposed to the full GNLSE that was solved numerically to obtain Fig. 3. Furthermore, in accordance with rule 5 the champion soliton is effectively created by scattering of an extended low-amplitude DW packet. We therefore consider scattering of a monochromatic DW at a soliton and take advantage of the QM scattering theory. To keep things simple we finally neglect the Raman effect. It only has a small influence on the DW scattering as such [21, 14], but makes the analytic theory much more cumbersome. Our quantification adequately describes the instantaneous formation of champion solitons, and it directly applies to Raman-free gas-filled fibers [5]. We proceed to our main theoretical issue and derive adiabatic ordinary differential equations (ODEs) for the soliton parameters.

3 Basic equations

Let ω_a and ω_b be two fixed reference frequencies, with negative and positive GVD respectively, such that the corresponding group velocities are equal (Fig. 2). We then define the common delay variable

$$\tau = t - \beta'(\omega_a)z = t - \beta'(\omega_b)z,$$

which is suitable for both the soliton and the DW. The soliton carrier frequency is permanently shifted in the course of interaction with the DW (Fig. 3b), let it be

$$\omega_s(z) = \omega_a + \nu(z)$$

where $\nu(z)$ is a yet unknown function with $\nu(0) = 0$. The dispersion function $\mathfrak{D}(\varpi)$ is taken from Eq. (3) for $z = 0$, i.e., in what follows

$$\mathfrak{D}(\varpi) = \beta(\omega_a + \varpi) - \beta(\omega_a) - \beta'(\omega_a)\varpi. \tag{5}$$

The frequency of the incoming DW is denoted by $\omega_b+\Omega$, a constant offset Ω is given in advance, as opposed to both $\nu(z)$ and the frequency of the reflected DW.

We describe the soliton by its envelope $\psi_a(z,\tau)$ yielded by a GNLSE that is centered at ω_a

$$i\partial_z\psi_a + \sum_{j=2}^{J} \frac{\beta_a^{(j)}}{j!}(i\partial_\tau)^j\psi_a + \frac{n_{2a}}{c}(\omega_a + i\partial_\tau)(|\psi_a|^2 + 2|\psi_b|^2)\psi_a = 0. \tag{6}$$

The envelope $\psi_b(z,\tau)$ of the DW is yielded by a GNLSE that is centered at ω_b

$$i\partial_z\psi_b + \sum_{j=2}^{J} \frac{\beta_b^{(j)}}{j!}(i\partial_\tau)^j\psi_b + \frac{n_{2b}}{c}(\omega_b + i\partial_\tau)(|\psi_b|^2 + 2|\psi_a|^2)\psi_b = 0. \tag{7}$$

Here $\beta_a^{(j)} = \beta^{(j)}(\omega_a)$ and $\beta_b^{(j)} = \beta^{(j)}(\omega_b)$, denote derivatives of $\beta(\omega)$ at the reference frequencies, it is presupposed that $\beta_a'' < 0$ and $\beta_b'' > 0$. The exact value of J is unimportant, provided that it is large enough to capture the higher-order-dispersion (HOD). Parameters $n_{2a,2b}$ quantify nonlinear refraction at the reference frequencies. Equations (6) and (7) are coupled by the cross-phase-modulation terms [1].

A significant observation is that the HOD terms with $j \geq 3$ and the self-steepening derivative $i\partial_\tau$ of the nonlinear term are important for a short powerful soliton described by Eq. (6) and less important for a weak long-duration DW described by Eq. (7). Our calculations show that Eq. (6) that is coupled with the following reduction of Eq. (7)

$$i\partial_z\psi_b - \frac{\beta_b''}{2}\partial_\tau^2\psi_b + \frac{2\omega_b}{c}n_{2b}|\psi_a|^2\psi_b = 0, \tag{8}$$

adequately describes soliton evolution under influence of the DW. Equation (8) is, of course, equivalent to the standard QM Schrödinger equation. Moreover, quadratic approximation of $\beta(\omega)$ near the reference frequency ω_b reduces the implicit relation (2) between the in/out frequencies to the following explicit form

$$\omega_i = \omega_b + \Omega, \qquad \omega_o = \omega_b - \Omega + 2\frac{\beta'(\omega_a+\nu) - \beta_a'}{\beta_b''}. \tag{9}$$

In the rest of this section we proceed as follows.

3.1 First, we split Eq. (6) into two parts: the main part, a NLSE that yields the soliton, and the perturbation part. This happens to be the most non-trivial step.

3.2 Then we use the splitting to derive ODEs for the soliton parameters with the soliton perturbation theory [24], as summarized in Appendix A. Here important conclusions are gained in spite of the fact that the ODEs depend on the yet unknown ψ_b.

3.3 The soliton solution for $|\psi_a|^2$ is then inserted into Eq. (8) and ψ_b is calculated using a small extension of the standard QM scattering theory (Appendix B). Finally, ψ_b is used to obtain the self-consistent ODEs for the soliton parameters.

3.1 Splitting of Eq. (6)

To apply the soliton perturbation theory we would like to split ψ_a into two parts: a soliton part with the yet unknown slowly varying parameters and a small rest part. In other words, the GNLSE (6) should be split into a suitable NLSE and the perturbation term. The simplest way of splitting is to replace Eq. (6) by

$$i\partial_z\psi_a - \frac{\beta_a''}{2}\partial_\tau^2\psi_a + \frac{\omega_a}{c}n_{2a}|\psi_a|^2\psi_a = \text{R.H.S.}, \tag{10}$$

where all unwanted terms are simply put on the right-hand-side and considered as a perturbation.

Unfortunately we have found that a perturbation theory that starts from Eq. (10) does not capture soliton compression. The reason is that the fundamental soliton described by the left-hand-side of Eq. (10) always experiences the same value β_a'' of GVD. This comes into conflict with the change of $\beta''(\omega_a+\nu)$ yielded by the permanently increasing soliton frequency shift $\nu(z)$. An additional argument comes from numerics: soliton compression is effective only if the frequency shifted soliton experiences a considerably smaller GVD (see Fig. 2a and rule 4).

To overcome this difficulty we now return to Eq. (6) and derive a new GNLSE in which the carrier frequency is not fixed. Note that if we perform a monochromatic wave substitution $\psi_a \sim e^{i(\kappa z-\varpi\tau)}$ in the linearized Eq. (6), we arrive to the dispersion relation $\kappa = \mathfrak{D}(\varpi)$, where

$$\mathfrak{D}(\varpi) = \sum_{j=2}^{J}\frac{\beta_a^{(j)}}{j!}\varpi^j,$$

in accordance with the definition (5). For a z-dependent offset $\nu(z)$ we naturally replace

$$e^{i\mathfrak{D}(\varpi)z - i\varpi\tau} \qquad \text{by} \qquad e^{i\int_0^z \mathfrak{D}(\nu)dz - i\nu\tau}.$$

These observations suggest the substitution

$$\psi_a(z,\tau) = \psi(z,\tau)e^{i\int_0^z \mathfrak{D}(\nu)dz - i\nu\tau}, \tag{11}$$

where $\psi(z,\tau)$ is the new envelope that is adjusted to the carrier frequency $\omega_a+\nu(z)$. For the present, $\nu(z)$ is just an arbitrary function, $\nu(z)$ will be equated to the soliton frequency shift later on.

The definition (11) is now inserted into the GNLSE (6) in which calculation of the dispersion operator term is the only difficulty. We profit from the identity

$$\sum_{j=2}^{J}\frac{\beta_a^{(j)}}{j!}(i\partial_\tau)^j=\mathfrak{D}(i\partial_\tau)=\sum_{j=0}^{J}\frac{\mathfrak{D}^{(j)}(\nu)}{j!}(i\partial_\tau-\nu)^j,$$

where both sides simply provide two Taylor expansions of $\mathfrak{D}(i\partial_\tau)$. The identity holds because the derivative operator $i\partial_\tau$ and multiplication by $\nu(z)$ commute with each other. Now substitution of Eq. (11) into Eq. (6) becomes trivial because

$$\sum_{j=2}^{J}\frac{\beta_a^{(j)}}{j!}(i\partial_\tau)^j\psi_a=e^{i\int_0^z\mathfrak{D}(\nu)dz-i\nu\tau}\sum_{j=0}^{J}\frac{\mathfrak{D}^{(j)}(\nu)}{j!}(i\partial_\tau)^j\psi,$$

and we derive that

$$i\partial_z\psi+\sum_{j=1}^{J}\frac{\mathfrak{D}^{(j)}(\nu)}{j!}(i\partial_\tau)^j\psi+\tau\frac{d\nu}{dz}\psi+\frac{n_{2a}}{c}(\omega_a+\nu+i\partial_\tau)(|\psi|^2+2|\psi_b|^2)\psi=0,\quad(12)$$

where $\nu(z)$ is arbitrary but small and slow frequency offset from ω_a. Equation (12) is the required GNLSE with the varying carrier frequency.

The coefficients in Eq. (12) are functions of the soliton frequency shift, which change as the soliton propagates along the fiber. It captures the decrease in GVD with the increase in ν. Moreover, Eq. (12) refers to the exact GVD at the varying carrier frequency because $\mathfrak{D}''(\nu)=\beta''(\omega_a+\nu)$, such that the HOD terms are less important, as compared to Eq. (6). We take $J=4$ in what follows. Another feature of Eq. (12) is the $\tau(d\nu/dz)\psi$ term. Noteworthy, similar terms appear in QM if one transforms the standard Schrödinger equation to an accelerated frame, they are related to the inertial forces [34].

Equation (12) is now split as follows

$$i\partial_z\psi+i[\beta'(\omega_a+\nu)-\beta_a']\partial_\tau\psi-\frac{\beta''(\omega_a+\nu)}{2}\partial_\tau^2\psi+\frac{\omega_a+\nu}{c}n_{2a}|\psi|^2\psi=R[\psi],\quad(13)$$

the perturbation term reading

$$R[\psi]=-\sum_{j=3,4}\frac{\beta^{(j)}(\omega_a+\nu)}{j!}(i\partial_\tau)^j\psi-\tau\frac{d\nu}{dz}\psi$$
$$-\frac{n_{2a}}{c}i\partial_\tau|\psi|^2\psi-\frac{2n_{2a}}{c}(\omega_a+\nu+i\partial_\tau)|\psi_b|^2\psi.\quad(14)$$

Equation (13) is the basis for further analysis of soliton evolution. It is coupled with the DW equation

$$i\partial_z\psi_b-\frac{\beta_b''}{2}\partial_\tau^2\psi_b+\frac{2\omega_b}{c}n_{2b}|\psi|^2\psi_b=0,\quad(15)$$

which follows from Eq. (8) due to Eq. (11).

3.2 Preliminary ODEs

We are interested in a special solution of Eq. (13), where $\psi(z,\tau)$ is approximated by a soliton with varying parameters, as yielded by the soliton perturbation theory in Appendix A. According to Eq. (50), we consider the trial function

$$\phi = \sqrt{\frac{|\beta''(\omega_a+\nu)|c}{(\omega_a+\nu)n_{2a}}}\frac{e^{i\Theta}}{\sigma\cosh\frac{\tau-\mathcal{T}}{\sigma}}, \tag{16}$$

which includes soliton duration $\sigma(z)$, phase $\Theta(z)$, and delay $\mathcal{T}(z)$. The phase equation is decoupled from the others, it is ignored in what follows. By construction, we equate $\nu(z)$ in Eq. (13) to the soliton frequency offset $\omega_s(z)-\omega_a$, that is why the trial function (16) does not contain the frequency shift [compare Eq. (47) to Eq. (50)]. A full set (49) of ODEs for the soliton parameters is then replaced by a more simple set (51). From Eq. (52) we obtain equations (17) and (18) for the duration and frequency offset

$$\frac{d}{dz}\frac{\beta''(\omega_a+\nu)}{(\omega_a+\nu)\sigma} = -\frac{n_{2a}}{c}(R[\phi], i\phi), \tag{17}$$

$$(R[\phi], \partial_\tau\phi) = 0, \tag{18}$$

and the equation for the delay:

$$\frac{d\mathcal{T}}{dz} = \beta'(\omega_a+\nu) - \beta'_a - \frac{(\omega_a+\nu)\sigma}{\beta''(\omega_a+\nu)}\frac{n_{2a}}{c}(R[\phi], i[\tau-\mathcal{T}]\phi). \tag{19}$$

Here $R[\phi]$ is calculated by Eq. (14) and the scalar product $(\cdot,\cdot)$ is defined in Eq. (43). Equations (17–19) are supplemented by the natural initial conditions

$$\sigma(0)=\sigma_0, \quad \nu(0)=0, \quad \mathcal{T}(0)=0, \tag{20}$$

and should return $\sigma(z)$, $\nu(z)$, and $\mathcal{T}(z)$.

A cumbersome but straightforward calculation of the scalar products reduces Eq. (17) and (18) to the form

$$\frac{d}{dz}\frac{\beta''(\omega_a+\nu)}{(\omega_a+\nu)\sigma} = -\frac{n_{2a}^2}{c^2}\int_{-\infty}^{\infty}|\psi_b|^2\left(\partial_\tau|\phi|^2\right)d\tau, \tag{21}$$

$$\frac{\beta''(\omega_a+\nu)}{(\omega_a+\nu)^2\sigma}\frac{d\nu}{dz} = -\frac{n_{2a}^2}{c^2}\int_{-\infty}^{\infty}|\psi_b|^2\left(\partial_\tau|\phi|^2\right)d\tau. \tag{22}$$

Here an important feature is that, even without knowledge of $|\psi_b|^2$, we derive an integral of motion

$$\frac{\beta''(\omega_a+\nu)}{(\omega_a+\nu)^2\sigma} = \text{const},$$

such that soliton duration is expressed in terms of the frequency offset

$$\sigma = \frac{\beta''(\omega_a+\nu)}{\beta''_a}\frac{\sigma_0}{(1+\nu/\omega_a)^2}. \tag{23}$$

In particular, we can ignore Eq. (21) and deal with only one Eq. (22). Moreover, it becomes clear that both increase in ν (Fig. 3b) and decrease in $\beta''(\omega_a+\nu)$ (Fig. 2a) contribute to the soliton compression.

To derive the ODE for the soliton delay, we evaluate the scalar product in Eq. (19) and obtain

$$\frac{d\mathcal{T}}{dz} = \mathcal{B} - \frac{(\omega_a+\nu)\sigma}{\beta''(\omega_a+\nu)} \frac{2n_{2a}^2}{c^2} \int_{-\infty}^{\infty} |\psi_b|^2 \left[|\phi^2| + \frac{\tau - \mathcal{T}}{2} \partial_\tau |\phi^2| \right] d\tau, \tag{24}$$

where

$$\mathcal{B} = \beta'(\omega_a+\nu) - \beta'_a - \frac{\beta''(\omega_a+\nu)}{(\omega_a+\nu)\sigma^2} + \frac{\beta'''(\omega_a+\nu)}{6\sigma^2}, \tag{25}$$

describes delay evolution through dispersion and self-steepening. For instance, when the DW is absent or too weak to modify the soliton, we obtain that velocity V of a soliton with the unchanged duration σ_0 and carrier frequency ω_a is determined by the relation

$$\frac{1}{V} = \beta'_a + \mathcal{B}|_{\nu=0} = \beta'_a - \frac{\beta''_a}{\omega_a\sigma_0^2} + \frac{\beta'''_a}{6\sigma_0^2}. \tag{26}$$

Here, the first correction to β'_a results from the self-steepening effect [25]. The second correction appears due to Cherenkov DW emitted by the soliton [24, 30].

The parameter V quantifies how the soliton peak is delayed, it does not describe the total pulse, which contains a non $\cosh^{-1}$ part and emitted DWs. Note, that there is no unique interpretation of the group velocity in a weakly nonlinear system [46]. Nevertheless, with Eq. (26) we can refine Eq. (2) for frequency of the DW scattered at a soliton. For the calculation in Fig. 3 the scattered and Cherenkov waves are at 538.7 THz and 853.5 THz respectively when derived with Eq. (2), and at 544.3 THz and 858.5 THz when derived with the refined soliton velocity from Eq. (26).

In the following we will find that $|\psi_b|^2$ naturally depends on the variable

$$\zeta = \frac{1 - \tanh\frac{\tau-\mathcal{T}}{\sigma}}{2}. \tag{27}$$

With Eq. (16) and (27), we transform Eq. (22) to

$$\frac{\sigma}{\omega_a+\nu}\frac{d\nu}{dz} = \frac{4n_{2a}}{c}\int_0^1 |\psi_b|^2(2\zeta-1)d\zeta, \tag{28}$$

and Eq. (24) to

$$\frac{d\mathcal{T}}{dz} = \mathcal{B} + \frac{4n_{2a}}{c}\int_0^1 |\psi_b|^2[1-(2\zeta-1)\operatorname{artanh}(2\zeta-1)]d\zeta. \tag{29}$$

Equations (28) and (29) require a proper expression for $|\psi_b|^2$ to render self-consistent ODEs for $\nu(z)$ and $\mathcal{T}(z)$. We now calculate $|\psi_b|^2$ by substituting ϕ from Eq. (16) for ψ in Eq. (15) and using the QM scattering theory.

3.3 Spatial version of the scattering theory

In this section we deal with the Eq. (15) in which $|\psi|^2$ is approximated by the soliton solution (16) such that

$$i\partial_z\psi_b - \frac{\beta_b''}{2}\partial_\tau^2\psi_b + \frac{U_0}{\cosh^2\frac{\tau-\mathcal{T}}{\sigma}}\psi_b = 0, \tag{30}$$

with

$$U_0 = \frac{2|\beta''(\omega_a+\nu)|}{\sigma^2}\frac{\omega_b}{\omega_a+\nu}\frac{n_{2b}}{n_{2a}}. \tag{31}$$

The involved soliton parameters are slow functions of z and Eq. (30) is treated in an adiabatic manner: we assume that ψ_b succeeds to adjust itself to the actual values of U_0, $\mathcal{T}$, and $d\mathcal{T}/dz$. Equation (30) is mathematically equivalent to the QM Eq. (53), as summarized in Table 1 of Appendix B.

Note, that QM scattering is most effective when kinetic energy, which corresponds to the incoming wave, is smaller than the potential barrier height, $k^2/(2m) \le U_0$, where for the sake of brevity $\hbar = 1$ and we recall that the soliton is initially at rest in the reference frame. In optical notations $\beta_b''\Omega^2/2 \le U_0$, i.e., the reasonable initial DW offset Ω and soliton duration σ_0 are subject to the inequality

$$\Omega \le \frac{2}{\sigma_0}\sqrt{\frac{|\beta_a''|}{\beta_b''}\frac{\omega_b}{\omega_a}\frac{n_{2b}}{n_{2a}}}, \tag{32}$$

which quantifies rule 2 and explains why the inequality $|\beta_a''| \gg \beta_b''$ (rule 4) is important. Another immediate observation is that event horizons created by optical solitons are imperfect due to the unavoidable QM tunneling effect.

Now we apply Table 1 of Appendix B to derive solutions of Eq. (30). Away from the soliton, the DWs are described by

$$\psi_b = e^{i(\kappa_\Omega z \mp \Omega\tau)} \quad \text{with} \quad \kappa_\Omega = \frac{\beta_b''}{2}\Omega^2,$$

cf., Eq. (54). The solution of Eq. (30), as yielded by scattering of a monochromatic DW, is parametrized by the DW frequency and will be denoted by $\psi_{b\Omega}$. The following asymptotic properties

$$\begin{aligned}
\psi_{b\Omega}(z,\tau)|_{\tau\to-\infty} &= e^{i(\kappa_\Omega z-\Omega\tau)} + B^*(\Omega)e^{i(\kappa_{\underline{\Omega}} z+\underline{\Omega}\tau)},\\
\psi_{b\Omega}(z,\tau)|_{\tau\to+\infty} &= A^*(\Omega)e^{i(\kappa_\Omega z-\Omega\tau)},
\end{aligned}$$

are required by Eq. (55). Here $A^*(\Omega)$ and $B^*(\Omega)$ appear because Eq. (30) is equivalent to the complex conjugated QM Eq. (53). The frequency $\underline{\Omega}$ differs from Ω because as the DW scatters, the soliton changes its velocity

$$\underline{\Omega} = \Omega - \frac{2}{\beta_b''}\frac{d\mathcal{T}}{dz}. \tag{33}$$

cf., Eq. (56). Approximating $d\mathcal{T}/dz$ by $\mathcal{B}$ from Eq. (25), we obtain for the frequency of the scattered DW

$$\omega_o = \omega_b - \underline{\Omega} = \omega_b - \Omega + \frac{2\mathcal{B}}{\beta_b''}, \tag{34}$$

which accounts for the influence of both HOD and self-steepening, unlike Eq. (9).

The solution to the scattering problem for Eq. (30) is expressed in terms of the hypergeometric function F, as given by Eq. (61) in which we change to optical notations in accordance with the Table 1

$$|\psi_{b\Omega}(z,\tau)|^2 = \frac{\sinh^2(\pi\bar{\Omega}\sigma)}{\cosh^2(\pi s)+\sinh^2(\pi\bar{\Omega}\sigma)} \times \left| F\left(\frac{1}{2}-i\bar{\Omega}\sigma+is, \frac{1}{2}-i\bar{\Omega}\sigma-is, 1-i\bar{\Omega}\sigma, \zeta\right)\right|^2,$$

where the first factor equals the DW transmission coefficient and

$$\bar{\Omega} = \Omega - \frac{1}{\beta_b''}\frac{d\mathcal{T}}{dz}, \qquad s = \frac{1}{2}\sqrt{\frac{8\sigma^2 U_0}{\beta_b''}-1}.$$

The parameters ζ and U_0 are defined by Eq. (27) and (31) respectively. Similar to Eq. (33) we approximate $d\mathcal{T}/dz$ by $\mathcal{B}$ from Eq. (25) in the definition of $\bar{\Omega}$ in what follows.

Before proceeding we note that, following the usual QM formulation, the amplitude of the incoming DW in the scattering problem was artificially set to 1. Let μ denote the ratio of the true DW power to the maximal power of the fundamental soliton at $z=0$ taken from Eq. (16)

$$(\text{incoming DW power}) = \mu\frac{|\beta_a''|c}{\sigma_0^2\omega_a n_{2a}}, \tag{35}$$

where our perturbation theory requires $\mu \ll 1$. The factor (35) is used to rescale $|\psi_{b\Omega}|^2$. The latter quantity is then inserted into Eq. (28) and (29) for the self-consistent ODEs for $\nu(z)$ and $\mathcal{T}(z)$.

4 Adiabatic equations

We finally formulate our main result: ODEs that describe how an optical soliton with the varying carrier frequency $\omega_a+\nu(z)$ is modified by scattering of a small-amplitude DW with the given frequency $\omega_b+\Omega$, where the reference frequencies $\omega_{a,b}$ are different but $\beta_a' = \beta_b'$ (Fig. 2). Two equations for the frequency offset $\nu(z)$ and for the soliton delay in the reference frame $\mathcal{T}(z)$ are required, because soliton duration $\sigma(z)$ is expressed by Eq. (23). The DW transmission coefficient reads

$$\begin{gathered}\mathfrak{T} = \frac{\sinh^2(\pi\bar{\Omega}\sigma)}{\cosh^2(\pi s)+\sinh^2(\pi\bar{\Omega}\sigma)}, \qquad \bar{\Omega} = \Omega - \frac{\mathcal{B}}{\beta_b''}, \\ s = \frac{1}{2}\sqrt{16\frac{|\beta''(\omega_a+\nu)|}{\beta_b''}\frac{\omega_b}{\omega_a+\nu}\frac{n_{2b}}{n_{2a}}-1},\end{gathered} \tag{36}$$

where $\mathcal{B}$ is given by Eq. (25). Soliton parameters are yielded by the following ODEs, one for the frequency offset

$$\frac{\sigma}{\omega_a+\nu}\frac{d\nu}{dz} = \frac{4\mu\mathfrak{T}}{\omega_a L_d}\int_0^1 F(\mathfrak{a},\mathfrak{b},\mathfrak{c},\zeta)(2\zeta-1)d\zeta, \tag{37}$$

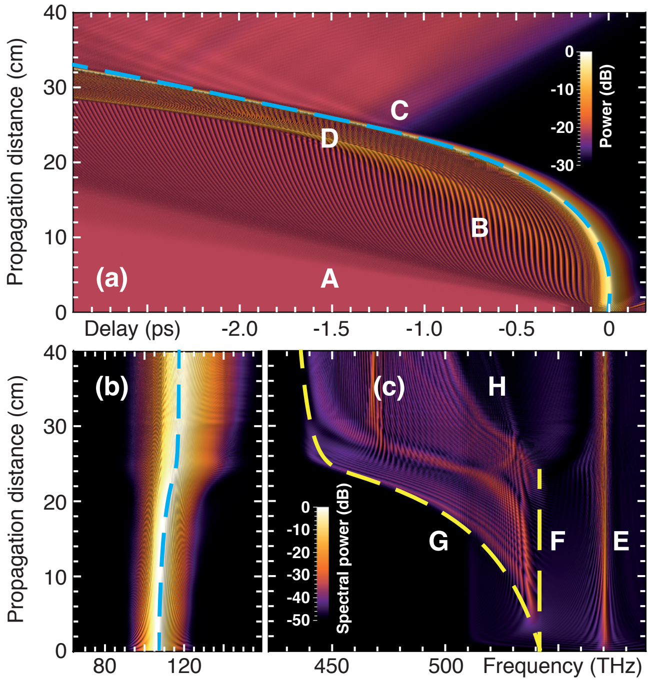

Figure 4: (a) space-time representation of the scattered DW and accelerated soliton. (b,c) spectral representation of the soliton and the DWs respectively. Density plots of power (a) and spectral power (b,c) are yielded by the full GNLSE. The imposed dashed lines are derived from the adiabatic ODEs. They indicate: soliton trajectory in (a), soliton carrier frequency in (b), and interval of frequencies for the scattered DWs in (c). See text for parameters and explanations of the patterns A–H.

and one for the duration

$$\frac{d\mathcal{T}}{dz} = \mathcal{B} + \frac{4\mu\mathfrak{T}}{\omega_a L_d} \int_0^1 F(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \zeta)[1 - (2\zeta - 1)\,\mathrm{artanh}(2\zeta - 1)]d\zeta. \tag{38}$$

The parameters of the hypergeometric function F are given by

$$\mathfrak{a}, \mathfrak{b} = \frac{1}{2} - i\bar{\Omega}\sigma \pm is, \qquad \mathfrak{c} = 1 - i\bar{\Omega}\sigma,$$

the quantity $L_d = \sigma_0^2/|\beta_a''|$ denotes dispersion length for the initial soliton, and the dimensionless parameter μ quantifies intensity of the DW, see Eq. (35). The initial conditions are specified in Eq. (20). Equations (37) and (38) are referred to as adiabatic ODEs in what follows.

Although the adiabatic ODEs seem to be complicated, they are solved within a few seconds and provide useful information on soliton behavior after the integrals have been tabulated. For instance, consider a low-amplitude DW with $\mu = 0.02$ that is scattered by a fundamental soliton with $\sigma_0 = 40\,\text{fs}$. The initial combination of carrier frequencies is given by Eq. (4), group delay and GVD are shown in Fig. 2.

Figure 4 compares predictions of Eq. (37) and (38) to a numerical solution of the full GNLSE in which n_2 is approximated by a constant [26]. Initially we superimpose the soliton and the incoming DW. The correct combination of the reflected and the transmitted DWs is then self-organized

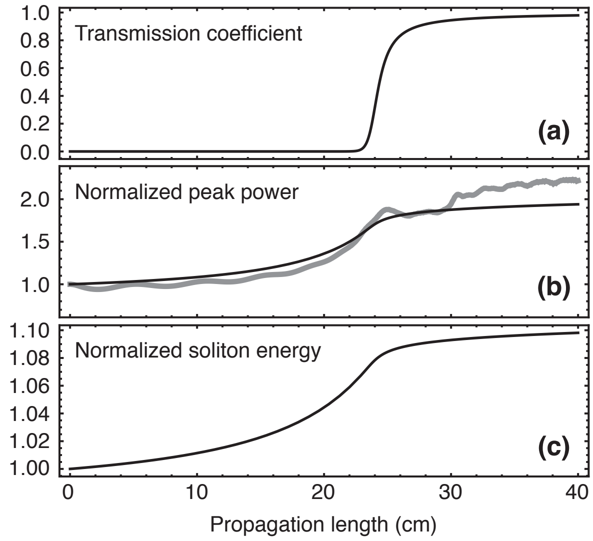

Figure 5: Evolution of the soliton parameters along the fiber is shown for the calculation in Fig. 4. The thin lines result from the adiabatic ODEs, the thick line is from the numerical solution of the full GNLSE. Soliton energy and peak power are normalized by their initial values.

as the system evolves. The soliton trajectory is shown in Fig. 4(a), it is well reproduced by the adiabatic ODEs (dashed line). Moreover, we get insight into all patterns observed in Fig. 4(a). (A) is the yet unperturbed DW, (B) shows interference of the incoming and the scattered DWs, (C) is a point at which the compressed soliton becomes transparent for the DWs and to a reasonable approximation does not change anymore. The pattern (D) is explained later on in Fig. 6.

Soliton carrier frequency is shown in Fig. 4(b), it is also well reproduced by the adiabatic equations (dashed line). Note that z in Eq. (37) can be rescaled to absorb μ. Therefore a weaker DW finally yields the same soliton frequency shift and the same peak power, but requires a longer propagation length. A very weak DW may yield a champion soliton that seems to appear from nowhere.

Spectral window for the DWs is shown in Fig. 4(c) and has a complex structure. Line (E) represents carrier frequency of the incoming DW. Line (F) indicates the DW reflected by the soliton that is initially motionless in the reference frame, as results from Eq. (2). This frequency was calculated in [49, 19, 41] and measured in [19, 45]. In our setting the soliton is all but motionless: the frequency of the reflected DW quickly changes as the soliton moves along the trajectory shown in Fig. 4(a). The adiabatic ODEs and Eq. (34) provide line (G). The latter accurately quantifies the range of DW frequencies that are observed in numerics. The depletion zone (H) is explained later on in Fig. 6.

An easy access to frequencies and soliton trajectory does not exhaust the list of benefits of the adiabatic ODEs. One can now follow all soliton parameters. For instance, evolution of the transmission coefficient, soliton energy, and peak power versus propagation length is shown in Fig. 5.

The transmission coefficient for a soliton that is motionless in the reference frame was estimated in Ref. [9] using two coupled NLSEs. Equation (36) provides a better insight on $\mathfrak{T}|_{z=0}$ because it

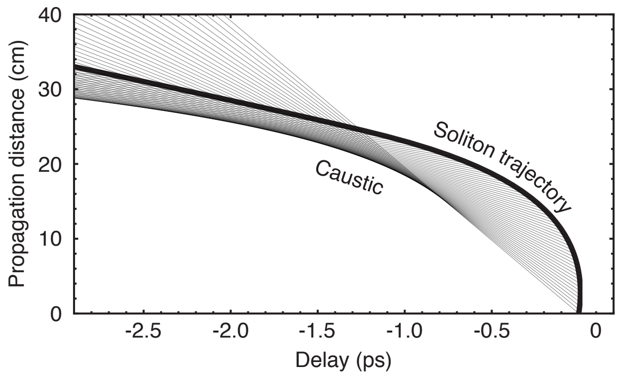

Figure 6: Soliton trajectory (thick line) and geometric rays for the reflected DWs (thin lines) are plotted for the calculation in Fig. 4. The caustic structure corresponds to that in Fig. 4(a). Note that some rays can intersect the soliton trajectory once again, DWs then experience their second scattering.

accounts for the self-steepening and HOD. Moreover, the adiabatic ODEs quantify the increase in the transmission coefficient with z as shown in Fig. 5(a). For the case at hand, scattering of the DW is switched off at $z \approx 25$ cm in accordance with Fig. 4(a).

Figure 5(b) shows increase in the soliton peak power as derived from the adiabatic ODEs (thin line) and from the numerical solution of the full GNLSE (thick line). The agreement is not as good as that for the frequencies and delay. This is to be expected since, e.g., oscillations of the soliton amplitude [the wavy thick line Fig. 4(b)] cannot be properly captured by the soliton perturbation theory [31]. Nevertheless, the adiabatic ODEs yield the correct trend and provide a reasonable estimate of the final peak power. Note that a two-fold increase in the peak power is accompanied by only 10% increase in the soliton energy, as shown in Fig. 4(c). This is a manifestation of the fact that soliton amplification occurs via a relatively small carrier frequency shift that is imposed on a very steep GVD profile.

To conclude this section we explain how our adiabatic model implies a more complete interpretation of the numerical results. Namely, the adiabatic ODEs provide the soliton trajectory $\mathcal{T}(z)$ and, moreover, Eq. (34) provides the frequency of the reflected DW at each point of this trajectory. It is then possible to plot rays for the reflected DWs in the sense of geometrical optics (Fig. 6). Pattern (D) in Fig. 4(a) is immediately recognized as a caustic created by the reflected DWs. Another observation is that the accelerated soliton catches up with the already reflected DWs, which are then scattered once again. That is why the depletion zone (H) in Fig. 4(c) appears. The second scattering is beyond the scope of this study, what explains the underestimation of the final soliton peak power by the adiabatic ODEs as in Fig. 5(b).

5 Conclusions

In conclusion, not only an inhomogeneity can scatter an imposed light field, but also the scattered light can efficiently modify the inhomogeneity. A dramatic example is given by an optical soliton that suddenly changes its amplitude because of interaction with a small DW packet. We quantify the change in soliton parameters by taking advantage of the soliton perturbation theory and QM scattering theory.

The main challenge is that the standard soliton perturbation theory does not capture soliton compression for the case at hand. To overcome the difficulty one should either account for the higher-order perturbations or develop a modified propagation equation in which all coefficients vary with the soliton carrier frequency. Using the latter approach we derived a set of ODEs that yield slow changes in soliton duration, delay, and frequency due to the interaction with a low-amplitude DW.

Predictions of the adiabatic ODEs (37) and (38) are found to be reasonably close to the solution of the full propagation equation. Search of parameters that yield a desired soliton behavior is then greatly simplified by using ODEs instead of GNLSE. Moreover, effective soliton manipulation requires careful tuning of the initial conditions, as summarized in Section 2 by empirical rules derived by trial and error. The adiabatic approach addresses this problem. For instance, the suitable DWs are confined by a fairly simple inequality (32).

Finally, adiabatic ODEs capture system features that are particularly difficult to address with the full GNLSE, such as geometric rays for the scattered DWs. An impressive example is given by the caustic structure that was first derived from the adiabatic ODEs in Fig. 6 and then recognized in Fig. 4(a). Another such feature is disappearance of the main scattered DW (F) in Fig. 4(c). The reason is that the involved DW experiences multiple scattering at the soliton.

A Soliton perturbation theory

Soliton perturbation theory is used to study localized solutions of a "slightly" modified NLSE or any other equation with stable solitary solutions that slowly change because of perturbations. To keep it simple, we avoid approaches that make use of NLSE integrability [29] or require full analysis of soliton perturbations [37]. Instead, we follow a simple scheme that utilizes Lagrangian structure of the NLSE [24].

We consider the general equation

$$\frac{\delta}{\delta \Psi^*} \int_{-\infty}^{\infty} \mathcal{L}\, dx dt = \mathcal{R}, \tag{39}$$

where both the Lagrangian density $\mathcal{L}$ and the perturbation $\mathcal{R}$ depend on a (complex) field $\Psi(x,t)$ and its space and time derivatives. Furthermore, we consider a trial function $\Psi = \Psi_C(x)$ with a free real parameter C. Informally speaking, $\Psi_C(x)$ is a good initial guess for the solitary solution of Eq. (39). The true soliton is then approximated by introducing a suitable $C = C(t)$. To this end, $\Psi_C(x)$ is inserted into the expression for action

$$\int_{-\infty}^{\infty} \mathcal{L}|_{\Psi=\Psi_C}\, dx dt = \int_{-\infty}^{\infty} L(C, \dot{C})\, dt, \tag{40}$$

where the new Lagrangian $L(C, \dot{C})$ is obtained by integration over space and $\dot{C}$ means dC/dt.

When $\mathcal{R}$ in Eq. (39) vanishes, $C(t)$ is derived from the standard Lagrangian equation

$$\frac{\partial L}{\partial C}-\frac{d}{dt}\frac{\partial L}{\partial \dot{C}}=0, \tag{41}$$

which ensures extremum of action on a sub-space covered by the trial functions. The latter equation is trivially generalized for trial functions that depend on several parameters. In the best case, $\Psi_C(x)$ with $C(t)$ derived from Eq. (41) is an exact solitary solution of the unperturbed Eq. (39).

For a small but non-vanishing $\mathcal{R}$ in Eq. (39), Eq. (41) for $C(t)$ should be revised. To this end we equate two expressions for the variation of action: the full one that follows from Eq. (39) and the approximate one yielded by the trial functions

$$\int_{-\infty}^{\infty}(\mathcal{R}\delta\Psi^*+\text{c.c.})dxdt=\int_{-\infty}^{\infty}\left(\frac{\partial L}{\partial C}-\frac{d}{dt}\frac{\partial L}{\partial \dot{C}}\right)\delta C dt.$$

Here again we require that the latter equation is valid at least on a sub-space covered by the trial functions, i.e., for $\Psi=\Psi_C$ and $\delta\Psi=(\partial\Psi_C/\partial C)\delta C$ with the revised, yet unknown $C(t)$ and an arbitrary variation δC. The sufficient condition is given by the equation [24]

$$\frac{\partial L}{\partial C}-\frac{d}{dt}\frac{\partial L}{\partial \dot{C}}=\int_{-\infty}^{\infty}\left(\mathcal{R}|_{\Psi=\Psi_C}\frac{\partial\Psi_C^*}{\partial C}+\text{c.c.}\right)dx, \tag{42}$$

which generalizes Eq. (41) and yields the revised $C(t)$.

Equation (42) is the main result of the soliton perturbation theory for Eq. (39). It is trivially generalized for the case of several parameters and quantifies influence of a small perturbation on natural evolution of soliton phase, position, and amplitude. Note that Eq. (42) does not tell the accuracy of the approximation, but simply provides a reasonable choice of $C(t)$.

In optical applications $\Psi(x,t)$ is replaced by the envelope $\psi(z,\tau)$ with the evolution variable z and delay "coordinate" τ. We introduce the abbreviation

$$(\psi_1,\psi_2)=\frac{1}{2}\int_{-\infty}^{\infty}(\psi_1\psi_2^*+\psi_1^*\psi_2)d\tau, \tag{43}$$

for the scalar product that appears in soliton perturbation theory. The shape of the trial function is suggested by the fact that the unperturbed NLSE

$$i\partial_z\psi+iS_0\partial_\tau\psi+\frac{\Delta_0}{2}\partial_\tau^2\psi+\gamma_0|\psi|^2\psi=0,$$

with constant coefficients S_0, Δ_0, and γ_0 has an exact solitary solution of the form

$$\psi=\sqrt{\frac{\Delta_0}{\gamma_0}}\frac{e^{-i\mathcal{W}(\tau-\mathcal{T})+i\Theta}}{\sigma\cosh\frac{\tau-\mathcal{T}}{\sigma}},\qquad \Delta_0\gamma_0>0, \tag{44}$$

with the parameters: soliton duration $\sigma(z)$, phase $\Theta(z)$, frequency shift $\mathcal{W}(z)$, and delay $\mathcal{T}(z)$ being yielded by the following ODEs

$$\begin{aligned}\frac{d\sigma}{dz}&=0, & \frac{d\Theta}{dz}&=\frac{\Delta_0}{2}(\sigma^{-2}+\mathcal{W}^2),\\ \frac{d\mathcal{W}}{dz}&=0, & \frac{d\mathcal{T}}{dz}&=S_0-\Delta_0\mathcal{W}.\end{aligned} \tag{45}$$

Now we consider a perturbed NLSE with the varying coefficients

$$i\partial_z\psi + iS_z\partial_\tau\psi + \frac{\Delta_z}{2}\partial_\tau^2\psi + \gamma_z|\psi|^2\psi = R[\psi], \tag{46}$$

where S_z, Δ_z, and γ_z slowly depend on z. The perturbation term, which appears on the right-hand-side, may depend on both ψ and ψ^*, their derivatives, etc. Equation (46) is combined with the solitary trial function motivated by Eq. (44)

$$\psi_s = \sqrt{\frac{\Delta_z}{\gamma_z}}\frac{e^{-i\mathcal{W}(\tau-\mathcal{T})+i\Theta}}{\sigma\cosh\frac{\tau-\mathcal{T}}{\sigma}}, \tag{47}$$

where the revised ODEs for the soliton parameters are of interest. To derive them we reformulate Eq. (46) in the spirit of Eq. (39)

$$\frac{\delta}{\delta\psi^*}\int_{-\infty}^{\infty}\mathcal{L}\,dzd\tau = R[\psi],$$

where the Lagrangian density reads

$$\mathcal{L} = \frac{i}{2}(\psi^*\partial_z\psi - \psi\partial_z\psi^*) + \frac{iS_z}{2}(\psi^*\partial_\tau\psi - \psi\partial_\tau\psi^*) - \frac{\Delta_z}{2}|\partial_\tau\psi|^2 + \frac{\gamma_z}{2}|\psi|^4.$$

By analogy with Eq. (40), we replace ψ by ψ_s, integrate the above $\mathcal{L}$ over $d\tau$, and calculate the Lagrangian

$$L = \frac{2\Delta_z}{\gamma_z}\left[-\frac{1}{\sigma}\frac{d\Theta}{dz} - \frac{\mathcal{W}}{\sigma}\frac{d\mathcal{T}}{dz} + \frac{S_z\mathcal{W}}{\sigma} - \frac{\Delta_z\mathcal{W}^2}{2\sigma} + \frac{\Delta_z}{6\sigma^3}\right].$$

ODEs for the soliton parameters are derived by analogy with Eq. (42) and can be reduced to the form

$$\begin{gathered}
\frac{d}{dz}\frac{\Delta_z}{\gamma_z\sigma} = (R[\psi_s], \partial_\Theta\psi_s), \qquad \frac{d}{dz}\frac{\Delta_z\mathcal{W}}{\gamma_z\sigma} = (R[\psi_s], \partial_\mathcal{T}\psi_s),\\
\frac{d\mathcal{T}}{dz} - S_z + \Delta_z\mathcal{W} = -\frac{\gamma_z\sigma}{\Delta_z}(R[\psi_s], \partial_\mathcal{W}\psi_s),\\
\frac{d\Theta}{dz} - \frac{\Delta_z}{2}(\sigma^{-2} + \mathcal{W}^2) = \frac{\gamma_z\sigma}{\Delta_z}(R[\psi_s], [\sigma\partial_\sigma + \mathcal{W}\partial_\mathcal{W}]\psi_s),
\end{gathered} \tag{48}$$

where the scalar product is defined by Eq. (43).

The previous ODEs (45) are a special case of the system just derived. For constant coefficients in the NLSE (46), the ODEs (48) reduce to the standard equations of the soliton perturbation theory [24]. The generalization for variable coefficients naturally applies to dispersion managed fibers and to the GNLSE with the variable carrier frequency that is used in the present paper.

To conclude this Appendix we note that the phase equation is usually decoupled from the others and can be ignored when $|\psi_s|^2$ is the only quantity of interest. Using Eq. (47) we simplify derivatives of ψ_s with respect to soliton parameters and write the relevant ODEs (48) as follows

$$\begin{gathered}
\frac{d}{dz}\frac{\Delta_z}{\gamma_z\sigma} = (R[\psi_s], i\psi_s),\\
\frac{\Delta_z}{\gamma_z\sigma}\frac{d\mathcal{W}}{dz} = -(R[\psi_s], [\partial_\tau + i\mathcal{W}]\psi_s),\\
\frac{d\mathcal{T}}{dz} - S_z + \Delta_z\mathcal{W} = \frac{\gamma_z\sigma}{\Delta_z}(R[\psi_s], i[\tau - \mathcal{T}]\psi_s).
\end{gathered} \tag{49}$$

The latter equations together with the definition (43) of the scalar product and (47) of ψ_s yield the main result of this Appendix for the soliton-like solution of Eq. (46).

We are especially interested in a situation when the soliton frequency shift is already absorbed by a properly chosen propagation equation and one can set $\mathcal{W}(z) \equiv 0$. The trial function ψ_s from Eq. (47) is then replaced by

$$\phi = \psi_s|_{\mathcal{W}=0} = \sqrt{\frac{\Delta_z}{\gamma_z}} \frac{e^{i\Theta}}{\sigma \cosh \frac{\tau - \mathcal{T}}{\sigma}}, \tag{50}$$

and ODEs (49) are simplified to the form

$$\begin{aligned} \frac{d}{dz} \frac{\Delta_z}{\gamma_z \sigma} &= (R[\phi], i\phi), \qquad (R[\phi], \partial_\tau \phi) = 0, \\ \frac{d\mathcal{T}}{dz} - S_z &= \frac{\gamma_z \sigma}{\Delta_z} (R[\phi], i[\tau - \mathcal{T}]\phi). \end{aligned} \tag{51}$$

To apply the ODEs (51) to Eq. (13) one should set

$$S_z = \beta'(\omega_a + \nu) - \beta'_a, \qquad \Delta_z = -\beta''(\omega_a + \nu), \qquad \gamma_z = \frac{\omega_a + \nu}{c} n_{2a}. \tag{52}$$

B Scattering at a moving soliton

QM scattering theory, as given in standard textbooks [32], applies to a stationary potential well, while we deal with a moving barrier. Let us consider the one-particle Schrödinger equation for the wave function $\Psi(x,t)$ in one spatial dimension taking for brevity $\hbar = 1$

$$i\partial_t \Psi = -\frac{1}{2m} \partial_x^2 \Psi + U(x - ut)\Psi, \tag{53}$$

where we allow for a displacement of the soliton-like potential $U(x)$ with constant velocity u. The forward (backward) wave solutions

$$\Psi = e^{i(\pm kx - \omega_k t)} \quad \text{with} \quad \omega_k = \frac{k^2}{2m}, \tag{54}$$

are suitable for Eq. (53) well away from the soliton. We are interested in scattering of the forward wave.

Specifically, the solution we are looking for is parametrized by the wave-number k and has the following asymptotic properties

$$\begin{aligned} \Psi_k(x,t)|_{x\to-\infty} &= e^{i(kx - \omega_k t)} + B(k) e^{i(-\underline{k}x - \omega_{\underline{k}} t)}, \\ \Psi_k(x,t)|_{x\to+\infty} &= A(k) e^{i(kx - \omega_k t)}, \end{aligned} \tag{55}$$

where $A(k)$ and $B(k)$ quantify transmission and reflection respectively, and the new wave-vector

$$\underline{k} = k - 2mu \tag{56}$$

describes change of momentum due to reflection by a moving barrier. Equation (56) can be derived similarly as Eq. (1). To solve the scattering problem we perform the following change of variables

$$\Psi_{k+mu}(x,t) = \Phi(\xi,t)e^{i(mux-\omega_{mu}t)}, \qquad \xi = x - ut.$$

In other words, we perform a suitable Galilei transform to obtain a stationary barrier. Indeed, Eq. (53) changes to a standard Schrödinger equation for $\Phi(\xi,t)$

$$i\partial_t\Phi = -\frac{1}{2m}\partial_\xi^2\Phi + U(\xi)\Phi, \tag{57}$$

with the familiar behavior for $\xi \to \pm\infty$

$$\begin{aligned}\Phi(\xi,t)|_{\xi\to-\infty} &= e^{i(k\xi-\omega_k t)} + b(k)e^{i(-k\xi-\omega_k t)},\\ \Phi(\xi,t)|_{\xi\to+\infty} &= a(k)e^{i(k\xi-\omega_k t)},\end{aligned}$$

where

$$a(k) = A(k+mu), \qquad b(k) = B(k+mu).$$

In the next step we take

$$\Phi(\xi,t) = \varphi(\xi)e^{-i\omega_k t},$$

and obtain the differential equation

$$\frac{d^2\varphi}{d\xi^2} + [k^2 - 2mU(\xi)]\varphi = 0, \tag{58}$$

with the asymtotics

$$\varphi(\xi) = \begin{cases} e^{ik\xi} + b(k)e^{-ik\xi}, & \xi \to -\infty,\\ a(k)e^{ik\xi}, & \xi \to +\infty,\end{cases} \tag{59}$$

which is a standard formulation of the scattering problem in QM.

Specifically, we use the following soliton-like potential barrier

$$U(\xi) = \frac{U_0}{\cosh^2(\xi/\ell)}, \tag{60}$$

where U_0 and ℓ are positive constants. The solution of the problem (58)–(60) is given by [32]

$$\varphi(\xi) = a(k)\left(e^{\xi/\ell} + e^{-\xi/\ell}\right)^{ik\ell} F\left(\frac{1}{2} - ik\ell + is, \frac{1}{2} - ik\ell - is, 1 - ik\ell, \tilde{\xi}\right),$$

where

$$\tilde{\xi} = \frac{1-\tanh(\xi/\ell)}{2}, \qquad s = \frac{1}{2}\sqrt{8m\ell^2U_0 - 1},$$
$$a(k) = \frac{\Gamma\left(\frac{1}{2} - ik\ell + is\right)\Gamma\left(\frac{1}{2} - ik\ell - is\right)}{\Gamma(1-ik\ell)\Gamma(-ik\ell)}.$$

Ψ	t	x	$x-ut$	m^{-1}	ℓ	u	k	ω
ψ_b^*	z	τ	$\tau-\mathcal{T}$	β_b''	σ	$d\mathcal{T}/dz$	Ω	κ

Table 1: Replacement rules that transform the QM scattering problem to the optical scattering problem.

Γ and F refer to the gamma and hypergeometric functions respectively, the expression

$$|a(k)|^2 = \frac{\sinh^2(\pi k\ell)}{\cosh^2(\pi s) + \sinh^2(\pi k\ell)},$$

quantifies the transmission coefficient. Now we return to the original $\Psi_k(x,t)$ and obtain

$$|\Psi_k(x,t)|^2 = \frac{\sinh^2(\pi\bar{k}\ell)}{\cosh^2(\pi s) + \sinh^2(\pi\bar{k}\ell)} \times \left| F\left(\frac{1}{2} - i\bar{k}\ell + is, \frac{1}{2} - i\bar{k}\ell - is, 1 - i\bar{k}\ell, \tilde{x}\right)\right|^2, \quad (61)$$

where the first factor yields $|A(k)|^2$ and

$$\tilde{x} = \frac{1 - \tanh[(x-ut)/\ell]}{2}, \qquad \bar{k} = k - mu.$$

The solution (61) of the QM scattering problem is transformed to optical notations using Table 1.

References

[1] G. P. Agrawal. *Nonlinear Fiber Optics*. Academic, New York, 4 edition, 2007.

[2] N. Akhmediev, A. Ankiewicz, and M. Taki. Waves that appear from nowhere and disappear without a trace. *Phys. Lett. A*, 373(2):675–678, February 2009.

[3] N. Akhmediev and M. Karlsson. Cherenkov radiation emitted by solitons in optical fibers. *Phys. Rev. A*, 51(3):2602–2607, 1995.

[4] N. N. Akhmediev and V. I. Korneev. Modulation instability and periodic solutions of the nonlinear Schrödinger equation. *Theor. Math. Phys.*, 69(2):1089–1093, 1986.

[5] T. Balciunas, C. Fourcade-Dutin, G. Fan, T. Witting, A. A. Voronin, A. M. Zheltikov, F. Gerome, G. G. Paulus, A. Baltuska, and F. Benabid. A strong-field driver in the single-cycle regime based on self-compression in a kagome fibre. *Nature Comm.*, 6(6117), Jan 2015.

[6] B. M. Bolotovskii and S. N. Stolyarov. Reflection of light from a moving mirror and related problems. *Sov. Phys. Usp.*, 32(9):813–827, September 1989.

[7] M. Born and E. Wolf. *Principles of Optics*. Cambridge University Press, 7 edition, 1999.

[8] R. W. Boyd. *Nonlinear Optics*. Academic, New York, 3 edition, 2008.

[9] A. Chabchoub and F. König. Efficient frequency shifting of dispersive waves at solitons. *Opt. Express*, 20(5):5538–5546, Feb 2012.

[10] A. Demircan, Sh. Amiranashvili, C. Brée, Ch. Mahnke, F. Mitschke, and G. Steinmeyer. Rogue events in the group velocity horizon. *Sci. Rep.*, 2(850):850, November 2012.

[11] A. Demircan, Sh. Amiranashvili, C. Brée, Ch. Mahnke, F. Mitschke, and G. Steinmeyer. Rogue wave formation by accelerated solitons at an optical event horizon. *Appl. Phys. B*, 115(3):343–354, August 2014.

[12] A. Demircan, Sh. Amiranashvili, C. Brée, U. Morgner, and G. Steinmeyer. Adjustable pulse compression scheme for generation of few-cycle pulses in the midinfrared. *Opt. Lett.*, 39(9):2735–2738, May 2014.

[13] A. Demircan, Sh. Amiranashvili, C. Brée, U. Morgner, and G. Steinmeyer. Supercontinuum generation by multiple scatterings at a group velocity horizon. *Opt. Express*, 22(4):3866–3879, February 2014.

[14] A. Demircan, Sh. Amiranashvili, C. Brée, and G. Steinmeyer. Compressible octave spanning supercontinuum generation by two-pulse collisions. *Phys. Rev. Lett.*, 110(23):233901, June 2013.

[15] A. Demircan, Sh. Amiranashvili, and G. Steinmeyer. Controlling light by light with an optical event horizon. *Phys. Rev. Lett.*, 106(16):163901, April 2011.

[16] J. M. Dudley, G. Genty, and S. Coen. Supercontinuum generation in photonic crystal fiber. *Rev. Mod. Phys.*, 78(4):1135–1184, 2006.

[17] J. M. Dudley, G. Genty, and B. J. Eggleton. Harnessing and control of optical rogue waves in supercontinuum generation. *Opt. Express*, 16(6):3644–3651, March 2008.

[18] A. I. D'yachenko, V. E. Zakharov, A. N. Pushkarev, V. F. Shvets, and V. V. Yan'kov. Soliton turbulence in nonintegrable wave systems. *Sov. Phys. JETP*, 69(6):1144–1147, December 1989.

[19] A. Efimov, A. V. Yulin, D. V. Skryabin, J. C. Knight, N. Joly, F. G. Omenetto, A. J. Taylor, and P. Russell. Interaction of an optical soliton with a dispersive wave. *Phys. Rev. Lett.*, 95(21):213902, 2005.

[20] M. Erkintalo, G. Genty, and J. M. Dudley. Giant dispersive wave generation through soliton collision. *Opt. Lett.*, 35(5):658–660, 2010.

[21] A. V. Gorbach and D. V. Skryabin. Bouncing of a dispersive pulse on an accelerating soliton and stepwise frequency conversion in optical fibers. *Opt. Express*, 15(22):14560, Oct 2007.

[22] A. V. Gorbach and D. V. Skryabin. Light trapping in gravity-like potentials and expansion of supercontinuum spectra in photonic-crystal fibres. *Nature Photonics*, 1(11):653–657, 2007.

[23] A. V. Gorbach and D. V. Skryabin. Theory of radiation trapping by the accelerating solitons in optical fibers. *Phys. Rev. A*, 76(5):053803, 2007.

[24] A. Hasegawa. *Optical Solitons in Fibers*. Springer, 1980.

[25] H. A. Haus and E. P. Ippen. Group velocity of solitons. *Opt. Lett.*, 26(21):1654–1656, 2001.

[26] A. Husakou. *Nonlinear phenomena of ultrabroadband radiation in photonic crystal fibers and hollow waveguides*. PhD thesis, Physical Dept., Freie Univ. Berlin, Berlin, Germany, 2002.

[27] A. V. Husakou and J. Herrmann. Supercontinuum generation of higher-order solitons by fission in photonic crystal fibers. *Phys. Rev. Lett.*, 87(20):203901, 2001.

[28] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev, and J. M. Dudley. The peregrine soliton in nonlinear fibre optics. *Nature Physics*, 6(10):790–795, 2010.

[29] Y. S. Kivshar and B. A. Malomed. Dynamics of solitons in nearly integrable systems. *Rev. Mod. Phys.*, 61(4):763–915, October 1989.

[30] Y. Kodama. Optical solitons in a monomode fiber. *J. Statistical Phys.*, 39(5-6):597, 1985.

[31] E. A. Kuznetsov, A. V. Mikhailov, and I. A. Shimokhin. Nonlinear interaction of solitons and radiation. *Physica D: Nonlinear Phenomena*, 87(1-4):201–215, October 1995.

[32] L. D. Landau and E. M. Lifshitz. *Quantum Mechanics*. Pergamon, 2 edition, 1965.

[33] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*. Pergamon, 4 edition, 1975.

[34] Asher Peres. *Quantum Theory: Concepts and Methods*, volume 72 of *Fundamental Theories of Physics*. Kluwer, 2002.

[35] T. G. Philbin, C. Kuklewicz, S. Robertson, S. Hill, F. Konig, and U. Leonhardt. Fiber-optical analog of the event horizon. *Science*, 319(5868):1367–1370, 2008.

[36] A. Picozzi, J. Garnier, T. Hansson, P. Suret, S. Randoux, G. Millot, and D. N. Christodoulides. Optical wave turbulence: Towards a unified nonequilibrium thermodynamic formulation of statistical nonlinear optics. *Phys. Rep.*, 542(1):1–132, September 2014.

[37] H.-G. Purwins, H. U. Bödeker, and Sh. Amiranashvili. Dissipative solitons. *Adv. Phys.*, 59(5):485–701, September 2010.

[38] N. N. Rosanov. Transformation of electromagnetic radiation at moving inhomogeneities of a medium. *JETP Letters*, 88(8):501–504, December 2009.

[39] N. N. Rosanov, N. V. Vysotina, and A. N. Shatsev. Forward light reflection from a moving inhomogeneity. *JETP Lett.*, 93(6):308–312, 2011.

[40] E. Rubino, A. Lotti, F. Belgiorno, S. L. Cacciatori, A. Couairon, U. Leonhardt, and D. Faccio. Soliton-induced relativistic-scattering and amplification. *Sci. Rep.*, 2(932):932, December 2012.

[41] D. Skryabin and A. V. Yulin. Theory of generation of new frequencies by mixing of solitons and dispersive waves in optical fibers. *Phys. Rev. E*, 72(1):016619, 2005.

[42] D. V. Skryabin and A. V. Gorbach. Colloquium: Looking at a soliton through the prism of optical supercontinuum. *Rev. Mod. Phys.*, 82(2):1287–1299, 2010.

[43] D. R. Solli, C. Ropers, P. Koonath, and B. Jalali. Optical rogue waves. *Nature*, 450(13):1054–1057, 2007.

[44] P. K. A. Wai, H. H. Chen, and Y. C. Lee. Radiations by "solitons" at the zero group-dispersion wavelength of single-mode optical fibers. *Phys. Rev. A*, 41(1):426–439, 1990.

[45] K. E. Webb, M. Erkintalo, Y. Xu, N. G. R. Broderick, J. M. Dudley, G. Genty, and S. G. Murdoch. Nonlinear optics of fibre event horizons. *Nature Communications*, 5(4969), September 2014.

[46] G. B. Whitham. *Linear and nonlinear waves*. John Wiley & Sons, 1974.

[47] J. Yang. *Nonlinear Waves in Integrable and Nonintegrable Systems*. SIAM, 2010.

[48] A. V. Yulin, R. Driben, B. A. Malomed, and D. V. Skryabin. Soliton interaction mediated by cascaded four wave mixing with dispersive waves. *Opt. Express*, 21(12):14481–14486, Jun 2013.

[49] A. V. Yulin, D. V. Skryabin, and P. St. J. Russell. Four-wave mixing of linear waves and solitons in fibers with higher-order dispersion. *Opt. Lett.*, 29(20):2411–2413, Oct 2004.

[50] V. E. Zakharov, A. Pushkarev, V. F. Shvets, and V. V. Yan'kov. Soliton turbulence. *JETP Lett.*, 48(2):79–82, July 1988.

[51] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP*, 34(1):62–69, 1972.