Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Construction of generalized pendulum equations with prescribed maximum number of limit cycles of the second kind

Klaus R. Schneider[1], Alexander Grin[2]

submitted: June 9, 2016

[1] Weierstraß-Institut
Mohrenstraße 39
10117 Berlin, Germany
E-Mail: Klaus.Schneider@wias-berlin.de

[2] Yanka Kupala State University of Grodno,
Ozheshko Street 22
230023 Grodno, Belarus
E-Mail: grin@grsu.by

No. 2272
Berlin 2016

2010 *Mathematics Subject Classification.* 34C05, 34C23 .

Key words and phrases. generalized pendulum equation, limit cycles of the second kind, limit cycle of multiplicity three, bifurcation behavior, Dulac-Cherkas function.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. Consider a class of planar autonomous differential systems with cylindric phase space which represent generalized pendulum equations. We describe a method to construct such systems with prescribed maximum number of limit cycles which are not contractible to a point (limit cycles of the second kind). The underlying idea consists in employing Dulac-Cherkas functions. We also show how this approach can be used to control the bifurcation of multiple limit cycles.

1. INTRODUCTION

We consider on a cylinder $\mathcal{Z} := \{(\varphi, y) : \varphi \in [0, 2\pi], y \in \mathbb{R}\}$ the generalized pendulum system

$$\frac{d\varphi}{dt} = y, \quad \frac{dy}{dt} = \sum_{j=0}^{l} h_j(\varphi, \mu) y^j, \quad l \geq 3 \tag{1.1}$$

depending on the real parameter $\mu \in \mathcal{I}$, where $\mathcal{I}$ is some interval. We assume that the functions $h_j : \mathbb{R} \times \mathcal{I} \to \mathbb{R}$, $0 \leq j \leq l$, are continuous, and additionally continuously differentiable and 2π-periodic in the first variable. We denote by f_l the corresponding vector field

$$f_l = \Big(y, \sum_{j=0}^{l} h_j(\varphi, \mu) y^j\Big).$$

Moreover we suppose

$$h_l(\varphi, \mu) \neq 0 \quad \text{for} \quad (\varphi, \mu) \in [0, 2\pi] \times \mathcal{I}. \tag{1.2}$$

The most difficult problem in the qualitative investigation of system (1.1) is the localization and estimation of the number of limit cycles which represent isolated closed orbits of (1.1) with finite primitive period.

It is well known that we have to distinguish two kinds of limit cycles of (1.1) on $\mathcal{Z}$. A limit cycle Γ on $\mathcal{Z}$ is called a limit cycle of the first kind, if Γ is contractible to a point on $\mathcal{Z}$, Γ is called a limit cycle of the second kind if Γ surrounds the cylinder $\mathcal{Z}$, that is, it is not contractible to a point [1, 2].

For the investigation of limit cycles of the first kind, the well-known methods for planar autonomous systems can be applied. Especially, the existence of a limit cycle of the first kind of (1.1) on $\mathcal{Z}$ requires the existence of an equilibrium of (1.1) on $\mathcal{Z}$. In contrast to that fact, the existence of a limit cycle of the second kind on $\mathcal{Z}$ does not need the existence of any equilibrium. If we study the behavior of (1.1) in dependence on the parameter μ then it may happen that there are some critical values of the parameter μ related to a qualitative change of the phase portrait of (1.1). These values are called bifurcation points. For example, they may be related to the occurrence of new equilibria or limit cycles. An important but difficult problem is to determine the number of limit cycles in dependence on the parameter μ. A well-known tool to prove the absence of limit cycles is the Bendixson criterion which has been extended by H. Dulac (see, e.g. [10]). A further development is due to L. Cherkas who introduced a class of functions - nowadays also called Dulac-Cherkas functions - which can be used to estimate the number of limit cycles in planar systems and to determine their localization and stability [3, 7].

In the papers [4, 5, 6] the method of Dulac-Cherkas functions has been extended to the study of limit cycles of the second kind.

In this paper, our goal is to show how Dulac-Cherkas functions can be used to construct systems (1.1) having a prescribed number of limit cycles of the second kind and to control their bifurcations.

The paper is organized as follows. In section 2 we recall some basic facts about Dulac-Cherkas functions. Section 3 contains the description of our general idea. By using functions $\Psi(\varphi, y, \mu)$ which are quadratic polynomials in y as Dulac-Cherkas functions in Section 4 we construct for the case $l = 3$ systems (1.1) which have at most one or three limit cycles. To derive systems (1.1) with more limit cycles, in Section 5 we deal with the construction of Dulac-Cherkas functions $\Psi(\varphi, y, \mu)$ which are polynomials in y of degree four in the case $l = 5$.

2. Preliminaries

We consider on the cylinder $\mathcal{Z}$ the system

$$\frac{d\varphi}{dt} = P(\varphi, y, \mu), \quad \frac{dy}{dt} = Q(\varphi, y, \mu), \quad f := (P, Q) \tag{2.1}$$

under the assumption

(A_1). $P, Q : \mathcal{Z} \times \mathcal{I} \to \mathbb{R}$ are continuous, continuously differentiable in the first two variables and 2π-periodic in the first variable.

Let $\mathcal{D}$ be a subregion of $\mathcal{Z}$.

Definition 2.1. *Suppose hypothesis (A_1) to be valid. A function $B : \mathcal{Z} \times \mathcal{I} \to \mathbb{R}$ with the same smoothness as P, Q and having the properties*

(i). $B(\varphi, y, \mu) = B(\varphi + 2\pi, y, \mu) \quad \forall(\varphi, y, \mu) \in \mathcal{D} \times \mathcal{I}$,
(ii). $div(Bf) := \frac{\partial(BP)}{\partial\varphi} + \frac{\partial(BQ)}{\partial y} = (gradB, f) + Bdivf \geq 0 \ \ (\leq 0)$
in $\mathcal{D}$ for $\mu \in \mathcal{I}$, where $div(Bf)$ vanishes only on a subset of $\mathcal{D}$ of measure zero

is called a Dulac function of system (2.1) in $\mathcal{D}$ for $\mu \in \mathcal{I}$.

The following result can be found in [2].

Theorem 2.2. *Suppose hypothesis (A_1) to be valid. Let B be a Dulac function of (2.1) in the bounded region $\mathcal{D} \subset \mathcal{Z}$ for $\mu \in \mathcal{I}$. If the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ is connected and contractible to a point, then (2.1) has no limit cycle of the first kind in $\mathcal{D}$. In case that $\partial\mathcal{D}$ consists of two closed curves Δ_1 and Δ_2 on $\mathcal{Z}$ which do not meet and which are not contractible to a point, then (2.1) has no limit cycle of the first kind in $\mathcal{D}$ and at most one limit cycle of the second kind in $\mathcal{D}$.*

Now we give a generalization of a Dulac function which can be found in [6]. It is basically due to L. Cherkas [3], hence we call it Dulac-Cherkas function.

Definition 2.3. *Suppose hypothesis (A_1) to be valid. A function $\Psi : \mathcal{Z} \times \mathcal{I} \to \mathbb{R}$ having the same smoothness as P, Q and with the properties*

(i). $\Psi(\varphi, y, \mu) = \Psi(\varphi + 2\pi, y, \mu) \quad \forall(\varphi, y, \mu) \in \mathcal{D} \times \mathcal{I}$.

(ii). *For $\mu \in \mathcal{I}$ the set*

$$\mathcal{W}(\mu) := \{(\varphi, y) \in \mathcal{D} : \Psi(\varphi, y, \mu) = 0\}$$

has measure zero.

(iii). *There is a real number $\kappa \neq 0$ such that for $\mu \in \mathcal{I}$*

(2.2) $$\Phi(\varphi, y, \mu, \kappa) := (grad\Psi, f) + \kappa\Psi div f \geq 0 \ \ (\leq 0) \quad \text{in } \mathcal{D},$$

where the set

$$\mathcal{V}_\kappa(\mu) := \{(\varphi, y) \in \mathcal{D} : \Phi(\varphi, y, \mu, \kappa) = 0\}$$

has the properties

(a). *$\mathcal{V}_\kappa(\mu)$ has measure zero.*

(b). *If $\Gamma(\mu)$ is a limit cycle of (2.1), then it holds $\Gamma(\mu) \cap V_\kappa(\mu) \neq \Gamma(\mu)$.*

(iv).

(2.3) $$(grad\Psi, f)_{|\mathcal{W}(\mu)} \neq 0$$

is called a Dulac-Cherkas function of system (2.1) in $\mathcal{D}$ for $\mu \in \mathcal{I}$.

Remark 2.4. *If the inequalities in (2.2) hold strictly, then also condition (2.3) is valid.*

Remark 2.5. *If Φ does not depend on y, that is*

$$\Phi(\varphi, y, \mu, \kappa) = \Phi_0(\varphi, \mu, \kappa)$$

and if Φ_0 vanishes only in finitely many points $\varphi_i(\mu, \kappa)$, then the conditions on the set $\mathcal{V}_\kappa(\mu)$ are fulfilled.

Remark 2.6. *From (2.3) it follows that the curves belonging to the set $\mathcal{W}(\mu)$ are crossed transversally by the trajectories of system (2.1).*

In what follows we focus on limit cycles of the second kind. Therefore, we assume for what follows

(A_2). The boundary of the considered region $\mathcal{D}$ consists of two closed curves Δ_u (upper closed curve) and Δ_l (lower closed curve) surrounding the cylinder $\mathcal{Z}$ and satisfying $\Delta_u \cap \Delta_u = \emptyset$.

The following results have been proved in [6]. They show how the topological structure of $\mathcal{W}(\mu)$ influences the topological structure of the limit cycles of the second kind of system (2.1).

Theorem 2.7. *Suppose the hypotheses (A_1) and (A_2) to be valid. Let Ψ be a Dulac-Cherkas function of (2.1) in $\mathcal{D}$ for $\mu \in \mathcal{I}$. If the set $\mathcal{W}(\mu)$ is empty, then system (2.1) has at most one limit cycle of the second kind in $\mathcal{D}$.*

Now we assume

(A_3).The set $\mathcal{W}(\mu)$ consists in $\mathcal{D}$ of s isolated closed curves (ovals) $\mathcal{W}_1(\mu), \mathcal{W}_2(\mu), ..., \mathcal{W}_s(\mu)$ surrounding the cylinder $\mathcal{Z}$ and which do not touch the boundaries Δ_u and Δ_l.

Without loss of generality we may assume the following ordering of these curves: $\mathcal{W}_i(\mu)$ is located on $\mathcal{Z}$ above $\mathcal{W}_{i+1}(\mu)$. With this ordering we associate the following notation: The region on $\mathcal{Z}$ between $\mathcal{W}_i(\mu)$ and $\mathcal{W}_{i+1}(\mu)$ is denoted by $\mathbb{A}_i(\mu)$, the region between Δ_u and $\mathcal{W}_1(\mu)$ is denoted by $\mathbb{A}_0(\mu)$, the region between $\mathcal{W}_s(\mu)$ and Δ_l is denoted by $\mathbb{A}_s(\mu)$. Fig. 1 illustrates the case $s = 2$.

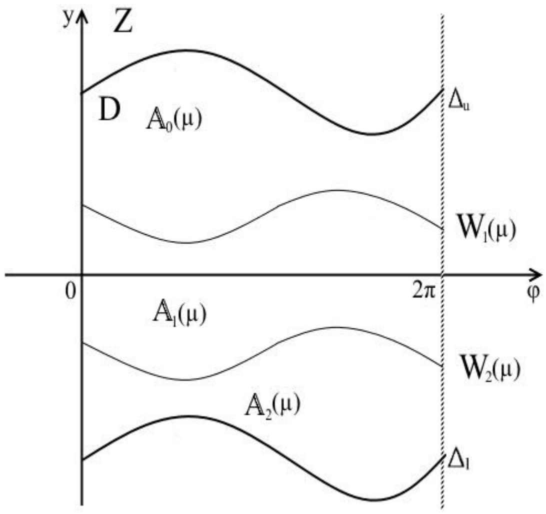

Fig.1. Regions $\mathbb{A}_i$ in the case $s = 2$.

Theorem 2.8. *Assume the hypotheses $(A_1) - (A_3)$ to be valid, and that $\mathcal{D}$ contains no equilibrium of (2.1). Then system (2.1) has at least $s-1$ but at most $s+1$ limit cycles of the second kind in $\mathcal{D}$, more precisely, the region $\mathbb{A}_i(\mu), i = 1, ..., s-1$, contains a unique limit cycle $\Gamma_i(\mu)$ of the second kind, each of the regions $\mathbb{A}_0(\mu)$ and $\mathbb{A}_s(\mu)$ may contain a unique limit cycle of the second kind. Furthermore, the limit cycle $\Gamma_i(\mu)$ in $\mathbb{A}_i(\mu)$ is hyperbolic and asymptotically stable (unstable) if*

$$\kappa\Phi(\varphi, y, \mu, \kappa)\Psi(\varphi, y, \mu) \; < \; 0 \; (> 0) \quad \textit{in} \quad \mathbb{A}_i(\mu).$$

3. General Idea

As we mentioned in the introduction, our goal is to show how Dulac-Cherkas functions can be used to construct systems (1.1) with a given number of limit cycles of the second kind and to control their bifurcations.

For this purpose we suppose that the given Dulac-Cherkas function has the structure

$$\Psi(\varphi, y, \mu) = \sum_{j=0}^{n} \Psi_j(\varphi, \mu) y^j, \quad n \geq 1, \tag{3.1}$$

where the functions $\Psi_j : \mathbb{R} \times \mathcal{I} \to \mathbb{R}, 0 \leq j \leq n$, are continuous and twice continuously differentiable and 2π-periodic in the first variable, and where

$$\Psi_n(\varphi, \mu) \neq 0 \quad \text{for} \quad (\varphi, \mu) \in [0, 2\pi] \times \mathcal{I}. \tag{3.2}$$

According to this assumption the corresponding function $\Phi(\varphi, y, \mu)$ determined by (2.2) has the structure

(3.3)
$$\Phi(\varphi, y, \mu, \kappa) = \sum_{i=0}^{m} \Phi_i(\varphi, \mu, \kappa) y^i, \quad m = n + l - 1.$$

To get an explicit representation of the coefficient functions Φ_i, we rewrite the natural number l in (1.1) in the form $l = 3 + s$. Then, using (1.1) and (3.1) we get

(3.4)
$$\begin{aligned} \Phi &\equiv (grad\Psi, f_l) + \kappa\Psi div f_l \\ &= A_0 + \kappa B_0 + \sum_{k=1}^{n+1} \Big(\frac{\partial \Psi_{k-1}}{\partial \varphi} + A_k + \kappa B_k\Big) y^k + \sum_{k=n+2}^{n+2+s} (A_k + \kappa B_k) y^k, \end{aligned}$$

where

(3.5)
$$A_k = \sum_{\substack{i+j=k, \\ 0 \le j \le n-1, \\ 0 \le i \le 3+s=l}} (j+1)\Psi_{j+1} h_i, \; B_k = \sum_{\substack{i+j=k, \\ 0 \le j \le n-1, \\ 0 \le i \le 3+s=l}} (i+1) h_{i+1} \Psi_j.$$

Under our assumptions $n \geq 1, l \geq 3$ we can represent the coefficients Φ_m, Φ_1, Φ_0 in the explicit form

(3.6)
$$\begin{aligned} \Phi_m &= (n + l\kappa) h_l(\varphi, \mu) \Psi_n(\varphi, \mu), \\ \Phi_1 &= \Psi_0'(\varphi, \mu) + 2k h_2(\varphi, \mu) \Psi_0(\varphi, \mu) \\ &\quad + (\kappa + 1) h_1(\varphi, \mu) \Psi_1(\varphi, \mu) + 2 h_0(\varphi, \mu) \Psi_2(\varphi, \mu), \\ \Phi_0 &= \Psi_1(\varphi, \mu) h_0(\varphi, \mu) + \kappa \Psi_0(\varphi, \mu) h_1(\varphi, \mu), \end{aligned}$$

where the prime denotes the differentiation with respect to φ.

For Ψ to be a Dulac-Cherkas function of (1.1) in $\mathcal{D}$ for $\mu \in \mathcal{I}$ it is sufficient that one of the inequalities

(3.7)
$$\Phi(\varphi, y, \mu, \kappa) \geq 0, \; \Phi(\varphi, y, \mu, \kappa) \leq 0$$

is fulfilled for $(\varphi, y, \mu) \in \mathcal{D} \times \mathcal{I}$.

Our key idea is to choose the constant κ and the functions $h_j, 0 \leq j \leq l$, in such a way that some of the coefficient functions Φ_i vanish identically such that Φ takes a structure which permits us in a simple way to derive conditions guaranteeing that Φ is positive or negative definite.

Since one of our goals is to show that Dulac-Cherkas functions can be used to control the existence of limit cycles of the second kind, we have to take into account that by Theorem 2.8 the existence of at least two limit cycles of the second kind for system (1.1) requires that the set $\mathcal{W}(\mu)$ contains at least one oval. From that reason we consider in the sequel functions Ψ in (3.1) for $n = 2$ and $n = 4$ with the property that the set $\mathcal{W}(\mu)$ has two or four ovals, respectively.

In what follows we demonstrate our key idea in the cases $l = 5, n = 2$, $l = 3, n = 2$, and $l = 5, n = 4$.

In the case $l = 5, n = 2$, the functions $\Phi_k, 0 \leq k \leq 6$, read according to (3.4) as follows

$$\begin{aligned}
\Phi_6 &= (2+5\kappa)h_5\Psi_2,\\
\Phi_5 &= (2+4\kappa)h_4\Psi_2 + (1+5\kappa)h_5\Psi_1,\\
\Phi_4 &= (2+3\kappa)h_3\Psi_2 + (1+4\kappa)h_4\Psi_1 + 5\kappa h_5\Psi_0,\\
\Phi_3 &= (2+2\kappa)h_2\Psi_2 + (1+3\kappa)h_3\Psi_1 + 4\kappa h_4\Psi_0 + \Psi_2',\\
\Phi_2 &= (2+1\kappa)h_1\Psi_2 + (1+2\kappa)h_2\Psi_1 + 3\kappa h_3\Psi_0 + \Psi_1',\\
\Phi_1 &= (2+0\kappa)h_0\Psi_2 + (1+1\kappa)h_1\Psi_1 + 2\kappa h_2\Psi_0 + \Psi_0',\\
\Phi_0 &= \qquad\qquad\qquad\quad (1+0\kappa)h_0\Psi_1 + 1\kappa h_1\Psi_0.
\end{aligned} \tag{3.8}$$

Our goal is to vanish $\Phi_6, ..., \Phi_1$ identically by a suitable choice of κ and of the functions h_i. Taking into account (3.2) we get from (3.8)

$$\begin{aligned}
\kappa &= -\frac{2}{5},\\
h_4 &= -\frac{(1+5\kappa)h_5\Psi_1}{(2+4\kappa)\Psi_2},\\
h_3 &= -\frac{(1+4\kappa)h_4\Psi_1 + 5\kappa h_5\Psi_0}{(2+3\kappa)\Psi_2},\\
h_2 &= -\frac{(1+3\kappa)h_3\Psi_1 + 4\kappa h_4\Psi_0 + \Psi_2'}{(2+2\kappa)\Psi_2},\\
h_1 &= -\frac{(1+2\kappa)h_2\Psi_1 + 3\kappa h_3\Psi_0 + \Psi_1'}{(2+1\kappa)\Psi_2},\\
h_0 &= -\frac{(1+1\kappa)h_1\Psi_1 + 2\kappa h_2\Psi_0 + \Psi_0'}{(2+0\kappa)\Psi_2}.
\end{aligned} \tag{3.9}$$

As a consequence we obtain

$$\Phi\Big(\varphi, y, \mu, -\frac{2}{5}\Big) = \Phi_0\Big(\varphi, \mu, -\frac{2}{5}\Big) = \Psi_1(\varphi,\mu)h_0(\varphi,\mu) - \frac{2}{5}\Psi_0(\varphi,\mu)h_1(\varphi,\mu) \tag{3.10}$$

such that the inequalities (3.7) read now

$$\Phi_0\Big(\varphi, \mu, -\frac{2}{5}\Big) \geq 0,\ \ \Phi_0\Big(\varphi, \mu, -\frac{2}{5}\Big) \leq 0 \tag{3.11}$$

that is, they do not depend on y.

The case $l = 3, n = 2$ can be treated analogously using the relations above. We obtain

$$
\begin{aligned}
\Phi_4 &= (2+3\kappa)h_3\Psi_2, \\
\Phi_3 &= (2+2\kappa)h_2\Psi_2 + (1+3\kappa)h_3\Psi_1 + \Psi_2', \\
\Phi_2 &= (2+1\kappa)h_1\Psi_2 + (1+2\kappa)h_2\Psi_1 + 3\kappa h_3\Psi_0 + \Psi_1', \\
\Phi_1 &= (2+0\kappa)h_0\Psi_2 + (1+1\kappa)h_1\Psi_1 + 2\kappa h_2\Psi_0 + \Psi_0', \\
\Phi_0 &= \qquad\qquad\qquad\quad (1+0\kappa)h_0\Psi_1 + 1\kappa h_1\Psi_0.
\end{aligned} \tag{3.12}
$$

Our goal is to vanish $\Phi_4, ..., \Phi_1$ identically by a suitable choice of κ and the functions h_i. Taking into account (3.2) we get from (3.12)

$$
\begin{aligned}
\kappa &= -\frac{2}{3}, \\
h_2 &= -\frac{(1+3\kappa)h_3\Psi_1 + \Psi_2'}{(2+2\kappa)\Psi_2}, \\
h_1 &= -\frac{(1+2\kappa)h_2\Psi_1 + 3\kappa h_3\Psi_0 + \Psi_1'}{(2+1\kappa)\Psi_2}, \\
h_0 &= -\frac{(1+1\kappa)h_1\Psi_1 + 2\kappa h_2\Psi_0 + \Psi_0'}{(2+0\kappa)\Psi_2}.
\end{aligned} \tag{3.13}
$$

The inequalities (3.7) have the form as in (3.11), where Φ_0 is defined by

$$
\Phi_0\Big(\varphi, \mu, -\frac{2}{3}\Big) = \Psi_1(\varphi,\mu)h_0(\varphi,\mu) - \frac{2}{3}\Psi_0(\varphi,\mu)h_1(\varphi,\mu).
$$

Next we consider the case $l = 5, n = 4$. Here, the functions $\Phi_k, 0 \le k \le 8$, read according to (3.4) as follows

$$
\begin{aligned}
\Phi_8 =& (4+5\kappa)h_5\Psi_4, \\
\Phi_7 =& (4+4\kappa)h_4\Psi_4 + (3+5\kappa)h_5\Psi_3, \\
\Phi_6 =& (4+3\kappa)h_3\Psi_4 + (3+4\kappa)h_4\Psi_3 + (2+5\kappa)h_5\Psi_2, \\
\Phi_5 =& (4+2\kappa)h_2\Psi_4 + (3+3\kappa)h_3\Psi_3 + (2+4\kappa)h_4\Psi_2 + (1+5\kappa)h_5\Psi_1 \\
& + \Psi_4', \\
\Phi_4 =& (4+1\kappa)h_1\Psi_4 + (3+2\kappa)h_2\Psi_3 + (2+3\kappa)h_3\Psi_2 + (1+4\kappa)h_4\Psi_1 \\
& + 5\kappa h_5\Psi_0 + \Psi_3', \\
\Phi_3 =& (4+0\kappa)h_0\Psi_4 + (3+1\kappa)h_1\Psi_3 + (2+2\kappa)h_2\Psi_2 + (1+3\kappa)h_3\Psi_1 \\
& + 4\kappa h_4\Psi_0 + \Psi_2', \\
\Phi_2 =& (3+0\kappa)h_0\Psi_3 + (2+1\kappa)h_1\Psi_2 + (1+2\kappa)h_2\Psi_1 + 3\kappa h_3\Psi_0 + \Psi_1', \\
\Phi_1 =& (2+0\kappa)h_0\Psi_2 + (1+1\kappa)h_1\Psi_1 + 2\kappa h_2\Psi_0 + \Psi_0', \\
\Phi_0 =& (1+0\kappa)h_0\Psi_1 + 1\kappa h_1\Psi_0.
\end{aligned} \tag{3.14}
$$

Taking into account (1.2) and (3.2) we get from (3.14) that Φ_8 vanishes identically if we choose

$$\kappa = -\frac{4}{5}. \tag{3.15}$$

From (3.14) and (3.2) we get further that we can use the functions $h_0, \ldots, h_4$ to vanish the functions $\Phi_3, \ldots, \Phi_7$ accordingly.

$$\begin{aligned}
h_4 &= -\frac{(3+5\kappa)h_5\Psi_3}{(4+4\kappa)\Psi_4}, \\
h_3 &= -\frac{(2+5\kappa)h_5\Psi_2 + (3+4\kappa)h_4\Psi_3}{(4+3\kappa)\Psi_4}, \\
h_2 &= -\frac{(1+5\kappa)h_5\Psi_1 + (2+4\kappa)h_4\Psi_2 + (3+3\kappa)h_3\Psi_3 + \Psi_4'}{(4+2\kappa)\Psi_4}, \\
h_1 &= -\frac{5\kappa h_5\Psi_0 + (1+4\kappa)h_4\Psi_1 + (2+3\kappa)h_3\Psi_2 + (3+2\kappa)h_2\Psi_3 + \Psi_3'}{(4+1\kappa)\Psi_4}, \\
h_0 &= -\frac{4\kappa h_4\Psi_0 + (1+3\kappa)h_3\Psi_1 + (2+2\kappa)h_2\Psi_2 + (3+1\kappa)h_1\Psi_3 + \Psi_2'}{4\Psi_4}.
\end{aligned} \tag{3.16}$$

Thus, the inequalities in (3.7) read

$$\begin{aligned}
&\Phi\Big(\varphi, y, \mu, -\frac{4}{5}\Big) \equiv \\
&\Phi_2\Big(\varphi, \mu, -\frac{4}{5}\Big)y^2 + \Phi_1\Big(\varphi, \mu, -\frac{4}{5}\Big)y + \Phi_0\Big(\varphi, \mu, -\frac{4}{5}\Big) \geq 0 \ (\leq 0),
\end{aligned} \tag{3.17}$$

where we can use the corresponding discriminant in order to derive conditions that $\Phi(\varphi, y, \mu, \kappa)$ is definite.

Another possibility to simplify the form of the inequalities in (3.7) is, at first, to determine κ, h_4, h_3, h_2 and h_0 as above and reduce $\Phi(\varphi, y, \mu)$ to the following form

$$\begin{aligned}
&\Phi\Big(\varphi, y, \mu, -\frac{4}{5}\Big) = \Phi_4\Big(\varphi, \mu, -\frac{4}{5}\Big)y^4 \\
&+ \Phi_2\Big(\varphi, \mu, -\frac{4}{5}\Big)y^2 + \Phi_1\Big(\varphi, \mu, -\frac{4}{5}\Big)y + \Phi_0\Big(\varphi, \mu, -\frac{4}{5}\Big) \geq 0 \ (\leq 0).
\end{aligned} \tag{3.18}$$

Then under the additional assumption

$$\Psi_1(\varphi, \mu) \neq 0 \qquad \text{for} \quad (\varphi, \mu) \in [0, 2\pi] \times \mathcal{I}$$

we can determine h_1 by

$$h_1 = -\frac{2h_0\Psi_2 - \frac{8}{5}h_2\Psi_0 + \Psi_0'}{(1 - \frac{4}{5})\Psi_1} = -\frac{10h_0\Psi_2 - 8h_2\Psi_0 + 5\Psi_0'}{\Psi_1}$$

from the identity $\Phi_1(\varphi, \mu, -\frac{4}{5}) \equiv 0$. Thus, the inequalities in (3.7) can be written as follows

$$\Phi_4\Big(\varphi, \mu, -\frac{4}{5}\Big)y^4 + \Phi_2\Big(\varphi, \mu, -\frac{4}{5}\Big)y^2 + \Phi_0\Big(\varphi, \mu, -\frac{4}{5}\Big) \geq 0 \ (\leq 0), \tag{3.19}$$

where the sign of y plays no role, and we can use different methods to guarantee that Φ is definite.

In all these cases we will choose the functions $\Psi_j(\varphi,\mu)$ in such a way that the corresponding set $\mathcal{W}(\mu)$ consists of a prescribed number of ovals and that the function $\Phi(\varphi,y,\mu,\kappa)$ satisfies one of the inequalities (3.11) or (3.17) or (3.19). Then under the assumptions of Theorem 2.8 the constructed system (1.1) has not more than a prescribed number of limit cycles of the second kind in $\mathcal{D}$.

4. Construction of systems (1.1) in the case $l = 3$ having not more than three limit cycles of the second kind

Now we apply the described method to construct systems

$$\frac{d\varphi}{dt} = y,\ \frac{dy}{dt} = h_0(\varphi,\mu) + h_1(\varphi,\mu)y + h_2(\varphi,\mu)y^2 + h_3(\varphi,\mu)y^3 \tag{4.1}$$

having not more than three limit cycles of the second kind. For this purpose we use for Ψ the ansatz

$$\Psi(\varphi,y,\mu) = \Psi_0(\varphi,\mu) + \Psi_1(\varphi,\mu)y + \Psi_2(\varphi,\mu)y^2, \tag{4.2}$$

where the functions Ψ_i satisfy the assumptions formulated at the beginning of section 3. According to our treatment in section 3, we have the case $l = 3, n = 2$. Thus, the set

$$\mathcal{W}(\mu) := \{(\varphi,y) \in \mathcal{D} : \Psi(\varphi,y,\mu) = 0\}$$

consists of at most two ovals surrounding the cylinder and the function Φ introduced in (3.3) represents a polynomial of degree $m = 4$, where the coefficient functions Φ_i are defined in (3.12). If the constant κ and the functions $h_i, 0 \le i \le 2$, satisfy the relations (3.13), then the functions Φ_4, Φ_3, Φ_2 and Φ_1 vanish identically and the inequalities (3.7) have the form (3.11). In our case we have

$$\Phi_0\Big(\varphi,\mu,-\frac{2}{3}\Big) = \Psi_1(\varphi,\mu)h_0(\varphi,\mu) - \frac{2}{3}\Psi_0(\varphi,\mu)h_1(\varphi,\mu), \tag{4.3}$$

where the functions h_0, h_1 and h_2 are defined by

$$h_2 = \frac{3}{2\Psi_2}\big(h_3\Psi_1 - \Psi_2'\big), \tag{4.4}$$

$$h_1 = \frac{1}{4\Psi_2}\Big[\frac{3\Psi_1}{2\Psi_2}\big(h_3\Psi_1 - \Psi_2'\big) + 6h_3\Psi_0 - 3\Psi_1'\Big], \tag{4.5}$$

$$h_0 = \frac{1}{2\Psi_2}\Big(\frac{3h_3\Psi_0\Psi_1}{2\Psi_2} - \frac{h_3\Psi_1^3}{8\Psi_2^2} - \Psi_0' + \frac{\Psi_1\Psi_1'}{4\Psi_2} + \frac{\Psi_1^2\Psi_2'}{8\Psi_2^2} - \frac{2\Psi_0\Psi_2'}{\Psi_2}\Big). \tag{4.6}$$

From (4.3), (4.5) and (4.6) we get

$$\Phi_0 = \frac{1}{2\Psi_2}\Big(\frac{2h_3\Psi_0\Psi_1^2}{2\Psi_2} - \frac{3\Psi_0\Psi_1\Psi_2'}{2\Psi_2} - \frac{h_3\Psi_1^4}{8\Psi_2^2} + \frac{\Psi_1^3\Psi_2'}{8\Psi_2^2} + \frac{\Psi_1^2\Psi_1'}{4\Psi_2} - \Psi_0'\Psi_1 + \Psi_1'\Psi_0 - 2h_3\Psi_0^2\Big). \tag{4.7}$$

Using these relations we have the following result.

Theorem 4.1. *Let the functions $\Psi_0, \Psi_1, \Psi_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous and twice continuously differentiable and 2π-periodic in the first variable, let the functions h_2, h_1 and h_0 be defined by (4.4), (4.5) and (4.6), respectively, let $h_3 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be any continuous function, continuously differentiable and 2π-periodic in the first variable and such that the relation (1.2) is satisfied. If there exists an interval $\mathcal{I}$ and a region $\mathcal{D}$ on $\mathcal{Z}$ whose boundary consists of two closed curves which surround the cylinder and do not meet such that for $\mu \in \mathcal{I}$*

- (i) *the function $\Phi_0(\varphi, \mu, \frac{2}{3})$ defined in (4.7) has the same sign in $\mathcal{D}$ and vanishes only in finitely many points $\varphi_i(\mu)$,*
- (ii) $$(grad\Psi, f)_{|\mathcal{W}(\mu)} \neq 0,$$
- (iii) *system (4.1) has no equilibrium in $\mathcal{D}$*

then system (4.1) has in $\mathcal{D}$ at most three limit cycles of the second kind for $\mu \in \mathcal{I}$.

Proof. Under the assumptions of Theorem 4.1 and taking into account the Remarks 2.4 and 2.5, the function $\Psi(\varphi, y, \mu)$ defined by (4.2) is for $\mu \in \mathcal{I}$ a Dulac-Cherkas function on the whole cylinder and the corresponding set $\mathcal{W}(\mu)$ contains at most two ovals surrounding the cylinder. Applying Theorem 2.8 the proof is complete.

□

In the following we choose the functions Ψ_0, Ψ_1, Ψ_2 in a special way in order to construct a system (4.1) having for $\mu = \mu_*$ a limit cycle of the second kind with multiplicity three exhibiting different bifurcation behavior for increasing and decreasing parameter μ.

The underlying idea is to construct a Dulac-Cherkas function $\Psi(\varphi, y, \mu)$ with the property that the topological structure of the corresponding set $\mathcal{W}(\mu)$ changes when the parameter μ crosses some critical value μ_*. We will exploit this property to construct a system (4.1) having μ_* as a bifurcation point related to a change of the number of limit cycles of the second kind.

We put

$$\Psi_2(\varphi, \mu) \equiv 1, \tag{4.8}$$

$$\Psi_1(\varphi, \mu) := 2(1 + \mu\cos\varphi), \tag{4.9}$$

$$\Psi_0(\varphi, \mu) := (1 + \mu\cos\varphi)^2 - \mu = \frac{1}{4}\Psi_1^2 - \mu. \tag{4.10}$$

Hence we have

$$\Psi(\varphi, y, \mu) = (y + 1 + \mu\cos\varphi)^2 - \mu \tag{4.11}$$

and the set

$$\mathcal{W}(\mu) := \{(\varphi, y) \in \mathcal{Z} : \Psi(\varphi, y, \mu) = 0\}$$

consists for $\mu > 0$ of the two ovals

(4.12) $$\mathcal{W}_1(\mu) := \{(\varphi, y) \in \mathcal{Z} : y + 1 + \mu\cos\varphi - \sqrt{\mu} = 0\},$$

(4.13) $$\mathcal{W}_2(\mu) := \{(\varphi, y) \in \mathcal{Z} : y + 1 + \mu\cos\varphi + \sqrt{\mu} = 0,$$

for $\mu = 0$ from the oval

(4.14) $$\mathcal{W}_0 := \{(\varphi, y) \in \mathcal{Z} : y = -1\},$$

and $\mathcal{W}(\mu)$ is empty for $\mu < 0$. Thus, we have a change in the topological structure of the set $\mathcal{W}(\mu)$ when μ crosses zero.

Now we determine a class of systems (4.1) for which the function $\Psi(\varphi, y, \mu)$ in (4.11) is a Dulac-Cherkas function.

By (4.8) - (4.10) and using the relations

$$\Psi_2' \equiv 0,\ 2\Psi_0' = \Psi_1\Psi_1',\ \Psi_1' = -2\mu\sin\varphi$$

we obtain from (4.3) - (4.7) for the functions $h_i, i = 0, 1, 2,$ and for Φ_0 the relations

(4.15) $$h_2(\varphi, \mu) = 3h_3(\varphi, \mu)(1 + \mu\cos\varphi),$$

(4.16) $$h_1(\varphi, \mu) = \frac{3}{2}\big[2h_3(\varphi, \mu)(1 + \mu\cos\varphi)^2 + \mu(\sin\varphi - h_3(\varphi, \mu)\big],$$

(4.17) $$h_0(\varphi, \mu) = \frac{1}{2}(1 + \mu\cos\varphi)\big[2h_3(\varphi, \mu)(1 + \mu\cos\varphi)^2 + \mu(\sin\varphi - 3h_3(\varphi, \mu)\big],$$

$$\Phi_0\Big(\varphi, \mu, -\frac{2}{3}\Big) \equiv \mu^2\big(\sin\varphi - h_3(\varphi, \mu)\big).$$

Thus, under the assumption that $h_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, continuously differentiable and 2π-periodic in the first variable and obeys the condition

(4.18) $$|h_3(\varphi, \mu)| > 1 \quad \forall(\varphi, \mu),$$

one of the inequalities 2.2 holds strictly, and we can conclude that the function Ψ in (4.11) is a Dulac-Cherkas function for the system

$$\begin{aligned}\frac{d\varphi}{dt} =& y,\\ \frac{dy}{dt} =& \frac{1}{2}(1+\mu\cos\varphi)\big[2h_3(\varphi,\mu)(1+\mu\cos\varphi)^2+\mu(\sin\varphi-3h_3(\varphi,\mu))\big]\\ &+\frac{3}{2}\big[2h_3(\varphi,\mu)(1+\mu\cos\varphi)^2+\mu(\sin\varphi-h_3(\varphi,\mu)\big]y\\ &+3h_3(\varphi,\mu)(1+\mu\cos\varphi)y^2+h_3(\varphi,\mu)y^3,\end{aligned} \tag{4.19}$$

on $\mathcal{Z}$ for all $\mu \in \mathbb{R}\setminus\{0\}$.
For $\mu = 0$, system (4.19) takes the form

$$\frac{d\varphi}{dt} = y, \quad \frac{dy}{dt} = h_3(\varphi,\mu)(y+1)^3. \tag{4.20}$$

Under the condition (4.18), the second equation has the unique equilibrium $y = -1$ which is an equilibrium of multiplicity three. Hence, we can conclude that system (4.19) has for $\mu = 0$ a unique closed trajectory which is a limit cycle

$$\Gamma(0) := \{(\varphi, y) \in \mathcal{Z} : y = -1\}$$

of the second kind of multiplicity three. Its stability depends on the sign of $h_3(\varphi,\mu)$.

Our goal is to study the bifurcation behavior of $\Gamma(0)$ when μ crosses the value 0 by means of Theorem 2.8. A basic assumption of that theorem is the non-existence of an equilibrium. It is obvious that any equilibrium $(\varphi_0(\mu), 0)$ of system (4.19) satisfies the equation

$$(1+\mu\cos\varphi)\big[2h_3(\varphi,\mu)(1+\mu\cos\varphi)^2+\mu(\sin\varphi-3h_3(\varphi,\mu)\big]=0. \tag{4.21}$$

The following lemmata are obvious.

Lemma 4.2. *The first factor in (4.21) has no root for $|\mu| < 1$, and there is a sufficiently small positive number $\mu_1 < 1$ such that the second factor has no real root for $|\mu| < \mu_1$. Thus, system (4.21) has no real real root for $|\mu| < \mu_1$.*

Lemma 4.3. *To given μ there is a sufficiently large positive number $C_0(\mu)$ such that all closed curves $y = C$ and $y = -C$ on $\mathcal{Z}$ with $C \geq C_0(\mu)$ are crossed by the trajectories of (4.19) transversally and in the direction of increasing $|y|$.*

If we denote by $\mathbb{A}(\mu)$ the region on $\mathcal{Z}$ bounded by the curves $y = C_0(\mu)$ and $y = -C_0(\mu)$, then we get from Lemma 4.3 by applying the Poincaré-Bendixson Theorem the following corollary.

Corollary 4.4. *System (4.19) has for $|\mu| < \mu_1$ in $\mathbb{A}(\mu)$ at least one limit cycle.*

By apply Theorem 2.8, Lemma 4.2, and Lemma 4.3 we get the following result.

Theorem 4.5. *From the limit cycle $\Gamma(0)$ of multiplicity three of the system (4.19) there bifurcates a unique simple limit cycle for decreasing μ, there bifurcate three simple limit cycles for increasing μ .*

5. CONSTRUCTION OF SYSTEMS (1.1) IN THE CASE $l = 5$ HAVING NOT MORE THAN FOUR LIMIT CYCLES OF THE SECOND KIND

We consider system (2.1) in the case $l = 5$

$$\begin{aligned} &\frac{d\varphi}{dt} = y, \\ &\frac{dy}{dt} = h_0(\varphi) + h_1(\varphi)y + h_2(\varphi)y^2 + h_3(\varphi)y^3 + h_4(\varphi)y^4 + h_5(\varphi)y^5. \end{aligned} \tag{5.1}$$

In the following subsection we construct a Dulac-Cherkas function such that system (5.1) has a unique limit cycle of the second kind, in the last subsection we construct a Dulac-Cherkas function such that system (5.1) has exactly four limit cycles of the second kind.

5.1. **Construction of a system (1.1) in the case $l = 5, n = 2$ having a unique limit cycle of the second kind.** We use the ansatz

$$\Psi(\varphi, y) = \Psi_0(\varphi) + \Psi_1(\varphi)y + \Psi_2(\varphi)y^2, \tag{5.2}$$

where

$$\Psi_2(\varphi) = \frac{1}{2},\ \Psi_1(\varphi) = 1, \Psi_0(\varphi) = \cos\varphi - 10. \tag{5.3}$$

In that case the curve $\mathcal{W}$ consists of the two ovals

$$\mathcal{W}_1 := \{(\varphi, y) \in Z : y = -1 + \sqrt{21 - 2\cos\varphi}\},$$
$$\mathcal{W}_2 := \{(\varphi, y) \in Z : y = -1 - \sqrt{21 - 2\cos\varphi}\}.$$

The region bounded by $\mathcal{W}_1$ and $\mathcal{W}_2$ is denoted by $\mathbb{A}_1$, the regions on $\mathcal{Z}$ located above $\mathcal{W}_1$ and below $\mathcal{W}_2$ are denoted by $\mathbb{A}_0$ and $\mathbb{A}_2$, respectively. Obviously, we have

$$\Psi(\varphi, y) < 0 \text{ in } \mathbb{A}_1. \tag{5.4}$$

If we determine the constant κ and the functions h_i according to (3.9), where we additionally set $h_5(\varphi) \equiv 1$, then we obtain

$$\kappa = -\frac{2}{5}, \tag{5.5}$$

$$h_0(\varphi) \equiv \frac{6203}{8} - \frac{305\cos\varphi}{2} + \frac{15\cos^2\varphi}{2} + \sin\varphi, \tag{5.6}$$

$$h_1(\varphi) \equiv \frac{5395}{8} - \frac{285\cos\varphi}{2} + \frac{15\cos^2\varphi}{2},$$

$$h_2(\varphi) \equiv -\frac{295}{2} + 15\cos\varphi,$$

$$h_3(\varphi) \equiv -\frac{85}{2} + 5\cos\varphi,$$
$$h_4(\varphi) \equiv 5.$$

The corresponding system (5.1) reads explicitly

$$
\begin{aligned}
\frac{d\varphi}{dt} =& y,\\
\frac{dy}{dt} =& \frac{6203}{8} - \frac{305\cos\varphi}{2} + \frac{15\cos^2\varphi}{2} + \sin\varphi\\
&+ \Big(\frac{5395}{8} - \frac{285\cos\varphi}{2} + \frac{15\cos^2\varphi}{2}\Big)y\\
&+ \Big(-\frac{295}{2} + 15\cos\varphi\Big)y^2 + \Big(-\frac{85}{2} + 5\cos\varphi\Big)y^3 + 5y^4 + y^5.
\end{aligned}
\tag{5.7}
$$

From (3.10) we get

$$
\Phi\Big(\varphi, y, -\frac{2}{5}\Big) = \Phi_0(\varphi)
$$

$$
\equiv \frac{27783}{8} - \frac{3969\cos\varphi}{4} + \frac{189\cos^2\varphi}{2} + \sin\varphi - 3\cos^3\varphi.
$$

The relation

$$
\Phi_0(\varphi) > 0 \qquad \forall\varphi \tag{5.8}
$$

can be easily verified. Thus, we can conclude that the function

$$
\Psi(\varphi, y) = \cos\varphi - 10 + y + 0.5y^2
$$

is a Dulac-Cherkas function for system (5.7) on $\mathcal{Z}$. From (5.6) we get that the function h_0 is positive for all φ. That implies that system (5.7) has no equilibrium on $\mathcal{Z}$. Thus, we get from Theorem 2.8 that system (5.7) has at least one limit cycle of the second kind. Furthermore, we obtain from (5.8), (5.5), (5.4)

$$
\kappa\Phi_0(\varphi)\Psi(\varphi, y) > 0 \text{ in } \mathbb{A}_1. \tag{5.9}
$$

Hence, system (5.7) has a unique limit cycle of the second kind in $\mathbb{A}_1$.
We denote by $\mathbb{N}$ the curve defined by

$$
\begin{aligned}
\mathbb{N} :=& \{(\varphi, y) \in \mathcal{Z} : \frac{dy}{dt} = \frac{6203}{8} - \frac{305\cos\varphi}{2} + \frac{15\cos^2\varphi}{2} + \sin\varphi\\
&+ (\frac{5395}{8} - \frac{285\cos\varphi}{2} + \frac{15\cos^2\varphi}{2})y\\
&+ (-\frac{295}{2} + 15\cos\varphi)y^2 + (-\frac{85}{2} + 5\cos\varphi)y^3 + 5y^4 + y^5 = 0\}.
\end{aligned}
\tag{5.10}
$$

It is obvious that any limit cycle of system (5.1) must either cut the curve $\mathbb{N}$ or be a subset of that curve. Using the method of Sturmian chains it can be shown that the curve $\mathbb{N}$ consists of a unique branch located in $\mathbb{A}_1$. Thus, using this fact, relation (5.9) and Theorem 2.8 we have the following result.

Theorem 5.1. *System (5.7) has a unique limit cycle of the second kind. This limit cycle is hyperbolic, orbitally unstable and located in the region* $\mathbb{A}_1$.

5.2. **Construction of a system in the case $l = 5$, $n = 4$ having exactly four limit cycles.**

We use for $\Psi(\varphi, y)$ the ansatz

$$\Psi(\varphi, y) = \sum_{j=0}^{4} \Psi_j(\varphi) y^j \tag{5.11}$$

with

$$\begin{aligned} &\Psi_0(\varphi) \equiv \frac{1}{10}, \Psi_1(\varphi) \equiv 0, \Psi_2(\varphi) = \frac{\cos\varphi}{10} - 2, \\ &\Psi_3(\varphi) \equiv 0, \Psi_4(\varphi) \equiv \frac{1}{2} \end{aligned} \tag{5.12}$$

such that we have

$$\Psi(\varphi, y) = \frac{1}{10} + \Big(\frac{\cos\varphi}{10} - 2\Big) y^2 + \frac{1}{2} y^4. \tag{5.13}$$

According to (3.3), $\Phi(\varphi, y)$ can be represented in the form

$$\Phi(\varphi, y) = \sum_{j=0}^{8} \Phi_j(\varphi) y^j, \tag{5.14}$$

where the functions $\Phi_j(\varphi)$ are defined by (3.14). If we set $h_5(\varphi) \equiv -10$ and determine $\kappa, h_0, ..., h_4$ by (3.15) and (3.16) we obtain

$$\begin{aligned} &\kappa = -\frac{4}{5}, h_4(\varphi) \equiv 0, h_3(\varphi) = 50 - \frac{5}{2}\cos\varphi, h_2(\varphi) \equiv 0, \\ &h_1(\varphi) = -\frac{55}{2} + \frac{5}{2}\cos\varphi - \frac{1}{16}\cos^2\varphi, h_0(\varphi) = \frac{1}{20}\sin\varphi. \end{aligned} \tag{5.15}$$

Using these relations, the corresponding system (5.1) reads

$$\begin{aligned} &\frac{d\varphi}{dt} = y, \\ &\frac{dy}{dt} = \frac{\sin\varphi}{20} + \big(-\frac{55}{2} + \frac{5\cos\varphi}{2} - \frac{\cos^2\varphi}{16}\big) y + \big(50 - \frac{5\cos\varphi}{2}\big) y^3 - 10 y^5, \end{aligned} \tag{5.16}$$

and from (3.14) we get

$$\begin{aligned} \Phi(\varphi, y) = &\frac{11}{5} - \frac{\cos\varphi}{5} + \frac{\cos^2\varphi}{200} + \Big(-\frac{\sin\varphi}{5} + \frac{\cos\varphi\sin\varphi}{100}\Big) y \\ &+ \Big(54 - \frac{87\cos\varphi}{10} + \frac{9\cos^2\varphi}{20} - \frac{3\cos^3\varphi}{400}\Big) y^2. \end{aligned} \tag{5.17}$$

Using the discriminant it can be shown by simple calculations that $\Phi(\varphi, y)$ is strictly positive for all φ and y. Thus, according to Remark 2.4, $\Psi(\varphi, y)$ is a Dulac-Cherkas function of system

(5.16) on $\mathcal{Z}$.
The set $\mathcal{W}$ consists of four closed curves $\mathcal{W}_i$ surrounding the cylinder $\mathcal{Z}$:

$$\mathcal{W}_1 := \{(\varphi, y) \in \mathcal{Z} : y = \sqrt{2 - \frac{\cos\varphi}{10} + \frac{\sqrt{\cos^2\varphi - 40\cos\varphi + 380}}{10}}\},$$

$$\mathcal{W}_2 := \{(\varphi, y) \in \mathcal{Z} : y = \sqrt{2 - \frac{\cos\varphi}{10} - \frac{\sqrt{\cos^2\varphi - 40\cos\varphi + 380}}{10}}\},$$

$$\mathcal{W}_3 := \{(\varphi, y) \in \mathcal{Z} : y = -\sqrt{2 - \frac{\cos\varphi}{10} - \frac{\sqrt{\cos^2\varphi - 40\cos\varphi + 380}}{10}}\},$$

$$\mathcal{W}_4 := \{(\varphi, y) \in \mathcal{Z} : y = -\sqrt{2 - \frac{\cos\varphi}{10} + \frac{\sqrt{\cos^2\varphi - 40\cos\varphi + 380}}{10}}\}.$$

As usually, we denote the region bounded by $\mathcal{W}_i$ and $\mathcal{W}_{i+1}$ by $\mathbb{A}_i$. It follows from (5.16) that there is a sufficiently large number C such that $\frac{dy}{dt} < 0$ on the closed curve $y = C$ and $\frac{dy}{dt} > 0$ on the closed curve $y = -C$. The region bounded by C and $\mathcal{W}_1$ will be denoted by $\mathbb{A}_0$, and region bounded by $\mathcal{W}_4$ and $-C$ will be denoted by $\mathbb{A}_4$. Since there is no equilibrium in the regions $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_3, \mathbb{A}_4$, by using Theorem 2.8 we get the following result.

Theorem 5.2. *System (5.16) has in the regions $\mathbb{A}_0$ and $\mathbb{A}_4$ a unique hyperbolic limit cycle of the second kind which is asymptotically orbitally stable, and in the regions $\mathbb{A}_1$ and $\mathbb{A}_3$ a unique hyperbolic limit cycle of the second kind which is orbitally unstable.*

Theorem 5.3. *The region $\mathbb{A}_2$ contains no limit cycle and no homoclinic curve of the second kind of system (5.16).*

Proof. The region $\mathbb{A}_2$ contains the closed curve $y = 0$ on which there are located two equilibria of system (5.16): the saddle $(0, 0)$ and and the stable node $(\pi, 0)$. The eigenvalues corresponding to the saddle have different signs, they read as $\lambda_1 \approx -24.9395, \lambda_2 \approx 0.0020$. Thus, the corresponding saddle value $\lambda = \lambda_1 + \lambda_2$ is negative.
In the same way as we have proved Theorem 4.1 in [] we can prove that any limit cycle of the second kind of system (5.16) cannot meet the closed curve $y = 0$. We denote the region on $\mathcal{Z}$ bounded by the closed curves $\mathcal{W}_2$ and $y = 0$ by $\mathbb{A}_2^+$, and by $\mathbb{A}_2^-$ the region bounded by the closed curves $y = 0$ and $\mathcal{W}_3$. Now we prove that there is no limit cycle of the second kind of system (5.16) in $\mathbb{A}_2^+$. First we note that $\Psi(\varphi, y) > 0$ in $\mathbb{A}_2$. This follows immediately from (5.13). Thus, $\Psi(\varphi, y)^{1/\kappa}$ is a Dulac function in $\mathbb{A}_2$ (see Lemma 2.7 in [6]), and by Theorem 2.2 in [6] there is at most one closed orbit of the second kind of system (5.16) in $\mathbb{A}_2^+$ (either a limit cycle or a homoclinic orbit of the second kind). Now we assume that there is a limit cycle Γ of the second kind in $\mathbb{A}_2^+$. Since $\Phi(\varphi, y)$ is strictly positive in $\mathbb{A}_2$ and $\kappa < 0$ by (5.15), we get from Theorem 2.8 that Γ is asymptotically orbitally stable. Next we denote by S_2^+ the separatrix of the saddle $(0, 0)$ located in the region $\mathbb{A}_2^+$ and having $(0, 0)$ as ω-limit set. Now we look for the

α-limit set of the separatrix S_2^+. For this purpose we consider the curve $\mathbb{N}$ defined by

$$\begin{aligned}\mathbb{N} :=&\{(\varphi, y) \in \mathcal{Z} : \frac{dy}{dt} = \frac{\sin\varphi}{20} + \big(- \frac{55}{2} + \frac{5\cos\varphi}{2} - \frac{\cos^2\varphi}{16}\big)y \\ &+ \big(50 - \frac{5\cos\varphi}{2}\big)y^3 - 10y^5 = 0\}.\end{aligned} \tag{5.18}$$

It can be shown that the maximum of the branch $\mathbb{N}_2$ of the curve $\mathbb{N}$ located in $\mathbb{A}_2$ satisfies $|y| < 0.0004$. That means the branch $\mathbb{N}_2$ passes the points $(-\pi, 0), (0, 0)$ and has the same sign as the curve $y = 1/20 \sin\varphi$. Since we can prove that the straight line $y = \frac{0.008}{\pi}\varphi - 0.0004$ in the interval $-\pi \leq \varphi \leq -\pi/2$ is a curve without contact, we can conclude that the α-limit set of the separatrix S_2^+ is not located in $\mathbb{A}_2^-$. Thus, there must be an invariant set in $\mathbb{A}_2^+$ which is not orbitally stable and which is the α-limit set of the separatrix S_2^+. Since the equilibrium $(-\pi, 0)$ is a stable focus, only a homoclinic orbit $\mathbb{H}$ of the second kind to the saddle $(0, 0)$, or an unstable limit cycle Γ_u of the second kind are possible α-limit sets of the separatrix S_2^+. But each of these possibilities leads to the existence of a second closed orbit in $\mathbb{A}_2^+$ which contradicts to the fact that at most one closed orbit can exists in $\mathbb{A}_2^+$. Thus, our assumption of the existence of a stable limit cycle of the second kind leads to a contradiction. The same proof works also in the region $\mathbb{A}_2^-$. □

From the Theorems 5.2 and 5.3 we get

Theorem 5.4. *System (5.16) has exactly four limit cycles of the second kind. They are hyperbolic and located in the regions $\mathbb{A}_i$, $i = 0, 1, 3, 4,$. The limit cycles in the region $\mathbb{A}_0$ and $\mathbb{A}_4$ are asymptotically orbitally stable, the other ones are unstable.*

REFERENCES

[1] A.A. ANDRONOV, A.A. VITT, S.E. KHAJKIN, *The theory of oscillations*, Nauka, Moscow, 1981.

[2] N.N. BAUTIN, E.A. LEONTOVICH, *Methods and rules for the qualitative study of dynamical systems on the plane*, Nauka, Moscow, 1990.

[3] L.A. CHERKAS, *Dulac function for polynomial autonomous systems on a plane*, Differential Equations **33** (1997), 692 - 701.

[4] L.A. CHERKAS, A.A. GRIN, *Dulac functions for dynamical systems on the cylinder* (in Russian), Bulletin of Grodno State University, **2** (2007), 3 - 8.

[5] L. A. CHERKAS, A. A. GRIN, *Limit cycle function of the second kind for autonomous systems on the cylinder*, Differential Equations **47** (2011), 462 - 470.

[6] L.A. CHERKAS, A.A. GRIN, K.R. SCHNEIDER, *A new approach to study limit cycles on a cylinder*, Dynamics of continuous, discrete and impulsive systems Series A: Mathematical Analysis **18** (2011), 839 - 851.

[7] A.A. GRIN, K.R. SCHNEIDER, *On some classes of limit cycles of planar dynamical systems*, Dynamics of Continuous, Discrete and Impulsive Systems, ser. A, **14** (2007), 641 - 656.

[8] A.A. GRIN, K.R. SCHNEIDER, *On the construction of a class of generalized Kukles systems having at most one limit cycle*, Journal of Mathematical Analysis and Applications **408**, (2013), 484 - 497.

[9] A.A. GRIN, K.R. SCHNEIDER, *Study of the bifurcation of a multiple limit cycle of the second kind by means of a Dulac-Cherkas function: a case study* WIAS Preprint 2226 (2016).

[10] L. PERKO, *Differential Equations and Dynamical Systems*, New York, Springer, Third Edition 2001.