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**Infinite hierarchy of nonlinear Schrödinger equations and  
their solutions**

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## Abstract

We study the infinite integrable nonlinear Schrödinger equation (NLSE) hierarchy beyond the Lakshmanan-Porsezian-Daniel equation which is a particular (fourth-order) case of the hierarchy. In particular, we present the generalized Lax pair and generalized soliton solutions, plane wave solutions, AB breathers, Kuznetsov-Ma breathers, periodic solutions and rogue wave solutions for this infinite order hierarchy. We find that 'even' order equations in the set affect phase and 'stretching factors' in the solutions, while 'odd' order equations affect the velocities. Hence 'odd' order equation solutions can be real functions, while 'even' order equation solutions are always complex.

## 1 Introduction

It is well-known that the 1-d fundamental nonlinear Schrödinger equation (NLSE) is integrable [1]. This fact has allowed the achievement of significant progress in the analysis of nonlinear optics, water waves, BECs and many other fields of nonlinear physics. The possibility to write solutions of the NLSE in analytical form stimulated numerous experimental works in these areas. Initial developments in soliton solutions has been strengthened recently by the advances in breather solutions. Various families of solutions have been presented in [2].

Although the NLSE is one of the fundamental equations in physics, it is not the only one which is integrable. In particular, various extensions of the NLSE are known. For example, Painlevé analysis of deformed NLS and Hirota equations has been given in [3]. Kano [4] considered small perturbations of the NLSE that allowed him to keep the modified equation nearly integrable. Such extensions expand the areas of applicability of integrable equations and provide efficient ways for application of evolution equations in practice. For example, they may help to clarify the physics of wave blow-up and collapse phenomena [5], as higher intensities require higher-order terms to be included.

In the present work, we provide an extension of the NLSE to infinite order equations that comprise the NLSE hierarchy. Namely, we consider extensions of the NLSE where additional terms can have arbitrarily large coefficients. This extension creates the infinite hierarchy of equations that are integrable with infinite number of arbitrary real coefficients. The additional terms in the equation include higher-order dispersion of all orders and higher-order dispersion of nonlinear terms. The arbitrariness of coefficients allow us to go well beyond simple NLSE. We define the invariant integrands of the NLSE as:

$$p_{j+1} = \psi \frac{\partial}{\partial t} \left( \frac{p_j}{\psi} \right) + \sum_{j_1+j_2=j} (p_{j_1} p_{j_2}), \quad (1)$$

where  $j = 1, 2, 3, \dots, \infty$  and  $j_1$  and  $j_2$  are non-zero positive integers which add up to  $j$ , noting that order is important. For example, if  $j = 1$  there are no such integers and so the right summation is zero, while for  $j = 4$ , we have  $(j_1, j_2) = (1, 3), (3, 1)$  and  $(2, 2)$ , so that:

$$\sum_{j_1+j_2=4} (p_{j_1}p_{j_2}) = 2p_1p_3 + p_2^2.$$

We take  $p_1 = |\psi|^2$  to start with. Hence, the first few functionals are:

$$\begin{aligned} p_2 &= \psi\psi_t^* \\ p_3 &= |\psi|^4 + \psi\psi_{tt}^* \\ p_4 &= \psi [\psi_t(\psi^*)^2 + 4\psi_t^*|\psi|^2 + \psi_{ttt}^*]. \end{aligned} \quad (2)$$

With this formulation, all signs are positive. Now, we define the  $j^{\text{th}}$  operator in the NLS hierarchy as

$$K_j(\psi, \psi^*) = (-1)^j \frac{\delta}{\delta \psi^*} \left[ \int p_{j+1} dt \right] \quad (3)$$

where we have taken the functional derivative of the invariant to get the higher order operator. Again, all signs are positive in each  $K_j$ . For example,  $K_2 = \psi_{tt} + 2\psi|\psi|^2$ , which is easily recognizable as an NLSE operator.

For higher orders, the  $j^{\text{th}}$  operator ( $j \geq 3$ ) can be presented in the form:

$$\begin{aligned} K_j &= \frac{\partial^j \psi}{\partial t^j} + 2j |\psi|^2 \frac{\partial^{j-2} \psi}{\partial t^{j-2}} + [1 + (-1)^j] \psi^2 \frac{\partial^{j-2} \psi^*}{\partial t^{j-2}} \\ &+ j(j-3) \psi \psi_t^* \frac{\partial^{j-3} \psi}{\partial t^{j-3}} + \dots \end{aligned} \quad (4)$$

There are only 2 terms when  $j = 3$ , as the last 2 terms reduce to zero:  $K_3 = \psi_{ttt} + 6|\psi|^2\psi_t$ . For  $j \geq 4$ , the next term to be added is:

$$2[j-1 - (-1)^j] \psi \psi_t \frac{\partial^{j-3} \psi^*}{\partial t^{j-3}}.$$

For  $j \geq 5$ , the next term to be added is:

$$j(j-1) \psi_t \psi^* \frac{\partial^{j-3} \psi}{\partial t^{j-3}}.$$

Finally, the term with no derivative in  $K_j$  is

$$\frac{j!}{[(j/2)!]^2} \psi |\psi|^j \quad (5)$$

if  $j$  is even,  $j = 2, 4, 6, \dots$ , and zero if  $j$  is odd.

The equation which includes the whole infinite hierarchy is:

$$F[\psi(x, t)] = i\psi_x + \sum_{j=1}^{\infty} (\alpha_{2j} K_{2j} - i \alpha_{2j+1} K_{2j+1}) = 0, \quad (6)$$

where each coefficient  $\alpha_j$ ,  $j = 2, 3, 4, 5, \dots, \infty$  is an arbitrary real number. In all expressions here,  $x$  is the propagation variable and  $t$  is transverse variable (time in a moving frame), with the function  $|\psi(x, t)|$  being the envelope of the waves.

In the book [2] and many papers, including [6], [7], [8], we have taken  $\alpha_2 = \frac{1}{2}$ . This normalization has certain convenient features. For example, rogue wave triplets with this scaling are circular rather than elliptical in the  $(x, t)$ -plane [9]. On the other hand, some authors, e.g. [1, 4], set  $\alpha_2 = 1$ . Any value of  $\alpha_2$  can be used in our present work [10–13], including zero. Hence, our solutions cover equations like  $\psi_x - \alpha_3(\psi_{ttt} + 6|\psi|^2\psi_t) = 0$ , which do not involve the basic NLSE operator at all. The latter is a significant advance over previous works.

Thus, the whole equation takes the form:

$$\begin{aligned}
F[\psi(x, t)] &= i\psi_x \\
&+ \alpha_2 K_2[\psi(x, t)] - i\alpha_3 K_3[\psi(x, t)] \\
&+ \alpha_4 K_4[\psi(x, t)] - i\alpha_5 K_5[\psi(x, t)] \\
&+ \alpha_6 K_6[\psi(x, t)] - i\alpha_7 K_7[\psi(x, t)] \\
&+ \alpha_8 K_8[\psi(x, t)] - i\alpha_9 K_9[\psi(x, t)] \\
&+ \dots = 0,
\end{aligned} \tag{7}$$

where the combined operator  $F[\psi(x, t)]$  represents the whole hierarchy of integrable equations.

In the lowest, second order, we obtain the fundamental nonlinear Schrödinger equation:

$$i\psi_x + \alpha_2 K_2 = i\psi_x + \alpha_2(\psi_{tt} + 2\psi|\psi|^2) = 0,$$

Keeping additionally the third order operator  $K_3$ , we obtain the Hirota equation:

$$i\psi_x + \alpha_2(\psi_{tt} + 2\psi|\psi|^2) - i\alpha_3[\psi_{ttt} + 6|\psi|^2\psi_t] = 0.$$

In the next generalization, we keep  $K_4$  as the fourth order ( $j = 4$ ) operator. It is known as the LPD operator (starting with fourth order derivative):

$$\begin{aligned}
K_4[\psi(x, t)] &= \psi_{tttt} + 8|\psi|^2\psi_{tt} + 6\psi|\psi|^4 + 4\psi|\psi_t|^2 \\
&+ 6\psi_t^2\psi^* + 2\psi^2\psi_{tt}^*
\end{aligned} \tag{8}$$

Continuing the process, we can keep  $K_5$  as the fifth order ( $j = 5$ ), i.e. quintic operator (starting with fifth order derivative):

$$\begin{aligned}
K_5[\psi(x, t)] &= \psi_{ttttt} + 10|\psi|^2\psi_{ttt} + 30|\psi|^4\psi_t + 10\psi\psi_t\psi_{tt}^* \\
&+ 10\psi\psi_t^*\psi_{tt} + 20\psi^*\psi_t\psi_{tt} + 10\psi_t^2\psi_t^*
\end{aligned}$$

This expression can be written in a shorter form:

$$\begin{aligned}
K_5[\psi(x, t)] &= \psi_{ttttt} + 10|\psi|^2\psi_{ttt} + 10(\psi|\psi_t|^2)_t \\
&+ 20\psi^*\psi_t\psi_{tt} + 30|\psi|^4\psi_t
\end{aligned} \tag{9}$$

The quintic equation has been considered, in a different context, by Hoseini and Marchant [14]. Further,  $K_6$  is the sixth order ( $j = 6$ ), i.e. sextic, operator (starting with sixth order derivative):

$$\begin{aligned}
K_6[\psi(x, t)] &= \psi_{ttttt} & (10) \\
&+ [60\psi^*|\psi_t|^2 + 50(\psi^*)^2\psi_{tt} + 2\psi_{ttt}^*] \psi^2 \\
&+ \psi [12\psi^*\psi_{ttt} + 8\psi_t\psi_{tt}^* + 22|\psi_{tt}|^2] \\
&+ \psi [18\psi_{ttt}\psi_t^* + 70(\psi^*)^2\psi_t^2] + 20(\psi_t)^2\psi_{tt}^* \\
&+ 10\psi_t [5\psi_{tt}\psi_t^* + 3\psi^*\psi_{ttt}] + 20\psi^*\psi_{tt}^2 \\
&+ 10\psi^3 [(\psi_t^*)^2 + 2\psi^*\psi_{tt}^*] + 20\psi|\psi|^6.
\end{aligned}$$

We present the heptic and octic operators in the Appendix.

We repeat, the coefficients  $\alpha_j$  are arbitrary real constants. They do not have to be small. This allows us to go well beyond the simple extension of the NLSE with corrective and perturbative terms. Particular case when only  $\alpha_3$  is nonzero, the equation is known as Hirota equation [6, 15]. Furthermore, when only  $\alpha_4$  is nonzero, the equation is known as the 'Lakshmanan - Porsezian - Daniel' (LPD) equation [16–18]. In this case, the coefficients within the  $K_4$  operator (8) were found using Painlevé analysis of the equation describing the Heisenberg spin chain. Thus, particular cases in the hierarchy have physical relevance. The equation when two coefficients  $\alpha_3$  and  $\alpha_4$  are arbitrary has been considered earlier in [7, 8]. In particular, soliton solutions of this equation were given in [7], while rogue wave solutions were presented in [8]. In those papers,  $\alpha_4$  is denoted by  $\gamma$ . The KdV is studied in [19, 20].

We believe that the sextic, heptic and octic operators of the NLS hierarchy are presented here for the first time. Although we do not present here the ninth order operator  $K_9[\psi]$ , with coefficient  $\alpha_9$  to save space, the results we give for first-order solitons and rogue waves does include it and all higher orders to infinity.

## 2 General observations

**Scaling.** If we have a solution  $\psi(x, t; \alpha_2, \alpha_3, \alpha_4, \dots)$  of the full equation, then we can generate a scaled solution by multiplying the function by an arbitrary real constant,  $c$ , multiplying  $t$  by  $c$ , leaving  $x$  unchanged and multiplying each  $\alpha_j$  in the solution by  $c^j$ . Hence the new solution is  $c\psi(x, ct; c^2\alpha_2, c^3\alpha_3, c^4\alpha_4, \dots)$ . If all  $\alpha_j = 0$  for  $j \geq 3$ , i.e we have the fundamental NLSE only, then the scaling  $\alpha_2 \rightarrow c^2\alpha_2$  is equivalent to scaling  $x$  by a factor of  $c^2$ , thus agreeing with the well-known scaling of NLSE solutions (e.g. see [21] and Eq. (2.3) of Ref. [2]). However, when more operators are included in the equation, it is important to note that the  $\alpha_j$ 's in the solution are scaled, not the variable  $x$ . This will be clear from the solutions analyzed in this paper. This scaling is not trivial, and so we retain the  $c$  factors throughout the solutions, for ease of use.

**Odd-numbered equations.** First, we make some general observations. If all even-labelled coefficients are zero, i.e.  $\alpha_{2n} = 0$ ,  $n = 1, 2, 3, \dots, \infty$  then we have

$$\psi_x = \sum_{j=1}^{\infty} \alpha_{2j+1} K_{2j+1}.$$

These can have real-valued solutions. For example, the first such equation is:  $\psi_x = \alpha_3(\psi_{ttt} + 6|\psi|^2\psi_t)$ . If we assume that  $\psi = f(y)$  is a real even function where  $y = t + x v_3$ , then for a localised solution ( $f(y) \rightarrow 0$  for  $y \rightarrow \infty$ ), we have  $v_3 f' = \alpha_3 (f'' + 2f^3)$ . For convenience, we set  $f(0) = 1$ . This shows that  $v_3 = \alpha_3$  and  $f'(y) = f \sqrt{1 - f^2}$ . Hence  $f = \text{sech}(y) = \text{sech}(t + x v_3)$ .

Similarly, the  $j = 2$  equation, with  $y = t + x v_5$  reduces to  $v_5 f' = \alpha_5 (f_{5y} + 10f^2 f_{3y} + 30f^4 f' + 40f f' f_{2y} + 10f_y^3)$ . Then  $v_5 = \alpha_5$  and  $f = \text{sech}(t + x v_5)$ . This pattern will be seen later with more complicated solutions. We will be able to plot  $\psi$  rather than just  $|\psi|$  for these solutions.

**Even-numbered equations.** If all odd-labelled coefficients are zero, i.e.  $\alpha_{2n+1} = 0$ ,  $n = 1, 2, 3, \dots, \infty$  then the equation becomes:

$$i\psi_x + \sum_{j=1}^{\infty} \alpha_{2j} K_{2j} = 0.$$

Now the solutions take the form  $\psi = e^{i\phi x} g(t)$ . In the NLSE case, when  $j = 1$ , for the localised solution which is even in  $t$ , we have  $\alpha_2(g_t^2 + g^4) = \phi g$ . For convenience, we now take  $g(0) = 1$ . This shows that  $\phi = \alpha_2$  and  $g'(t) = g \sqrt{1 - g^2}$ . Hence  $g = \text{sech}(t)$  and  $\psi = e^{i\alpha_2 x} \text{sech}(t)$ .

Similarly, for the  $j = 2$  equation, we have  $\alpha_4 (g_{4t} + 8g^2 g_{2t} + 6g^5 + 10g g_t^2 + 2g^2 g_{2t}) = \phi g$ , and solution  $\psi = e^{i\alpha_4 x} \text{sech}(t)$ . Again, this structure will be seen for other types of solutions.

**Plane wave solutions.** In order to illustrate the usefulness of the approach, we start with the simplest plane wave solution of the extended NLS equation. If a solution  $\psi$  is independent of  $t$ , then we see from Eq.(5) that

$$i\psi_x + \psi \sum_{n=1}^{\infty} \binom{2n}{n} \alpha_{2n} |\psi|^{2n} = 0,$$

where  $\binom{2n}{n}$  is a binomial coefficient. So

$$i\psi_x + 2\alpha_2 \psi |\psi|^2 + 6\alpha_4 \psi |\psi|^4 + 20\alpha_6 \psi |\psi|^6 + \dots = 0.$$

Thus, for the unit-background forward-propagating plane wave solution to Eq.(7),  $\psi_p = \exp(i\phi x)$ , we have (with  $j = 2n$ ):

$$\phi = 2\alpha_2 + 6\alpha_4 + 20\alpha_6 + \dots = \sum_{n=1}^{\infty} \binom{2n}{n} \alpha_{2n} = \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \alpha_{2n}$$

Thus, for an arbitrary background of the plane wave, we can write the solution as

$$\begin{aligned} \psi_p &= c \exp \left( i x c^2 \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \alpha_{2n} c^{2n-2} \right) \\ &= c \exp [i x c^2 (2\alpha_2 + 6c^2 \alpha_4 + 20c^4 \alpha_6 \\ &\quad + 70c^6 \alpha_8 + 252c^8 \alpha_{10} + \dots)], \end{aligned} \tag{11}$$

recalling that  $\alpha_2$  does not have to be  $1/2$ .

Here  $c$  is the arbitrary amplitude of the plane wave and the series in (4) contain even coefficients of Eq.(7). The simple nature of the scaling is apparent, with arbitrary background level  $c$  causing each coefficient  $\alpha_{2n}$  to be multiplied by  $c^{2n}$ . The expression (4) represents the solution of the equation (7) of any order up to infinite one. The presence of only even terms in this expression is related to the fact that we deal with the forward propagating wave. Any skewness in the  $(x, t)$ -plane would result in addition of odd terms. We do not present this case as this would go beyond the simplicity of our illustrative example. In order to construct more complicated solutions of Eq.(7), we have to find its Lax pair. These solutions can also be found for the equation with an infinite number of terms.

**First order soliton solutions.** The first order soliton of Eq.(7), taking  $\alpha_2$  and all other coefficients  $\alpha_j$  to be arbitrary, is

$$\psi_s = c \exp(ix\phi_s) \operatorname{sech}(ct + xv_s) \quad (12)$$

where the phase is:

$$\begin{aligned} \phi_s &= \sum_{n=1}^{\infty} \alpha_{2n} c^{2n} \\ &= c^2(\alpha_2 + c^2\alpha_4 + c^4\alpha_6 + c^6\alpha_8 + c^8\alpha_{10} + \dots), \end{aligned} \quad (13)$$

and where the velocity is:

$$\begin{aligned} v_s &= \sum_{n=1}^{\infty} \alpha_{2n+1} c^{2n+1}, \\ &= c^3(\alpha_3 + c^2\alpha_5 + c^4\alpha_7 + c^6\alpha_9 + \dots). \end{aligned} \quad (14)$$

The background level,  $c$  is arbitrary. It is clear from the expression that velocity depends on third, fifth, seventh and ninth order coefficients,  $\alpha_{2n+1}$ , while the phase depends on the fourth, sixth and eighth order coefficients  $\alpha_{2n}$ . Plainly, for unit background, each term has unit coefficient. When only  $\alpha_3, \alpha_4$  are nonzero, it reduces to a result in [7]. So this solution applies for infinitely many orders in the original equation. It confirms and generalizes the brief derivations on odd and even-numbered equations above.

### 3 Generalized rogue waves and related solutions

Again, we allow for all operator coefficients ( $\alpha_j, j = 3, 4, 5, \dots, \infty$ ) to be arbitrary. Then:

$$\psi(x, t) = c \left[ 4 \frac{1 + 2i B_r x}{D(x, t)} - 1 \right] e^{i\phi_r x} \quad (15)$$

where  $D(x, t) = 1 + 4B_r^2 x^2 + 4(ct + v_r x)^2$ , and

$$B_r = \sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2} \alpha_{2n} c^{2n} \quad (16)$$



$$\begin{aligned}
&= 2c^2(\alpha_2 + 6c^2\alpha_4 + 30c^4\alpha_6 + 140c^6\alpha_8 \\
&+ 630c^8\alpha_{10} + \dots).
\end{aligned}$$

Here  $c$  is the arbitrary background level. The coefficient  $\phi$  in the exponential factor is then equal to

$$\begin{aligned}
\phi_r &= c^2 \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \alpha_{2n} c^{2n-2} \\
&= 2c^2(\alpha_2 + 3c^2\alpha_4 + 10c^4\alpha_6 + 35c^6\alpha_8 \\
&+ 126c^8\alpha_{10} + \dots).
\end{aligned} \tag{17}$$

Finally, the velocity is

$$\begin{aligned}
v_r &= \sum_{n=1}^{\infty} \frac{(2n+1)!}{(n!)^2} \alpha_{2n+1} c^{2n+1} \\
&= 2c^3(3\alpha_3 + 15c^2\alpha_5 + 70c^4\alpha_7 \\
&+ 315c^6\alpha_9 + 1386c^8\alpha_{11} + \dots).
\end{aligned} \tag{18}$$

The velocity clearly depends only on the coefficients of odd-order operators: Hirota operator with  $v = 6c^3\alpha_3$  when the other  $\alpha'_j$ s are zero, the 5<sup>th</sup> order operator (quintic, with  $v = 30c^5\alpha_5$  when the other  $\alpha'_j$ s are zero), the 7<sup>th</sup> order operator (heptic, with  $v = 140c^7\alpha_7$  when the other  $\alpha'_j$ s are zero), etc. We note that the exponential factor,  $\phi$ , and stretching factor,  $B_r$ , here depend only on the coefficients of even-order operators. When  $\alpha_j = 0$ , for all  $j > 4$ , it reduces to a result in [8].

If we have only even-numbered equations, then  $\phi_r, B_r$  are non-zero, and we obtain complex-valued, zero-velocity, solutions resembling that of the NLSE (which is the  $\alpha_2$  case). An example is given in Fig. 1.

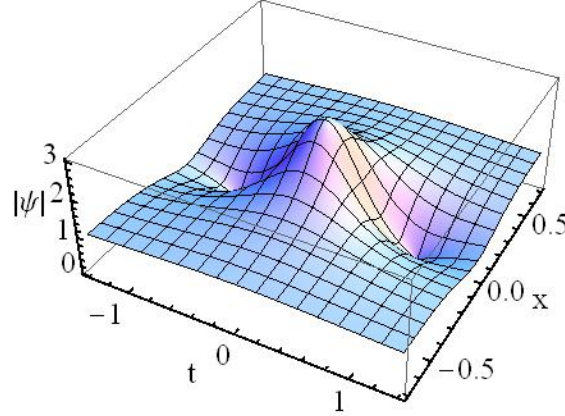


Figure 1: Plot of the rogue wave, Eq.(15), solution of Eq.(7), with  $c = 1$ ,  $\alpha_4 = \frac{1}{4}$ , and all other  $\alpha_j$ 's zero.

If we have only odd-numbered equations (see section 2), then  $\phi_r = B_r = 0$ , and we obtain the real-valued solution

$$\psi(x, t) = c \left[ \frac{4}{4(ct + v_r x)^2} - 1 \right].$$

Then, along the diagonal line  $ct + v_r x = 0$ , we have  $\psi(x, t) = 3c$ . So, this solution resembles a moving soliton on a background (though the shape is different from the 'sech' function), and does not have the single peak which is a feature of solutions of the full equation which contains at least one even-labelled term. An example is given in Fig.2. Earlier works, e.g. [6], included both  $\alpha_2$  and  $\alpha_3$  terms, and hence found rogue waves with a single maximum.

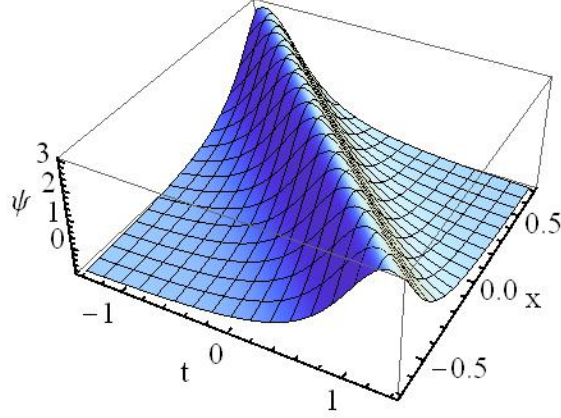


Figure 2: Plot of the moving soliton on a background, Eq.(15), with  $c = 1$ ,  $\alpha_5 = \frac{1}{16}$ , and all other  $\alpha_j$ 's zero.

If we have at least one even-numbered equation with at least one odd-numbered equation, the resulting solution looks like an NLSE rogue wave [22], with the central part having a velocity (see [6]). An example is given in Fig.3. We stress that this is a remarkably simple result for an equation that can contain hundreds of terms, each with various derivatives. It could help explain the appearance of rogue waves in a multitude of physical, biological, financial and social situations, going well beyond the  $j = 3, 4$  cases that have been previously analyzed.

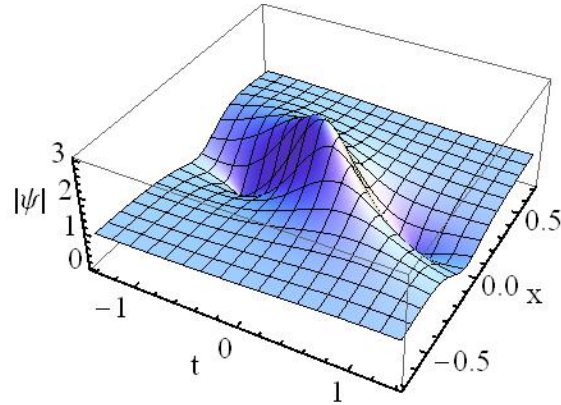


Figure 3: Plot of the rogue wave, Eq.(15), with  $c = 1$ ,  $\alpha_4 = 1/4$ ,  $\alpha_5 = \frac{1}{16}$ , and all other  $\alpha_j$ 's zero.

## 4 Generalized Akhmediev breathers and related solutions

The basic NLSE breather explains the evolution of modulation instability (e.g. see sect.3.7 of [2]). Here, we consider all odd and even-order equations. The odd-order equations basically modify the breather velocity when compared to the basic NLSE breather, as is already known for the Hirota ( $\alpha_3 \neq 0$ ) case. On the other hand, the even-order equations basically modify the phase of the basic NLSE breather and introduce a 'stretching factor' in  $x$  in the non-phase part of the solution. We take arbitrary background, i.e. any real  $c$ , while noting that the scaling is relatively simple once the  $c = 1$  case is known. Thus, the general breather on arbitrary background  $c$  is:

$$\psi_b = c e^{ix\phi} \left( 1 + \frac{\kappa [\kappa C(x) + i\sqrt{4 - \kappa^2} S(x)]}{\sqrt{4 - \kappa^2} \cos[\kappa(ct + v_b x)] - 2C(x)} \right) \quad (19)$$

where

$$C(x) = \cosh \left( B_b \kappa \sqrt{1 - \frac{\kappa^2}{4}} x \right),$$

$$S(x) = \sinh \left( B_b \kappa \sqrt{1 - \frac{\kappa^2}{4}} x \right)$$

with  $\kappa$  being an arbitrary real frequency in the range of modulation instability, i.e.  $0 < \kappa < 2$ .

Now, the velocity is:

$$v_b = \sum_{n=1}^{\infty} \alpha_{2n+1} c^{2n+1} \frac{(2n+1)!}{n!} \left( \sum_{r=0}^n \frac{(-1)^r \kappa^{2r} r!}{(n-r)!(2r+1)!} \right).$$

We now sum the series on the right, obtaining a closed form result:

$$v_b = \sum_{n=1}^{\infty} \alpha_{2n+1} c^{2n+1} \frac{(2n+1)!}{(n!)^2} {}_2F_1 \left( 1, -n; \frac{3}{2}; \frac{\kappa^2}{4} \right), \quad (20)$$

where  ${}_2F_1$  means hypergeometric function [23]. In our range, if  $\kappa$  is small, then this function can be approximated:

$${}_2F_1 \left( 1, -n; \frac{3}{2}; \frac{\kappa^2}{4} \right) \approx 1 - \frac{n}{6} \kappa^2 + \frac{n}{60} (n-1) \kappa^4 + \dots \quad (21)$$

In fact, for any  $\kappa$  in our range,  ${}_2F_1 \left( 1, -n; \frac{3}{2}; \frac{\kappa^2}{4} \right)$  is exactly a polynomial in  $\kappa$  with  $n+1$  terms, with the highest power being  $\kappa^{2n}$ . Thus:

$$\begin{aligned} v_b = & \alpha_3 c^3 (6 - \kappa^2) + \alpha_5 c^5 (30 - 10\kappa^2 + \kappa^4) \\ & + \alpha_7 c^7 (140 - 70\kappa^2 + 14\kappa^4 - \kappa^6) + \alpha_9 c^9 (630 \\ & - 420\kappa^2 + 126\kappa^4 - 18\kappa^6 + \kappa^8) + \alpha_{11} c^{11} (2772 \\ & - 2310\kappa^2 + 924\kappa^4 - 198\kappa^6 + 22\kappa^8 - \kappa^{10}) + \dots \end{aligned}$$

For  $\kappa \rightarrow 0$ , from Eqs.(20) and (21), we have

$$v_b = \sum_{n=1}^{\infty} \alpha_{2n+1} c^{2n+1} \frac{(2n+1)!}{(n!)^2} = 6\alpha_3 c^3 + 30\alpha_5 c^5 \\ + 140\alpha_7 c^7 + 630\alpha_9 c^9 + 2772\alpha_{11} c^{11} + \dots$$

agreeing with rogue wave result Eq.(18). Thus, the velocity given by Eq.(18) is the low-frequency ( $\kappa \rightarrow 0$ ) limit (for any  $c$ ) of Eq.(20).

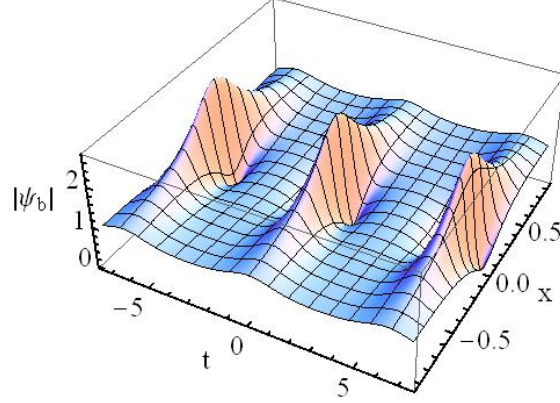


Figure 4: Plot of the complex Akhmediev breather, Eq.(19), solution of Eq.(7), with  $\kappa = 1$ ,  $c = 1$ ,  $\alpha_4 = \frac{1}{4}$ , and all other  $\alpha_j$ 's zero.

Similarly, the 'stretching factor' is given by:

$$B_b = 2 \sum_{n=0}^{\infty} \alpha_{2n+2} c^{2n+2} \frac{(2n+1)!}{(n!)^2} {}_2F_1 \left( 1, -n; \frac{3}{2}; \frac{\kappa^2}{4} \right) \\ = 2 [\alpha_2 c^2 + \alpha_4 c^4 (6 - \kappa^2) + \alpha_6 c^6 (30 - 10\kappa^2 + \kappa^4) \\ + \alpha_8 c^8 (140 - 70\kappa^2 + 14\kappa^4 - \kappa^6) + \alpha_{10} c^{10} (630 \\ - 420\kappa^2 + 126\kappa^4 - 18\kappa^6 + \kappa^8) + \alpha_{12} c^{12} (2772 \\ - 2310\kappa^2 + 924\kappa^4 - 198\kappa^6 + 22\kappa^8 - \kappa^{10}) + \dots]$$

The  $\kappa \rightarrow 0$  limit is

$$B_b(\kappa = 0) = 2 \sum_{n=0}^{\infty} \alpha_{2n+2} c^{2n+2} \frac{(2n+1)!}{(n!)^2}$$

agreeing with rogue wave result Eq.(16). The phase is:

$$\phi = \sum_{n=1}^{\infty} \alpha_{2n} c^{2n} \frac{(2n)!}{(n!)^2} = 2(\alpha_2 c^2 + 3\alpha_4 c^4 + 10\alpha_6 c^6 + \dots).$$

Note that the phase matches that of the plane wave solution, Eq.(13), and the rogue wave, Eq.(17). Thus, the rogue wave Eq.(15) can also be obtained as the low-frequency ( $\kappa \rightarrow 0$ ) limit of the breather given by Eq.(19).

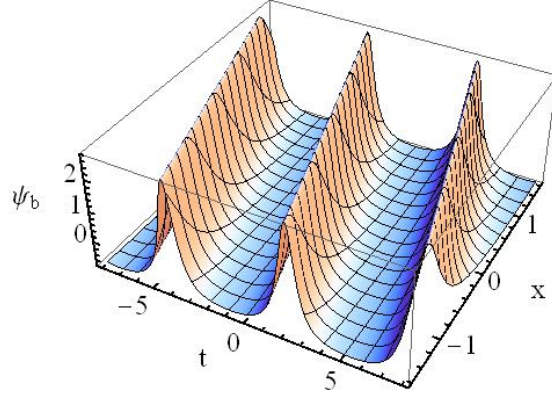


Figure 5: Plot of the real-valued non-breathing solution related to the Akhmediev breathers, Eq.(19). It is a solution of Eq.(7), with  $\kappa = 1, c = -1, \alpha_5 = \frac{1}{16}$ , and all other  $\alpha_j$ 's zero.

We have  $|\phi(0, 0)| = |c|(1 + \sqrt{4 - \kappa^2})$ ; this decreases from a maximum of  $3|c|$  when  $\kappa = 0$  to a minimum of  $|c|$  when  $\kappa = 2$ . Again, if we have only even numbered equations, then  $v_b = 0$ , and the breather solution resembles that of the NLSE. An example is given in Fig.4.

If we have only odd numbered equations, then  $\phi_b = B_b = 0$ , and the solution  $\psi_b(x, t)$  of Eq. (19) becomes real-valued. An example is given in Fig.5. It does not 'breathe' and so we can describe it as a solution related to a breather. If at least one even and one odd coefficient are nonzero, then  $\phi_b, B_b, v_b$  are all nonzero. The example of this solution is shown in Fig.6. It is similar to the one in Fig.4 but has nonzero velocity.

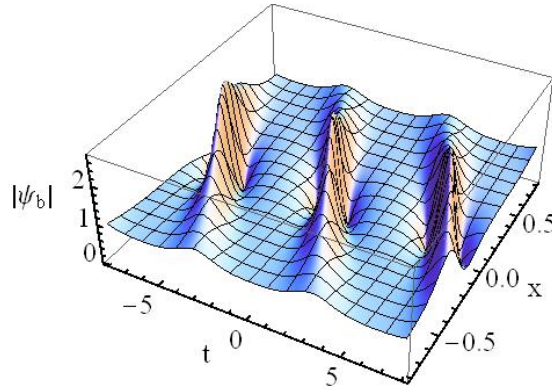


Figure 6: Plot of the complex Akhmediev breather, Eq.(19), solution of Eq.(7), with  $\kappa = 1, c = 1, \alpha_4 = \frac{1}{4}, \alpha_5 = \frac{1}{5}$ , and all other  $\alpha_j$ 's zero.

Beyond the NLSE solution, only the Hirota case [6] when  $v_b = \alpha_3(6 - \kappa^2)$  and 4-th order case [24] were previously known. This new solution can provide a considerable extension of applicability to problems of modulation instability in physics, chemistry, etc.

## 5 Generalized Kuznetsov-Ma breathers and moving solitons

The NLSE Kuznetsov-Ma breather is given by Eq.(3.63) of [2]. The generalized Kuznetsov-Ma breather can be written as:

$$\psi_m = c \sqrt{2} e^{ix(\phi_m)} \times \frac{2(1-a_1)C_s(x) - \sqrt{2}\sqrt{a_1}C_m(x,t) + 2i\sqrt{1-2a_1}S_m(x)}{\sqrt{2}C_m(x,t) - 2\sqrt{a_1}C_s(x)} \quad (22)$$

where

$$\begin{aligned} C_m(x,t) &= \cosh [2\sqrt{1-2a_1}(ct + v_m x)], \\ C_s(x) &= \cos (2\sqrt{1-2a_1}B_m x), \\ S_m(x) &= \sin (2\sqrt{1-2a_1}B_m x), \end{aligned}$$

with  $a_1$  being an arbitrary real number within the interval  $0 < a_1 < \frac{1}{2}$ . The velocity is:

$$v_m = \sum_{n=1}^{\infty} 4^n \alpha_{2n+1} c^{2n+1} \left( 1 + \sum_{r=1}^n \frac{(2r-1)!! a_1^r}{r!} \right) \quad (23)$$

This can be written in closed form:

$$\begin{aligned} v_m &= \sum_{n=1}^{\infty} 4^n \alpha_{2n+1} c^{2n+1} \left[ \frac{1}{\sqrt{1-2a_1}} \right. \\ &\quad \left. - a_1^{n+1} \frac{(2n+1)!!}{(n+1)!} {}_2F_1 \left( 1, n + \frac{3}{2}; n + 2; 2a_1 \right) \right] \quad (24) \end{aligned}$$

Thus, we have:

$$\begin{aligned} v_m &= 4c^3(a_1+1)\alpha_3 + 8c^5(3a_1^2+2a_1+2)\alpha_5 \\ &\quad + 32c^7(5a_1^3+3a_1^2+2a_1+2)\alpha_7 + 32c^9(35a_1^4 \\ &\quad + 20a_1^3+12a_1^2+8a_1+8)\alpha_9 + 128c^{11}(63a_1^5 \\ &\quad + 35a_1^4+20a_1^3+12a_1^2+8a_1+8)\alpha_{11} + \dots \end{aligned}$$

The result in Eq.(24) can also be expressed as:

$$\begin{aligned} v_m &= \sum_{n=1}^{\infty} \frac{2^{n-1} \alpha_{2n+1} c^{2n+1}}{n! \sqrt{1-2a_1}} \times \\ &\quad \left[ 2^{n+1} n! - (2n+1)!! \mathcal{B}_{1/2}(2a_1, n+1) \right], \quad (25) \end{aligned}$$

where the incomplete beta function  $\mathcal{B}_z(a, b)$  is defined as  $\int_0^z t^a (1-t)^{b-1} dt$ .

For the upper point in the parameter range, i.e. for  $a_1 \rightarrow \frac{1}{2}$ , we have

$$\begin{aligned} \lim_{a_1 \rightarrow \frac{1}{2}} v_m &= \sum_{n=1}^{\infty} c^{2n+1} \frac{(2n+1)! \alpha_{2n+1}}{(n!)^2} \\ &= 6c^3 \alpha_3 + 30c^5 \alpha_5 + 140c^7 \alpha_7 \\ &\quad + 630c^9 \alpha_9 + 2772c^{11} \alpha_{11} + \dots, \quad (26) \end{aligned}$$

again agreeing with rogue wave result Eq.(18).

The stretching factor,  $B_m$  is given by:

$$B_m = 2 \sum_{n=0}^{\infty} 4^n \alpha_{2n+2} c^{2n+2} \left[ \frac{1}{\sqrt{1-2a_1}} - a_1^{n+1} \frac{(2n+1)!!}{(n+1)!} {}_2F_1 \left( 1, n + \frac{3}{2}; n+2; 2a_1 \right) \right] \quad (27)$$

Further,

$$\lim_{a_1 \rightarrow \frac{1}{2}} B_m = 2c^2 \alpha_2 + 12c^4 \alpha_4 + 60c^6 \alpha_6 + 280c^8 \alpha_8 + 1260c^{10} \alpha_{10} + \dots,$$

i.e. it is the rogue wave result of Eq.(16). The phase is:

$$\begin{aligned} \phi_m &= \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \alpha_{2n} c^{2n} (2a_1)^n \\ &= 2a_1 (2c^2 \alpha_2 + 12a_1 c^4 \alpha_4 + 80a_1^2 c^6 \alpha_6 + 560a_1^3 c^8 \alpha_8 + 4032a_1^4 c^{10} \alpha_{10} + \dots), \end{aligned} \quad (28)$$

recalling that we usually set  $\alpha_2 = \frac{1}{2}$ . For the upper point in the parameter range

$$\lim_{a_1 \rightarrow \frac{1}{2}} \phi_m = 2c^2 \alpha_2 + 6c^4 \alpha_4 + 20c^6 \alpha_6 + 70c^8 \alpha_8 + 252c^{10} \alpha_{10} + \dots.$$

i.e. it is the rogue wave result of Eq.(17).

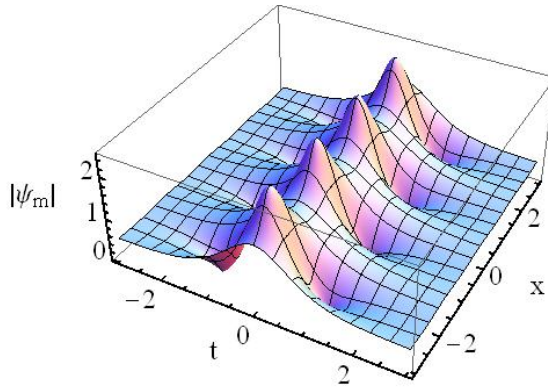


Figure 7: Plot of the Kuznetsov-Ma breather solution given by Eq.(22). It is a solution of Eq.(7), with  $a_1 = \frac{1}{8}$ ,  $c = 1$ ,  $\alpha_4 = \frac{1}{4}$ , and all other  $\alpha_j$ 's zero.

So if we only consider odd-order equations, i.e. those with coefficients  $\alpha_3, \alpha_5, \alpha_7$  etc., then only the velocity changes, while the stretching factor,  $B_m$  and phase,  $\phi_m$ , are zero. This makes  $\psi_m$  of Eq.(22) real. If we only consider even-order equations, i.e. those with coefficients  $\alpha_2, \alpha_4, \alpha_6$

etc., then only the stretching factor,  $B_m$  and the phase,  $\phi_m$ , change, while velocity remains equal to zero. Thus, the rogue wave given by Eq.(15) can also be obtained as the upper parameter ( $a_1 \rightarrow 1/2$ ) limit of the Kuznetsov-Ma breather of Eq.(22). Again here, if we have only even numbered equations, then  $v_m = 0$ , and the breather solution resembles that of the NLSE. An example is given in Fig.7.

If we have only odd numbered equations, then  $\phi_m = B_m = 0$ , and the solution  $\psi_m(x, t)$  of Eq.(22) becomes real-valued. An example is given in Fig.8. In contrast to the odd-equations Akhmediev breathers, these contain no trigonometric functions and hence do not feature periodicity. We describe them as solutions related to the Kuznetsov-Ma breather. They resemble a moving soliton on a background, like the rogue wave shown in Fig.2.

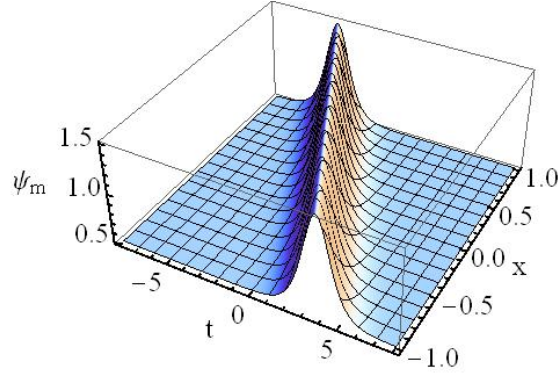


Figure 8: Plot of the non-breathing solution, Eq.(22). It is a solution of Eq.(7), with  $a_1 = \frac{1}{8}$ ,  $c = -1$ ,  $\alpha_5 = \frac{1}{5}$ , and all other  $\alpha_j$ 's zero. The background level is  $\frac{1}{2}$  and the maximum value is  $\frac{3}{2}$ .

## 6 Periodic solutions

**Elliptic dn solutions.** The NLSE  $dn$  solution has been given in Eq.(3.65) of [2]. We are now in a position to give periodic solutions of the full infinite equation, where we recall that  $\alpha_2$  and all other coefficients  $\alpha_j$  are arbitrary. It is given by:

$$\psi_s = c \exp(ix\phi_d) \operatorname{dn}(ct + xv_e, m), \quad (29)$$

where  $dn$  is a Jacobi elliptic function [23], with real modulus  $m$  such that  $0 < m < 1$ . For definition of  $m$ , we have  $\operatorname{dn}(y, m) = 1 - \frac{1}{2}m y^2 + \dots$ . The phase term is:

$$\phi_d = \sum_{n=1}^{\infty} \alpha_{2n} c^{2n} \frac{(2n)!}{(n!)^2} {}_2F_1(-n, -n; -2n; m). \quad (30)$$

We find that this can be expressed in terms of  $P_n$ , the set of orthogonal Legendre polynomials of the first kind:

$$\phi_d = \sum_{n=1}^{\infty} \alpha_{2n} c^{2n} m^n P_n \left( \frac{2}{m} - 1 \right) \quad (31)$$



These well-known polynomials are  $P_1(y) = y$ ,  $P_2(y) = \frac{1}{2}(3y^2 - 1)$ ,  $P_3(y) = \frac{1}{2}y(5y^2 - 3)$ ,  $P_4(y) = \frac{1}{8}(35y^4 - 30y^2 + 3)$ ,  $P_5(y) = \frac{1}{8}y(63y^4 - 70y^2 + 15)$ , etc. Thus:

$$\begin{aligned}\phi_d &= (2 - m)c^2\alpha_2 + (6 - 6m + m^2)c^4\alpha_4 \\ &+ (20 - 30m + 12m^2 - m^3)c^6\alpha_6 + (70 - 140m \\ &+ 90m^2 - 20m^3 + m^4)c^8\alpha_8 + (252 - 630m \\ &+ 560m^2 - 210m^3 + 30m^4 - m^5)c^{10}\alpha_{10} + \dots\end{aligned}$$

Further, the velocity is:

$$v_e = \sum_{n=1}^{\infty} \alpha_{2n+1} c^{2n+1} \frac{(2n)!}{(n!)^2} {}_2F_1(-n, -n; -2n; m) \quad (32)$$

Similarly, this can be simplified to

$$v_e = \sum_{n=1}^{\infty} \alpha_{2n+1} c^{2n+1} m^n P_n\left(\frac{2}{m} - 1\right) \quad (33)$$

where  $P_n$  is a member of the same set of orthogonal Legendre polynomials of the first kind. So

$$\begin{aligned}v_e &= (2 - m)c^3\alpha_3 + (6 - 6m + m^2)c^5\alpha_5 + (20 \\ &- 30m + 12m^2 - m^3)c^7\alpha_7 + (70 - 140m \\ &+ 90m^2 - 20m^3 + m^4)c^9\alpha_9 + (252 - 630m \\ &+ 560m^2 - 210m^3 + 30m^4 - m^5)c^{11}\alpha_{11} + \dots\end{aligned}$$

As with most solutions, the even order equations affect the phase, while the odd order equations affect the velocity. In this case, the solution functions have the same form, though the set of coefficients (the  $\alpha_j$ 's) differ. If we have only odd-label equations, then  $\phi_d = 0$  and solution of eq.(29) is real.

If  $m = 1$  we have  $P_n(1) = 1$ , and

$$c e^{i\phi_d x} \operatorname{dn}(ct + v_e x, 1) = c e^{i\phi_s x} \operatorname{sech}(t + v_s x),$$

since  $\phi_d(m = 1) = \sum_{n=1}^{\infty} \alpha_{2n}$ , agreeing with Eq.(13), and  $v_e = \sum_{n=1}^{\infty} \alpha_{2n+1}$ , agreeing with Eq.(14). Thus, we have reproduced the fundamental 'sech' soliton result covering all operators.

**Elliptic cn solutions.** The NLSE cn solution has been given by Eq.(3.66) of [2]. We now give the elliptic cn solution of the full infinite equation. We can write it in a convenient way using hyperbolic functions, as follows:

$$\psi_s = \frac{c}{\sqrt{2}} \coth(\zeta) e^{i\phi_c x} \operatorname{cn}\left[\frac{ct + xv_c}{\sinh(\zeta)}, \frac{1}{2} \cosh^2(\zeta)\right] \quad (34)$$

where  $\operatorname{cn}$  is a Jacobi elliptic function [23], with  $\zeta$  real. With our modulus definition,  $\operatorname{cn}(y, m) = 1 - \frac{1}{2}y^2 + \frac{1}{6}(\frac{1}{4} + m)y^4 + \dots$ . The phase term can be expressed in terms of  $P_n$ , the set of

orthogonal Legendre polynomials of the first kind:

$$\begin{aligned}
\phi_c &= \sum_{n=1}^{\infty} \alpha_{2n} c^{2n} \sinh^{-2n}(\zeta) P_n(\sinh^2(\zeta)) \\
&= \alpha_2 c^2 + \frac{\alpha_4}{2} c^4 [3 \sinh^4(\zeta) - 1] \operatorname{csch}^4(\zeta) \\
&\quad + \frac{\alpha_6}{2} c^6 [5 \sinh^6(\zeta) - 3 \sinh^2(\zeta)] \operatorname{csch}^6(\zeta) \\
&\quad + \frac{\alpha_8}{8} c^8 [35 \sinh^8(\zeta) - 30 \sinh^4(\zeta) + 3] \operatorname{csch}^8(\zeta) \\
&\quad + \frac{\alpha_{10}}{8} c^{10} [63 \sinh^{10}(\zeta) - 70 \sinh^6(\zeta) \\
&\quad + 15 \sinh^2(\zeta)] \operatorname{csch}^{10}(\zeta) + \dots
\end{aligned} \tag{35}$$

This can be re-expressed in more compact form:

$$\begin{aligned}
\phi_c &= \alpha_2 c^2 + \frac{\alpha_4}{2} c^4 [3 - \operatorname{csch}^4(\zeta)] + \frac{\alpha_6}{2} c^6 [5 - 3 \operatorname{csch}^4(\zeta)] \\
&\quad + \frac{\alpha_8}{8} c^8 [35 - 30 \operatorname{csch}^4(\zeta) + 3 \operatorname{csch}^8(\zeta)] \\
&\quad + \frac{\alpha_{10}}{8} c^{10} [63 - 70 \operatorname{csch}^4(\zeta) + 15 \operatorname{csch}^8(\zeta)] + \dots
\end{aligned}$$

Similarly, the velocity is:

$$\begin{aligned}
v_c &= \sum_{n=1}^{\infty} \alpha_{2n+1} c^{2n+1} \sinh^{-2n}(\zeta) P_n(\sinh^2(\zeta)) \\
&= \alpha_3 c^3 + \frac{\alpha_5}{2} c^5 [3 - \operatorname{csch}^4(\zeta)] + \frac{\alpha_7}{2} c^7 [5 - 3 \operatorname{csch}^4(\zeta)] \\
&\quad + \frac{\alpha_9}{8} c^9 [35 - 30 \operatorname{csch}^4(\zeta) + 3 \operatorname{csch}^8(\zeta)] \\
&\quad + \frac{\alpha_{11}}{8} c^{11} [63 - 70 \operatorname{csch}^4(\zeta) + 15 \operatorname{csch}^8(\zeta)] + \dots
\end{aligned} \tag{36}$$

where  $P_n$  is a member of the same set of orthogonal Legendre polynomials.

On the other hand, the solution can be written without hyperbolic functions:

$$\psi_s = \frac{c}{\sqrt{2}} \sqrt{s+1} e^{i\phi_c x} \operatorname{cn}[\sqrt{s}(ct + xv_c), \frac{1}{2}(1+s^{-1})] \tag{37}$$

Then

$$\begin{aligned}
\phi_c &= \sum_{n=1}^{\infty} \alpha_{2n} c^{2n} s^n P_n\left(\frac{1}{s}\right) \\
&= c^2 \alpha_2 + \frac{\alpha_4}{2} c^4 (3 - s^2) + \frac{\alpha_6}{2} c^6 (5 - 3s^2) \\
&\quad + \frac{\alpha_8}{8} c^8 (35 - 30s^2 + 3s^4) \\
&\quad + \frac{\alpha_{10}}{8} c^{10} (63 - 70s^2 + 15s^4) + \dots
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
v_c &= \sum_{n=1}^{\infty} \alpha_{2n+1} c^{2n+1} s^n P_n \left( \frac{1}{s} \right) \\
&= c^3 \alpha_3 + \frac{\alpha_5}{2} c^5 (3 - s^2) + \frac{\alpha_7}{2} c^7 (5 - 3 s^2) \\
&\quad + \frac{\alpha_9}{8} c^9 (35 - 30 s^2 + 3 s^4) \\
&\quad + \frac{\alpha_{11}}{8} c^{11} (63 - 70 s^2 + 15 s^4) + \dots
\end{aligned} \tag{39}$$

Again, the even order equations affect the phase, while the odd order equations affect the velocity. If we have only odd-label equations, then  $\phi_c = 0$  and solution of Eq.(34) is real. In this case, the solutions have the same form, with the set of coefficients (the  $\alpha_j$ 's) being different. If  $s = 1$ , i.e  $\sinh(\zeta) = 1$ , we have  $P_n(1) = 1$ , so the solution given in Eq.(37) reduces to Eq.(12), viz.  $c e^{i\phi_s x} \operatorname{sech}(t + v_s x)$ , as was the case in section 6.

## 7 The case of $x$ -dependent coefficients

**Solitons.** We have considered the coefficients to be constants, but we now allow them to vary on propagation, so that  $\alpha_m = \alpha_m(x)$ . In a fiber, this would correspond to different sections possessing different physical and optical characteristics. For example, suppose that just one of the coefficients, viz.  $\alpha_{2j}(x)$  in Eq.(6) is non-zero. Then

$$i\psi_x + \alpha_{2j}(x) K_{2j}(x, t) = 0,$$

for a particular  $j$ . We note that  $K_{2j}(x, t)$  contains no derivatives with respect to  $x$ . We now transform to a new variable,  $X$ , such that

$$\frac{dX}{dx} = \alpha_{2j}(x) \quad \text{i.e.} \quad X = \int \alpha_{2j}(x) dx.$$

Then  $i\psi_X + K_{2j}(X, t) = 0$ . Here the coefficient is a constant, viz. unity, and we can use the constant-coefficient solutions already found, simply by making the replacement:  $\alpha_{2j}x \rightarrow \int \alpha_{2j}(x) dx$ . The velocities, stretching factors and phases are modified in this way for all the solutions given above. For example, if we take  $j = 1$ , we have the NLS only,  $K_2 = \psi_{tt} + 2|\psi|^2 \psi$ . The soliton solution is, from Eq.(12),  $\psi = \exp[i\alpha_2 x] \operatorname{sech}(t)$ . When  $\alpha_2 = \alpha_2(x)$ , we have  $\psi = \exp[i \int \alpha_2(x) dx] \operatorname{sech}(t)$ .

We can generalize this by allowing all operator coefficients to be non-zero and to be functions of  $x$ . Then the soliton solution of maximum amplitude  $c$  is

$$\begin{aligned}
\psi_m &= c \exp \left[ i \sum_{n=1}^{\infty} c^{2n} J_{2n}(x) \right] \\
&\quad \times \operatorname{sech} \left[ t + \sum_{n=1}^{\infty} c^{2n+1} J_{2n+1}(x) \right],
\end{aligned} \tag{40}$$

where  $J_m(x) = \int \alpha_m(x) dx$ . If each coefficient is constant, then  $J_m(x) = \alpha_m x$ , and

$$\psi_m = c \exp[ix \sum_{n=1}^{\infty} c^{2n} \alpha_{2n}] \operatorname{sech}[t + x \sum_{n=1}^{\infty} c^{2n+1} \alpha_{2n+1}],$$

as in Eqs.(13) and (14).

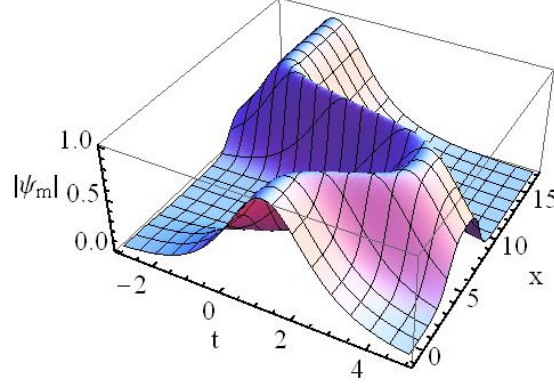


Figure 9: Plot of soliton, Eq.(40), moving under the influence of operators with variable coefficients.

To plot an example, let us use Gaussian functions to switch the operators 'on' and 'off' during soliton propagation. We set  $\alpha_2(x) = \exp[-\frac{1}{2}x^2]$ ,  $\alpha_3(x) = -\exp[-\frac{1}{2}(x-3)^2]$ ,  $\alpha_4(x) = \exp[\frac{1}{2}(x-6)^2]$  and  $\alpha_5(x) = 2\exp[-\frac{1}{2}(x-9)^2]$ . Hence  $J_2 = \sqrt{\frac{\pi}{2}}\operatorname{erf}(\frac{x}{\sqrt{2}})$ , etc. We plot the solution, from Eq.(40), in Fig.9. Clearly, the 3-rd order operator, mediated by  $\alpha_3$  moves the soliton towards the right, while the 5-th order operator, mediated by  $\alpha_5$  moves the soliton towards the left. The other 2 operators affect phase only, and not velocity. In parts where the Gaussians are almost zero, the soliton propagates with unchanged velocity and phase.

Using Eq.(15), or varying  $\alpha_2(x)$ , the NLS unit-background rogue wave becomes

$$\psi(x, t) = \left[ 4 \frac{1 + 4iJ_2(x)}{1 + 4t^2 + 16J_2^2(x)} - 1 \right] e^{2iJ_2(x)}. \quad (41)$$

**Kuznetsov-Ma breathers.** We now consider a Kuznetsov-Ma breather, but allow for variable coefficients. We take

$$\alpha_2(x) = \gamma \operatorname{sech}^2\left(\frac{x-10}{8}\right), \quad \alpha_3(x) = -\frac{\gamma}{4} \operatorname{sech}^2\left(\frac{x-50}{8}\right),$$

$$\alpha_4(x) = \frac{\gamma}{6} \operatorname{sech}^2\left(\frac{x-90}{8}\right)$$

We can still use the result of Eq.(22), again with each  $\alpha_n x$  with  $\int \alpha_n(x) dx$ . Here we take  $a_1 = 1/8$ . Thus  $v_m$  of Eq.(24) is replaced by:  $v_m = -9\gamma \tanh\left(\frac{x-50}{8}\right)$ ,  $B_m$  of Eq.(27) is replaced by:  $B_m = 4\gamma \left[ 4 \tanh\left(\frac{x-10}{8}\right) + 3 \tanh\left(\frac{x-90}{8}\right) \right]$ , while  $\phi_m$  of Eq.(28) is replaced by:

$$\phi_m = 4\gamma \tanh\left(\frac{x-10}{8}\right) + \frac{\gamma}{2} \tanh\left(\frac{x-90}{8}\right).$$

This example, with  $\gamma = \frac{1}{2}$ , is shown in Fig.10. In this figure, the angled propagation (at around  $x = 50$ ) is due to non-zero velocity ( $v_m$ ) being introduced by the third-order operator, with coefficient  $\alpha_3(x)$ , as this operator differs strongly from zero only around  $x = 50$ . The breather peaks, around  $x = 10$  and  $x = 90$ , are due to the influence of coefficients  $\alpha_2(x)$  and  $\alpha_4(x)$ , respectively, as these coefficients differ substantially from zero only near these values of  $x$ . Hence, as in Fig.7, peaks occur in these regions.

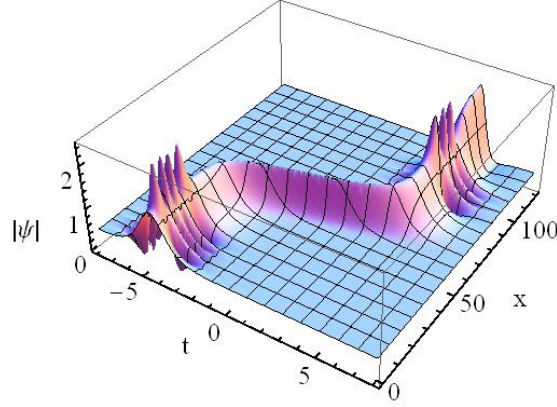


Figure 10: Plot of Kuznetsov-Ma breather in the case of variable coefficients. The pattern shows the influence of 3 operators with coefficients which vary on propagation.

## 8 Conclusion

In conclusion, we have presented the infinite integrable NLSE hierarchy beyond the Lakshmanan-Porsezian-Daniel equation, which is a particular fourth-order case of the hierarchy. Specifically, we have presented explicit forms of the equations and given generalized soliton solutions, plane wave solutions, Akhmediev breathers, Kuznetsov-Ma breathers, periodic and rogue wave solutions for this infinite order hierarchy. We have found that 'even' order equations in the set affect phase and 'stretching factors' in the solutions, while 'odd' order equations affect the velocities. Hence 'odd' order equation solutions can be real functions, while 'even' order equation solutions are always complex. Of special interest is the possibility of using variable coefficients in the hierarchy to influence evolution dynamics. Examples of such evolution are given.

## 9 Appendix

Following eqn. (10), we now present  $K_7$ , the seventh order ( $j = 7$ ), i.e. heptic, operator (starting with seventh order derivative):

$$\begin{aligned}
K_7[\psi] &= \psi_{tttttt} + 70\psi_{tt}^2\psi_t^* + 112\psi_t|\psi_{tt}|^2 \\
&+ 98|\psi_t|^2\psi_{ttt} + 70\psi^2[\psi_t[(\psi_t^*)^2 + 2\psi^*\psi_{tt}^*] \\
&+ \psi^*(2\psi_{tt}\psi_t^* + \psi_{ttt}\psi^*)] + 28\psi_t^2\psi_{ttt}^* \\
&+ 14\psi[\psi^*(20|\psi_t|^2\psi_t + \psi_{ttttt}) + 3\psi_{ttt}\psi_{tt}^* \\
&+ 2\psi_{tt}\psi_{ttt}^* + 2\psi_{tttt}\psi_t^* + \psi_t\psi_{tttt}^* + 20\psi_t\psi_{tt}(\psi^*)^2] \\
&+ 140|\psi|^6\psi_t + 70\psi_t^3(\psi^*)^2 \\
&+ 14(5\psi_{tt}\psi_{ttt} + 3\psi_t\psi_{tttt})\psi^*. \tag{42}
\end{aligned}$$

There is an infinite number of higher-order operators. The highest one that we provide here is  $K_8$ , which is the eighth order ( $j = 8$ ), i.e. octic, operator (starting with eighth order derivative):

$$\begin{aligned}
K_8[\psi] &= \psi_{ttttttt} + 14\psi^3[40|\psi_t|^2(\psi^*)^2 + 20\psi_{tt}(\psi^*)^3 \\
&+ 2\psi_{ttt}^*\psi^* + 3(\psi_{tt}^*)^2 + 4\psi_t^*\psi_{ttt}^*] \\
&+ \psi^2[28\psi^*(14\psi_{tt}\psi_{tt}^* + 11\psi_{ttt}\psi_t^* + 6\psi_t\psi_{ttt}^*) \\
&+ 238\psi_{tt}(\psi_t^*)^2 + 336|\psi_t|^2\psi_{tt}^* + 560\psi_t^2(\psi^*)^3 \\
&+ 98\psi_{tttt}(\psi^*)^2 + 2\psi_{ttttt}^*] + 2\psi\{21\psi_t^2[9(\psi_t^*)^2 \\
&+ 14\psi^*\psi_{tt}^*] + \psi_t[728\psi_{tt}\psi_t^*\psi^* + 238\psi_{ttt}(\psi^*)^2 \\
&+ 6\psi_{tttt}^*] + 34|\psi_{ttt}|^2 + 36\psi_{tttt}\psi_{tt}^* + 22\psi_{tt}\psi_{tttt}^* \\
&+ 20\psi_{tttt}\psi_t^* + 161\psi_{tt}^2(\psi^*)^2 + 8\psi_{ttttt}\psi^*\} \\
&+ 182\psi_{tt}|\psi_{tt}|^2 + 308\psi_{tt}\psi_{ttt}\psi_t^* + 252\psi_t\psi_{ttt}\psi_{tt}^* \\
&+ 196\psi_t\psi_{tt}\psi_{ttt}^* + 168\psi_t\psi_{tttt}\psi_t^* + 42\psi_t^2\psi_{ttt}^* \\
&+ 14\psi^*(30\psi_t^3\psi_t^* + 4\psi_{tttt}\psi_t + 5\psi_{ttt}^2 + 8\psi_{tt}\psi_{tttt}) \\
&+ 490\psi_t^2\psi_{tt}(\psi^*)^2 \\
&+ 140\psi^4\psi^*[(\psi_t^*)^2 + \psi^*\psi_{tt}^*] + 70\psi|\psi|^8. \tag{43}
\end{aligned}$$

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