

**Weierstraß-Institut**  
**für Angewandte Analysis und Stochastik**  
**Leibniz-Institut im Forschungsverbund Berlin e. V.**

Preprint

ISSN 2198-5855

**Constrained evolution**  
**for a quasilinear parabolic equation**

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submitted: February 24, 2016

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No. 2232  
Berlin 2016



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2010 *Mathematics Subject Classification.* 35K59, 35K20, 34H05, 80M50, 93B52.

*Key words and phrases.* Feedback control, quasilinear parabolic equation, monotone nonlinearities, convex sets.

PC and GG gratefully acknowledge some financial support from the GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica) and the IMATI – C.N.R. Pavia.

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## ABSTRACT

In the present contribution, a feedback control law is studied for a quasilinear parabolic equation. First, we prove the well-posedness and some regularity results for the Cauchy–Neumann problem for this equation, modified by adding an extra term which is a multiple of the subdifferential of the distance function from a closed convex set  $K$  of  $L^2(\Omega)$ . Then, we consider convex sets of obstacle or double-obstacle type, and we can act on the factor of the feedback control in order to be able to reach the convex set within a finite time, by proving rigorously this property.

## 1 Introduction

A noteworthy interest has arisen in the mathematical literature of the last twenty years for sliding mode control (SMC) problems. SMC is considered as a main tool for the systematic design of robust controllers for nonlinear complex dynamical systems operating under uncertainty. The advantage of SMC is the separation of the motion of the overall system in independent partial components with a lower dimension.

The design of feedback control systems with sliding modes is related to the selection of suitable control functions enforcing motions along ad-hoc manifolds. Hence, a manifold of lower dimension (called the sliding manifold) has to be identified such that the original system, restricted to this sliding manifold, has a desired behavior; then, one acts on the system through the control in order to constrain the evolution on it, that is, to design an SMC law that forces the trajectories of the system to reach the sliding surface and remain on it (see, e.g., [10] and references therein).

Known methods developed for ODEs (cf., e.g., [8]) have been recently extended to the control of infinite-dimensional dynamical systems. For instance, we mention some papers dealing with SMC for semilinear PDEs: [4] deals with the stabilization problem of a one-dimensional unstable heat conduction system (rod) ruled by a parabolic partial differential equation with a Dirichlet type actuator from one of the boundaries; in [7], an SMC law is studied for a class of parabolic systems where the control acts through a Neumann boundary condition; a delay-independent SMC strategy was proposed in [14] to control a class of quasilinear parabolic PDE systems with time-varying delay.

The recent paper [2], in which two of the three authors of this note are involved, faces different kinds of SMC problems for a standard phase field system. This system couples two parabolic equations in terms of the variables temperature and order parameter. Sliding manifolds are considered both for a linear combination of variables and just for the order parameter. It is shown that the chosen SMC laws force the system to reach within finite time the sliding manifold. In particular, the control law is nonlocal in space for two of the examined problems. When reporting the related results during a conference, the third author of this note observed that it was analogously interesting, from the point of view of control problems, to force variables to reach not a single elected manifold but instead a closed convex subset of the space in which the variables still can move.

We started to think about it and, in order to develop this idea, decided to argue first

on a single nonlinear equation, of course an evolutionary equation, and of parabolic type like

$$\partial_t \vartheta - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) = f \quad \text{in } Q := \Omega \times (0, T), \quad (1.1)$$

which fits into a well-established subject (let us quote some monographs, i.e., [1, 3, 5, 9, 11, 12, 15]). With the aim of discussing existence and dynamics of solutions in the framework of the space  $L^2(\Omega)$ , and being interested to reach a closed convex  $K \subset L^2(\Omega)$  in finite time, a feedback control is added to the equation (1.1) by considering

$$\partial_t \vartheta - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) + \rho \partial d_K(\vartheta) \ni f, \quad (1.2)$$

where  $\partial d_K$  is the subdifferential of the distance function  $d_K$  associated with  $K$  and  $\rho$  is a positive parameter, to be suitably chosen in order to force the solution to enter the convex set (if it is not already inside).

We point out that (1.2) has the structure of an evolution inclusion (cf. [1, 3]) but is not a standard variational inequality constraining the solution to stay inside the convex set. On the other hand, one may exert the control on (1.2) via the parameter  $\rho$ : we can imagine that the larger the coefficient  $\rho$  is, the faster the solution will enter the convex.

In our approach, we can deal with convex sets in  $L^2(\Omega)$  of obstacle and double-obstacle type. We are able to treat these cases: of course, the analysis is not straightforward, as the reader will see, and it will also become clear why obstacle convex sets are suitable for us. This paper is a first attempt to approach a somehow new field of investigation, and so we ask the reader to be generous: indeed, to the best of our knowledge, at the moment we are not aware of other similar essays.

We complement (1.2) with Neumann homogeneous boundary conditions and initial conditions, then we discuss existence and uniqueness of the solution to the initial-boundary value problem without any restriction on  $\rho > 0$  and on the nonempty closed convex  $K$  of  $L^2(\Omega)$ . Then, we focus on convex sets of obstacle type and prove that for a sufficiently large  $\rho$  the solution  $\vartheta$  will reach the convex set in finite time.

The paper is organized as follows. In the next section, we list our assumptions, state the problem in a precise form and present our results. In Section 3, we introduce and solve an approximating problem which is useful to construct a solution to the problem at hand. The proofs of our results are then given in Sections 4 and 5, mainly, since the Appendix is just devoted to establish a technical lemma.

## 2 Statement of the problem and results

In this section, we describe the problem under study and present our results. First, we assume  $\Omega$  to be a smooth bounded open subset of  $\mathbb{R}^d$ . Now, we specify the assumptions on the structure of our system. As for  $\kappa$ , we assume that

$$\kappa : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is continuous, nonnegative and bounded,} \quad (2.1)$$

and set

$$\kappa_* := \inf \kappa, \quad \kappa^* := \sup \kappa \quad \text{and} \quad G(r) := \int_0^r \kappa(s) ds \quad \text{for } r \in \mathbb{R}. \quad (2.2)$$

The last condition defines the function  $G : \mathbb{R} \rightarrow \mathbb{R}$ , and we suppose that

$$G \text{ is strictly increasing.} \quad (2.3)$$

This assumption is satisfied if and only if the set where the nonnegative function  $\kappa$  vanishes has an empty interior, and it clearly holds if  $\kappa_* > 0$ . In this case, the inverse function  $G^{-1}$  is Lipschitz continuous and not only continuous on its domain ( $\inf G, \sup G$ ). Next, even for a future convenience, we introduce the spaces

$$H := L^2(\Omega) \quad \text{and} \quad V := H^1(\Omega), \quad (2.4)$$

and we endow them with their standard norms. It is understood that  $H$  is embedded in  $V^*$ , the dual space of  $V$ , in the usual way, i.e., such that  $\langle u, v \rangle = \int_{\Omega} uv \, dx$  for every  $u \in H$  and  $v \in V$ , where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $V^*$  and  $V$ . Furthermore, we list our assumptions and notations regarding the convex set:

$$K \text{ is a nonempty, closed, and convex subset of } H \quad (2.5)$$

$$d_K : H \rightarrow H \text{ is the distance function associated to } K \quad (2.6)$$

$$P_K : H \rightarrow H \text{ is the projection operator on } K \quad (2.7)$$

$$Q_K := I_H - P_K, \text{ where } I_H : H \rightarrow H \text{ is the identity map.} \quad (2.8)$$

So, (2.5)–(2.8) are related to each other as follows: for  $v \in H$ , we have

$$P_K v \in K \quad \text{and} \quad d_K(v) = \|Q_K v\|_H \leq \|v - z\|_H \quad \text{for every } z \in K. \quad (2.9)$$

Concerning the data, we in principle assume that

$$f \in L^2(0, T; H) \quad \text{and} \quad \vartheta_0 \in V. \quad (2.10)$$

At this point, we can state the problem under investigation: given a real number  $\rho > 0$ , we look for a triplet  $(\vartheta, u, \sigma)$  satisfying the regularity properties

$$\vartheta \in H^1(0, T; V^*) \cap L^\infty(0, T; H) \quad (2.11)$$

$$u \in H^1(0, T; H) \cap L^\infty(0, T; V) \quad (2.12)$$

$$\sigma \in L^\infty(0, T; H) \quad (2.13)$$

and solving the problem

$$\begin{aligned} \langle \partial_t \vartheta(t), v \rangle + \int_{\Omega} \nabla u(t) \cdot \nabla v + \rho \int_{\Omega} \sigma(t) v &= \int_{\Omega} f(t) v \\ \text{for a.a. } t \in (0, T) \text{ and every } v \in V & \end{aligned} \quad (2.14)$$

$$u = G(\vartheta) \quad \text{a.e. in } Q \quad \text{and} \quad \sigma(t) \in \partial d_K(\vartheta(t)) \quad \text{for a.a. } t \in (0, T) \quad (2.15)$$

$$\vartheta(0) = \vartheta_0. \quad (2.16)$$

We observe that the system (2.14)–(2.15) is the variational formulation of the differential inclusion (1.2) complemented with the no-flux boundary condition for  $\nabla u$  (i.e., for  $\kappa(\vartheta)\nabla\vartheta$  whenever the chain rule can be applied). Moreover, we notice that (2.11) implies that  $\vartheta \in C^0([0, T]; V^*)$  (and even that  $\vartheta$  is a weakly continuous  $H$ -valued function) so that the initial condition (2.16) makes sense. Here is our first result.

**Theorem 2.1.** *Under the assumptions and notations (2.1)–(2.10), suppose that  $\kappa_* > 0$ . Then, for every  $\rho > 0$ , problem (2.14)–(2.16) has at least one solution  $(\vartheta, u, \sigma)$  satisfying (2.11)–(2.13) and*

$$\vartheta \in H^1(0, T; H) \cap L^\infty(0, T; V). \quad (2.17)$$

Moreover, there is only one such solution if  $\kappa$  is a positive constant.

We can deal with the degenerate case  $\kappa_* = 0$  only for convex sets of obstacle or double-obstacle type. Namely, we suppose that

$$I \text{ is a closed nonempty interval} \quad (2.18)$$

$$K := \{v \in H : v(x) \in I \text{ for a.a. } x \in \Omega\}. \quad (2.19)$$

In this case, the projection on  $K$  is a pointwise projection, i.e., for  $v \in H$  and almost every  $x \in \Omega$ , it holds that

$$(P_K v)(x) \text{ is the projection of } v(x) \text{ on the interval } I. \quad (2.20)$$

Moreover, we have to reinforce our assumptions by postulating that

$$f \in L^1(0, T; L^p(\Omega)) \quad \text{and} \quad \vartheta_0 \in L^p(\Omega) \quad \text{for some } p > 2. \quad (2.21)$$

**Theorem 2.2.** *Besides the assumptions and notations (2.1)–(2.10), suppose that (2.18)–(2.19) and (2.21) hold. Then, for every  $\rho > 0$ , problem (2.14)–(2.16) has at least one solution  $(\vartheta, u, \sigma)$  satisfying (2.11)–(2.13) as well as*

$$\vartheta \in L^\infty(0, T; L^p(\Omega)). \quad (2.22)$$

The main result of our paper is the next one. It holds true for the particular class (2.18)–(2.19) of convex subsets. However, the degenerate case  $\kappa_* = 0$  is allowed as well. We ensure the existence of a solution  $(\vartheta, u, \sigma)$  whose component  $\vartheta$  approaches and eventually reaches the convex set  $K$  in a finite time  $T^* < T$ , provided that the parameter  $\rho$  is large enough. Indeed, from the statement it follows that the condition

$$\rho > \rho^* + \frac{d_K(\vartheta_0)}{T} \quad \text{where} \quad \rho^* := \|f\|_{L^\infty(0, T; H)} \quad (2.23)$$

implies  $T^* < T$ . Moreover, the speed of approach is  $\rho - \rho^*$ , at least. The precise meaning of the theorem relies on the following observation, which follows from the regularity of  $\vartheta$  specified by (2.11):

$$\vartheta \text{ is an } H\text{-valued weakly continuous function.} \quad (2.24)$$

Namely, the continuous representative  $\vartheta \in C^0([0, T]; V^*)$  satisfies  $\vartheta(t) \in H$  for every  $t \in [0, T]$ , and  $\vartheta$  is continuous from  $[0, T]$  to  $H$  endowed with its weak topology.

**Theorem 2.3.** *Under the assumptions and notations (2.1)–(2.10) and (2.18)–(2.19), suppose either  $\kappa_* > 0$  or (2.21). Furthermore, assume that*

$$\rho^* := \|f\|_{L^\infty(0, T; H)} < +\infty. \quad (2.25)$$

Then, for every  $\rho > \rho^*$ , there exists a solution  $(\vartheta, u, \sigma)$  to problem (2.14)–(2.16) with the following properties:

$$i) \quad \text{if } \vartheta_0 \in K, \quad \text{then } \vartheta(t) \in K \text{ for every } t \in [0, T] \quad (2.26)$$

$$ii) \quad \text{if } \vartheta_0 \notin K, \quad \text{there exists } T^* \in (0, T] \text{ satisfying } T^* \leq \frac{d_K(\vartheta_0)}{\rho - \rho^*} \quad \text{such that}$$

$$\frac{d}{dt} d_K(\vartheta(t)) \leq -(\rho - \rho^*) \quad \text{in the sense of distributions on } (0, T^*) \quad (2.27)$$

$$\vartheta(t) \in K \quad \text{for every } t \in [0, T] \text{ such that } t > T^*. \quad (2.28)$$

In particular, in the case *ii*), the function  $t \mapsto d_K(\vartheta(t))$  is strictly decreasing on  $[0, T^*]$ .

We close this section with a list of denotations and tools. Throughout the paper,  $\|\cdot\|_X$  denotes the norm in the generic Banach space  $X$  or in a power thereof. However, we simply write  $\|\cdot\|_p$  for the standard norm in  $L^p(\Omega)$ . Moreover, we repeatedly use the denotation

$$Q_t := \Omega \times (0, t) \quad \text{for } 0 < t \leq T \quad (2.29)$$

as well as the Young inequalities

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{and} \quad ab \leq \theta a^{\frac{1}{\theta}} + (1 - \theta)b^{\frac{1}{1-\theta}}$$

$$\text{for every } a, b \geq 0, \quad \delta > 0, \quad \text{and } \theta \in (0, 1) \quad (2.30)$$

and Hölder's inequality. Furthermore, we account for the compact embedding  $V \subset H$ . Finally, we follow a general rule to denote constants: the small-case symbol  $c$  stands for different constants which depend only on  $\Omega$ , on the final time  $T$ , the structure of the problem and on the constants and the norms of the functions involved in the assumptions of our statements. A symbol like  $c_\delta$  signals that the constant can depend also on the parameter  $\delta$ . Hence, the meaning of  $c$  (or  $c_\delta$ ) might change from line to line and even within the same chain of equalities or inequalities.

### 3 Approximation

In this section, we introduce an approximating problem which depends on the parameters  $\varepsilon, \alpha \in (0, 1)$  and is useful to establish some parts of our results. We could have decided to take  $\alpha = \varepsilon$  to reach the same goal. However, we think that the choice of two different parameters could prove to be more suitable for the numerical treatment. From one side, we replace the function  $\kappa$  by a strictly positive  $\kappa_\alpha$  in order to ensure uniform parabolicity. On the other hand, we regularize the subdifferential  $\partial d_K$ . However, for the sake of simplicity, we often avoid stressing the dependence on both parameters in the notation and write, e.g.,  $\vartheta_\varepsilon$  instead of  $\vartheta_{\varepsilon, \alpha}$ . We introduce the functions  $\kappa_\alpha$  and  $G_\alpha$  as follows:

$$\kappa_\alpha := \kappa \quad \text{if } \kappa_* > 0, \quad \text{and} \quad \kappa_\alpha := \kappa + \alpha \quad \text{if } \kappa_* = 0, \quad (3.1)$$

$$G_\alpha(r) := \int_0^r \kappa_\alpha(s) ds \quad \text{for } r \in \mathbb{R}. \quad (3.2)$$

Moreover, let  $d_K^\varepsilon : H \rightarrow \mathbb{R}$  and  $Dd_K^\varepsilon : H \rightarrow H$  be the Moreau–Yosida regularizations of the nondifferentiable function  $d_K$  and of its subdifferential  $\partial d_K$ . Thus, for  $v \in H$ , we have that

$$d_K^\varepsilon(v) := \inf_{z \in H} \left( d_K(z) + \frac{1}{2\varepsilon} \|z - v\|_H^2 \right) \quad (3.3)$$

$$Dd_K^\varepsilon(v) \text{ is the gradient of } d_K^\varepsilon \text{ at } v, \text{ i.e., the unique element of } \partial d_K^\varepsilon(v). \quad (3.4)$$

The statement (3.4) means that the map  $H \ni z \mapsto (Dd_K^\varepsilon(v), z)_H$  is the Fréchet derivative of  $d_K^\varepsilon$  at  $v$  ( $d_K^\varepsilon$  is Fréchet differentiable, indeed). We recall that the subdifferential  $\partial d_K^\varepsilon$ , which we identify with the single-valued map  $Dd_K^\varepsilon$ , actually is the Yosida regularization of  $\partial d_K$ , thus Lipschitz continuous with Lipschitz constant  $1/\varepsilon$  (see, e.g., [3, p. 28 and Prop. 2.11, p. 39]). These maps can be given explicitly, as shown in the next lemma. As we could not find precise references on it, we proved the result in the Appendix.

**Lemma 3.1.** *Let  $H$  be a Hilbert space. With the assumptions and notations (2.6)–(2.8) and (3.3)–(3.4), the formulas*

$$Dd_K^\varepsilon(v) = \frac{Q_K v}{\max\{\varepsilon, d_K(v)\}} \quad (3.5)$$

and

$$d_K^\varepsilon(v) = \int_0^{d_K(v)} \min\{s/\varepsilon, 1\} ds \quad (3.6)$$

hold true for every  $v \in H$ .

At this point, we introduce the approximating problem. It consists in finding a triplet  $(\vartheta_\varepsilon, u_\varepsilon, \sigma_\varepsilon)$  satisfying

$$\vartheta_\varepsilon, u_\varepsilon \in H^1(0, T; H) \cap L^\infty(0, T; V) \quad \text{and} \quad \sigma_\varepsilon \in L^\infty(0, T; H) \quad (3.7)$$

and solving the variational problem

$$\int_\Omega \partial_t \vartheta_\varepsilon(t) v + \int_\Omega \nabla u_\varepsilon(t) \cdot \nabla v + \rho \int_\Omega \sigma_\varepsilon(t) v = \int_\Omega f(t) v$$

for a.a.  $t \in (0, T)$  and every  $v \in V$  (3.8)

$$u_\varepsilon = G_\alpha(\vartheta_\varepsilon) \quad \text{a.e. in } Q \quad \text{and} \quad \sigma_\varepsilon(t) = Dd_K^\varepsilon(\vartheta_\varepsilon(t)) \quad \text{for a.a. } t \in (0, T) \quad (3.9)$$

$$\vartheta_\varepsilon(0) = \vartheta_0. \quad (3.10)$$

**Theorem 3.2.** *Under the assumptions and notations (2.1)–(2.10) and (3.1)–(3.4), for every  $\varepsilon, \alpha \in (0, 1)$ , problem (3.8)–(3.10) has a unique solution  $(\vartheta_\varepsilon, u_\varepsilon, \sigma_\varepsilon)$  satisfying (3.7).*

The rest of the section is devoted to prove this well-posedness result. We first establish the existence of a solution via a fixed point argument. Concerning the symbols  $\vartheta$ ,  $u$  and  $\sigma$  we often use, we point out that they have nothing to do with the original problem (2.14)–(2.16), which is out of interest at the moment.



**Existence.** For a given  $\bar{\vartheta} \in L^2(Q)$ , we look for a solution  $\vartheta \in H^1(0, T; H) \cap L^\infty(0, T; V)$  to the problem

$$\begin{aligned} \int_{\Omega} \partial_t \vartheta(t) v + \int_{\Omega} \kappa_{\alpha}(\vartheta) \nabla \vartheta(t) \cdot \nabla v &= \int_{\Omega} (f(t) - \rho \sigma(t)) v \\ \text{for a.a. } t \in (0, T) \text{ and every } v \in V & \end{aligned} \quad (3.11)$$

$$\vartheta(0) = \vartheta_0, \quad (3.12)$$

where  $\sigma(t) := Dd_K^\varepsilon(\bar{\vartheta}(t))$  for a.a.  $t \in (0, T)$ . As  $\kappa_{\alpha}$  is a continuous function such that  $\alpha \leq \kappa_{\alpha} \leq \kappa^* + 1$ , problem (3.11)–(3.12) has a unique solution  $\vartheta$  satisfying the prescribed regularity. Moreover, by testing (3.11) with  $v = \vartheta(t)$ , and noting that  $\|\sigma(t)\|_H \leq 1$  by (3.5), we immediately obtain that

$$\begin{aligned} \frac{1}{2} \|\vartheta(t)\|_H^2 &\leq \frac{1}{2} \|\vartheta_0\|_H^2 + \int_0^t (\|f(s)\|_H + \rho) \|\vartheta(s)\|_H ds \\ &\leq \frac{1}{2} \|\vartheta_0\|_H^2 + \|f\|_{L^2(0, T; H)}^2 + \rho^2 T + \frac{1}{2} \int_0^t \|\vartheta(s)\|_H^2 ds. \end{aligned}$$

By applying the Gronwall lemma, we deduce a bound in  $L^\infty(0, T; H)$  and infer that

$$\|\vartheta\|_{L^2(0, T; H)} \leq R \quad (3.13)$$

for some constant  $R$  depending only on the data,  $T$  and  $\rho$ . At this point, we denote by  $\mathcal{B}^2$  and  $\mathcal{B}^\infty$  the closed unit balls of  $L^2(0, T; H)$  and  $L^\infty(0, T; H)$ , respectively, set  $\mathcal{K} := R\mathcal{B}^2$  and define the maps  $\mathcal{S} : \mathcal{K} \rightarrow \mathcal{B}^\infty$  and  $\mathcal{F} : \mathcal{K} \rightarrow \mathcal{K}$  by setting for  $\bar{\vartheta} \in \mathcal{K}$

$$\mathcal{S}(\bar{\vartheta}) := \sigma \quad \text{given by } \sigma(t) := Dd_K^\varepsilon(\bar{\vartheta}(t)) \quad \text{for a.a. } t \in (0, T) \quad (3.14)$$

$$\vartheta := \mathcal{F}(\bar{\vartheta}) \in H^1(0, T; H) \cap L^\infty(0, T; V) \text{ is the unique solution to (3.11)–(3.12).} \quad (3.15)$$

We verify that we can apply the Schauder fixed point theorem to  $\mathcal{F}$  with respect to the strong topology of  $L^2(0, T; H)$ . Clearly,  $\mathcal{K}$  is nonempty, bounded, convex and closed. Next, if  $\bar{\vartheta} \in \mathcal{K}$  and  $\vartheta := \mathcal{F}(\bar{\vartheta})$ , then  $\vartheta$ ,  $u := G_{\alpha}(\vartheta)$  and  $\sigma := \mathcal{S}(\bar{\vartheta})$  satisfy  $\partial_t \vartheta - \Delta u = f - \rho \sigma$  in the sense of distributions on  $Q$ , in principle, then a.e. in  $Q$  since  $\partial_t \vartheta$  and the right-hand side belong to  $L^2(Q)$ . Moreover,  $u$  satisfies the homogeneous Neumann boundary condition. By multiplying the above equation by  $\partial_t u = \partial_t \vartheta / \kappa_{\alpha}(\vartheta)$  and also recalling that  $\kappa_{\alpha} \leq \kappa^* + 1$  and that  $\|\sigma\|_{L^\infty(0, T; H)} \leq 1$ , we easily obtain

$$\|\partial_t \vartheta\|_{L^2(0, T; H)} + \|\nabla u\|_{L^\infty(0, T; H)} \leq c, \quad \text{whence also } \|\nabla \vartheta\|_{L^\infty(0, T; H)} \leq c_{\alpha}$$

since  $\nabla \vartheta = \nabla u / \kappa_{\alpha}(\vartheta)$  and  $\kappa_{\alpha} \geq \alpha$ . We conclude that

$$\|\vartheta\|_{H^1(0, T; H) \cap L^\infty(0, T; V)} \leq c_{\alpha}. \quad (3.16)$$

By the Aubin-Lions lemma (see, e.g., [9, Thm. 5.1, p. 58]), we see that  $\mathcal{F}(\mathcal{K})$  is relatively compact in  $L^2(0, T; H)$ . Finally, we check that  $\mathcal{F}$  is continuous. Let  $\bar{\vartheta}_n, \bar{\vartheta} \in \mathcal{K}$  be such that  $\bar{\vartheta}_n \rightarrow \bar{\vartheta}$  strongly in  $L^2(Q)$ , and set  $\sigma_n := \mathcal{S}(\bar{\vartheta}_n)$  and  $\vartheta_n := \mathcal{F}(\bar{\vartheta}_n)$ . Then,  $\sigma_n$  converge to  $\sigma := \mathcal{S}(\bar{\vartheta})$  strongly in  $L^2(0, T; H)$  since  $Dd_K^\varepsilon$  is Lipschitz continuous on  $H$ . Furthermore, estimate (3.16) holds for  $\vartheta_n$ . Therefore, we have (for a subsequence)

$$\begin{aligned} \vartheta_n \rightarrow \vartheta & \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \\ & \quad \text{strongly in } L^2(0, T; H) \text{ and a.e. in } Q \end{aligned}$$

for some  $\vartheta \in H^1(0, T; H) \cap L^\infty(0, T; V)$  which necessarily belongs to  $\mathcal{K}$ . Since  $\kappa_\alpha$  is continuous and bounded, we infer that  $\kappa_\alpha(\vartheta_n)$  converges to  $\kappa_\alpha(\vartheta)$  strongly in  $L^p(Q)$  for every  $p \in [1, +\infty)$ . Thus, it is straightforward to deduce that  $\vartheta$  solves (3.11)–(3.12), i.e., that  $\vartheta = \mathcal{F}(\vartheta)$ , and that the convergence holds for the whole sequence  $\{\vartheta_n\}$ . Therefore,  $\mathcal{F}$  is continuous and we conclude that it has at least a fixed point. Now, if  $\vartheta_\varepsilon$  is a fixed point of  $\mathcal{F}$  and we set  $u_\varepsilon := G_\alpha(\vartheta_\varepsilon)$  and  $\sigma_\varepsilon := Dd_K^\varepsilon(\vartheta_\varepsilon)$ , one easily sees that the triplet  $(\vartheta_\varepsilon, u_\varepsilon, \sigma_\varepsilon)$  satisfies (3.7) and it is clear that it is a solution to problem (3.8)–(3.10).

**Uniqueness.** Let  $(\vartheta_1, u_1, \sigma_1)$  and  $(\vartheta_2, u_2, \sigma_2)$  be two solutions of problem (3.8)–(3.10) satisfying the regularity requirement (3.7). We write (3.8) for both of them, take the difference, and integrate with respect to time. We have for almost every  $s \in (0, T)$  and every  $v \in V$

$$\int_{\Omega} (\vartheta_1(s) - \vartheta_2(s)) v + \int_{\Omega} \nabla(1 * (u_1 - u_2))(s) \cdot \nabla v = -\rho \int_{\Omega} (1 * (\sigma_1 - \sigma_2))(s) v$$

with the general notation  $(1 * v)(s) := \int_0^s v(\tau) d\tau$ . Now, we choose  $v = (u_1 - u_2)(s)$  and integrate over  $(0, t)$  with respect to  $s$ . We obtain

$$\int_{Q_t} (\vartheta_1 - \vartheta_2)(u_1 - u_2) + \frac{1}{2} \int_{\Omega} |(1 * \nabla(u_1 - u_2))(t)|^2 = -\rho \int_{Q_t} (1 * (\sigma_1 - \sigma_2))(u_1 - u_2).$$

We recall that  $\alpha \leq G'_\alpha \leq \kappa^* + 1$  and ignore the second term on the left-hand side, which is nonnegative. Furthermore, we owe to the Young inequality. We deduce that

$$\alpha \int_0^t \|(\vartheta_1 - \vartheta_2)(s)\|_H^2 ds \leq \frac{\alpha}{2} \int_0^t \|(\vartheta_1 - \vartheta_2)(s)\|_H^2 ds + \frac{c}{\alpha} \int_{Q_t} |1 * (\sigma_1 - \sigma_2)|^2. \quad (3.17)$$

Now, we use the Hölder inequality and account for the  $(1/\varepsilon)$ -Lipschitz continuity of  $Dd_K^\varepsilon$  (as a map from  $H$  into itself). We have for every  $s \in [0, T]$

$$\begin{aligned} \|(1 * (\sigma_1 - \sigma_2))(s)\|_H^2 &= \|\int_0^s (\sigma_1 - \sigma_2)(\tau) d\tau\|_H^2 \\ &\leq c \int_0^s \|(\sigma_1 - \sigma_2)(\tau)\|_H^2 d\tau \leq c_\varepsilon \int_0^s \|(\vartheta_1 - \vartheta_2)(\tau)\|_H^2 d\tau \end{aligned}$$

and deduce that

$$\int_{Q_t} |1 * (\sigma_1 - \sigma_2)|^2 \leq c_\varepsilon \int_0^t \left( \int_0^s \|(\vartheta_1 - \vartheta_2)(\tau)\|_H^2 d\tau \right) ds.$$

Coming back to (3.17) and applying the Gronwall lemma, we conclude that  $\vartheta_1 = \vartheta_2$ , whence also  $u_1 = u_2$  and  $\sigma_1 = \sigma_2$ .

## 4 Well-posedness

This section deals with Theorems 2.1 and 2.2. In order to prove the statements regarding existence, we start from the solution  $(\vartheta_\varepsilon, u_\varepsilon, \sigma_\varepsilon)$  to the approximating problem (3.8)–(3.10) and perform a number of a priori estimates in which all of the occurring constants  $c > 0$  will be independent of both  $\varepsilon$  and  $\alpha$ .

**First a priori estimate.** We test (3.8) by  $\vartheta_\varepsilon$  and have

$$\frac{1}{2} \int_{\Omega} |\vartheta_\varepsilon(t)|^2 + \int_{Q_t} \nabla u_\varepsilon \cdot \nabla \vartheta_\varepsilon = \frac{1}{2} \int_{\Omega} |\vartheta_0|^2 + \int_{Q_t} (f - \rho \sigma_\varepsilon) \vartheta_\varepsilon.$$

We recall that  $\|\sigma_\varepsilon\|_{L^\infty(0,T;H)} \leq 1$  (cf. (3.5)) and observe that there hold, a.e. in  $Q$ ,

$$\begin{aligned} |u_\varepsilon| &= |G_\alpha(\vartheta_\varepsilon)| \leq (\kappa^* + 1) |\vartheta_\varepsilon| \\ \nabla u_\varepsilon \cdot \nabla \vartheta_\varepsilon &= \frac{1}{\kappa_\alpha(\vartheta_\varepsilon)} |\nabla u_\varepsilon|^2 \geq \frac{1}{\kappa^* + 1} |\nabla u_\varepsilon|^2 \\ \nabla u_\varepsilon \cdot \nabla \vartheta_\varepsilon &= \kappa_\alpha(\vartheta_\varepsilon) |\nabla \vartheta_\varepsilon|^2 \geq \kappa_* |\nabla \vartheta_\varepsilon|^2. \end{aligned}$$

Hence, by also owing to the Gronwall lemma, we easily deduce that

$$\|\vartheta_\varepsilon\|_{L^\infty(0,T;H)} + \|u_\varepsilon\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c \quad (4.1)$$

$$\|\vartheta_\varepsilon\|_{L^2(0,T;V)} \leq c \quad \text{provided that } \kappa_* > 0. \quad (4.2)$$

By comparison in the variational equation (2.14), we infer from (4.1) that

$$\|\partial_t \vartheta_\varepsilon\|_{L^2(0,T;V^*)} \leq c. \quad (4.3)$$

**Second a priori estimate.** We notice that (2.14) can be written as

$$\partial_t \vartheta_\varepsilon - \operatorname{div}(\kappa_\alpha(\vartheta_\varepsilon) \nabla \vartheta_\varepsilon) = f - \rho \sigma_\varepsilon \quad \text{a.e. in } Q,$$

with the homogeneous Neumann boundary condition for  $u_\varepsilon$ . By multiplying by  $\partial_t u_\varepsilon$ , integrating over  $Q_t$ , and applying the Hölder and Young inequalities, we obtain

$$\int_{Q_t} \frac{1}{\kappa_\alpha(\vartheta_\varepsilon)} |\partial_t u_\varepsilon|^2 + \frac{1}{2} \int_{\Omega} |\nabla u_\varepsilon(t)|^2 \leq c + \frac{1}{2(\kappa^* + 1)} \int_{Q_t} |\partial_t u_\varepsilon|^2.$$

As  $\kappa_\alpha(\vartheta_\varepsilon) \leq \kappa^* + 1$ , we deduce on account of (4.1) that

$$\|u_\varepsilon\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq c. \quad (4.4)$$

Moreover, from the identities

$$\partial_t \vartheta_\varepsilon = \frac{\partial_t u_\varepsilon}{\kappa_\alpha(\vartheta_\varepsilon)} \quad \text{and} \quad \nabla \vartheta_\varepsilon = \frac{\nabla u_\varepsilon}{\kappa_\alpha(\vartheta_\varepsilon)}$$

we infer that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq c \quad \text{provided that } \kappa_* > 0. \quad (4.5)$$

**Convergence.** At this point, by standard compactness results (in particular, we owe to the Aubin-Lions lemma proved, e.g., in [9, Thm. 5.1, p. 58]), we deduce that

$$\begin{aligned} \vartheta_\varepsilon &\rightarrow \vartheta && \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \\ &&& \text{and strongly in } L^2(0,T;V^*) \end{aligned} \quad (4.6)$$

$$\begin{aligned} u_\varepsilon &\rightarrow u && \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \\ &&& \text{strongly in } L^2(0,T;H) \text{ and a.e. in } Q \end{aligned} \quad (4.7)$$

$$\sigma_\varepsilon \rightarrow \sigma \quad \text{weakly star in } L^\infty(0,T;H) \quad (4.8)$$

for some triplet  $(\vartheta, u, \sigma)$ , as  $(\varepsilon, \alpha)$  tends to  $(0, 0)$ , at least for a subsequence. Moreover, we also have

$$\begin{aligned} \vartheta_\varepsilon \rightarrow \vartheta \quad & \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \\ & \text{strongly in } L^2(0, T; H) \text{ and a.e. in } Q \quad \text{provided that } \kappa_* > 0. \end{aligned} \quad (4.9)$$

Thus, it is clear that  $(\vartheta, u, \sigma)$  is a solution to (2.14)–(2.16) satisfying the regularity requirements stated in Theorems 2.1 and 2.2, with the exception of (2.22), whenever we prove that  $u = G(\vartheta)$  a.e. in  $Q$  and that  $\sigma(t) \in \partial d_K(\vartheta(t))$  for a.a.  $t \in (0, T)$ . At this point, we have to distinguish the different cases corresponding to the above statements.

**Conclusion of the existence proof in the uniformly parabolic case.** We complete the proof of the existence part of Theorem 2.1 by assuming  $\kappa_* > 0$ . Thus,  $\kappa_\alpha = \kappa$  and  $G_\alpha = G$ . Thanks to the pointwise convergence (a.e.) given by (4.7) and (4.9) and to the continuity of  $G$ , we immediately deduce that  $u = G(\vartheta)$  a.e. in  $Q$ . As for the second condition in (2.15), we owe to the strong convergence (4.9) and apply, e.g., [1, Lemma 2.3, p. 38] to the maximal monotone operator induced on  $L^2(0, T; H)$  by  $\partial d_K$ .

On the contrary, for the degenerate case allowed in Theorem 2.2, some more work has to be done. Concerning the relation  $u = G(\vartheta)$  that we have to prove, we observe that  $G_\alpha(r) \geq G(r)$  for  $r \geq 0$  and  $G_\alpha(r) \leq G(r)$  for  $r \leq 0$  since  $\kappa \leq \kappa_\alpha$ . It follows that  $\inf G_\alpha = -\infty \leq \inf G$  and  $\sup G_\alpha = +\infty \geq \sup G$ , i.e., the domain  $D(G_\alpha^{-1})$  of  $G_\alpha^{-1}$  includes the domain  $D(G^{-1})$  of  $G^{-1}$ .

**Lemma 4.1.** *The following convergence holds true:*

$$G_\alpha^{-1}(s) \rightarrow G^{-1}(s) \quad \text{uniformly in every compact subset of } D(G^{-1}). \quad (4.10)$$

*Proof.* We first establish the pointwise convergence. This trivially holds if  $s = 0$ . Assume  $s > 0$ . Then,  $G_\alpha^{-1}(s) > G_{\alpha'}^{-1}(s) > G^{-1}(s)$  for  $\alpha' \in (0, \alpha)$  since  $G(r) < G_{\alpha'}(r) < G_\alpha(r)$  for every  $r > 0$ . Thus the limit  $\ell$  of  $G_\alpha^{-1}(s)$  as  $\alpha \searrow 0$  exists and satisfies  $\ell \geq \ell_0 := G^{-1}(s)$ . It follows that the constant  $s = G_\alpha(G_\alpha^{-1}(s))$  converges to  $G(\ell)$ , i.e., that  $s = G(\ell)$ . As  $G(\ell_0) = s$  and  $G$  is one-to-one by (2.3), we conclude that  $\ell = \ell_0$ . Thus, we have proved that  $G_\alpha^{-1}(s)$  converges to  $G^{-1}(s)$  pointwise in the interval  $D(G^{-1}) \cap [0, +\infty)$ . Since the convergence is monotone and the limit  $G^{-1}$  is continuous, the convergence is uniform on every compact subset (Dini's theorem). As the case of negative values of  $s$  is similar, (4.10) is proved.  $\square$

At this point, we can go on and show that  $u = G(\vartheta)$  a.e. in  $Q$ . From Lemma 4.1 and the pointwise convergence (4.7) of  $u_\varepsilon$  to  $u$  we infer that  $\vartheta_\varepsilon = G_\alpha^{-1}(u_\varepsilon)$  converges to  $G^{-1}(u)$  a.e. in  $Q$ . Since  $\vartheta_\varepsilon$  converges to  $\vartheta$  weakly in  $L^2(Q)$  by (4.6), we conclude that (see, e.g., [9, Lemme 1.3, p. 12])  $G^{-1}(u) = \vartheta$ , i.e.,  $u = G(\vartheta)$ , a.e. in  $Q$ . As a by-product, there holds the convergence

$$\vartheta_\varepsilon \rightarrow \vartheta \quad \text{a.e. in } Q, \quad (4.11)$$

and we use it to prove that  $\sigma(t) \in \partial d_K(\vartheta(t))$  for a.a.  $t \in (0, T)$ . Here, we owe to assumptions (2.18)–(2.19) and (2.21) on the convex  $K$  and on the data. For convenience, we set

$$p_I : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is the projection on } I \quad \text{and} \quad q_I := I_{\mathbb{R}} - p_I,$$

where  $I_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$  is the identity map, so that (2.20) reads  $(P_K v)(x) = p_I(v(x))$  a.e. in  $\Omega$ , for every  $v \in H$ . Hence, (3.5) becomes

$$(Dd_K^\varepsilon(v))(x) = \frac{q_I(v(x))}{\max\{\varepsilon, d_K(v)\}} \quad \text{a.e. in } \Omega, \quad \text{for every } v \in H. \quad (4.12)$$

**A new a priori estimate.** We define the truncation operator  $T_n : \mathbb{R} \rightarrow \mathbb{R}$  by setting  $T_n(r) := \max\{-n, \min\{r, n\}\}$  for  $r \in \mathbb{R}$  and use the notation  $(r)^q := |r|^q \text{sign } r$  (with  $\text{sign } 0 = 0$ ) for  $r \in \mathbb{R}$  and  $q > 0$ . Next, we take  $\vartheta_* \in I$ , set  $\zeta := \vartheta_\varepsilon - \vartheta_*$  for convenience and test (2.14) by  $(T_n(\zeta))^{p-1}$ . By integrating over  $Q_t$  and recalling that  $p > 2$ , we obtain that

$$\begin{aligned} & \int_{\Omega} T_{n,p}(\zeta(t)) + (p-1) \int_{Q_t} \kappa_\alpha(\vartheta_\varepsilon) |T_n(\zeta)|^{p-2} T_n(\zeta) |\nabla \vartheta_\varepsilon|^2 + \rho \int_{Q_t} \sigma_\varepsilon (T_n(\zeta))^{p-1} \\ &= \int_{\Omega} T_{n,p}(\zeta(0)) + \int_{Q_t} f (T_n(\zeta))^{p-1}, \end{aligned} \quad (4.13)$$

where we have set

$$T_{n,p}(r) := \int_0^r (T_n(s))^{p-1} ds \quad \text{for } r \in \mathbb{R}.$$

We observe that  $pT_{n,p}(r) \geq |T_n(r)|^p$  for every  $r \in \mathbb{R}$ . Indeed, if  $0 \leq r \leq n$ , we have  $pT_{n,p}(r) = r^p = |T_n(r)|^p$ ; if  $r > n$ , then  $pT_{n,p}(r) \geq pT_{n,p}(n) = n^p = |T_n(r)|^p$ . On the other hand, both  $T_{n,p}$  and  $|T_n|^p$  are even functions. Therefore, we have

$$\int_{\Omega} T_{n,p}(\zeta(t)) \geq \frac{1}{p} \|T_n(\zeta(t))\|_p^p.$$

The second term of (4.13) is nonnegative. For the third one, we note that the following pairs of functions share their signs:  $q_I(\vartheta_\varepsilon)$  and  $\zeta$  since  $\vartheta_* \in I$ ;  $\sigma_\varepsilon$  and  $q_I \vartheta_\varepsilon$  thanks to (4.12);  $(T_n(\zeta))^{p-1}$  and  $\zeta$  by our definition of  $(\cdot)^{p-1}$ . Thus, the same holds for  $\sigma_\varepsilon$  and  $(T_n(\zeta))^{p-1}$ , so that their product is nonnegative. To deal with the right-hand side, we use assumption (2.21) on  $\vartheta_0$  and  $f$ . First, we notice that  $T_{n,p}(r) \leq |r|^p$  for every  $r \in \mathbb{R}$  so that the first integral can be estimated from above by  $\|\vartheta_0 - \vartheta_*\|_p^p$ . Next, we account for the Hölder inequality and treat the last term as follows:

$$\begin{aligned} & \int_{Q_t} f (T_n(\zeta))^{p-1} \leq \int_0^t \|f(s)\|_p \|(T_n(\zeta(s)))^{p-1}\|_{p'} \\ &= \int_0^t \|f(s)\|_p (\|T_n(\zeta(s))\|_p^p)^{1/p'} ds \leq c \int_0^t \|f(s)\|_p (1 + \|T_n(\zeta(s))\|_p^p) ds. \end{aligned}$$

Therefore, coming back to (4.13) and recalling that  $f \in L^1(0, T; L^p(\Omega))$ , we can apply the Gronwall lemma and deduce that  $\|T_n(\zeta)\|_{L^\infty(0, T; L^p(\Omega))} \leq c$ , whence immediately

$$\|\vartheta_\varepsilon\|_{L^\infty(0, T; L^p(\Omega))} \leq c. \quad (4.14)$$

**Conclusion of the proof of Theorem 2.2.** Estimate (4.14) ensures the further regularity (2.22) for the function  $\vartheta$  given by (4.6). On the other hand, the pointwise convergence (4.11), our assumption  $p > 2$ , (4.14), and the Egorov theorem imply that

$$\vartheta_\varepsilon \rightarrow \vartheta \quad \text{strongly in } L^2(Q). \quad (4.15)$$

Thus, the strong convergence (4.9) already established in the uniformly parabolic case holds also in the present one. Therefore, we can combine it with the weak convergence of  $\sigma_\varepsilon$  to  $\sigma$  in  $L^2(Q)$  ensured by (4.8) and proceed as before, i.e., we apply, e.g., [1, Lemma 2.3, p. 38] to the maximal monotone operator induced on  $L^2(0, T; H)$  by  $\partial d_K$ . We conclude that  $\sigma(t) \in \partial d_K(\vartheta(t))$  for a.a.  $t \in (0, T)$ , and the proof of Theorem 2.2 is complete.  $\square$

**Uniqueness.** We prove the last sentence of Theorem 2.1 by assuming that  $\kappa$  is a positive constant. We pick two solutions  $(\vartheta_i, u_i, \sigma_i)$ ,  $i = 1, 2$ , write (2.14) for both of them, and test the difference by  $\vartheta_1 - \vartheta_2$ . We obtain

$$\frac{1}{2} \int_{\Omega} |(\vartheta_1 - \vartheta_2)(t)|^2 + \kappa \int_{Q_t} |\nabla(\vartheta_1 - \vartheta_2)|^2 + \rho \int_{Q_t} (\sigma_1 - \sigma_2)(\vartheta_1 - \vartheta_2) = 0.$$

All of the terms on the left-hand side are nonnegative, the third one since  $\sigma_i(t) \in \partial d_K(\vartheta_i(t))$  for a.a.  $t \in (0, T)$  and  $\partial d_K : H \rightarrow 2^H$  is monotone. We immediately deduce that  $\vartheta_1 = \vartheta_2$ , whence also  $u_1 = u_2$ . By comparison in (2.14), we infer that  $\sigma_1 = \sigma_2$  as well.  $\square$

## 5 Proof of Theorem 2.3

It suffices to prove that the solution  $(\vartheta, u, \sigma)$  constructed in the previous section satisfies the conditions of the statement. Therefore, we keep the notation already used and the boundedness and convergence specified in the above proofs (the latter for a not relabeled subsequence  $(\varepsilon, \alpha) \rightarrow (0, 0)$ , as usual). In fact, we only need to know that  $(\vartheta, u, \sigma)$  is a solution to (2.14)–(2.16) satisfying the regularity conditions (2.11)–(2.13) and to account for

$$\|\vartheta_\varepsilon\|_{L^\infty(0, T; H)} \leq c \quad \text{and} \quad \vartheta_\varepsilon \rightarrow \vartheta \quad \text{strongly in } L^2(Q). \quad (5.1)$$

**The differential inequality.** We test (3.8) by  $\sigma_\varepsilon(t) = Dd_K^\varepsilon(\vartheta_\varepsilon(t))$ , integrate over  $\Omega$ , and obtain, for a.a.  $t \in (0, T)$ ,

$$\frac{d}{dt} d_K^\varepsilon(\vartheta_\varepsilon(t)) + \int_{\Omega} \kappa_\alpha(\vartheta_\varepsilon(t)) \nabla \vartheta_\varepsilon(t) \cdot \nabla \sigma_\varepsilon(t) + \rho \|\sigma_\varepsilon(t)\|_H^2 = \int_{\Omega} f(t) \sigma_\varepsilon(t). \quad (5.2)$$

We observe that the second term on the left-hand side is nonnegative on account of (4.12). Indeed, for almost all  $x \in \Omega$ , we have

$$\sigma_\varepsilon(x, t) = \frac{q_I(\vartheta_\varepsilon(x, t))}{\max\{\varepsilon, d_K(\vartheta_\varepsilon(t))\}}, \quad \text{whence} \quad \nabla \sigma_\varepsilon(x, t) = \frac{q_I'(\vartheta_\varepsilon(x, t)) \nabla \vartheta_\varepsilon(x, t)}{\max\{\varepsilon, d_K(\vartheta_\varepsilon(t))\}}.$$

On the other hand, both  $\kappa_\alpha$  and  $q_I'$  are nonnegative functions. As for the right-hand side, we recall assumption (2.25) on  $f$  and that  $\|\sigma_\varepsilon(t)\|_H \leq 1$  for a.a.  $t \in (0, T)$  by (3.9) and (3.5). Therefore, we deduce from (5.2) that

$$\frac{d}{dt} d_K^\varepsilon(\vartheta_\varepsilon(t)) + \rho \|\sigma_\varepsilon(t)\|_H^2 \leq \rho^* := \|f\|_{L^\infty(0, T; H)}. \quad (5.3)$$

Notice that the definition of  $\rho^*$  agrees with (2.25). We observe that the formulas (3.5)–(3.6) imply that  $\|Dd_K^\varepsilon(v)\|_H = 1$  if  $d_K(v) > \varepsilon$  and that  $d_K(v) > \varepsilon$  if and only if  $d_K^\varepsilon(v) >$

$\varepsilon/2$ . Hence, we deduce from (5.3) that the functions  $\psi_\varepsilon \in W^{1,1}(0, T)$  and  $\varphi_\varepsilon \in L^\infty(0, T)$ , defined by

$$\psi_\varepsilon(t) := d_K^\varepsilon(\vartheta_\varepsilon(t)) \quad \text{and} \quad \varphi_\varepsilon(t) := \|\sigma_\varepsilon(t)\|_H^2, \quad (5.4)$$

satisfy

$$\psi'_\varepsilon(t) + \rho \varphi_\varepsilon(t) \leq \rho^* \quad \text{for a.a. } t \in (0, T) \quad (5.5)$$

$$\varphi_\varepsilon(t) = 1 \quad \text{a.e. in the set where } \psi_\varepsilon > \varepsilon/2. \quad (5.6)$$

**Lemma 5.1.** *Let  $\psi \in W^{1,1}(0, T)$  and  $\varphi \in L^\infty(0, T)$  and assume that for some positive constant  $\rho, \delta, \gamma$  the following conditions hold*

$$\psi'(t) + \rho \varphi(t) \leq \rho - \delta \quad \text{for a.a. } t \in (0, T)$$

$$\varphi(t) = 1 \quad \text{a.e. in the set where } \psi > \gamma.$$

*Then, we have  $\psi(t) \leq \gamma$  for every  $t \in [0, T]$  if  $\psi(0) \leq \gamma$ . Moreover, if  $\psi(0) > \gamma$ , then there exists some  $T_\gamma \in (0, T]$  satisfying  $T_\gamma \leq (\psi(0) - \gamma)/\delta$  such that*

$$\psi' \leq -\delta \quad \text{a.e. in } (0, T_\gamma) \quad \text{and} \quad \psi(t) \leq \gamma \quad \text{for every } t \in [0, T] \text{ such that } t > T_\gamma. \quad (5.7)$$

*Proof.* We start with the following statement:

$$\text{if } t_0 \in [0, T) \text{ and } \psi(t_0) \leq \gamma, \text{ then } \psi(t) \leq \gamma \text{ for every } t \in [t_0, T]. \quad (5.8)$$

By contradiction, assume that there exists some  $t \in (t_0, T]$  such that  $\psi(t) > \gamma$  and consider the set  $A := \{s \in (t_0, t] : \psi(\tau) > \gamma \text{ for every } \tau \in (s, t]\}$ . Then,  $A$  is nonempty and  $t_1 := \inf A$  satisfies  $t_1 \geq t_0$ , whence  $\psi(t_1) = \gamma$ . Moreover,  $t_1 < t$  and  $\psi(\tau) > \gamma$  for every  $\tau \in (t_1, t]$ . From the assumptions, it follows that  $\psi' \leq -\delta$  a.e. in  $(t_1, t)$ . Thus,  $\psi(t) \leq \psi(t_1) - \delta(t - t_1) < \gamma$ , a contradiction. Therefore, (5.8) is established. From (5.8) we deduce the first sentence of the statement regarding the case  $\psi(0) \leq \gamma$ . Assume now that  $\psi(0) > \gamma$ , and set  $T_\gamma := \sup\{t \in (0, T] : \psi(s) > \gamma \text{ for every } s \in [0, t]\}$ . Then,  $T_\gamma > 0$  and  $\psi(s) > \gamma$  for every  $s \in [0, T_\gamma)$ . Our assumptions imply that  $\varphi = 1$  and  $\psi' \leq -\delta$  a.e. in  $(0, T_\gamma)$ . Thus, the first part of (5.7) is proved. Let us pass to the second one, by assuming that  $T_\gamma < T$ . Then,  $\psi(T_\gamma) = \gamma$ , whence  $\psi(t) \leq \gamma$  for every  $t \in [T_\gamma, T]$ , by (5.8).  $\square$

At this point, we can continue the proof of Theorem 2.3. We assume that  $\rho > \rho^*$  and prepare some material. Then, we prove the sentences *i*) and *ii*) of the statement.

**Preliminary remarks.** By recalling (5.5)–(5.6), we apply Lemma 5.1 to the functions (5.4) with  $\delta = \rho - \rho^*$  and  $\gamma = \varepsilon/2$ , and set  $T_{\varepsilon, \alpha}^* := T_\gamma$  if  $d_K^\varepsilon(\vartheta_0) > \varepsilon/2$ . Next, we observe that [3, Prop. 2.11, p. 39] implies

$$d_K^\varepsilon(\vartheta_0) \rightarrow d_K(\vartheta_0). \quad (5.9)$$

Now, we recall the strong convergence (5.1) and deduce that

$$\vartheta_\varepsilon(t) \rightarrow \vartheta(t) \quad \text{strongly in } H \quad \text{for a.a. } t \in (0, T). \quad (5.10)$$

Note that we can establish pointwise strong convergence only in the uniformly parabolic case (the weak star convergence (4.9) implies strong convergence in  $C^0([0, T]; H)$ , indeed),

while, in general, we just have the almost everywhere strong convergence (5.10). We prove that (5.10) implies that

$$d_K^\varepsilon(\vartheta_\varepsilon(t)) \rightarrow d_K(\vartheta(t)) \quad \text{for a.a. } t \in (0, T). \quad (5.11)$$

For a fixed  $t$  for which the strong convergence (5.10) holds, by setting  $v_\varepsilon := \vartheta_\varepsilon(t)$  and  $v := \vartheta(t)$  for brevity, we have from (3.6) that

$$d_K(\vartheta(t)) - d_K^\varepsilon(\vartheta_\varepsilon(t)) = \int_0^{d_K(v)} (1 - \min\{s/\varepsilon, 1\}) ds - \int_{d_K(v_\varepsilon)}^{d_K(v_\varepsilon)} \min\{s/\varepsilon, 1\} ds.$$

The first integral tends to zero by the Lebesgue dominated convergence theorem, while the absolute value of the second one is estimated by  $|d_K(v_\varepsilon) - d_K(v)| \leq \|v_\varepsilon - v\|_H$ .

**First case.** Suppose that  $\vartheta_0 \in K$ . Then,  $d_K(\vartheta_0) = 0$ , whence also  $d_K^\varepsilon(\vartheta_0) = 0$ . Thus,  $d_K^\varepsilon(\vartheta_\varepsilon(t)) \leq \varepsilon/2$  for every  $t \in [0, T]$ . Hence, from (5.11), we infer that  $d_K(\vartheta(t)) = 0$ , i.e.,  $\vartheta(t) \in K$ , for a.a.  $t \in (0, T)$ . In order to extend this to the whole interval  $[0, T]$ , we owe to (2.24) and to the fact that the convex set  $K$  is weakly closed. Therefore, the property  $\vartheta(t) \in K$  proved for a dense subset of  $[0, T]$  holds true for whole interval.

**Second case.** Let us now suppose that  $\vartheta_0 \notin K$ . Thus  $d_K(\vartheta_0) > 0$  and we can assume that  $\varepsilon < d_K(\vartheta_0)$ . Hence, we also have  $d_K^\varepsilon(\vartheta_0) > \varepsilon/2$  and the time  $T_{\varepsilon, \alpha}^* \in (0, T]$  is well defined. We set

$$T^* := \liminf_{(\varepsilon, \alpha) \rightarrow (0, 0)} T_{\varepsilon, \alpha}^*. \quad (5.12)$$

Thus,  $T^* \in [0, T]$ , and we prove that it satisfies the conditions of Theorem 2.3. From the lemma and (5.9), we have

$$T_{\varepsilon, \alpha}^* \leq \frac{d_K^\varepsilon(\vartheta_0)}{\rho - \rho^*}, \quad \text{whence} \quad T^* \leq \frac{d_K(\vartheta_0)}{\rho - \rho^*}, \quad (5.13)$$

i.e., the upper bound of the statement. We now show that  $T^* > 0$  and argue by contradiction, i.e., we assume that  $T_{\varepsilon, \alpha}^*$  tends to zero for a subsequence. From the inequality  $d_K^\varepsilon(\vartheta_\varepsilon(t)) \leq \varepsilon/2$  for  $t \geq T_{\varepsilon, \alpha}^*$  and (5.11), we deduce that  $d_K(\vartheta(t)) = 0$  for a.a.  $t \in (0, T)$ . Hence, we have  $\vartheta(t) \in K$  for a.a.  $t \in (0, T)$  and (2.24) yields  $\vartheta_0 \in K$ , a contradiction. Hence, we can consider the nonempty interval  $(0, T^*)$  and prove (2.27). To this end, we take any  $t_0 \in (0, T^*)$ . We can assume that  $T_{\varepsilon, \alpha}^* > t_0$ , so that the first sentence of (5.7) implies

$$\frac{d}{dt} d_K^\varepsilon(\vartheta_\varepsilon(t)) \leq -(\rho - \rho^*) \quad \text{for a.a. } t \in (0, t_0). \quad (5.14)$$

Now, we fix  $\bar{\vartheta} \in K$  and have  $d_K^\varepsilon(\vartheta_\varepsilon(t)) \leq d_K(\vartheta_\varepsilon(t)) \leq \|\vartheta_\varepsilon(t) - \bar{\vartheta}\|_H \leq c$  for a.a.  $t \in (0, T)$  thanks to the first condition in (5.1). Hence, by accounting for (5.11) once more, we derive that  $d_K^\varepsilon(\vartheta_\varepsilon(\cdot))$  converges to  $d_K(\vartheta(\cdot))$  strongly in  $L^2(0, T)$ , whence the convergence in the sense of distributions follows for their derivatives, and the inequality (5.14) is conserved in the limit. As  $t_0$  is arbitrary, (2.27) is proved. Finally, we show (2.28) by assuming  $T^* < T$ . Take any  $t_0 \in (T^*, T)$ . We can assume that  $T_{\varepsilon, \alpha}^* < t_0$  (for a subsequence). Hence, we have  $d_K^\varepsilon(\vartheta_\varepsilon(t)) \leq \varepsilon/2$  for every  $t \in [t_0, T]$ . It follows that  $d_K(\vartheta(t)) = 0$  for a.a.  $t \in (t_0, T)$ , by (5.11). As  $t_0$  is arbitrary, we have  $\vartheta(t) \in K$  for a.e.  $t \in (T^*, T)$ , and we conclude that  $\vartheta(t) \in K$  for every  $t \in [T^*, T]$ , by the weak continuity (2.24). This completes the proof.  $\square$



## 6 Appendix

In this section, we prove Lemma 3.1. We denote by  $(\cdot, \cdot)$  the scalar product of  $H$  and write  $\|\cdot\|$  instead of  $\|\cdot\|_H$ . We start proving that the function  $d_K$  is Fréchet differentiable at any point  $v \in H \setminus K$  and that its gradient is given by

$$Dd_K(v) = \frac{Q_K v}{\|Q_K v\|_H}. \quad (6.1)$$

This easily follows from [6, Prop. 2.2]. Indeed, we immediately deduce from this result that  $2Q_K$  is the Fréchet derivative of the map  $v \mapsto \varphi(v) := \|Q_K v\|^2$ . Now, we read  $d_K$  as the square root of  $\varphi$  and assume that  $v \in H \setminus K$ , i.e.,  $\varphi(v) > 0$ . Then,  $\varphi > 0$  in a neighborhood of  $v$ , whence  $d_K$  is Fréchet differentiable at  $v$  and (6.1) follows from applying the chain rule:

$$Dd_K(v) = \frac{1}{2} (\varphi(v))^{-1/2} D\varphi(v) = \frac{1}{2} (d_K(v))^{-1} 2Q_K v = \frac{Q_K v}{\|Q_K v\|}.$$

On the contrary, the proof of the rest of the lemma needs some work. Assume first  $v \in K$ . Then,  $v$  is a minimum point for  $d_K$  and  $d_K^\varepsilon$ , whence  $0 \in \partial d_K(v)$  and  $Dd_K^\varepsilon(v) = 0$ . On the other hand, we have  $Q_K v = 0$ , and thus (3.5) holds true in this case. Assume now  $d_K(v) > \varepsilon$ . Then, the point

$$\xi := \frac{Q_K(v)}{d_K(v)}$$

satisfies  $\|\xi\| = 1$  and  $v - \varepsilon\xi \notin K$ . Hence, (3.5) reduces to  $Dd_K^\varepsilon(v) = \xi$  and thus means that  $Dd_K(v - \varepsilon\xi) = \xi$  (by the definition of the Yosida regularization). Therefore, we prove this fact. We set  $\lambda := 1 - \varepsilon/d_K(v)$  and observe that  $\lambda > 0$  and that

$$v - \varepsilon\xi - P_K v = Q_K v - \frac{\varepsilon}{d_K(v)} Q_K v = \lambda Q_K v = \lambda(v - P_K v). \quad (6.2)$$

Then we have, for every  $z \in K$ ,

$$(v - \varepsilon\xi - P_K v, z - P_K v) = \lambda(v - P_K v, z - P_K v) \leq 0.$$

As  $P_K v \in K$ , this shows that  $P_K(v - \varepsilon\xi) = P_K(v)$ . By applying (6.1) to  $v - \varepsilon\xi$  and recalling (6.2) once more, we obtain

$$Dd_K(v - \varepsilon\xi) = \frac{v - \varepsilon\xi - P_K v}{\|v - \varepsilon\xi - P_K v\|} = \frac{\lambda Q_K v}{\|\lambda Q_K v\|} = \xi.$$

Hence, the desired equality is established under the assumption  $d_K(v) > \varepsilon$ . As  $d_K^\varepsilon$ ,  $Q_K$  and  $d_K$  are continuous, (3.5) holds also if  $d_K(v) = \varepsilon$ , and we consider the last case, i.e.,  $0 < d_K(v) < \varepsilon$ . We first prove that

$$d_K^\varepsilon(v) = \frac{1}{2\varepsilon} (d_K(v))^2 \quad \text{if } 0 < d_K(v) < \varepsilon. \quad (6.3)$$

It is well known that the infimum in the definition (3.3) is a minimum. We first look for a minimum point  $z \notin K$ . Then, in view of (6.1),  $z$  has to satisfy

$$\frac{z - v}{\varepsilon} + \frac{Q_K z}{\|Q_K z\|} = 0, \quad \text{that is,} \quad v = z + \varepsilon \frac{Q_K z}{\|Q_K z\|}.$$

It easily follows that  $P_K v = P_K z$ . Therefore, we have that

$$d_K(v) = \|v - P_K v\| = \|v - P_K z\| = \left(1 + \frac{\varepsilon}{\|Q_K z\|}\right) \|z - P_K z\| = d_K(z) + \varepsilon > \varepsilon,$$

while we were assuming that  $d_K(v) < \varepsilon$ . Therefore, every minimum point  $z$  has to belong to  $K$ , and we have that

$$d_K^\varepsilon(v) = \min_{z \in K} \frac{1}{2\varepsilon} \|z - v\|^2 = \frac{1}{2\varepsilon} \|v - P_K v\|^2,$$

so that (6.3) is proved. Since the set  $\{v \in H : 0 < d_K(v) < \varepsilon\}$  is open, we can differentiate (6.3) by applying the chain rule and (6.1), and deduce that (3.5) holds also in this case. To conclude the proof, we have to derive (3.6). Now observe that this identity trivially holds if  $v \in K$  and that  $K$  is nonempty. It thus suffices to prove that both sides of the identity have the same Fréchet gradient. To this end, assume first that  $v \notin K$ . By differentiating the right-hand side at  $v$  with the chain rule and applying (6.1) and (3.5), we obtain

$$\min\{d_K(v)/\varepsilon, 1\} Dd_K(v) = \min\{d_K(v)/\varepsilon, 1\} \frac{Q_K v}{d_K(v)} = \frac{Q_K v}{\max\{\varepsilon, d_K(v)\}} = Dd_K^\varepsilon(v).$$

Assume now that  $v$  belongs to  $K$  and take any  $h \in H$  satisfying  $\|h\| \leq \varepsilon$ . Then, we have  $d_K(v+h) \leq \|h\| \leq \varepsilon$ , and we infer that

$$0 \leq \int_0^{d_K(v+h)} \min\{s/\varepsilon, 1\} ds = \int_0^{d_K(v+h)} \frac{s}{\varepsilon} ds = \frac{1}{2\varepsilon} (d_K(v+h))^2 \leq \frac{1}{2\varepsilon} \|h\|^2.$$

Thus, the Fréchet gradient of the right-hand side of (3.6) at  $v$  is zero. On the other hand, we also have  $Dd_K^\varepsilon(v) = 0$  in this case. This completes the proof.  $\square$

## Acknowledgments

PC and GG gratefully acknowledge some financial support from the GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica) and the IMATI – C.N.R. Pavia.

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