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Properties of the solutions of delocalised coagulation and inception problems with outflow boundaries

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Abstract

Well posedness is established for a family of equations modelling particle populations undergoing delocalised coagulation, advection, inflow and outflow in a externally specified velocity field. Very general particle types are allowed while the spatial domain is a bounded region of d-dimensional space for which every point lies on exactly one streamline associated with the velocity field. The problem is formulated as a semi-linear ODE in the Banach space of bounded measures on particle position and type space. A local Lipschitz property is established in total variation norm for the propagators (generalised semi-groups) associated with the problem and used to construct a Picard iteration that establishes local existence and global uniqueness for any initial condition. The unique weak solution is shown further to be a differentiable or at least bounded variation strong solution under smoothness assumptions on the parameters of the coagulation interaction. In the case of one spatial dimension strong differentiability is established even for coagulation parameters with a particular bounded variation structure in space. This one dimensional extension establishes the convergence of the simulation processes studied in [Patterson, *Stoch. Anal. Appl.* 31, 2013] to a unique and differentiable limit.

1 Introduction

Smoluchowski [18] introduced equations for the concentrations of particles of different sizes undergoing coagulation in a spatially homogeneous population.

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t,y) = \frac{1}{2}\sum_{y'<y} K(y,y')c(t,y')c(t,y-y') - c(t,y)\sum_{y'} K(y,y')c(t,y'), \tag{1}$$

where $c(t,y)$ is the concentration of particles of size y at time t and K is a symmetric function defining the 'reaction' rates. The Smoluchowski coagulation equations can be regarded as describing a system of binary reactions involving an infinite number of species, but with a very structured, although non-sparse set of rates and (1) abstractly written $\dot{c} = R(c)$. The model therefore extends naturally to a reaction–transport problem for spatially inhomogeneous populations of coagulating particles of the general form $\dot{c} + \mathcal{A}c = R(c)$ for some transport operator $\mathcal{A}$.

Since coagulation is a binary reaction in which every possible pair of particles may coagulate, the equations are, even in the spatially homogeneous case, non-linear and more significantly non-local in particle size (size may here be generalised to 'type'). The first existence results for the Smoluchowski coagulation equation and its extensions were based on convergent sub-sequences of approximating stochastic processes. The first convergence result of this kind with simple diffusive transport of particles is due to Lang and Xanh [8], generalisations were achieved by Norris [12, 11], Wells [19] and Yaghouti et al. [21]. This is quite a natural approach, because the equations are based on a microscopic stochastic model and related stochastic processes have also proved fruitful for numerical purposes going back to Marcus [9] and Gillespie [5].

The results just mentioned are essentially compactness results and say nothing about uniqueness of the limiting trajectories, much less of uniqueness for the solutions to the Smoluchowski equation and its extensions. Convergence and uniqueness were proved together by Guiaş [6] who modelled diffusion as a random walk on a lattice and used a more functional analytic approach. Going further in this direction one is led to regard the Smoluchowski equation and its extensions as an ODE on a Banach space and to proceed via a locally Lipschitz source term and a Picard iteration method to show existence and uniqueness in some functional setting. The general strategy is presented in chapters 5&6 of [17]. Applications to Smoluchowski problems are given by [20, 1, 3] and the works cited therein.

Especially when approaching the Smoluchowski equation from the point of view of stochastic particle systems it is natural to think of measure valued solutions. A particle system is identified with its empirical measure and thus instead of functional solutions one is led to look at measure valued solutions in a weak setting. To give the concrete example that will be the focus of this work: A solution (with a given initial condition) is a flow of measures μ_t on positions in $\mathcal{X}$ and particle types (sizes and potentially additional details) in $\mathcal{Y}$ satisfying

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{X}\times\mathcal{Y}} & f(x,y)\mu_t(\mathrm{d}x,\mathrm{d}y)\\ &= \int_{\mathcal{X}\times\mathcal{Y}} u_t(x)\cdot\nabla f(x,y)\mu_t(\mathrm{d}x,\mathrm{d}y) + \int_{\mathcal{X}\times\mathcal{Y}} f(x,y)I_t(\mathrm{d}x,\mathrm{d}y)\\ &\quad + \frac{1}{2}\int_{(\mathcal{X}\times\mathcal{Y})^2}\left[f(x_1,y_1+y_2) - f(x_1,y_1) - f(x_2,y_2)\right]\\ &\qquad K(y_1,y_2)h(x_1,x_2)\mu_t(\mathrm{d}x_1,\mathrm{d}y_1)\mu_t(\mathrm{d}x_2,\mathrm{d}y_2) \qquad (2)\end{aligned}$$

for all f in a class of functions D to be specified below. Here the problem has been moved from the strong formulation of (1) to a weak setting; a transport operator $u_t\cdot\nabla$ (the dual of the $\mathcal{A}$ mentioned above) has been introduced and the delocalisation of the coagulation specified via a function h, which may be regarded as a mollifier. A particle source term I_t has also been added, which is relevant for many real-world applications as discussed later.

Signed measures can be regarded as Banach space under a wide range of norms and equation (2) interpreted as a Banach space valued ODE and Picard-like fixed point strategies introduced. An important insight of the monograph [7] was to exploit duality of linear operators and norms between measures and appropriate spaces of test functions in pursuit of this programme. In this way one performs most calculations for operators on test function spaces, which are a little easier to work with than operators on spaces of measures. Measure valued solutions are also the topic of [11], which also uses a linear operator approach, but uses approximation rather than duality arguments and deals with unbounded coagulation kernels.

All the work discussed so far deals with diffusing particles (contrast (2)) and solutions either with a zero gradient boundary conditions, which excludes outflow or defined on the whole of $\mathbb{R}^d$ so that outflow is thereby excluded. For numerical reasons motivated by applications in engineering, the present author has been interested in the Smoluchowski equation with advective transport and a delocalised coagulation interaction [15, 10]. In particular for engineering applications particle gain and loss terms are important—industrial equipment is designed to take in material, alter it and then send it on either as waste or product. This gives the problem as formulated in [15, 10] and other applied works a different structure to those studied in previous mathematical works. For example, individual particles experience irreversible processes, but nevertheless the system is expected to reach a steady state in the large time limit under a wide range of conditions. Measure valued processes (which can be interpreted as particle processes) with an inflow term although no interaction were also studied in [4].

For (2) specific problem an initial existence result via the compactness of approximating stochastic processes was given in [14]. In that work however convergence of the approximating processes could not be proved, only sequential compactness, because the number of distinct limit points was unknown. This was not only mathematically frustrating, but also a major obstacle hindering the numerical analysis of the associated simulation methods.

The purpose of the present work is to establish uniqueness of measure valued solutions for (2). Additionally Lipschitz continuity in the initial conditions is shown and the same Picard iteration method that proves uniqueness of solutions provides a purely analytic existence proof. The result can thus be characterised as one of "well posedness". Formally there are some new existence results—the assumption of only one spatial dimension in [14] is relaxed, but with the assumptions used in this work the proof in that paper could easily be extended. The existence of a differentiable strong solution is of interest, because it opens the way to a study of the way in which the solution approaches a solution of the corresponding equation with a local coagulation interaction, see for example [16].

2 Statement of Main Results

In order to make a precise statement it is first necessary to go into details regarding the various objects appearing in (2). The basic spaces are the particle position and type spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively. The type space, which carries information about the mass and any other internal details of a particle is assumed to be a locally compact, second countable Hausdorff space on which coagulation is represented by a commutative + operator. The particle position space $\mathcal{X}$ is assumed to be a simply connected, relatively compact subset of $\mathbb{R}^d$, which is equipped with Lebesgue measure and a derivative ∇. Both $\mathcal{X}$ and $\mathcal{Y}$ are given their respective Borel σ-algebras and $\mathcal{X} \times \mathcal{Y}$ is given the product topology and σ-algebra.

Throughout this work $\mathbb{R}^d$ will be given the usual Euclidean norm, which will be written $|\cdot|$. Linear operators L between two normed spaces $(A, \|\cdot\|_A)$ $(B, \|\cdot\|_B)$ are given the operator norm

$$\|L\|_{A\to B} := \sup_{x\in A:\ \|x\|_A=1} \|Lx\|_B .$$

2.1 Properties of the Flow and Spatial Domain

Velocity field

Particles are assume to be transported in a time dependent velocity field u_t defined on $\overline{\mathcal{X}}$ the closure of $\mathcal{X}$ such that $u \in C\left(\mathbb{R}^+, C^2\left(\overline{\mathcal{X}}, \mathbb{R}^d\right)\right)$, satisfying

- $\|u\|_\infty := \sup_{t\in\mathbb{R}^+, x\in\overline{\mathcal{X}}} |u_t(x)| < \infty$,
- $\|\nabla \cdot u\|_\infty := \sup_{t\in\mathbb{R}^+, x\in\overline{\mathcal{X}}} \left|\sum_{k=1}^d \frac{\partial}{\partial x_k} u_{k,t}(x)\right| < \infty$,
- $|||\nabla u||| := \sup_t \|\nabla u_t(x)\|_{\mathbb{R}^d\to\mathbb{R}^d} < \infty$ viewing the matrices ∇u_t as linear operators,
- $\|\nabla\nabla \cdot u\|_\infty := \sup_{t\in\mathbb{R}^+, x\in\overline{\mathcal{X}}} |\nabla(\nabla \cdot u_t(x))| < \infty$.

Boundaries

It is assumed that the spatial domain $\mathcal{X}$ is simply connected and has a regular boundary $\partial\mathcal{X}$ that can be decomposed into three parts, each with outward normal $n(x)$:

- Γ_{in} where $n(x) \cdot u_t(x) < 0$ for all $t \in \mathbb{R}^+$,
- Γ_{side} where $n(x) \cdot u_t(x) = 0$,
- Γ_{out} where $n(x) \cdot u_t(x) > 0$.

Further $\Gamma_{\text{in}} \subset \mathcal{X}$ but $\Gamma_{\text{side}}, \Gamma_{\text{out}} \subset \mathbb{R}^d \setminus \mathcal{X}$.

Flow Field

Define $\Phi_{s,t}(x)$ as the position at time t of a particle moving with the velocity field u starting from x at time s. It is assumed that

- There exists a $t_0 > 0$ such that, for all $t \geq 0$ and $x \in \mathcal{X}$ one has $\Phi_{t,t+t_0}(x) \notin \mathcal{X}$, that is, an upper bounded on the residence time.
- For every $t > 0$ and $x \in \mathcal{X}$ there exist unique $s(t,x)$, $\xi(t,x)$ such that $\Phi_{s(t,x),t}(\xi(t,x)) = x$ and either $s(t,x) = 0$ or $\xi(t,x) \in \Gamma_{\text{in}}$ (the possibility of both is not excluded). This defines a start position for each point in the flow and $\xi(t,x) = \Phi_{t,s(x)}(x)$.
- $s(t,x)$ and $\xi(t,x)$ are differentiable in x and $\|\nabla s\|_\infty := \sup_{t,x} |\nabla s(t,x)| < \infty$. A bound for the derivative of ξ is given in the appendix.
- The set $\Xi_t = \{x \in \mathcal{X} \colon \xi(t,x) \in \Gamma_{\text{in}}\}$ forms a differentiable $d-1$ dimensional manifold that divides $\mathcal{X}\setminus\Xi_t$ into two disjoint simply connected components.

2.2 Test Function Spaces

Definition 1. *Let $\mathcal{B}_{\mathrm{b}}(\mathcal{Y})$ be the space of bounded measurable functions on $\mathcal{Y}$ with the supremum (not essential supremum) norm, which will be written $\|\cdot\|_{\mathcal{Y}-\infty}$.*

Definition 2. *Let $B := \mathcal{B}_{\mathrm{b}}(\mathcal{X} \times \mathcal{Y})$ be the space of bounded measurable functions on $\mathcal{X} \times \mathcal{Y}$ with the supremum (not essential supremum) norm, which will be written $\|\cdot\|_B$.*

Definition 3. *Let B^d be the space of d-dimensional vector valued functions with components in B. This will be given the norm $\|f\|_{B^d} := \sup_{x,y} |f(x, y)|$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$.*

To handle the derivative in (2) and associated boundary condition introduce

Definition 4.

$$D := \left\{ f \in B \colon f \text{ differentiable}, \nabla f \in B^d, \lim_{x \to \Gamma_{\mathrm{out}}} \|f(x, \cdot)\|_{\mathcal{Y}-\infty} = 0 \right\}$$

The norm is

$$\|f\|_D = \|f\|_B + \|\nabla f\|_{B^d}.$$

This is an appropriate class of test functions to use in (2), because the derivative is well behaved. For a discussion of the boundary condition see [14], although that work imposes slightly stricter regularity conditions, which are here seen to be unnecessary.

2.3 Solution Spaces

A particle distribution is at a minimum a measure on the product of the particle position and type spaces, that is on $\mathcal{X} \times \mathcal{Y}$. The solution processes must accordingly take values in the following spaces, which are built from the space of measures on particle types $\mathcal{Y}$:

Definition 5. *Let $\left(\mathcal{M}(\mathcal{Y}), \|\cdot\|_{\mathcal{Y}-\mathrm{TV}}\right) =: \mathcal{M}(\mathcal{Y})_{\mathrm{TV}}$ be the normed space of signed bounded measures on $\mathcal{Y}$ with the total variation norm*

$$\|\mu\|_{\mathcal{Y}-\mathrm{TV}} := \sup_{f \neq 0} \frac{\left| \int_{\mathcal{Y}} f(y) \mu(\mathrm{d}y) \right|}{\|f\|_{\mathcal{Y}-\infty}}, \quad f \in \mathcal{B}_{\mathrm{b}}(\mathcal{Y}).$$

Definition 6. *Let $\mathcal{M} = \mathcal{M}(\mathcal{X} \times \mathcal{Y})$ be the vector space of bounded signed measures on $\mathcal{X} \times \mathcal{Y}$.*

Under reasonable assumptions one expects to find solutions to (2) that are absolutely continuous with respect to Lebesgue measure on $\mathcal{X}$; this leads to the following space (compare [11]).

Definition 7.

$$\mathcal{M}_{0,\infty} = \{\mu \in \mathcal{M} \colon \mu(\mathrm{d}x, \mathrm{d}y) = c(x, \mathrm{d}y)\mathrm{d}x, c \in L^\infty\left(\mathcal{X}, \mathcal{M}(\mathcal{Y})_{\mathrm{TV}}\right)\}$$

with the norm

$$\|c\|_{\mathcal{M}_{0,\infty}} = \operatorname*{ess\,sup}_x \|c(x, \cdot)\|_{\mathcal{Y}-\mathrm{TV}}$$

where a measure is identified with its density.

The B and D dual norms on $\mathcal{M}$ will play a role in this work

Definition 8. *Let $\mu \in \mathcal{M}$,*

$$\|\mu\|_{\mathrm{TV}} \equiv \|\mu\|_{B^\star} := \sup_{f \neq 0} \frac{\left| \int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \mu(\mathrm{d}x, \mathrm{d}y) \right|}{\|f\|_B}, \quad f \in B,$$

and

$$\|\mu\|_{D^\star} := \sup_{f \neq 0} \frac{\left| \int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \mu(\mathrm{d}x, \mathrm{d}y) \right|}{\|f\|_D}, \quad f \in D.$$

As the notation suggests, the $B^\star$ norm is the total variation norm on $\mathcal{M}$. For calculations the $B^\star$ point of view is emphasised, however the main results are stated in terms of TV. When dealing with processes the following abbreviation is useful

Definition 9. *Let $T > 0$ and $c \in L^\infty([0,T);\mathcal{M})$ then*

$$|||c|||_{B^\star} := \operatorname*{ess\,sup}_{t\in[0,T)} \|c(t,\cdot,\cdot)\|_{B^\star}.$$

2.4 Coagulation

It is now possible to set out the assumptions on the coagulation dynamics specified by K and h in (2). K is assumed to be non-negative and measurable with some bound $K_\infty > 0$ such that $\sup_{y_1,y_2} K(y_1,y_2) \le K_\infty$.

The delocalisation $h\colon \mathcal{X}^2 \to \mathbb{R}$ must be measurable and non-negative. For fixed $x_1 \in \mathcal{X}$ write h_{1,x_1} and h_{2,x_1} for the the functions given by $h_{1,x_1}(\cdot) = h(x_1,\cdot)$ and $h_{2,x_1}(\cdot) = h(\cdot,x_1)$. It will be assumed that neither K nor h are identically zero—this would lead to a trivial problem with no coagulation.

H1: $\|h_{i,x}\|_B \le C_1 \quad \forall x \in \mathcal{X},\ i = 1,2.$

H2: H1 holds and $h(x,x_2) = \sum_{j=1}^J \chi_{j,1}(x)\chi_{j,2}(x_2)$ with $\chi_{j,i}$ positive, and of special bounded variation (derivative in L^1 plus atoms) for all i and j with the number of atoms in the weak derivatives bounded. Further one has $\sup_{x,x_2} \sum_{j=1}^J \|\chi_{j,2}\|_B \int_r^t |\nabla\chi_{j,1}(\Phi_{r,s}(x),x_2)|\,\mathrm{d}s \le C_2 t_0 e^{|||\nabla u|||\min(t-r,t_0)}$ and its symmetric counterpart $\sup_{x_1,x} \sum_{j=1}^J \|\chi_{j,1}\|_B \int_r^t |\nabla\chi_{j,2}(x_1,\Phi_{r,s}(x))|\,\mathrm{d}s \le C_2 t_0 e^{|||\nabla u|||\min(t-r,t_0)}$

H3: H1 holds and the $h_{i,x}$ are in D with $\left\|\frac{\partial}{\partial\xi}h_{i,x}(\xi)\right\|_B \le C_2\ \forall x \in \mathcal{X}, i = 1,2$. It should be noted that H3 implies H2 (Proposition 60 is helpful here).

The function h parametrises the numerical methods that lie behind this work [15]. H2 is describes the case where the spatial domain is partitioned into cells and coagulation is only simulated between particles that are in the same cell. From a software point of view this is somewhat simpler than dealing with functions satisfying H3. In one dimension, which was the case simulated in [15], H2 is a weak integrability condition on the derivative of h.

2.5 Inception

Particles are added to the system with intensity given by signed measures $I_t \in \mathcal{M}$.

I1: $\sup_t \|I_t\|_{B^\star} < \infty$ and $I \in C([0,\infty),(\mathcal{M},\|\cdot\|_{D^\star}))$.

I2: I1 holds, the I_t are non-negative measures and for every $f \in B$

$$\int_{\mathcal{X}\times\mathcal{Y}} f(x,y)I_t(\mathrm{d}x,\mathrm{d}y) = \int_{\mathcal{X}\times\mathcal{Y}} f(x,y)I_{\mathrm{int}}(t,x,\mathrm{d}y)\mathrm{d}x + \int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} f(\xi,y)I_{\mathrm{bdry}}(t,\xi,\mathrm{d}y)\mathrm{d}\xi$$

with $I_{\mathrm{int}} \in C([0,\infty),\mathcal{M}_{0,\infty})$ also $I_{\mathrm{bdry}} \in C([0,\infty),L^\infty(\Gamma_{\mathrm{in}},\mathcal{M}(\mathcal{Y})_{\mathrm{TV}}))$ with the respective norms uniformly bounded for all time and with some $I_* > 0$ such that $\|I_{\mathrm{bdry}}(t,\xi,\cdot)\|_{\mathcal{Y}-\mathrm{TV}} \le I_* u_t(\xi)\cdot n(\xi)$ for all t and $\xi \in \Gamma_{\mathrm{in}}$.

I3: I2 holds, I_{bdry} has a time and space derivative so that $I_{\mathrm{bdry}} \in C^1([0,\infty)\times\Gamma_{\mathrm{in}},\mathcal{M}(\mathcal{Y})_{\mathrm{TV}})$ and I_{int} has an $\mathcal{X}$-derivative which is $\nabla I_{\mathrm{int}} \in C\left([0,\infty),(\mathcal{M}_{0,\infty})^d\right)$

2.6 Statements of the Theorems

These results progress from local existence and uniqueness of a measure valued solution to a global result and then existence followed by differentiability of a density for the measures.

Theorem 10. *Assume H2 or H3 holds and that $c_0 \in \mathcal{M}$, then there exists a $T = T(c_0)$ such there is a unique solution c_t to* (2) *in $L^\infty([0,T),(\mathcal{M},\|\cdot\|_{\mathrm{TV}}))$ with initial condition c_0 and this solution is in $C([0,T),(\mathcal{M},\|\cdot\|_{\mathrm{TV}}))\cap C^1((0,T),(\mathcal{M},\|\cdot\|_{D^\star}))$.*

Additionally, there is no time interval on which more than one TV-bounded solution exists for a given initial condition. If solutions exist on a common compact time interval for at two or more initial conditions, then the solutions are Lipschitz continuous with respect to the initial data in the TV-norm on this compact time interval.

In the physically reasonable setting of non-negative particle numbers, the previous result holds for all time:

Theorem 11. *The $T = T(c_0)$ from the previous theorem is ∞ if c_0 and the I_t are non-negative measures.*

Theorem 12. *Assume H2 or H3 holds, that c_0 is in the positive cone of $\mathcal{M}_{0,\infty}$, and that I2 is satisfied, then* (2) *has a unique solution, which is in $L^\infty([0,\infty),\mathcal{M}_{0,\infty})$ and therefore has a density in $L^\infty([0,\infty)\times\mathcal{X},\mathcal{M}(\mathcal{Y})_{\mathrm{TV}})$ starting from c_0.*

Theorem 13. *Assume that $c_0 \in W^{1,\infty}(\mathcal{X},\mathcal{M}(\mathcal{Y})_{\mathrm{TV}})$ is consistent with the boundary condition given below, that I3 is satisfied, and further that either H3 holds and $\mathcal{X}$ has a sufficiently regular boundary or $d = 1$, H2 holds and u is bounded away from 0, then* (2) *has a unique solution c with a density in $W^{1,\infty}([0,\infty)\times\mathcal{X},\mathcal{M}(\mathcal{Y})_{\mathrm{TV}})$, satisfying the boundary condition $-u_t(x)\cdot n(x)c(t,x,\mathrm{d}y) = I_{\mathrm{bdry}}(t,x,\mathrm{d}y) \quad \forall t \in \mathbb{R}^+,\ x \in \Gamma_{\mathrm{in}}$ and with initial condition c_0.*

As a corollary of the preceding two results an earlier result by the author, which demonstrated the existence of converging sub-sequences of stochastic approximations to solutions (2) can be extended to a full convergence result:

Theorem 14. *The stochastic jump processes studied in [14], which have $\mathcal{X} = [0,L)$ for some $L > 0$ and satisfy H2 and I3 converge to the unique solution of* (2) *and this weak solution is also a strong solution in the Sobolev space $W^{1,\infty}([0,\infty)\times[0,L),\mathcal{M}(\mathcal{Y})_{\mathrm{TV}})$ provided that the initial condition c_0 is in $W^{1,\infty}([0,L),\mathcal{M}(\mathcal{Y})_{\mathrm{TV}})$ with $u_0(0)c_0(0,\mathrm{d}y) = I_{\mathrm{bdry}}(0,0,\mathrm{d}y)$. Further one has $u_t(0)c(t,0,\mathrm{d}y) = I_{\mathrm{bdry}}(t,0,\mathrm{d}y)$ for all t.*

Proof. In [14] it was shown that every sequence of approximating processes has a sub-sequence converging to a solution of (2) (a compactness result). Theorem 11 shows that there is only one such limit point so one has convergence and Theorem 13 yields the differentiability. □

3 Dual Operator Estimates

Introduce the more compact notation $\langle f,\mu\rangle = \int_{\mathcal{X}\times\mathcal{Y}} f(x,y)\mu(\mathrm{d}x,\mathrm{d}y)$ for $f \in B$ and $\mu \in \mathcal{M}$. It is now helpful to seek a generator for the evolution given in (2), that is an operator A_t such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f,\mu_t\rangle = \langle A_t(f),\mu_t\rangle + \langle f,I_t\rangle\,. \tag{3}$$

This is in fact a dual generator, because it acts on the functions not the measures.

The author emphasises his dependence on Kolokoltsov [7] for the material in this section and the first half of the next. The first novelty in this section is the boundary condition associated with the finite domain and outflow, which required careful treatment, but is not covered by the existing work. Also the consideration of coefficients of bounded variation (H2) is essential to treating the motivating example from [14] and even under this relatively weak assumption differentiability of the solutions in one spatial dimension is established. An additional variation from [7] appears in Proposition 26 where some additional problem structure is exploited and enables the fixed point methods to be applied in the $B^\star$-norm, rather than the weaker $D^\star$-norm used in [7].

3.1 The Generators

Because (2) is quadratic in μ the same must be true of the expression $\langle A_t[\mu](f), \mu_t\rangle$, which is achieved by including the path $(\mu_r)_{r\in[0,t]}$ as a parameter of A_t. It is technically convenient to parametrise by the entire path, not just μ_t, because one eventually deals with propagators where the dependence cannot be expressed in terms of μ at any finite set of time points. One notes that $A_t[\mu] = U_t + H_t[\mu]$ where U is the transport operator and H is the coagulation operator.

Definition 15. *Let $\mu \in L^\infty\left([0,T), (\mathcal{M}, \|\cdot\|_{B^\star})\right)$ and assume H1 holds. The coagulation generator parametrised by μ is $H_t[\mu]\colon B \to B$ defined by*

$$H_t[\mu](f)(x,y) = \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}} h(x,x_2) f(x, y+y_2) K(y,y_2)\mu_t(\mathrm{d}x_2, \mathrm{d}y_2) \\ - \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}} f(x,y)\left[h(x,x_2)+h(x_2,x)\right] K(y,y_2)\mu_t(\mathrm{d}x_2,\mathrm{d}y_2) \quad (4)$$

for $t \in [0,T)$.

This is not the only possible definition for $H_t[\mu]$, other versions also yield the desired expression (the coagulation term from (2)) for $\langle H_t[\mu](f), \mu_t\rangle$. Each definition would lead to characterising the solutions as fixed points of a different mapping; the definition given here seems to be the one that minimises the technical difficulties in the following analysis.

Proposition 16. *Let $0 < T$ and $\mu \in L^\infty\left([0,T), \mathcal{M}\right)$ and assume H1 holds, then the operator norm of $H_t[\mu]$ as a mapping $B \to B$ satisfies*

$$\operatorname*{ess\,sup}_t \|H_t[\mu]\|_{B\to B} \leq \frac{3}{2} K_\infty C_1 |||\mu|||_{B^\star}.$$

Proof. Immediate. □

Definition 17. *Let $t \in \mathbb{R}$ and $f \in D$ then the transport generator $U_t\colon D \to B$ is given by*

$$U_t f(x,y) = u_t(x)\cdot \nabla f(x,y).$$

One can now define $A_t[\mu] = U_t + H_t[\mu]$ as a linear operator $D \to B$.

3.2 The Propagators

Propagators are generalisations of semi-groups to deal with time dependent generators. For a detailed discussion the reader is referred to [7, Chapter 2] or [17, Chapter 5]. The key idea (given in the dual setting appropriate to this section) is that a generator A_t generates a family of linear operators $A^{r,s}$ such that $A^{r,s}A^{s,t} = A^{r,t}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{s,t} = A^{s,t}A_t, \quad \frac{\mathrm{d}}{\mathrm{d}s}A^{s,t} = -A_s A^{s,t}. \quad (5)$$

The goal of this section is to construct such a family of propagators for the generator $A_t[\mu]$ from the previous section.

Definition 18. *Let $s,t \in \mathbb{R}$ and $f \in B$ and define the transport propagators $U^{t,s}\colon B \to B$ by*

$$U^{s,t} f(x,y) = \begin{cases} f\left(\Phi_{s,t}(x), y\right) & \Phi_{s,t}(x) \in \mathcal{X} \\ 0 & \textit{otherwise} \end{cases}$$

where Φ is the flow due to the velocity field u (see §2.1 and Appendix A).

Proposition 19. *Let $t \geq s$, then the transport propagator $U^{s,t}$ preserves D. The following operator norm estimates hold:*

$$\left\|U^{s,t}\right\|_{B \to B} \leq \mathbb{1}\left(t-s \leq t_0\right),$$

where $\mathbb{1}$ is an indicator function and

$$\left\|U^{s,t}\right\|_{D \to D} \leq e^{(t-s)|||\nabla u|||}\mathbb{1}\left(t-s \leq t_0\right).$$

Proof. The B-norm of f is immediate. Use the chain rule and Proposition 60 in the appendix for the derivative of f. □

The required propagator is now constructed as a perturbation of the transport propagator U by the bounded coagulation generator:

Definition 20. *Let $T > 0$, $0 \leq r \leq t \leq T$ and $\mu \in L^\infty\left([0,T), \mathcal{M}\right)$, and define (compare [7, Theorem 2.9]):*

$$A^{r,t}[\mu] \quad := \quad U^{r,t} \quad + \quad \sum_{m=1}^{\infty} \int_{r \leq s_1 \leq \ldots \leq s_m \leq t} U^{r,s_1} H_{s_1}[\mu] \cdots U^{s_{m-1},s_m} H_{s_m}[\mu] U^{s_m,t} \mathrm{d}s_1 \cdots \mathrm{d}s_m.$$

It is now necessary to establish estimates for the operator norm of A on B and D. For this it is shown that the infinite sum just given is absolutely convergent in both operator norms. During this analysis it is convenient to use some additional notation:

Definition 21. *Under the assumptions of Definition 20 let $f \in B$ and $t \geq 0$; define both $f^0_{r,t} := U^{r,t} f$ and*

$$f^m_{r,t} := \int_r^t U^{r,s} H_s[\mu] f^{m-1}_{s,t} \mathrm{d}s.$$

This allows one to write

$$A^{r,t}[\mu] f = \sum_{m=0}^{\infty} f^m_{r,t}. \tag{6}$$

Proposition 22. *Under the assumptions of Definitions 20&21*

$$\left\|f^m_{r,t}(x,\cdot)\right\|_{\mathcal{Y}-\infty} \leq \frac{1}{m!}\left(\frac{3}{2} K_\infty C_1 |||\mu|||_{B^\star}(t-r)\right)^m \left\|f\left(\Phi_{r,t}(x),\cdot\right)\right\|_{\mathcal{Y}-\infty},$$

which is zero for $t - r \geq t_0$ and

$$\left\|f^m_{r,t}\right\|_B \leq \frac{1}{m!}\left(\frac{3}{2} K_\infty C_1 |||\mu|||_{B^\star}(t-r)\right)^m \|f\|_B.$$

Proof. Proceed by induction. □

The B-operator norm estimate now follows:

Proposition 23. *Let $T > 0$, $0 \leq r \leq t < T$ and $\mu \in L^\infty\left([0,T), (\mathcal{M}, \|\cdot\|_{B^\star})\right)$ and assume H1 holds, then $A^{r,t}[\mu]$ is a locally bounded propagator on B satisfying*

$$\left\|A^{r,t}[\mu]\right\|_{B \to B} \leq e^{\frac{3}{2} K_\infty C_1 |||\mu|||_{B^\star}(t-r)} \mathbb{1}(t-r \leq t_0)$$

and $A^{r,t}[\mu] f$ is $\|\|_B$-continuous in t for every $f \in B$ and $t \geq r$ (this is known as 'strong continuity'). Further, for any $f \in D$ and almost all t

$$\frac{\mathrm{d}}{\mathrm{d}t} A^{r,t}[\mu] f = A^{r,t}[\mu] A_t[\mu] f,$$

where one recalls $A_t[\mu] = u_t \cdot \nabla + H_t[\mu]$.

Proof. The first part of the result follows from Proposition 22 and (6).

The (left) generator $A_t[\mu]$ can be found differentiating the series in Proposition 20 term by term and observing that the resulting series is again absolutely convergent. □

Differentiating with respect to r in Proposition 23 is not possible, because $A^{r,t}$ does not necessarily preserve D. This is addressed in the next few propositions by making stronger smoothness assumptions on h, the spatial delocalisation of the coagulation interaction introduced in §2.4 and used in the definition of H (Definition 15).

Proposition 24. *Under the assumptions of Definitions 20&21 and additionally assuming either H3 holds or H2 holds and μ is bounded for all (not just Lebesgue almost all t), for example because it is continuous*

$$\begin{aligned}\left\|\nabla f^m_{r,t}(x,\cdot)\right\|_{\mathcal{Y}-\infty} \leq{}& \left(\frac{3}{2}K_\infty C_1 |||\mu|||_{B^\star}\right)^m \frac{(t-r)^m}{m!}\left\|\nabla\left(f\left(\Phi_{r,t}(x),\cdot\right)\right)\right\|_{\mathcal{Y}-\infty}\\ &+ m\left(\frac{3}{2}K_\infty C_1 |||\mu|||_{B^\star}\right)^m \frac{(t-r)^{m-1}}{(m-1)!}\frac{C_2 t_0 e^{|||\nabla u|||(t-r)}}{C_1}\left\|f\left(\Phi_{r,t}(x),\cdot\right)\right\|_{\mathcal{Y}-\infty}\end{aligned}$$

and

$$\begin{aligned}\left\|\nabla f^m_{r,t}\right\|_{B^d} \leq{}& \left(\frac{3}{2}K_\infty C_1 |||\mu|||_{B^\star}\right)^m \frac{(t-r)^m}{m!} e^{|||\nabla u|||(t-r)}\left\|\nabla f\right\|_{B^d}\mathbb{1}(t-r\geq t_0)\\ &+ m\left(\frac{3}{2}K_\infty C_1 |||\mu|||_{B^\star}\right)^m \frac{(t-r)^{m-1}}{(m-1)!}\frac{C_2 t_0 e^{|||\nabla u|||(t-r)}}{C_1}\left\|f\right\|_B\mathbb{1}(t-r\geq t_0).\end{aligned}$$

Proof. The first inequality is established by induction making use of Proposition 22 for the terms in f. The second inequality introduces Proposition 60 to get an estimate for $\nabla\Phi_{r,t}(x)$ □

Proposition 25. *Let $T>0$, $0\leq r\leq t<T$, $\mu\in L^\infty\left([0,T),(\mathcal{M},\|\cdot\|_{B^\star})\right)$ and either H3 hold or H2 hold but with μ bounded for all (not just Lebesgue almost all) times, then $A^{r,t}[\mu]$ is a propagator on D and there is a $C_3\in\mathbb{R}$ such that*

$$\left\|A^{r,t}[\mu]\right\|_{D\to D}\leq e^{\left(|||\nabla u|||+\frac{3}{2}K_\infty C_1|||\mu|||_{B^\star}\right)(t-r)}C_3\mathbb{1}(t-r\leq t_0).$$

Further, for any $f\in D$ and almost all t

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{r,t}[\mu]f = A^{r,t}[\mu]A_t[\mu]f,\quad \frac{\mathrm{d}}{\mathrm{d}r}A^{r,t}[\mu]f = -A_r[\mu]A^{r,t}[\mu]f.$$

Proof. From Proposition 24 one sees that, for $f\in D$

$$\begin{aligned}\left\|\nabla\left(A^{r,t}[\mu]f\right)\right\|_{B^d}\leq{}& e^{\left(\frac{3}{2}K_\infty C_1|||\mu|||_{B^\star}+|||\nabla u|||\right)(t-r)}\mathbb{1}(t-r\leq t_0)\times\\ &\left(\left\|\nabla f\right\|_{B^d}+\frac{3K_\infty C_2 t_0|||\mu|||_{B^\star}}{2}\left(1+\frac{3}{2}K_\infty C_1|||\mu|||_{B^\star}(t-r)\right)\left\|f\right\|_B\right).\end{aligned}\tag{7}$$

For the f part of the D-norm use Proposition 23, the first statement of that proposition also established the boundary condition for D. Differentiation in r and t is performed term by term in the infinite sum from Definition 20. □

These results concerning the dual propagators are concluded by showing Lipschitz continuity in the measure valued path parameter.

Proposition 26. *Let $T>0$, suppose $\mu,\nu\in L^\infty\left([0,T),(\mathcal{M},\|\cdot\|_{B^\star})\right)$, $0\leq s\leq t<T$ and H1 holds, then*

$$\begin{aligned}&\left\|A^{s,t}[\mu]-A^{s,t}[\nu]\right\|_{B\to B}\\ &\qquad\leq\frac{3}{2}K_\infty C_1 e^{3K_\infty C_1\max(|||\mu|||_{B^\star},|||\nu|||_{B^\star})(t-s)}\mathbb{1}(t-s\leq t_0)\operatorname*{ess\,sup}_{r\in[s,t]}\left\|\mu_r-\nu_r\right\|_{B^\star}\mathrm{d}r.\end{aligned}$$

and

$$\left\|A^{s,t}[\mu]-A^{s,t}[\nu]\right\|_{B\to B} \leq \frac{3}{2}K_\infty C_1(t-s)e^{\frac{3}{2}K_\infty C_1\max(|||\mu|||_{B^\star},|||\nu|||_{B^\star})(t-s)}\mathbb{1}(t-s\leq t_0)\int_s^t \|\mu_r-\nu_r\|_{B^\star}\,\mathrm{d}r.$$

Proof. Write $M=\max(|||\mu|||_{B^\star},|||\nu|||_{B^\star})$ and show by induction that

$$\left\|U^{r,s_1}H_{s_1}[\mu]\cdots U^{s_{m-1},s_m}H_{s_m}[\mu]U^{s_m,t}-U^{r,s_1}H_{s_1}[\nu]\cdots U^{s_{m-1},s_m}H_{s_m}[\nu]U^{s_m,t}\right\|_{B\to B} \leq \frac{3}{2}K_\infty C_1\left(\frac{3}{2}K_\infty C_1 M\right)^{m-1}\sum_{j=1}^m \left\|\mu_{s_j}-\nu_{s_j}\right\|_{B^\star}. \quad (8)$$

□

This result exploits a small amount of additional problem structure to adapt the method set out in the proof of Theorem 2.12 in [7]. The key is that the parameterisation only affects the coagulation (H) part of the propagator, which has a bounded generator, while the transport (U) part of the propagator, which has an unbounded generator is independent of the parameterisation by μ and ν.

4 Operators on the Space of Measures

Under the duality pairing of B and $\mathcal{M}$ given by $\langle f,\mu\rangle=\int_{\mathcal{X}\times\mathcal{Y}} f\mu(\mathrm{d}x,\mathrm{d}y)$ as used above, (dual) operators $B\to B$ define (pre-dual) operators $\mathcal{M}\to\mathcal{M}$ with the same operator norms.

Definition 27. *Let $0\leq s\leq t<T$ and $\mu\in L^\infty([0,T),(\mathcal{M},\|\cdot\|_{B^\star}))$. For the pre-duals of $U^{s,t}$ and $A^{s,t}[\mu]$ write $\widetilde{U}^{t,s}$ and $\widetilde{A}^{t,s}[\mu]$ respectively and note the reversal of the time indices. For the pre-dual of $H_t[\mu]$ write $\widetilde{H}_t[\mu]$.*

It is emphasised that $A^{s,t}[\mu]$ acts on functions while $\widetilde{A}^{t,s}[\mu]$ acts on measures, but both are parameterised by a measure-valued path μ.

The existence of the dual operators and their norm estimates is immediate, see for example [7, Thrm 2.10]. The duality relations yield:

Proposition 28. *Let $0\leq s\leq t<T$, $\mu,\nu\in L^\infty([0,T),\mathcal{M})$ and assume H1 holds, then*

$$\left\|\widetilde{A}^{t,s}[\mu]\right\|_{\mathcal{M}\to\mathcal{M}}=\left\|A^{s,t}[\mu]\right\|_{B\to B}\leq e^{\frac{3}{2}K_\infty C_1|||\mu|||_{B^\star}(t-s)}\mathbb{1}(t-s\leq t_0),$$

along with

$$\left\|\widetilde{A}^{t,s}[\mu]-\widetilde{A}^{t,s}[\nu]\right\|_{\mathcal{M}\to\mathcal{M}}=\left\|A^{s,t}[\mu]-A^{s,t}[\nu]\right\|_{B\to B} \leq \frac{3}{2}K_\infty C_1 e^{3K_\infty C_1\max(|||\mu|||_{B^\star},|||\nu|||_{B^\star})(t-s)}\mathbb{1}(t-s\leq t_0)\int_{r\in[s,t]}\|\mu_r-\nu_r\|_{B^\star}\,\mathrm{d}r.$$

and

$$\left\|\widetilde{A}^{t,s}[\mu]-\widetilde{A}^{t,s}[\nu]\right\|_{\mathcal{M}\to\mathcal{M}}=\left\|A^{s,t}[\mu]-A^{s,t}[\nu]\right\|_{B\to B} \leq \frac{3}{2}K_\infty C_1(t-s)e^{\frac{3}{2}K_\infty C_1\max(|||\mu|||_{B^\star},|||\nu|||_{B^\star})(t-s)}\mathbb{1}(t-s\leq t_0)\operatorname*{ess\,sup}_{r\in[s,t]}\|\mu_r-\nu_r\|_{B^\star}.$$

Proof. Duality and Proposition 23. □

Proposition 29. *Let $0 \le s \le t < T$, $\mu \in L^\infty\left([0,T),(\mathcal{M},\|\cdot\|_{B^\star})\right)$, $c \in \mathcal{M}$, $f \in D$ and assume H1 holds, then for almost all t*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle f, \widetilde{A}^{t,s}[\mu]c\right\rangle = \left\langle A_t[\mu]f, \widetilde{A}^{t,s}[\mu]c\right\rangle.$$

Proof. Duality and Proposition 23 □

4.1 The Fixed Point Mapping

This section presents a Picard iteration method for (2) highlighting the roles of the $B^\star$ and $D^\star$ norms on the space of measures. The mapping that will be shown to have a fixed point is:

Definition 30. *Suppose $c_0 \in \mathcal{M}$, $0 \le t < T$, let $\mu \in L^\infty\left([0,T),(\mathcal{M},\|\cdot\|_{B^\star})\right)$ and suppose H1 holds. Define $\Psi_{c_0}\colon L^\infty\left([0,T),(\mathcal{M},\|\cdot\|_{B^\star})\right) \to L^\infty\left([0,T),(\mathcal{M},\|\cdot\|_{B^\star})\right)$ by*

$$\Psi_{c_0}(\mu)(t) = \widetilde{A}^{t,0}[\mu]c_0 + \int_0^t \widetilde{A}^{t,s}[\mu]I_s\mathrm{d}s. \tag{9}$$

Proposition 31. *Under the assumptions of Definition 30 one has $\Psi_{c_0}(\mu) \in C_{\mathrm{b}}\left([0,T],(\mathcal{M},\|\cdot\|_{B^\star})\right)$ with*

$$\|\Psi_{c_0}(\mu)(t)\|_{B^\star} \le e^{\frac{3}{2}K_\infty C_1 \|\|\mu\|\|_{B^\star}\min(t,t_0)}\left(\|c_0\|_{B^\star}\,\mathbb{1}\left(t-s \le t_0\right) + \frac{2\sup_s\|I_s\|_{B^\star}}{3K_\infty C_1\|\|\mu\|\|_{B^\star}}\right).$$

The time derivative exists for almost all $t \in (0,T)$ with

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Psi_{c_0}(\mu)(t)\right\|_{D^\star} \le \|I_t\|_{D^\star} + \left(\|u\|_\infty + \frac{3}{2}K_\infty C_1\|\mu_t\|_{B^\star}\right)\|\Psi_{c_0}(\mu)(t)\|_{B^\star} \tag{10}$$

and if $\mu \in C\left([0,T),(\mathcal{M},\|\cdot\|_{B^\star})\right)$ then $\Psi_{c_0}(\mu) \in C^1\left((0,T),(\mathcal{M},\|\cdot\|_{D^\star})\right)$.

Proof. $B^\star$ boundedness is a consequence of Proposition 28 and continuity follows from the continuity in t of $\widetilde{A}^{t,0}[\mu]$.

For the time derivative differentiate the formula in Definition 30, and use Proposition 29. □

Proposition 32. *Suppose $c_0 \in \mathcal{M}$, $T \in (0,\infty)$, H2 or H3 holds and $c\colon [0,T) \to (\mathcal{M},\|\cdot\|_{B^\star})$ is a bounded solution to* (2) *with initial condition c_0, then c is a fixed point of Ψ_{c_0}.*

Proof. Suppose c to be a solution of (2) and let $t \in [0,T)$, then using duality and Proposition 25 one finds

$$\frac{\partial}{\partial r}\left\langle f, \widetilde{A}^{t,r}[c]c_r\right\rangle = \left\langle f, \widetilde{A}^{t,r}[c]I_r\right\rangle. \tag{11}$$

Integrating over $r \in [0,t]$ completes the result. This (standard) argument can be found, for example, in [17, §5.1]. □

Proposition 32 is the only place where one requires H2 or H3 in the existence and uniqueness analysis. This is in order to invoke Proposition 25 and more fundamentally so that $A^{s,t}[\mu]$ preserves D; otherwise one cannot give meaning to $\frac{\mathrm{d}}{\mathrm{d}r}\widetilde{A}^{t,r}[\mu]$. Without this result it still follows that the mapping Ψ has unique fixed point with all the advertised properties (in particular solving (2)), but one cannot rule out the possibility that there are additional (possibly less regular) solutions to (2). These conclusions are stated more formally in Proposition 35 for which two preparatory results are needed.

Proposition 33. *Let $c_0 \in \mathcal{M}$, $M \in \mathbb{R}^+$ be large enough to satisfy*

$$M > \|c_0\|_{B^\star} + \frac{2\sup_s\|I_s\|_{B^\star}}{3K_\infty C_1 M},$$

define $E_M = \{\mu \in \mathcal{M}\colon \|\mu\|_{B^\star} \le M\}$ and assume H1 holds. Then there exists a $\tau_M > 0$ such that Ψ_{c_0} preserves $L^\infty\left([0,\tau_M),(E_M,\|\cdot\|_{B^\star})\right)$.

Proof. Let $r_M > 1$ be given by

$$r_M \left(\|c_0\|_{B^\star} + \frac{2\sup_s \|I_s\|_{B^\star}}{3K_\infty C_1 M} \right) = M \tag{12}$$

and suppose $\mu \in L^\infty\left([0,T),(E_M,\|\cdot\|_{B^\star})\right)$ for some $T > 0$. Use Definition 30 along with the operator norm estimate from Proposition 28 to see that, for $t < T$

$$\|\Psi_{c_0}(\mu)(t)\|_{B^\star} \le e^{\frac{3}{2}K_\infty C_1 M \min(t,t_0)} \frac{M}{r_M} \tag{13}$$

and so $\|\Psi_{c_0}(\mu)(t)\|_{B^\star} \le M$ if $\min(t,t_0) \le \frac{2\log r_M}{3K_\infty C_1 M}$. Hence it is sufficient to take $\tau_M = \frac{2\log r_M}{3K_\infty C_1 M}$ and if t_0, the maximum residence time for a particle, satisfies $t_0 \le \frac{2\log r_M}{3K_\infty C_1 M}$ then one may take $\tau_M = \infty$. □

Proposition 34. *Let $c_0 \in \mathcal{M}$ and E_M, τ_M be as in Proposition 33 and assume H1 holds, then there is a $\tau'_M \le \tau_M$ such that Ψ_{c_0} is a contraction on $L^\infty\left([0,\tau'_M),(E_M,\|\cdot\|_{B^\star})\right)$.*

Proof. Suppose μ and ν are in $L^\infty\left([0,\tau_M),(E_M,\|\cdot\|_{B^\star})\right)$ $f \in B$ and $t \in [0,\tau_M)$, then by Proposition 26

$$\begin{aligned}
&\|\Psi_{c_0}(\mu)(t) - \Psi_{c_0}(\nu)(t)\|_{B^\star} \\
&\qquad\qquad \le \left\| \widetilde{A}^{t,0}[\mu] - \widetilde{A}^{t,0}[\nu] \right\|_{B\to B} \|c_0\|_{B^\star} \mathbb{1}\left(t \le t_0\right) \\
&\qquad + \int_0^t \left\| \widetilde{A}^{t,s}[\mu] - \widetilde{A}^{t,s}[\nu] \right\|_{B\to B} \|I_s\|_{B^\star} \mathbb{1}\left(t - s \le t_0\right) \mathrm{d}s \\
&\qquad \le \frac{3}{2}K_\infty C_1 t e^{\frac{3}{2}K_\infty C_1 M t} \|c_0\|_{B^\star} \mathbb{1}(t \le t_0) \operatorname*{ess\,sup}_{r\in[0,t_M)} \|\mu_r - \nu_r\|_{B^\star} \\
&\qquad + \frac{3}{4}K_\infty C_1 \min(t^2, t_0^2) e^{\frac{3}{2}K_\infty C_1 M \min(t,t_0)} \sup_r \|I_r\|_{B^\star} \operatorname*{ess\,sup}_{r\in[0,t_M)} \|\mu_r - \nu_r\|_{B^\star}.
\end{aligned} \tag{14}$$

Hence for any $0 < r < 1$ one can find a $\tau'_M \le \tau_M$ such that

$$\sup_{t\in[0,\tau'_M)} \left\|\Psi_{c_0}(\mu^1)(t) - \Psi_{c_0}(\mu^2)(t)\right\|_{B^\star} \le r \operatorname*{ess\,sup}_{t\in[0,\tau'_M)} \left\|\mu^1_t - \mu^2_t\right\|_{B^\star}. \tag{15}$$

□

Proposition 35. *Let $c_0 \in \mathcal{M}$ and E_M be as in Proposition 33, τ'_M as in Proposition 34 and assume H1 holds, then* (2) *with initial condition c_0 has a solution on $[0,\tau'_M)$ and this solution is in $C_\mathrm{b}\left([0,\tau'_M),(E_M,\|\cdot\|_{B^\star})\right)$. If H2 or H3 hold this solution is unique.*

Proof. By Proposition 34 there is precisely one fixed point of Ψ_{c_0}, which by Proposition 31 is a solution of (2) with initial condition c_0. Proposition 33 shows that this solution is in $C_\mathrm{b}\left([0,t'_M),(E_M,\|\cdot\|_{B^\star})\right)$. By Proposition 32 every solution of (2) with initial condition c_0 is a fixed point of Ψ_{c_0} and thus is unique. □

Proposition 36. *Let $T > 0$, assume H1 and suppose Ψ_{μ_0} and Ψ_{ν_0} have fixed points μ and ν respectively. Write $M = \max\left(|||\mu|||_{B^\star}, |||\nu|||_{B^\star}\right)$, then there exists $C_4(M) > 0$ such that for $t \le T$*

$$\|\mu_t - \nu_t\|_{B^\star} \le \|\mu_0 - \nu_0\|_{B^\star} e^{\frac{3}{2}K_\infty C_1 M \min(t,t_0)} e^{C_4(M)t}$$

and thus at most one finite solution is possible for any given initial condition.

Proof. Since any solution must be a fixed point of Ψ for the appropriate initial condition

$$\begin{aligned}
\|\mu_t - \nu_t\|_{B^\star} &= \|\Psi_{\mu_0}(\mu)(t) - \Psi_{\nu_0}(\nu)(t)\|_{B^\star} \\
&\le \|\Psi_{\mu_0}(\mu)(t) - \Psi_{\mu_0}(\nu)(t)\|_{B^\star} + \|\Psi_{\mu_0}(\nu)(t) - \Psi_{\nu_0}(\nu)(t)\|_{B^\star}.
\end{aligned} \tag{16}$$

Now by Proposition 28 estimate the second term as follows

$$\|\Psi_{\mu_0}(\nu)(t) - \Psi_{\nu_0}(\nu)(t)\|_{B^\star} = \left\|\widetilde{A}^{t,0}[\nu]\,(\mu_0 - \nu_0)\right\|_{B^\star} \leq e^{\frac{3}{2}K_\infty C_1 M t}\mathbb{1}\,(t \leq t_0)\,\|\mu_0 - \nu_0\|_{B^\star}\,. \quad (17)$$

For the first term using Proposition 28 one finds

$$\begin{aligned}
&\|\Psi_{\mu_0}(\mu)(t) - \Psi_{\mu_0}(\nu)(t)\|_{B^\star} \\
&\qquad\leq \left\|\widetilde{A}^{t,0}[\mu] - \widetilde{A}^{t,0}[\nu]\right\|_{B\to B}\|\mu_0\|_{B^\star}\,\mathbb{1}\,(t \leq t_0) \\
&\qquad\quad + \int_0^t \left\|\widetilde{A}^{t,s}[\mu] - \widetilde{A}^{t,s}[\nu]\right\|_{B\to B}\|I_s\|_{B^\star}\,\mathbb{1}\,(t - s \leq t_0)\,\mathrm{d}s \\
&\qquad\leq \frac{3}{2}K_\infty C_1 e^{3K_\infty C_1 M t}\|\mu_0\|_{B^\star}\,\mathbb{1}(t \leq t_0)\int_{r\in[0,T)}\|\mu_r - \nu_r\|_{B^\star}\,\mathrm{d}r \\
&\qquad\quad + \frac{3}{2}K_\infty C_1 e^{3K_\infty C_1 M \min(t,t_0)}\min(t,t_0)\sup_r\|I_r\|_{B^\star}\int_{r\in[0,T)}\|\mu_r - \nu_r\|_{B^\star}\,. \quad (18)
\end{aligned}$$

so using Gronwall with

$$C_4(M) = \frac{3}{2}K_\infty C_1 e^{\frac{3}{2}K_\infty C_1 M t_0}\left(\left\|c_0^1\right\|_{B^\star} + t_0\sup_r\|I_r\|_{B^\star}\right) \quad (19)$$

one has

$$\|\mu_t - \nu_t\|_{B^\star} \leq \|\mu_0 - \nu_0\|_{B^\star}\,e^{\frac{3}{2}K_\infty C_1 M \min(t,t_0)}e^{C_4(M)t}. \quad (20)$$

□

Proof of Theorem 10. The existence of a solution on a small time interval is the conclusion of Proposition 35, this procedure may be iterated, but the time steps may decay so that a solution cannot necessarily be constructed for all time.

Proposition 32 establishes a representation for any solutions, should they exist. Using this representation boundedness and continuity in the $B^\star$-norm along with differentiability in the $D^\star$-norm were established in Proposition 31.

For compact subsets of the time interval on which a solution exists (which may be longer than the time interval for which this theorem proves existence), $B^\star$ Lipschitz continuity in the initial conditions and uniqueness are consequences of Proposition 36. □

4.2 Positive Measures

Write B^+ for the cone of non-negative functions in B and $\mathcal{M}^+, \mathcal{M}_{0,\infty}^+$ for the cone of non-negative measures in $\mathcal{M}$, respectively $\mathcal{M}_{0,\infty}$. These cones are of course not Banach spaces, but one would expect the physical solutions of any reaction–transport problem to remain in $\mathcal{M}^+$, if they start there. This is indeed the case and turns out to allow the local existence result for the coagulation–transport problem studied here to be extended to a global one, which along with the results already established makes the problem well posed.

Proposition 37. *Let $T > 0$ and $\mu \in L^\infty\left([0,T), (\mathcal{M}^+, \|\cdot\|_{B^\star})\right)$, then for $0 \leq s \leq t < T$ $A^{s,t}[\mu]$ is a positivity preserving on B, the same is true of $\widetilde{A}^{t,s}[\mu]$ on $\mathcal{M}$ and both operators are contractions on the respective positive cones, that is*

$$\left\|A^{s,t}[\mu]f\right\|_B \leq \|f\|_B,\ f \in B^+, \qquad \left\|\widetilde{A}^{t,s}[\mu]\nu\right\|_{B^\star} \leq \|\nu\|_{B^\star}\,,\ \nu \in \mathcal{M}^+.$$

Proof. A proof for the dual propagators on B suffices. For this note that $U^{s,t}$ is positivity preserving with B-operator norm 1. One further checks that $H_t[\mu]$ generates a positivity preserving propagator with operator norm at most 1 on B^+, which will be denoted $H^{s,t}[\mu]$. One can now approximate $A^{s,t}[\mu]$ by

$$U^{t,t_{m-1}}H^{t,t_{m-1}}[\mu]\cdots U^{t_1,t_2}H^{t_1,t_2}[\mu]U^{s,t_1}H^{s,t_1}[\mu], \quad t_i = s + i\frac{t-s}{m},\ i = 1,\dots m-1,\ m \in \mathbb{N} \quad (21)$$

which is a splitting, to see positivity is preserved and the operator norm is bounded above by 1. ◻

The key estimate from Proposition 31 can now be improved (recall t_0 is the maximum particle residence time from §2.1):

Proposition 38. *Assume H1 holds, $c_0 \in \mathcal{M}^+$ and $\mu \in L^\infty([0,T), \mathcal{M}^+)$ for $T \in [0,\infty)$ then*

$$\|\Psi_{c_0}(\mu)(t)\|_{B^\star} \leq \|c_0\|_{B^\star} \mathbb{1}(t \leq t_0) + \min(t, t_0) \sup_s \|I_s\|_{B^\star}.$$

Proof. This follows from Definition 30, and the norm estimates in Proposition 37. ◻

Proof of Theorem 11. One can take $M = \|c_0\|_{B^\star} + t_0 \sup_s \|I_s\|_{B^\star}$ and $t_M = \infty$ in Proposition 33. Proposition 34 then extends to show that Ψ_{c_0} is a contraction on $L^\infty([0,\infty), E_M \cap \mathcal{M}^+)$. ◻

4.3 Measures with Lebesgue Densities

One would of course like to prove that every measure valued to solution to (2) is in fact also a strong solution to an appropriate extension of (1). The main difficulty that has to be addressed in this section is the inflow of pre-existing particles through Γ_{in} which leads to I_t having a singular (with respect to Lebesgue measure on $\mathcal{X}$) part concentrated on Γ_{in}. In this section it is shown that under a mild time-regularity condition (I2) the advective transport smooths out the inception concentrated on Γ_{in} sufficiently for solutions to (2) to remain in $\mathcal{M}_{0,\infty}$. Shocks are of course preserved by advective transport, but what happens here is more like spraying paint onto a moving surface, as long as the surface keeps moving a thin layer of paint is deposited everywhere and no ridge (shock) is created.

Proposition 39. *Assume H1 holds, $0 \leq s \leq t < T$, $\mu \in L^\infty([0,T), (\mathcal{M}, \|\cdot\|_{B^\star}))$, then*

$$\widetilde{A}^{t,s}[\mu] = \widetilde{U}^{t,s} + \sum_{m=1}^{\infty} \int_{r \leq s_1 \leq \ldots \leq s_m \leq t} \widetilde{U}^{t,s_m} \widetilde{H}_{s_m}[\mu] \widetilde{U}^{s_m, s_{m-1}} \cdots \widetilde{H}_{s_1}[\mu] \widetilde{U}^{r,s_1} \mathrm{d}s_1 \cdots \mathrm{d}s_m.$$

Proof. For each m the term in the sum here is dual to the term with the same m in Definition 20. ◻

Proposition 40. *Assume H1 holds, $0 \leq s \leq t < T$, $\mu \in L^\infty([0,T), (\mathcal{M}, \|\cdot\|_{B^\star}))$ and $c \in \mathcal{M}_{0,\infty}$, then for any $\phi \in \mathcal{B}_\mathrm{b}(\mathcal{Y})$ and bounded measurable $f\colon \mathcal{X} \to \mathbb{R}$*

$$\int_{\mathcal{X}} f(x) \int_{\mathcal{Y}} \phi(y) \widetilde{U}^{t,s} c(x, \mathrm{d}y) \mathrm{d}x = \int_{\mathcal{X}} f(x) e^{-\int_s^t \nabla \cdot u_r(\Phi_{t,r}(x)) \mathrm{d}r} \int_{\mathcal{Y}} \phi(y) c(\Phi_{t,s}(x), \mathrm{d}y) \, \mathrm{d}x$$

and

$$\begin{aligned} \int_{\mathcal{Y}} \phi(y) \widetilde{H}_t[\mu] c(x, \mathrm{d}y) = &\int_{\mathcal{Y}} \frac{1}{2} \int_{\mathcal{X} \times \mathcal{Y}} \phi(y + y_2) h(x, x_2) K(y, y_2) \mu_t(\mathrm{d}x_2, \mathrm{d}y_2) c(x, \mathrm{d}y) \\ &- \int_{\mathcal{Y}} \frac{1}{2} \int_{\mathcal{X} \times \mathcal{Y}} \phi(y) \left[h(x, x_2) + h(x_2, x)\right] K(y, y_2) \mu_t(\mathrm{d}x_2, \mathrm{d}y_2) c(x, \mathrm{d}y). \end{aligned}$$

Proof. For the first statement, which concerns the transport propagator U, one makes the change of variable $x \leftrightarrow \Phi_{s,t}(x)$. Liouville's formula then gives the determinant of the Jacobian as $\left|\det \frac{\partial \Phi_{s,t}(x)}{\partial x}\right| = \exp \int_s^t \nabla \cdot u_t(\Phi_{t,r}(x)) \, \mathrm{d}r$. Alternatively one can approximate c by $\mathcal{X}$-differentiable functions (since the claim is only of an L^1 nature) and check the formula directly using Proposition 59.

For H_t use Definition 15; the important point is that the new measure also has a density with respect to Lebesgue measure on $\mathcal{X}$. ◻

Proposition 41. *Assume H1 holds, $0 \le s \le t < T$, $\mu \in L^\infty\left([0,T), (\mathcal{M}, \|\cdot\|_{B^\star})\right)$ then $\widetilde{A}^{t,s}[\mu]$ is a bounded propagator on $\mathcal{M}_{0,\infty}$ with*

$$\left\|\widetilde{A}^{t,s}[\mu]\right\|_{\mathcal{M}_{0,\infty}\to\mathcal{M}_{0,\infty}} \le e^{\left(\|\nabla\cdot u\|_\infty + \frac{3}{2}K_\infty C_1 \|\|\mu\|\|_{B^\star}\right)(t-s)} \mathbb{1}(t-s \le t_0).$$

Proof. From Proposition 40 one sees that

$$\left\|\widetilde{U}^{t,s}\right\|_{\mathcal{M}_{0,\infty}\to\mathcal{M}_{0,\infty}} \le e^{\|\nabla\cdot u\|_\infty (t-s)}$$

and

$$\left\|\widetilde{H}_t[\mu]\right\|_{\mathcal{M}_{0,\infty}\to\mathcal{M}_{0,\infty}} \le \frac{3}{2}K_\infty C_1 \|\|\mu\|\|_{B^\star}.$$

The proof now follows that of Proposition 23. □

The $\widetilde{U}$ and therefore also the $\widetilde{A}$ are not (norm-)continuous on $\mathcal{M}_{0,\infty}$. This can easily be seen by considering a small translation of a step function regarded as the density of a measure in $\mathcal{M}_{0,\infty}$. The eventual time continuity of the solutions will depend on having some $\mathcal{X}$-regularity for the densities of the measures in $\mathcal{M}_{0,\infty}$.

The next proposition provides a better norm estimate when the propagator is restricted to positive measures. This is then used in Propositions 43&44 to show that inception concentrated on the inflow boundary does not take the solution out of $\mathcal{M}_{0,\infty}$.

Proposition 42. *Assume H1 holds, $0 \le s \le t < T$, $\mu \in L^\infty\left([0,T), (\mathcal{M}^+, \|\cdot\|_{B^\star})\right)$ and let $c \in \mathcal{M}_{0,\infty}$ be a positive measure, then $\widetilde{A}^{t,s}[\mu]c$ is also a positive measure and*

$$\left\|\widetilde{A}^{t,s}[\mu]c\right\|_{\mathcal{M}_{0,\infty}} \le e^{\|\nabla\cdot u\|_\infty (t-s)} \mathbb{1}(t-s \le t_0) \|c\|_{\mathcal{M}_{0,\infty}}.$$

Proof. Since $\mathcal{M}_{0,\infty} \subset \mathcal{M}$ preservation of positivity is a consequence of Proposition 37 and Proposition 40 states the $\mathcal{M}_{0,\infty}$ is preserved. To proceed note that the propagator generated by $\widetilde{H}_t[\mu]$ preserves $\mathcal{M}^+_{0,\infty}$ not just $\mathcal{M}_{0,\infty}$ and coagulation reduces $c(x,\mathcal{Y}) = \|c(x,\cdot)\|_{\mathcal{Y}-\mathrm{TV}}$ for all $x \in \mathcal{X}$. Secondly $\widetilde{U}^{t,s}$ is positivity preserving and $\left\|\widetilde{U}^{t,s}\right\|_{\mathcal{M}_{0,\infty}} \le e^{\|\nabla\cdot u\|(t-s)}$ using the representation from Proposition 40 so the result now follows by the same splitting approximation as in the proof of Proposition 37.

□

Proposition 43. *Let $T > 0$, and $\nu\colon [0,T) \to \mathcal{M}$ be such that (note the reduction in the domain of integration accompanied by a change in the position of the time argument)*

$$\int_{\mathcal{X}\times\mathcal{Y}} f(x,y)\nu_t(\mathrm{d}x,\mathrm{d}y) = \int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} f(\xi,y)\nu(t,\xi,\mathrm{d}y)\mathrm{d}\xi \qquad \forall f \in B \quad \forall t \in [0,T).$$

Suppose further that there is a $\nu_* \in (0,\infty)$ such that $\sup_{t\in[0,T),\xi\in\Gamma_{\mathrm{in}}} \|\nu(t,\xi,\cdot)\|_{\mathcal{Y}-\mathrm{TV}} / u_t(\xi)\cdot n(\xi) \le \nu_*$, then $\int_0^t \widetilde{U}^{t,s}\nu_s \mathrm{d}s \in L^\infty\left([0,T), \mathcal{M}_{0,\infty}\right)$ and for all (not just almost all) $t < T$

$$\left\|\int_0^t \widetilde{U}^{t,s}\nu_s \mathrm{d}s\right\|_{\mathcal{M}_{0,\infty}} \le \nu_* e^{\|\nabla\cdot u\|_\infty \min(t,t_0)}.$$

Proof. Let $\xi \in \Gamma_{\mathrm{in}}$ and take an orthonormal basis for $\mathbb{R}^d$ at ξ given by $e_1 = n(\xi)$ the outward normal and $e_2,\dots,e_d \in \Gamma_{\mathrm{in}}$. With respect to this basis let the rows of the matrix $\nabla\Phi_{r,t}(x)\mid_{x=\xi}$ be $\partial_i\Phi_{r,t}(x)\mid_{x=\xi}$. Thus rows $2,\dots,d$ of this matrix are the same as rows $2,\dots,d$ of $\frac{\partial\Phi_{r,t}(\xi)}{\partial(r,\xi)}$ and using Proposition 59 the first row is

$$\frac{\partial}{\partial r}\Phi_{r,t}(\xi) = -\nabla\Phi_{r,t}(x)\mid_{x=\xi} u_r(\xi) = \\ -\sum_{i=1}^d \nabla\Phi_{r,t}(x)\mid_{x=\xi} e_i\left(e_i\cdot u_r(\xi)\right) = -\sum_{i=1}^d \partial_i\Phi_{r,t}(x)\mid_{x=\xi}\left(e_i\cdot u_r(\xi)\right), \quad (22)$$

which is $\pm\left(\partial_1 \Phi_{r,t}(x)\mid_{x=\xi}\right)\left(n(\xi)\cdot u_r(\xi)\right)$ plus a linear combination of the remaining rows. One thus has for $\xi \in \Gamma_{\rm in}$

$$\det\left(\frac{\partial \Phi_{r,t}(\xi)}{\partial(r,\xi)}\right) = -u_r(\xi)\cdot n(\xi)\det\left(\nabla \Phi_{r,t}(x)\right)\Big|_{x=\xi}. \tag{23}$$

Now let $f \in B$ with $f\left(\Phi_{r,t}(\xi), y\right) = 0$ for $\Phi_{r,t}(\xi) \notin \mathcal{X}$ as in the definition of $U^{r,t}$ so

$$\begin{aligned}\left\langle f, \int_s^t \widetilde{U}^{t,r}\nu_r \mathrm{d}r\right\rangle &= \int_s^t \int_{\Gamma_{\rm in}}\int_{\mathcal{Y}} f\left(\Phi_{r,t}(\xi), y\right)\nu(r,\xi,\mathrm{d}y)\mathrm{d}\xi\mathrm{d}r \\ &= \int_{\substack{x:\ x=\Phi_{r,t}(\xi)\\ r\in(s,t),\xi\in\Gamma_{\rm in}}} \det\left(\frac{\partial \Phi_{r,t}(\xi)}{\partial(r,\xi)}\right)^{-1}\int_{\mathcal{Y}} f\left(x,y\right)\nu(r,\xi,\mathrm{d}y)\mathrm{d}x \\ = \int_{\substack{x:\ x=\Phi_{r,t}(\xi)\\ r\in(s,t),\xi\in\Gamma_{\rm in}}} &\left|\det\left(\nabla\Phi_{r,t}(\xi)\right)^{-1}\right| \left\|f\left(x,\cdot\right)\right\|_{\mathcal{Y}-\infty}\left|u_r(\xi)\cdot n(\xi)\right|^{-1}\left\|\widehat{\nu}(x,\cdot)\right\|\mathrm{d}x \\ &\leq e^{\|\nabla\cdot u\|_\infty \min(t-s,t_0)}\nu_* \int_{\substack{x:\ x=\Phi_{r,t}(\xi)\\ r\in(s,t),\xi\in\Gamma_{\rm in}}} \left\|f\left(x,\cdot\right)\right\|_{\mathcal{Y}-\infty}\mathrm{d}x, \end{aligned} \tag{24}$$

where $\widehat{\nu}(x,\mathrm{d}y)$ is defined to be $\nu(r,\xi,\mathrm{d}y)$ for the unique r,ξ such that $\Phi_{r,t}(\xi) = x$. Proposition 60 in the Appendix provides the estimate for the determinant. ☐

Proposition 44. *Let $T > 0$ and $\mu \in L^\infty\left([0,T), (\mathcal{M}^+, \|\cdot\|_{B^\star})\right)$, then under the conditions of Proposition 43*

$$\left\|\int_0^t \widetilde{A}^{t,s}[\mu]\nu_s \mathrm{d}s\right\|_{\mathcal{M}_{0,\infty}\to\mathcal{M}_{0,\infty}} \leq \nu_* e^{\|\nabla\cdot u\|_\infty \min(t,t_0)}.$$

Proof. Use the series expansion from Proposition 39 and the $\mathcal{M}_{0,\infty}$-operator norm estimates from Proposition 42. ☐

Proof of Theorem 12. Theorem 11 provides the existence of a solution c. Proposition 32 shows that this solution satisfies

$$c_t = \widetilde{A}^{t,0}[c]c_0 + \int_0^t \widetilde{A}^{t,s}[c]I_s\mathrm{d}s \quad t \geq 0. \tag{25}$$

Propositions 42&44 show that this is in $\mathcal{M}_{0,\infty}$ for all times. The boundedness follows from the estimates in the same two propositions. ☐

In order to obtain a strong solution to (2) it is not sufficient that the measure valued solutions have a density with respect to Lebesgue measure on $\mathcal{X}$, this density should itself have a derivative.

4.4 Differentiability

One could proceed as in Proposition 25 to see that $\widetilde{A}^{t,s}[c]$ preserves measures with $\mathcal{X}$-differentiable densities except for possible jumps where s,t,x are such that $\Phi_{t,s}(x) \in \Gamma_{\rm in}$. This leaves two questions open—how to handle these jumps and secondly the treatment of the integral term from (25) and in particular the $I_{\rm bdry}$ part of I in that integral. The right approach to these tasks seems to be to introduce the space of measures with $\mathcal{X}$-bounded variation densities:

Definition 45.

$$\mathcal{M}_{\rm BV} = \left\{c \in \mathcal{M}_{0,\infty}\colon\ \left(\exists C = C(c)\right)\left(\forall f \in D^d\right)\left(\langle \nabla\cdot f, c\rangle \leq C\left\|f\right\|_{B^d}\right)\right\}.$$

This is equivalent to the existence of a measure $\nabla c \in \mathcal{M}^d$ (not necessarily in $\mathcal{M}_{0,\infty}$) such that $\langle \nabla f, c\rangle = -\langle f, \nabla c\rangle$.

Until now the notation $\langle f, \mu\rangle$ has been used for $\int_{\mathcal{X}\times\mathcal{Y}} f(x,y)\mu(\mathrm{d}x,\mathrm{d}y)$ for $f \in B$ and $\mu \in \mathcal{M}$. To consider derivatives it is necessary to move to vector valued functions and measures; to facilitate this the notation is extended so that for $g \in B^d$ and $\nu \in \mathcal{M}^d$

$$\langle g, \nu\rangle := \sum_{i=1}^{d} \int_{\mathcal{X}\times\mathcal{Y}} g_i(x,y)\nu_i(\mathrm{d}x,\mathrm{d}y). \tag{26}$$

Some more definitions are now needed for the proof that $\int_0^t \widetilde{A}^{t,s} I_s \mathrm{d}s$ and by extension the entire solution is $\mathcal{M}_{\mathrm{BV}}$. First recall $s(t,x)$ from §2.1, the time at which a particle travelling with the flow must have entered the domain in order to reach x at time t. Since t is fixed in the relevant places $s(x)$ will be written for brevity in numerous sub- and superscripts, the t should be understood.

Definition 46. *Let $f \in B, t > 0$ and $\mu \in C([0,t],\mathcal{M})$. Define $f^0_{r,t} = U^{r,t} f$, and $f^{m+1}_{r,t} = \int_r^t U^{r,s} H_s[\mu] f^m_{s,t} \mathrm{d}s$ as in Definition 21. The define $\tilde{f}$ by*

$$\tilde{f}^m_{r,t}(x,y) = f^m_{r,t}\left(\Phi_{t,r}(x), y\right) \mathbb{1}\left(\Phi_{t,r}(x) \in \mathcal{X}\right)$$

the operators $S^{r,t}\colon B \to B$ by

$$S^{r,t} f(x,y) = \sum_{m=0}^{\infty} \tilde{f}^m_{r,t}(x,y)$$

and finally the operator $\mathcal{S}^t\colon B \to B$ by

$$\mathcal{S}^t f(x,y) = S^{s(x),t} f(x,y) = \sum_{m=0}^{\infty} \tilde{f}^m_{s(x),t}(x,y).$$

This operator can also be regarded as acting on B^d by applying it componentwise.

Proposition 47. *Let $f \in B$ or $f \in B^d$, $t > 0$ and $\mu \in C([0,t],\mathcal{M})$, then*

$$\left\|\mathcal{G}^t f(x,\cdot)\right\|_{\mathcal{Y}-\infty} \leq e^{\frac{3}{2}K_\infty C_1 |||\mu|||_{B^\star}(t-s(x))} \left\|f(x,\cdot)\right\|_{\mathcal{Y}-\infty}$$

for all $x \in \mathcal{X}$ and the operator even preserves D.

Proof. This is an exercise in estimating the terms of the summation as in Propositions 22&24. □

Proposition 48. *Let $F \in D^d$, $t > 0$ and $\mu \in C([0,t],\mathcal{M})$, then the operator $\left(\nabla\cdot\mathcal{S}^t\right)\colon D^d \to B$ defined by*

$$\left(\nabla\cdot\mathcal{S}^t\right) F = \nabla\cdot\left(\mathcal{S}^t F\right) - \mathcal{S}^t\left(\nabla\cdot F\right)$$

satisfies

$$\left\|\left(\nabla\cdot\mathcal{S}^t\right) F(x,\cdot)\right\|_{\mathcal{Y}-\infty} \leq \left\|F(x,\cdot)\right\|_{\mathcal{Y}-\infty} \times \left(3K_\infty C_2 t_0 |||\mu|||_{B^\star} + \left\|\nabla s\right\|_\infty\right)\left(1 + \frac{3}{2}K_\infty C_1 |||\mu|||_{B^\star}\left(t - s(x)\right)\right) e^{\frac{3}{2}K_\infty C_1 |||\mu|||_{B^\star}(t-s(x))}$$

for all $x \in \mathcal{X}$. Of course $\mathcal{S}^t$ depends on μ, but since this will always be the unique solution to (2) *this detail is ignored in the notation.*

Proof. Define $\tilde{F}^m_{r,t}$ by replacing f with F throughout Definition 46 and let $f = \nabla\cdot F$ and let $\tilde{f}^m_{r,t}$ be as in Definition 46. By induction one establishes

$$\left\|\nabla\cdot\tilde{F}^m_{r,t}(x,\cdot) - \tilde{f}^m_{r,t}(x,\cdot)\right\|_{\mathcal{Y}-\infty} \leq \left\|F(x,\cdot)\right\|_{\mathcal{Y}-\infty} \times \frac{3}{2}K_\infty C_2 t_0 |||\mu|||_{B^\star} m \left(\frac{3}{2}K_\infty C_2 t_0 |||\mu|||_{B^\star}\right)^{m-1} \frac{\left(t - s(x)\right)^{m-1}}{(m-1)!}. \tag{27}$$

One then establishes a similar formula with r replaced by $s(x)$ and the result follows. □

Proposition 49. *Assume H2 or H3 holds, that c_0 is in the positive cone of $\mathcal{M}_{\mathrm{BV}}$, and that I3 is satisfied, then the unique solution c to* (2) *given by Theorem 12 satisfies $c_t \in \mathcal{M}_{\mathrm{BV}}\ \forall t \in \mathbb{R}^+$.*

Proof. Following (25) and §2.5 c_t can be decomposed as

$$c_t = \widetilde{A}^{t,0}[c]c_0 + \int_0^t \widetilde{A}^{t,s}[c]I_{\mathrm{int}}(s)\mathrm{d}s + \int_0^t \widetilde{A}^{t,s}[c]I_{\mathrm{bdry}}(s)\mathrm{d}s. \tag{28}$$

One checks in the same way as for the dual propagator in Proposition 24 that $\widetilde{A}^{t,s}[c]$ preserves $\mathcal{M}_{\mathrm{BV}}$, possibly introducing a new jump on the manifold $\{x \in \mathcal{X} \colon \Phi_{t,s}(x) \in \Gamma_{\mathrm{in}}\}$. This deals with the first two terms in the above representation; the third term is somewhat more challenging.

Let $F \in D^d$ and $t \in \mathbb{R}^+$, then it is sufficient to show that

$$\int_0^t \left\langle \nabla \cdot F, \widetilde{A}^{t,s}[c]I_{\mathrm{bdry}}(s) \right\rangle \mathrm{d}s = \int_0^t \int_{\Gamma_{\mathrm{in}}} \int_{\mathcal{Y}} \left(A^{s,t}[c]f\right)(\xi, y) I_{\mathrm{bdry}}(s, \xi, \mathrm{d}y)\mathrm{d}\xi\mathrm{d}s \tag{29}$$

is bounded by a constant times $\|F\|_{B^d}$. One can introduce a change of variables $(s,\xi) \leftrightarrow x$ where (t is fixed) $\Phi_{s,t}(\xi) = x$, so $\xi = \Phi_{t,s}(x)$ is the point where fluid reaching x at time t entered the domain and the time of entry was s. The determinant of the Jacobian for this transformation is the inverse of

$$\det \frac{\partial \Phi_{s,t}(\xi)}{\partial(s,\xi)} = -e^{-\int_s^t \nabla\cdot u_r(\Phi_{t,r}(x))\mathrm{d}r} u_s(\xi)\cdot n(\xi) \tag{30}$$

by Proposition 60 and the additional factor of $-u_s(\xi)\cdot n(\xi)$ comes from replacing the $\mathcal{X}$ direction perpendicular to Γ_{in} with s (ξ lives in the $d-1$ dimensional manifold Γ_{in}) so

$$\begin{aligned}\int_0^t \left\langle \nabla \cdot F, \widetilde{A}^{t,s}[c]I_{\mathrm{bdry}}(s) \right\rangle \mathrm{d}s = \\ -\int_{\mathcal{X}} \int_{\mathcal{Y}} \left(\mathcal{S}^t \nabla\cdot F\right)(x,y) e^{\int_s^t \nabla\cdot u_r(\Phi_{t,r}(x))\mathrm{d}r} \left(u_{s(x)}(\xi(x))\cdot n(\xi(x))\right)^{-1} I_{\mathrm{bdry}}(s(x), \xi(x), \mathrm{d}y)\mathrm{d}x \\ = \left\langle \mathcal{S}^t \nabla \cdot F, \nu_t \right\rangle\end{aligned} \tag{31}$$

where

$$\nu_t(x, \mathrm{d}y) = e^{\int_s^t \nabla\cdot u_r(\Phi_{t,r}(x))\mathrm{d}r} \left(u_{s(x)}(\xi(x))\cdot n(\xi(x))\right)^{-1} I_{\mathrm{bdry}}(s(x), \xi(x), \mathrm{d}y). \tag{32}$$

Writing

$$\mathcal{S}^t \nabla \cdot F = \nabla \cdot \left(\mathcal{S}^t F\right) - \left(\nabla \cdot \mathcal{S}^t\right) F \tag{33}$$

and checking that $\nu_t \in \mathcal{M}_{\mathrm{BV}}$ concludes the proof. □

To simplify the remainder of this section it will be assumed that $\mathcal{X} = [0, L) \times \Gamma_{\mathrm{in}}$ for some $L > 0$, that is, that $\mathcal{X}$ is a something rather like a cylinder. The results are expected to generalise, but this assumption avoids introducing technical conditions on $\mathcal{X}$. In particular $x \in \mathcal{X}$ can be written as $(x_1, x_2, \cdots, x_d)$ for $x_1 \in [0, L)$ and $(x_2, \cdots, x_d) \in \Gamma_{\mathrm{in}}$.

Proposition 50. *Assume H1 and I3 hold, $c \in L^\infty\left([0,\infty), \mathcal{M}_{0,\infty}\right)$ solves* (2) *with additionally $c \in W^{1,\infty}\left([0,\infty) \times \mathcal{X}, \mathcal{M}(\mathcal{Y})_{\mathrm{TV}}\right)$. Let $n(x)$ be the outward normal on Γ_{in}, then*

$$u_{1,t}(x)c(t,x,\mathrm{d}y) = -u_t(x)\cdot n(x)c(t,x,\mathrm{d}y) = I_{\mathrm{bdry}}(t,x,\mathrm{d}y) \quad \forall t \in \mathbb{R}^+,\ x \in \Gamma_{\mathrm{in}}.$$

Proof. Consider (2) with $f(x_1, x_2, \ldots, x_d) = \frac{\epsilon - x_1}{\epsilon}\mathbb{1}\{x_1 \le \epsilon\}$ as $\epsilon \to 0$. □

The existence of a one or more inverses to the divergence operator is necessary to avoid making statements about an empty set of functions in the remainder of this section.

Proposition 51. *Let $f \in D$, then there exists $g \in D^d$ such that $\nabla \cdot g \equiv f$ and $g \cdot n = 0$ on Γ_{side}, where n is the outward normal.*

Proof. Take $g(x_1, x_2, \ldots, x_d, y) = (g_1, g_2, \ldots, g_d)$ where

$$g_1(x_1, x_2, \ldots, x_d, y) = -\int_{x_1}^{L} f(\xi, x_2, \ldots, x_d, y)\mathrm{d}\xi \tag{34}$$

and $g_i \equiv 0$ for $i > 1$. This construction has a natural generalisation in terms of path integrals. It is not important exactly which end point on Γ_{out} is chosen because $f = 0$ all along this boundary. □

This representation is not in general unique. Consider for example the case where $f = 0$ on Γ_{side} and take integrals along lines perpendicular to the direction used in the above proof.

If c solves (2), $f \in D$ and $g\colon \mathcal{X} \to C_{\mathrm{b}}(\mathcal{Y})^d$ is differentiable with $\nabla \cdot g \equiv f$ and $g \cdot n = 0$ on Γ_{side} for normal vectors n, then applying the divergence theorem to (2) (at this stage in a purely formal calculation) suggests

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{X}\times\mathcal{Y}} g \cdot \nabla c \,\mathrm{d}x\mathrm{d}y &= \int_{\mathcal{X}\times\mathcal{Y}} u_t^\top (\nabla g)\nabla c \,\mathrm{d}x\mathrm{d}y - \int_{\mathcal{X}\times\mathcal{Y}} g^\top (\nabla u_t)\nabla c \,\mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\mathcal{X}\times\mathcal{Y}} g \cdot \nabla(\nabla \cdot u_t)c \,\mathrm{d}x\mathrm{d}y + \int_{\mathcal{X}\times\mathcal{Y}} g \cdot \nabla I_{\mathrm{int}} \,\mathrm{d}x\mathrm{d}y \\
&+ \int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} g \cdot n \frac{\partial}{\partial t} c \,\mathrm{d}x\mathrm{d}y - \int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} u_t^\top (\nabla g) n c \,\mathrm{d}x\mathrm{d}y - \int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} g \cdot n I_{\mathrm{int}} \mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} \nabla \cdot g I_{\mathrm{bdry}} \,\mathrm{d}x\mathrm{d}y + \int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} g^\top (\nabla u_t) n c \,\mathrm{d}x\mathrm{d}y \\
&+ \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}}\int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} K(y, y_2) g(x, y + y_2) \cdot n(x) \left(h(x, x_2) c_t(x, \mathrm{d}y)\right) c_t(x_2, \mathrm{d}y_2)\mathrm{d}x_2\mathrm{d}x \\
&- \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}}\int_{\mathcal{X}\times\mathcal{Y}} K(y, y_2) g(x, y + y_2) \cdot \nabla \left(h(x, x_2) c_t(x, \mathrm{d}y)\right) c_t(x_2, \mathrm{d}y_2)\mathrm{d}x_2\mathrm{d}x \\
- \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}}\int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} & K(y, y_2) g(x, y) \cdot n(x) \left(\left(h(x, x_2) + h(x_2, x)\right) c_t(x, \mathrm{d}y)\right) c_t(x_2, \mathrm{d}y_2)\mathrm{d}x_2\mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}}\int_{\mathcal{X}\times\mathcal{Y}} K(y, y_2) g(x, y) \cdot \nabla \left(\left(h(x, x_2) + h(x_2, x)\right) c_t(x, \mathrm{d}y)\right) c_t(x_2, \mathrm{d}y_2)\mathrm{d}x_2\mathrm{d}x.
\end{aligned} \tag{35}$$

Definition 52. *Define a norm on $\mathcal{M}^d$, which by a slight abuse of notation will also be referred to as the $B^\star$-norm by setting $\|\mu\|_{B^\star} = \sup_{f \in B^d\colon \|f\|_B = 1} |\langle f, \mu\rangle|$, for $\mu \in \mathcal{M}^d$.*

Conditions are now provided to make (35) rigorous, first by restricting the test functions to the interior of the domain, so that the boundary terms can be ignored and then proceeding to more general test functions:

Proposition 53. *Assume H2 and I3 hold and that $c \in L^\infty\left([0, \infty), \mathcal{M}_{0,\infty}\right)$ solves (2). Suppose further that $c_t \in \mathcal{M}_{\mathrm{BV}}$ for each t, so that there exist vector measures ν_t of finite total variation such that $\langle \nabla \cdot f, c_t\rangle = -\langle f, \nu_t\rangle$ for all $f \in D^d$. This means that $\nu \in L^\infty\left([0, \infty), (\mathcal{M}^d, \|\cdot\|_{B^\star})\right)$ and in particular for all $g \in C^1_{\mathrm{K}}\left(\mathcal{X}^\circ, \mathcal{B}_{\mathrm{b}}(\mathcal{Y})^d\right)$, the space of once continuously differentiable functions with compact support strictly contained in the interior $\mathcal{X}^\circ$ of $\mathcal{X}$ that also satisfy $\nabla \cdot g \in D$ (for example $g \in C^2_{\mathrm{K}}\left(\mathcal{X}^\circ, \mathcal{B}_{\mathrm{b}}(\mathcal{Y})^d\right)$) one has*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle g, \nu_t\rangle &= \left\langle (\nabla g)^\top u_t, \nu_t\right\rangle - \left\langle (\nabla u_t)^\top g, \nu_t\right\rangle - \langle g \cdot \nabla\left(\nabla \cdot u\right), c_t\rangle + \langle g, \nabla I_{\mathrm{int},t}\rangle + \left\langle \widehat{H_t[c]} g, \nu_t\right\rangle \\
&\quad + \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}}\int_{\mathcal{X}\times\mathcal{Y}} g(x, y + y_2) K(y, y_2) \widehat{c}_t(x, \mathrm{d}y) \cdot \left(\nabla_x h(x, \xi)(\mathrm{d}x)\right) c_t(\xi, \mathrm{d}y_2)\mathrm{d}\xi \\
&\quad - \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}}\int_{\mathcal{X}\times\mathcal{Y}} g(x, y) K(y, y_2) \widehat{c}_t(x, \mathrm{d}y) \cdot \left(\nabla_x \left[h(x, \xi) + h(\xi, x)\right](\mathrm{d}x)\right) c_t(\xi, \mathrm{d}y_2)\mathrm{d}\xi.
\end{aligned}$$

Here $\widehat{H_t[c]}$ is a bounded linear operator mapping $B \to B$ acting componentwise on g with $\left\|\widehat{H_t[c]}\right\|_{B\to B} = \|H_t[c]\|_{B\to B}$ and $\widehat{H_t[c]}g(x,y) = H_t[c]g(x,y)$ (recall H_t is specified in Definition 15) for all y and all x except possibly x at which h and c both have discontinuities, which is a set of ($\mathbb{R}^d$-Lebesgue) measure 0. Similarly $\|\widehat{c}\|_{\mathcal{M}_{0,\infty}} = \|c\|_{\mathcal{M}_{0,\infty}}$ with possible differences between c and $\widehat{c}$ on the same set of measure 0. Because h is only assumed to be of bounded variation, it only has a weak derivative; in the case of the weak derivative with respect to the first argument this is written $\nabla_x h(x,\xi)(\mathrm{d}x)$.

Proof. The boundary integrals on Γ vanish because g is zero here. Note that the product of two functions of bounded variation (c and the h in the definition of H) is itself of bounded variation, but the Leibniz product rule for differentiation has to be adapted slightly at points where both are discontinuous (yielding $\widehat{H}$ and $\widehat{c}$). The details follow from [2, Theorem 3.96 & Example 3.97]. In one dimension this amounts to adjustments to give left or right continuity at the jump points. □

The terms in the preceding expression can be grouped as follows (c is in this context known):

- Transport $\langle u_t^\top \nabla g, \nu_t\rangle = \langle U_t g, \nu_t\rangle$ (note ∇g is a matrix),
- linear reactions $-\langle g\nabla u_t, \nu_t\rangle + \left\langle \widehat{H_t[c]}g, \nu_t\right\rangle$,
- source terms, which are collected as a vector measure $\widehat{J}_t[c]$ so that, for $g \in C_{\mathrm{K}}\left(\mathcal{X}^\circ, \mathcal{B}_{\mathrm{b}}(\mathcal{Y})^d\right)$

$$\begin{aligned}\left\langle g, \widehat{J}_t[c]\right\rangle &= -\left\langle g\cdot\nabla\left(\nabla\cdot u\right), c_t\right\rangle + \langle g, \nabla I_{\mathrm{int},t}\rangle \\ &\quad + \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}}\int_{\mathcal{X}\times\mathcal{Y}} g(x, y+y_2)K(y,y_2)\widehat{c}_t(x,\mathrm{d}y)\left(\nabla_x h(x,\xi)(\mathrm{d}x)\right)c_t(\xi,\mathrm{d}y_2)\mathrm{d}\xi \\ &- \frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}}\int_{\mathcal{X}\times\mathcal{Y}} g(x,y)K(y,y_2)\widehat{c}_t(x,\mathrm{d}y)\left(\nabla_x\left[h(x,\xi)+h(\xi,x)\right](\mathrm{d}x)\right)c_t(\xi,\mathrm{d}y_2)\mathrm{d}\xi. \end{aligned} \tag{36}$$

This characterisation is however limited to functions with compact support in the interior of $\mathcal{X}$. It can only give information about how a solution changes within $\mathcal{X}$, it says nothing about what might happen on Γ_{in}. Including the boundary terms in the integration by parts/Gauss Theorem used for Proposition 53 yields the following additional terms. That these are the correct additional terms is part of the assertion of Proposition 57.

Definition 54. *Let $c \in L^\infty\left([0,\infty), \mathcal{M}_{0,\infty}\right)$ and define a vector measure $J_t[c]$ on $\mathcal{X}\times\mathcal{Y}$ by*

$$\begin{aligned}&\langle g, J_t[c]\rangle = \left\langle g, \widehat{J}_t[c]\right\rangle + \\ &\int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} g(x,y)\cdot n(x)\left(\frac{\partial}{\partial t}\frac{I_{\mathrm{bdry}}(t,x,\mathrm{d}y)}{u_t(x)\cdot n(x)} + I_{\mathrm{int}}(t,x,\mathrm{d}y) - \nabla_\Gamma\cdot\left(\frac{u_t(x)I_{\mathrm{bdry}}(t,x,\mathrm{d}y)}{u_t(x)\cdot n(x)}\right)\right)\mathrm{d}x \\ &-\frac{1}{2}\int_{\mathcal{X}\times\mathcal{Y}}\int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} K(y,y_2)g(x,y)\cdot n(x)\left((h(x,x_2)+h(x_2,x))\right)\frac{I_{\mathrm{bdry}}(t,x,\mathrm{d}y)}{u_t(x)\cdot n(x)}c_t(x_2,\mathrm{d}y_2)\mathrm{d}x_2\mathrm{d}x \\ &\quad + \int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}} g(x,y)\cdot\nabla_\Gamma I_{\mathrm{bdry}}(t,x,\mathrm{d}y)\mathrm{d}x - \int_{\Gamma_{\mathrm{in}}\times\mathcal{Y}}\frac{I_{\mathrm{bdry}}(t,x,\mathrm{d}y)}{u_t(x)\cdot n(x)}g(x,y)\cdot\left(\nabla_\Gamma n(x)\right)^\top u_t(x)\mathrm{d}x\end{aligned}$$

for $g \in B^d$ and where ∇_Γ is the derivative restricted to directions perpendicular to $n(x)$. Under the assumptions on $\mathcal{X}$ set out above $\nabla_\Gamma = (0, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d})$.

Definition 55. *For $c \in L^\infty\left([0,\infty), \mathcal{M}_{0,\infty}\right)$ define time dependent linear operators $\widetilde{G}_t[c]$ on $\mathcal{M}^d$ by $\left\langle g, \widetilde{G}_t[c]\nu\right\rangle = -\langle g\cdot\nabla u_t, \nu\rangle + \left\langle \widehat{H_t[c]}g, \nu\right\rangle$ for all $g \in B^d$.*

One can now compactly rewrite the equation from Proposition 53 as (compare (3))

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle g,\nu_t\rangle=\left\langle g,\widetilde{U}_t\nu_t\right\rangle+\left\langle g,\widetilde{G}_t[c]\nu_t\right\rangle+\langle g,J_t[c]\rangle \tag{37}$$

for all $g\in B^d$ such that $\nabla\cdot g\in D$. The additional terms introduced in Definition 54 are not seen by the smaller class of test functions used in Proposition 53.

Proposition 56. *Assume H1 holds and that $c\in L^\infty([0,\infty),\mathcal{M}_{0,\infty})$, then there is a strongly continuous, bounded propagator $\widetilde{V}^{t,s}[c]$ on $\mathcal{M}^d$ with*

$$\left\|\widetilde{V}^{t,s}[c]\right\|_{\mathcal{M}^d\to\mathcal{M}^d}\le e^{\left(\frac{3}{2}K_\infty C_1|||c|||_{B^\star}+|||\nabla u|||\right)\min(t-s,t_0)}$$

and for $f\in D$, $\mu\in\mathcal{M}^d$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle f,\widetilde{V}^{t,s}[c]\mu\right\rangle=\left\langle f,\left(\widetilde{U}_t+\widetilde{G}_t[c]\right)\widetilde{V}^{t,s}[c]\mu\right\rangle.$$

Proof. This follows the same perturbation argument as Proposition 23 since by duality $\left\|\widetilde{G}_t[c]\right\|_{\mathcal{M}^d\to\mathcal{M}^d}\le\frac{3}{2}K_\infty C_1|||c|||_{B^\star}+|||\nabla u|||$. □

Proposition 57. *Let I3 hold; assume further that either $d=1$, H2 holds and $\inf_{t,x}u_t(x)>0$ or H3 holds for general d; assume further that $c\in L^\infty([0,\infty),\mathcal{M}_{0,\infty})$, then (37) has a unique solution*

$$\nu_t=\widetilde{V}^{t,0}[c]\nu_0+\int_0^t\widetilde{V}^{t,s}[c]J_s[c]\mathrm{d}s\in C\left([0,\infty),\left(\mathcal{M}^d,\|\cdot\|_{B^\star}\right)\right)$$

with initial condition ν_0. This solution is in $L^\infty\left([0,\infty),\mathcal{M}^d_{0,\infty}\right)$ provided $\nu_0\in\mathcal{M}^d_{0,\infty}$ and thus (identifying the measure with its $\mathcal{X}$-density) also in $L^\infty\left([0,\infty)\times\mathcal{X},\mathcal{M}(\mathcal{Y})^d_{\mathrm{TV}}\right)$.

Proof. Existence and uniqueness are immediate for this linear problem. Continuity in the $B^\star$-norm follows from the strong continuity in t of $\widetilde{V}^{t,s}$.

That the propagators $\widetilde{V}^{t,s}$ preserve $\mathcal{M}^d_{0,\infty}$ can be seen by analogy with Proposition 41. Definition 54 expresses the $J_t[c]$ as a sum of $\widehat{J}_t[c]$ and a term concentrated on the inflow boundary. Under H3 $\widehat{J}_t[c]\in\mathcal{M}^d_{0,\infty}$ and so one argues as in Propositions 43&44 to show that $\int_0^t\widetilde{V}^{t,s}[c]J_s[c]\mathrm{d}s$ has a density with respect to Lebesgue measure on $\mathcal{X}$.

In the case when only H2 holds, then the x-derivatives of h in (36) may only exist in a distributional sense. However, under H2, the measure $\nabla_x h(x,\xi)(\mathrm{d}x)$ can be expressed as a sum of an absolutely continuous part with a bounded density and a finite number of atoms $\alpha_k(t)\delta_{a_k}$ with $\alpha_k(t)\in\mathbb{R}$, $a_k\in\mathcal{X}\subset\mathbb{R}$. When $d=1$ each of these atoms is like a simpler version of the boundary part of the inception measure, which in this case reduces under the assumption I2 (see § 2.5) to $I_{\mathrm{bdry}}(t,\mathrm{d}y)\delta_0(\mathrm{d}x)$. The boundedness (uniform in t and k) of the $\alpha_k(t)$ is immediate from the boundedness of K and c and since u is bounded away from 0 the analysis of Propositions 43&44 applies to show that for each k and all t

$$\int_0^t\widetilde{V}^{t,s}[c]\alpha_k(t)\delta_{a_k}\mathrm{d}s\in\mathcal{M}^d_{0,\infty} \tag{38}$$

with a global in time bound in the $\mathcal{M}_{0,\infty}$-norm. □

Proof of Theorem 13. For $d=1$ assume without loss of generality that $\mathcal{X}=[0,L)$ for some $L>0$ and $\Gamma_{\mathrm{in}}=\{0\}$. The boundary condition $c(t,0,\mathrm{d}y)=\frac{I_{\mathrm{bdry}}(t,0,\mathrm{d}y)}{u_t(0)}$ is given by Proposition 50. The presumed derivative ν from Proposition 57 is then used to construct

$$\widetilde{c}(t,x,\mathrm{d}y)=\frac{I_{\mathrm{bdry}}(t,0,\mathrm{d}y)}{u_t(0)}+\int_0^x\nu(t,\xi,\mathrm{d}y)\mathrm{d}\xi, \tag{39}$$

which is readily seen to be a strong solution to (2) and therefore to be in the same $L^\infty([0,\infty),\mathcal{M}_{0,\infty})$ equivalence class as c. Therefore (a version of) c is in $L^\infty\left([0,\infty),W^{1,\infty}(\mathcal{X},\mathcal{M}(\mathcal{Y})_{\mathrm{TV}})\right)$ and since $\frac{\mathrm{d}}{\mathrm{d}t}c$ can be expressed in terms of c and $\frac{\mathrm{d}}{\mathrm{d}x}c$ the result follows.

This argument does not generalise easily to more then one space dimension. However the existence of a weak derivative was shown in Proposition 49 and under H3 Proposition 57 shows that this weak derivative in fact has an L^∞ density. The boundary condition comes from Proposition 50. □

5 Discussion

This paper proves the well posedness of an equation for measures, modelling the creation and coagulation of particles in a flow, for example a flame, for which stochastic approximations were studied in [14]. In that work the existence of one or more non-negative solutions was proved under somewhat less general assumptions there by constructing the solutions as limits of stochastic approximations. The present work extends this result by showing that there is in fact only one solution to the equation for a given initial condition and thus that all limit points of the approximating sequence from [14] are the same and those approximations converge rather than merely having convergent sub-sequences. The present work incidentally provides an additional, less constructive proof of the existence of a solution to (2).

It is proved here and in [14] that solutions to (2) have a density with respect to Lebesgue measure on $\mathcal{X}$ and that this is uniformly bounded in time and in $\mathcal{X}$. The differentiability of the density is established here even for delocalisations that are of bounded variation, but only in one spatial dimension. This result does not extend in full generality to higher spatial dimensions—it is easy to imagine two parallel streams of particles that never mix and therefore not even continuity over the dividing line in the flow, much less differentiability, is to be expected. It seems therefore likely that the discontinuous, cell based delocalisation of the coagulation interaction used for numerical purposes in [15] is not well suited to more than one spatial dimension and that smoother delocalisations should be used. Similar methods have been used for the simulation of Boltzmann gases[13].

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A The flow field

Definition 58. *Let $s, t \in \mathbb{R}$ and define the flows $\Phi_{s,t}$ by*

$$\frac{\partial}{\partial t}\Phi_{s,t}(x) = u_t\left(\Phi_{s,t}(x)\right), \quad \Phi_{s,s}(x) = x.$$

Φ is a vector, so in more than one dimension it is necessary to distinguish between the matrix $\nabla\Phi$, which is the subject of the next two propositions and the divergence, a real number $\nabla \cdot \Phi$, which occurs in connection with the velocity field u.

Proposition 59.

$$\frac{\partial}{\partial s}\Phi_{s,t}(x) = -\nabla\Phi_{s,t}(x)u_s(x).$$

Proof.

$$\lim_{\delta\searrow 0}\frac{\Phi_{s,t}(x)-\Phi_{s-\delta,t}(x)}{\delta} = \lim_{\delta\searrow 0}\frac{\Phi_{s,t}(x)-\Phi_{s,t}\left(\Phi_{s-\delta,s}(x)\right)}{\delta}$$
$$= \lim_{\delta\searrow 0}\frac{\nabla\Phi_{s,t}(x)\left(x-\Phi_{s-\delta,s}(x)\right)}{\delta} = -\nabla\Phi_{s,t}(x)u_s(x). \quad (40)$$

The right sided limit is dealt with similarly. □

Proposition 60.

$$e^{-|||\nabla u|||(t-s)} \le \|\nabla\Phi_{s,t}(x)\|_{\mathbb{R}^d\to\mathbb{R}^d} \le e^{|||\nabla u|||(t-s)}.$$

and

$$\det\nabla\Phi_{s,t}(x) = e^{\int_s^t \nabla\cdot u_r(\Phi_{s,r}(x))\mathrm{d}r}, \qquad \det\nabla\Phi_{t,s}(x) = e^{-\int_s^t \nabla\cdot u_r(\Phi_{t,r}(x))\mathrm{d}r}.$$

Proof. For the first statement one has $\frac{\partial}{\partial t}\Phi_{s,t}(x) = u_t\left(\Phi_{s,t}(x)\right)$ so that, since u and therefore Φ are both smooth,

$$\frac{\partial}{\partial t}\nabla\Phi_{s,t}(x) = \nabla u_t\left(\Phi_{s,t}(x)\right)\nabla\Phi_{s,t}(x) \quad (41)$$

and the result follows by an application of Gronwall's inequality.

The result for the determinant is known as Liouville's formula. One checks by row operations that $\det\nabla u_t\left(\Phi_{s,t}(x)\right)\nabla\Phi_{s,t}(x) = \mathrm{Tr}(\nabla u)\det\left(\nabla\Phi_{s,t}(x)\right)$ and the result that follows by solving the resulting ODEs. □