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Grad-div stabilization for the evolutionary Oseen problem with inf-sup stable finite elements

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Abstract

The approximation of the time-dependent Oseen problem using inf-sup stable mixed finite elements in a Galerkin method with grad-div stabilization is studied. The main goal is to prove that adding a grad-div stabilization term to the Galerkin approximation has a stabilizing effect for small viscosity. Both the continuous-in-time and the fully discrete case (backward Euler method, the two-step BDF, and Crank–Nicolson schemes) are analyzed. In fact, error bounds are obtained that do not depend on the inverse of the viscosity in the case where the solution is sufficiently smooth. The bounds for the divergence of the velocity as well as for the pressure are optimal. The analysis is based on the use of a specific Stokes projection. Numerical studies support the analytical results.

1 Introduction

A considerable amount of papers have been recently written concerning the numerical approximation of the steady Oseen equations. These equations play a crucial role in the numerical simulation of the time-dependent incompressible Navier–Stokes equations. After having applied an implicit time discretization, a fixed point iteration can be used every time step to solve the resulting non-linear equations. In this context one has to approach a steady Oseen problem in every time step of this iteration. It is well known that the standard Galerkin method suffers from instabilities for small values of the viscosity. Stabilized methods have to be used to improve the numerical simulations. The well-known streamline upwind/Petrov-Galerkin (SUPG) method combined with the pressure-stabilization/Petrov-Galerkin (PSPG) method allows to achieve both stability and accuracy. A grad-div stabilization is usually included. The SUPG/PSPG/grad-div stabilized method applied to the steady Oseen equations was analyzed in [26], see also [24]. In [19], see also [4], the *hp* version of the stabilized SUPG/PSPG/grad-div method was analyzed for the same equations. Similar error bounds as in [26] were obtained. The h version of the method was revisited in [13] using conforming inf-sup stable elements. In [20], the reduced SUPG/grad-div stabilized scheme was studied again in the case of using inf-sup stable elements. A stabilized finite element formulation using orthogonal subscales was analyzed in [8]. All the results mentioned above concern the steady Oseen equations.

The analysis of the time-dependent Oseen equations can be seen as a first step towards the analysis of the evolutionary Navier–Stokes equations. However, in the case of the evolutionary Oseen equations the literature is rather scarce. The stabilized approach based on orthogonal subscales was described in [7] but not analyzed. The analysis of the method using time-dependent subscales can be found in [9]. Recently, in [10] the time-dependent Oseen problem was considered using Local Projection Stabilization (LPS) methods with stabilization of the streamline derivative together with grad-div stabilization. In the case of using methods of order k without compatibility condition, error bounds are obtained under a restriction on the mesh size: a certain measure for the mesh size should be of order of the square root of the viscosity, see [10, (35)] for details. In order to avoid the restriction on the mesh size for small viscosity, the authors of [10] consider pairs satisfying a certain element-wise compatibility condition between the discrete velocities on the fine mesh and on the projection space. Even in that case optimal error bounds for the pressure were not obtained in [10].

Several authors have previously studied the effect of grad-div stabilization. In [22] the author considers the approximation of the steady incompressible Navier–Stokes equations using both SUPG and grad-div stabilization. The grad-div stabilization is shown to enhance the accuracy of the solution and to improve the convergence of preconditioned iterations for the linearized Navier–Stokes problem if the corresponding stabilization parameter is not too large. In [23] the use of the grad-div term on the numerical solution of the Stokes equations is considered. The authors show that this stabilization improves the well-posedness of the continuous problem for small values of the viscosity. They also analyze the influence of this stabilization on the accuracy of the approximation. A refined analysis was presented in [16]. In [21] the grad-div stabilization is considered as a subgrid pressure model in the framework of variational multiscale methods. Some error estimates for the steady Oseen problem with grad-div stabilization are proved. In [10] a significant role of grad-div stabilization for inf-sup stable approximations is observed while a SUPG-type stabilization seems to be much less important. More precisely the authors of [10] say: "it turned out that the grad-div stabilization with a globally constant parameter set is essential for an improvement of local mass balance and gives always good results in our numerical experiments. Nevertheless, the theoretical foundation is not really convincing." With the theoretical results obtained in the present paper a theoretical foundation is added to the fact already observed about the improvement due to the grad-div stabilization. Even for the simulation of turbulent flows it was observed in [17, Fig. 3] that the addition of only the grad-div stabilization to the Galerkin approximation was sufficient for performing stable simulations.

Finally, the role of grad-div stabilization in preconditioning techniques should be mentioned. One of the advantages of such a formulation is its positive effect in the solution of the Schur complement problem. As it is claimed in [2], with this approach, the solution of the Schur complement is no longer the bottleneck of the iterative solution as it is the case in many block preconditioning approaches to the original system arising from the linearized Navier–Stokes equations, see also [3, 15]. The optimal value of the stabilization parameter is found in these works to be small. This fact is in agreement with the size $\mathcal{O}(1)$ found to be optimal in the present paper.

In the present paper, mixed finite element approximations to the time-dependent Oseen problem are analyzed using inf-sup stable pairs of finite element spaces and a grad-div stabilization. It is shown that the plain Galerkin approximations can be stabilized by adding only a grad-div stabilization term. Optimal error bounds with constants that do not depend on the viscosity parameter are obtained for the L^2 norm of the divergence of the velocity and the L^2 norm of the pressure, assuming the solution is smooth enough. In addition, an error bound for $\nu^{1/2}$ times the gradient of the velocity is proved that is optimal in the diffusion-dominated regime although it is a weak term in the convection-dominated regime.

The derived optimal error bounds for the L^2 norm of the divergence of the velocity and the L^2 norm of the pressure are global bounds that can only be applied to globally smooth solutions. In [12] local error estimates are obtained for the SUPG method applied to evolutionary convection-reaction-diffusion equations combined with the backward Euler scheme. The question of getting local error bounds for the method studied in this paper following the techniques in [12] will be the subject of future research.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded polyhedral domain with Lipschitz boundary $\partial\Omega$ and let $(0,T)$ be a time interval with $T < \infty$. The time-dependent Oseen problem, as a model problem for the linearized Navier–Stokes equations, reads as follows

$$\begin{aligned} \partial_t \tilde{\boldsymbol{u}} - \nu \Delta \tilde{\boldsymbol{u}} + (\boldsymbol{b}\cdot\nabla)\tilde{\boldsymbol{u}} + \nabla \tilde{p} &= \tilde{\boldsymbol{f}} && \text{in} \quad (0,T] \times \Omega, \\ \nabla \cdot \tilde{\boldsymbol{u}} &= 0 && \text{in} \quad [0,T] \times \Omega, \\ \tilde{\boldsymbol{u}} &= \boldsymbol{0} && \text{on} \quad [0,T] \times \partial\Omega, \\ \tilde{\boldsymbol{u}}(0,\boldsymbol{x}) &= \tilde{\boldsymbol{u}}_0(\boldsymbol{x}) && \text{in} \quad \Omega, \end{aligned} \tag{1}$$

where $\tilde{\boldsymbol{u}} : (0,T) \times \Omega \to \mathbb{R}^d$ and $\tilde{p} : (0,T) \times \Omega \to \mathbb{R}$ are the unknown velocity and pressure, $\nu > 0$ is the viscosity, $\tilde{\boldsymbol{f}} : (0,T) \times \Omega \to \mathbb{R}^d$ the external forces, and $\boldsymbol{b} : (0,T) \times \Omega \to \mathbb{R}^d$ a solenoidal vector field, i.e., $\nabla \cdot \boldsymbol{b} = 0$, with $\boldsymbol{b} \in L^\infty(0,T;L^\infty(\Omega)^d)$.

For the numerical analysis it is of advantage to perform a change of variables: $(\boldsymbol{u},p) = e^{-\alpha t}(\tilde{\boldsymbol{u}},\tilde{p})$ with $\alpha > 0$. A direct calculation shows that with this transformation one obtains a problem with a positive zeroth order term:

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + \alpha\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in} \quad (0,T] \times \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in} \quad [0,T] \times \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on} \quad [0,T] \times \partial\Omega, \\ \boldsymbol{u}(0,\boldsymbol{x}) &= \boldsymbol{u}_0(\boldsymbol{x}) && \text{in} \quad \Omega. \end{aligned} \tag{2}$$

The analysis can be applied to the new problem. Finally, one can transform back to the original variables. In the analysis, $\alpha = 1/T$ is chosen such that the error bounds only change in a multiplicative constant of size $e^{\alpha t} \le e$.

In this paper a grad-div scheme to approach problem (2) will be analyzed. The outline of the paper is as follows. Section 2 introduces the grad-div stabilization of the Galerkin approximation and some preliminaries are stated. Section 3 is devoted to the analysis of the continuous-in-time case. In Section 4 the fully discrete case is considered using the backward Euler method, the two-step backward differentiation formula (BDF2), and the Crank–Nicolson scheme as time integrators. Finally, some numerical studies which support the analytical results are presented in Section 5.

2 Preliminaries and notation

Using the function spaces, see [6, Section 1.2]

$$V = H_0^1(\Omega)^d, \quad Q = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : (q,1) = 0 \right\},$$

the weak formulation of problem (2) is: Find $(\boldsymbol{u},p) \in V \times Q$ such that for all $(\boldsymbol{v},q) \in V \times Q$,

$$(\partial_t \boldsymbol{u},\boldsymbol{v}) + \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + \alpha\boldsymbol{u},\boldsymbol{v}) - (\nabla\cdot\boldsymbol{v},p) + (\nabla\cdot\boldsymbol{u},q) = (\boldsymbol{f},\boldsymbol{v}). \tag{3}$$

Notice that

$$((\boldsymbol{b}\cdot\nabla)\boldsymbol{v},\boldsymbol{v}) = 0 \quad \forall \boldsymbol{v}\in V. \tag{4}$$

The Hilbert space $H^{\mathrm{div}} = \{\boldsymbol{u}\in L^2(\Omega)^d \ : \ \nabla\cdot\boldsymbol{u} = 0, \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}$ will be endowed with the inner product of $L^2(\Omega)^d$ and the space $V^{\mathrm{div}} = \{\boldsymbol{u}\in H_0^1(\Omega)^d \ : \ \nabla\cdot\boldsymbol{u} = 0\}$ with the inner product of $H_0^1(\Omega)^d$. Let $\Pi \ : L^2(\Omega)^d \to H^{\mathrm{div}}$ be the Leray projector that maps each function in $L^2(\Omega)^d$ onto its divergence-free part. The Stokes operator in Ω is given by

$$A \ : \ \mathcal{D}(A)\subset V^{\mathrm{div}} \to V^{\mathrm{div}}, \quad A = -\Pi\Delta, \quad \mathcal{D}(A) = H^2(\Omega)^d\cap V^{\mathrm{div}}.$$

The norm in $L^2(\Omega)$ for scalar-, vector-, and tensor-valued functions is denoted by $\|\cdot\|_0$ and the norm in $H^k(\Omega)$ by $\|\cdot\|_k$.

In the error analysis, the Poincaré–Friedrichs inequality

$$\|\boldsymbol{v}\|_0 \le C_{PF}\|\nabla\boldsymbol{v}\|_0 \quad \forall v\in V \tag{5}$$

will be used and also the inequality

$$\|\nabla\cdot\boldsymbol{v}\|_0 \le \|\nabla\boldsymbol{v}\|_0\,, \qquad \boldsymbol{v}\in H_0^1(\Omega)^d, \tag{6}$$

which follows from the identity, $\|\nabla\boldsymbol{v}\|_0^2 = \|\nabla\cdot\boldsymbol{v}\|_0^2 + \|\nabla\times\boldsymbol{v}\|_0^2$, a relation that can be obtained from the vector identity $\nabla\times(\nabla\times\boldsymbol{v}) = -\Delta\boldsymbol{v} + \nabla(\nabla\cdot\boldsymbol{v})$ after taking the inner product in $L^2(\Omega)^d$ with $\boldsymbol{v}$ and integrating by parts.

Let $V_h\subset V$ and $Q_h\subset Q$ be two families of finite element spaces that correspond to a family of partitions $\mathcal{T}_h$ of Ω into mesh cells with maximal diameter h. In this paper, only pairs of finite element spaces will be considered that satisfy the discrete inf-sup condition

$$\inf_{q_h\in Q_h}\sup_{\boldsymbol{v}_h\in V_h}\frac{(\nabla\cdot\boldsymbol{v}_h, q_h)}{\|\nabla\boldsymbol{v}_h\|_0\|q_h\|_0} \ge \beta_0 > 0. \tag{7}$$

It will be also assume that $V_h\subset H_0^1(\Omega)^d$ and $Q_h\subset L^2(\Omega)$ comprise piecewise polynomials of degrees at most k and l, respectively. It will be assumed that the meshes are quasi-uniform and that the following inverse inequality holds for each $v_h\in V_h$, see, e.g., [6, Theorem 3.2.6],

$$\|\boldsymbol{v}_h\|_{W^{m,q}(K)^d} \le C_{\mathrm{inv}} h_K^{l-m-d\left(\frac{1}{q'}-\frac{1}{q}\right)}\|\boldsymbol{v}_h\|_{W^{l,q'}(K)^d}, \tag{8}$$

where $0\le l\le m\le 1$, $1\le q'\le q\le\infty$, h_K is the size (diameter) of the mesh cell $K\in\mathcal{T}_h$, and $\|\cdot\|_{W^{m,q}(K)^d}$ is the norm in $W^{m,q}(K)^d$.

The space of discretely divergence-free functions is denoted by

$$V_h^{\mathrm{div}} = \{\boldsymbol{v}_h\in V_h \ : \ (\nabla\cdot\boldsymbol{v}_h, q_h) = 0 \quad \forall q_h\in Q_h\}\,.$$

The linear operator $A_h \ : \ V_h^{\mathrm{div}}\to V_h^{\mathrm{div}}$ is defined by

$$(A_h\boldsymbol{v}_h,\boldsymbol{w}_h) = (\nabla\boldsymbol{v}_h,\nabla\boldsymbol{w}_h) \quad \forall\boldsymbol{v}_h,\boldsymbol{w}_h\in V_h^{\mathrm{div}}. \tag{9}$$

Note that from this definition it follows that

$$\|A_h^{1/2}\boldsymbol{v}_h\|_0 = \|\nabla \boldsymbol{v}_h\|_0, \quad \|\nabla A_h^{-1/2}\boldsymbol{v}_h\|_0 = \|\boldsymbol{v}_h\|_0 \quad \forall \boldsymbol{v}_h \in V_h^{\mathrm{div}}. \tag{10}$$

Additionally, the linear operators $B_h \ : \ V_h^{\mathrm{div}}, \to V_h^{\mathrm{div}}$, given by

$$(B_h\boldsymbol{v}_h, \boldsymbol{w}_h) = (\nabla\cdot\boldsymbol{v}_h, \nabla\cdot\boldsymbol{w}_h) \quad \forall \boldsymbol{v}_h, \boldsymbol{w}_h \in V_h^{\mathrm{div}}, \tag{11}$$

and $D_h : L^2(\Omega) \to V_h^{\mathrm{div}}$, given by

$$(D_h q, \boldsymbol{v}_h) = (\nabla\cdot\boldsymbol{v}_h, q) \quad q \in L^2(\Omega), \forall \boldsymbol{v}_h \in V_h^{\mathrm{div}} \tag{12}$$

are defined. Finally, the so-called discrete Leray projection $\Pi_h^{\mathrm{div}} \ : \ L^2(\Omega)^d \to V_h^{\mathrm{div}}$ is introduced, which is the L^2 orthogonal projection of $L^2(\Omega)^d$ onto V_h^{div}

$$(\Pi_h^{\mathrm{div}}\boldsymbol{v}, \boldsymbol{w}_h) = (\boldsymbol{v}, \boldsymbol{w}_h) \quad \forall \boldsymbol{w}_h \in V_h^{\mathrm{div}}. \tag{13}$$

By definition, it is clear that the projection is stable in the L^2 norm: $\|\Pi_h^{\mathrm{div}}\boldsymbol{v}\|_0 \le \|\boldsymbol{v}\|_0$ for all $\boldsymbol{v} \in L^2(\Omega)^d$. Denote by π_h the L^2 projection of the pressure p in (2) onto Q_h. Then, for $l \ge 0$ and $0 \le m \le 1$ one has

$$\|p - \pi_h\|_m \le C h^{l+1-m}\|p\|_{l+1}, \quad p \in H^{l+1}(\Omega). \tag{14}$$

There exists an interpolation operator $I_h : H^1(\Omega) \to V_h$ that satisfies for all $\boldsymbol{v} \in H^l(\Omega)^d$ and all mesh cells $K \in \mathcal{T}_h$

$$\|\boldsymbol{v} - I_h\boldsymbol{v}\|_{0,K} + h_K|\boldsymbol{v} - I_h\boldsymbol{v}|_{1,K} \le C h_K^l \|\boldsymbol{v}\|_{l,\omega(K)}, \quad 1 \le l \le r+1, \tag{15}$$

where $\omega(K)$ denotes a certain local neighborhood of K, see [25].

To carry out the analysis, the Stokes problem

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{g} \quad \text{in} \quad \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on} \quad \partial\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in} \quad \Omega, \end{aligned} \tag{16}$$

will be considered. The standard Galerkin approximation $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ is the solution of the mixed finite element approximation to (16), given by

$$\begin{aligned} \nu(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}_h) + (\nabla p_h, \boldsymbol{v}_h) &= (\boldsymbol{g}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h, \\ (\nabla\cdot\boldsymbol{u}_h, q_h) &= 0 \quad \forall q_h \in Q_h. \end{aligned} \tag{17}$$

Following [14] one gets the estimates

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \le C\left(\inf_{\boldsymbol{v}_h\in V_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \nu^{-1}\inf_{q_h\in Q_h}\|p - q_h\|_0\right), \tag{18}$$

$$\|p - p_h\|_0 \le C\left(\nu\inf_{\boldsymbol{v}_h\in V_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \inf_{q_h\in Q_h}\|p - q_h\|_0\right), \tag{19}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \le Ch\left(\inf_{\boldsymbol{v}_h\in V_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \nu^{-1}\inf_{q_h\in Q_h}\|p - q_h\|_0\right). \tag{20}$$

It can be observed that the error bounds for the velocity depend on negative powers of ν.

For the purpose of analysis, it is useful to have a projection of $(\boldsymbol{u}, p)$ onto $V_h \times Q_h$ where the bounds for the velocity are uniform in ν. This goal can be achieved for smooth functions by choosing a special right-hand side in (16). For the Oseen problem, let $(\boldsymbol{u}, p)$ be the solution of (2) with $\boldsymbol{u} \in V \cap H^{k+1}(\Omega)^d$, $p \in Q \cap H^k(\Omega)$, $k \geq 1$, and define the right-hand side of the Stokes problem (16) by

$$\boldsymbol{g} = \boldsymbol{f} - \partial_t \boldsymbol{u} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} - \alpha \boldsymbol{u} - \nabla p. \tag{21}$$

Then $(\boldsymbol{u}, 0)$ is the solution of (16). Denoting the corresponding Galerkin approximation in $V_h \times Q_h$ by $(\boldsymbol{s}_h, l_h)$, one obtains from (18) – (20)

$$\|\boldsymbol{u} - \boldsymbol{s}_h\|_0 + h\|\boldsymbol{u} - \boldsymbol{s}_h\|_1 \leq Ch^{k+1}\|\boldsymbol{u}\|_{k+1}, \tag{22}$$

$$\|l_h\|_0 \leq C\nu h^k \|\boldsymbol{u}\|_{k+1}, \tag{23}$$

where the constant C does not depend on ν.

Remark 1 Assuming the necessary smoothness in time and considering (16) with

$$\boldsymbol{g} = \partial_t \left(\boldsymbol{f} - \partial_t \boldsymbol{u} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} - \alpha \boldsymbol{u} - \nabla p\right),$$

then one can derive an error bound of form (22) also for $\partial_t(\boldsymbol{u} - \boldsymbol{s}_h)$. In the same way, one can proceed for higher order derivatives in time.

The method that will be studied for solving the Oseen problem (2) is obtained by adding to the Galerkin method a control of the divergence constraint, the so-called grad-div stabilization: Find $(\boldsymbol{u}_h, p_h) \ : \ (0, T] \to V_h \times Q_h$ such that for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$ one has

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + A_\mu((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = (\boldsymbol{f}, \boldsymbol{v}_h), \tag{24}$$

where the initial discrete velocity is an appropriate approximation of $\boldsymbol{u}_0$ in V^h,

$$\begin{aligned} A_\mu((\boldsymbol{w}, r), (\boldsymbol{v}, q)) = {} & \nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{v}) + ((\boldsymbol{b} \cdot \nabla)\boldsymbol{w} + \alpha \boldsymbol{w}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, r) \\ & +(\nabla \cdot \boldsymbol{w}, q) + \mu(\nabla \cdot \boldsymbol{w}, \nabla \cdot \boldsymbol{v}), \end{aligned}$$

and $\mu \geq 0$ is a stabilization parameter, whose optimal asymptotic choice will be determined by the results of the numerical analysis.

3 Error analysis of the method in the continuous-in-time case

The proof of the error estimates is based on the comparison of the Galerkin approximation $(\boldsymbol{u}_h, p_h)$ in (24) with the approximation $(\boldsymbol{s}_h, l_h)$ of the Stokes equations with right-hand side (21). Let $\boldsymbol{e}_h = \boldsymbol{u}_h - \boldsymbol{s}_h \in V_h^{\mathrm{div}}$, then a straightforward calculation, using $\nabla \cdot \boldsymbol{u} = 0$, yields

$$\begin{aligned}&(\partial_t \boldsymbol{e}_h, \boldsymbol{v}_h) + A_\mu((\boldsymbol{e}_h, p_h - l_h), (\boldsymbol{v}_h, q_h))\\ &= (\partial_t(\boldsymbol{u} - \boldsymbol{s}_h), \boldsymbol{v}_h) + ((\boldsymbol{b} \cdot \nabla)(\boldsymbol{u} - \boldsymbol{s}_h) + \alpha(\boldsymbol{u} - \boldsymbol{s}_h), \boldsymbol{v}_h)\\ &\quad + \mu(\nabla \cdot (\boldsymbol{u} - \boldsymbol{s}_h), \nabla \cdot \boldsymbol{v}_h) + (\nabla p, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in V_h,\ q_h \in Q_h.\end{aligned} \tag{25}$$

Taking $(\boldsymbol{v}_h, q_h) = (\boldsymbol{e}_h, p_h - l_h)$ in (25), observing that $(\nabla p, \boldsymbol{e}_h) = -(p, \nabla \cdot \boldsymbol{e}_h) = -(p - \pi_h, \nabla \cdot \boldsymbol{e}_h)$ since $\boldsymbol{e}_h \in V_h^{\mathrm{div}}$, using (4), and applying the Cauchy–Schwarz inequality gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}_h\|_0^2 + \nu\|\nabla \boldsymbol{e}_h\|_0^2 + \alpha\|\boldsymbol{e}_h\|_0^2 + \mu\|\nabla \cdot \boldsymbol{e}_h\|_0^2\\ &\le \|\partial_t(\boldsymbol{u} - \boldsymbol{s}_h)\|_0\|\boldsymbol{e}_h\|_0 + \|\boldsymbol{b}\|_\infty\|\nabla(\boldsymbol{u} - \boldsymbol{s}_h)\|_0\|\boldsymbol{e}_h\|_0\\ &\quad + \alpha\|\boldsymbol{u} - \boldsymbol{s}_h\|_0\|\boldsymbol{e}_h\|_0 + (\mu\|\nabla \cdot (\boldsymbol{u} - \boldsymbol{s}_h)\|_0 + \|p - \pi_h\|_0)\|\nabla \cdot \boldsymbol{e}_h\|_0.\end{aligned}$$

With Young's inequality and hiding terms on the left-hand side, one obtains

$$\begin{aligned}&\frac{d}{dt}\|\boldsymbol{e}_h\|_0^2 + 2\nu\|\nabla \boldsymbol{e}_h\|_0^2 + \alpha\|\boldsymbol{e}_h\|_0^2 + \mu\|\nabla \cdot \boldsymbol{e}_h\|_0^2\\ &\le \frac{3}{\alpha}\left(\|\partial_t(\boldsymbol{u} - \boldsymbol{s}_h)\|_0^2 + \|\boldsymbol{b}\|_\infty^2\|\nabla(\boldsymbol{u} - \boldsymbol{s}_h)\|_0^2 + \alpha^2\|\boldsymbol{u} - \boldsymbol{s}_h\|_0^2\right)\\ &\quad + 2\mu\|\nabla \cdot (\boldsymbol{u} - \boldsymbol{s}_h)\|_0^2 + 2\mu^{-1}\|p - \pi_h\|_0^2.\end{aligned} \tag{26}$$

Assuming now

$$\begin{aligned}(\boldsymbol{u}, p) &\in L^2(0, t; H^{k+1}(\Omega)^d) \times L^2(0, t; H^{l+1}(\Omega)),\\ (\partial_t \boldsymbol{u}, \partial_t p) &\in L^2(0, t; H^k(\Omega)^d) \times L^2(0, t; H^l(\Omega)),\end{aligned} \tag{27}$$

applying the estimates (22) and (14), integrating (26) on $(0, t)$, and recalling that $\alpha = 1/T$, one gets

$$\begin{aligned}&\|\boldsymbol{e}_h(t)\|_0^2 + 2\nu\|\nabla \boldsymbol{e}_h\|_{L^2(0,t;L^2)}^2 + \alpha\|\boldsymbol{e}_h\|_{L^2(0,t;L^2)}^2 + \mu\|\nabla \cdot \boldsymbol{e}_h\|_{L^2(0,t;L^2)}^2\\ &\le \|\boldsymbol{e}_h(0)\|_0^2 + Ch^{2k}\Big((T + \mu)\|\boldsymbol{u}\|_{L^2(0,t;H^{k+1})}^2 + T\|\partial_t \boldsymbol{u}\|_{L^2(0,t;H^k)}^2\Big)\\ &\quad + C\mu^{-1}h^{2(l+1)}\|p\|_{L^2(0,t;H^{l+1})}^2,\end{aligned} \tag{28}$$

where $C = C\left(\|\boldsymbol{b}\|_{L^\infty(0,t;L^\infty)}\right)$.

Remark 2 It will be assumed that $\boldsymbol{u}_0 \in H^k(\Omega)^d$ and that the projection $\boldsymbol{s}_h$ is well defined at $t = 0$, which implies that compatibility conditions at $t = 0$ are assumed. Then, if, for example, $\boldsymbol{u}_h(0) = I_h \boldsymbol{u}_0$, the error $\boldsymbol{e}_h(0)$ can be bounded by

$$\|\boldsymbol{e}_h(0)\|_0 \le \|I_h \boldsymbol{u}_0 - \boldsymbol{u}_0\|_0 + \|\boldsymbol{u}_0 - \boldsymbol{s}_h(0)\|_0,$$

and then (15) and (22) can be applied.

Theorem 1 *Let $(\boldsymbol{u}, p) \in V \times Q$ be the solution of (3) and let $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ be the solution of (24). Assume that (27) and the conditions from Remark 2 hold. Then, the following error estimate holds for all $t \in (0, T]$*

$$\begin{aligned}
&\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|_0^2 + \nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|^2_{L^2(0,t;L^2)} + \alpha\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{L^2(0,t;L^2)}\\
&\quad +\mu\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\|^2_{L^2(0,t;L^2)}\\
&\le Ch^{2k}\left((T+\mu)\|\boldsymbol{u}\|^2_{L^2(0,t;H^{k+1})} + T\|\partial_t\boldsymbol{u}\|^2_{L^2(0,t;H^k)} + \|\boldsymbol{u}_0\|_k^2 + \|\boldsymbol{u}(t)\|_k^2\right)\\
&\quad +C\mu^{-1}h^{2l+2}\|p\|^2_{L^2(0,t;H^{l+1})}, \qquad (29)
\end{aligned}$$

where $C = C\left(\|\boldsymbol{b}\|_{L^\infty(0,t;L^\infty)}\right)$.

Proof The result is obtained by applying the triangle inequality to the left-hand side of (29) and using (28) and (22).

Remark 3 The most common situation for the choice of the finite element spaces consists in choosing the polynomial degree of the pressure one degree smaller than for the velocity, i.e., $l = k-1$. Then, it follows from (29) that the optimal error bound $\mathcal{O}(h^k)$ is obtained for $\mu = \mathcal{O}(1)$. In the case $k = l$, e.g., for the MINI element where $k = l = 1$, one can choose $\mu = \mathcal{O}(h)$ or even $\mu = \mathcal{O}(h^2)$. Which choice is the better one depends on the concrete situation, see the discussion in [16].

To facilitate the presentation of the further analysis, it will be assumed henceforth that $l = k-1$.

Lemma 1 *The following stability estimate holds for the discrete velocity*

$$\begin{aligned}
&\|\boldsymbol{u}_h(t)\|_0^2 + 2\nu\|\nabla\boldsymbol{u}_h\|^2_{L^2(0,t;L^2)} + \alpha\|\boldsymbol{u}_h\|^2_{L^2(0,t;L^2)} + 2\mu\|\nabla\cdot\boldsymbol{u}_h\|^2_{L^2(0,t;L^2)}\\
&\qquad \le \|\boldsymbol{u}_h(0)\|_0^2 + \frac{1}{\alpha}\|\boldsymbol{f}\|^2_{L^2(0,t;L^2)} \quad \forall t\in[0,T]. \qquad (30)
\end{aligned}$$

Proof Taking $(\boldsymbol{v}_h, q_h) = (\boldsymbol{u}_h, p_h)$ in (24) and using (4), it follows immediately that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_h\|_0^2 + \nu\|\nabla\boldsymbol{u}_h\|_0^2 + \alpha\|\boldsymbol{u}_h\|_0^2 + \mu\|\nabla\cdot\boldsymbol{u}_h\|_0^2 \le \|\boldsymbol{f}\|_0\|\boldsymbol{u}_h\|_0.$$

Applying Young's inequality on the right-hand side and integrating on $(0, t)$ gives (30).

Remark 4 From (30) it follows in particular that

$$\lim_{\mu\to\infty}\|\nabla\cdot\boldsymbol{u}_h\|^2_{L^2(0,t;L^2)} \le \lim_{\mu\to\infty}\frac{1}{2\mu}\left(\|\boldsymbol{u}_h(0)\|_0^2 + \frac{1}{\alpha}\|\boldsymbol{f}\|^2_{L^2(0,t;L^2)}\right) = 0.$$

This behavior is in agreement with the result of [5] where the authors show that the grad-div stabilized Taylor–Hood approximation to the evolutionary Navier–Stokes equations converges to the Scott–Vogelius solution as the stabilization parameter tends to infinity. The Scott–Vogelius element pair provides point-wise mass conservation. In the numerical tests of [5] the grad-div stabilized Taylor–Hood approximation with large stabilization parameter is shown to provide excellent mass conservation for the Navier–Stokes approximation.

The next step in the error analysis consists in obtaining a bound for the pressure error. This bound is derived as usual on the basis of the discrete inf-sup condition (7).

As a first step, a bound for $\|\partial_t \boldsymbol{e}_h\|_{-1}$ is needed. By definition, it is

$$\|\partial_t \boldsymbol{e}_h\|_{-1} = \sup_{\boldsymbol{\varphi}\in H_0^1(\Omega)^d, \boldsymbol{\varphi}\neq \boldsymbol{0}} \frac{|\langle \partial_t \boldsymbol{e}_h, \boldsymbol{\varphi}\rangle|}{\|\nabla \boldsymbol{\varphi}\|_0},$$

where $\langle\cdot,\cdot\rangle$ denotes the corresponding duality pairing. To start with, the bound of $\|\partial_t \boldsymbol{e}_h\|_{-1}$ is reduced to a bound of $\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0$. From [1, Lemma 3.11] it is known that

$$\|\partial_t \boldsymbol{e}_h\|_{-1} \le Ch\|\partial_t \boldsymbol{e}_h\|_0 + C\|A^{-1/2}\Pi\partial_t \boldsymbol{e}_h\|_0, \tag{31}$$

where Π is the Leray projector introduced in Section 2. Applying [1, (2.15)], one obtains

$$\|A^{-1/2}\Pi\partial_t \boldsymbol{e}_h\|_0 \le Ch\|\partial_t \boldsymbol{e}_h\|_0 + \|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0, \tag{32}$$

with A_h defined in (9). From (31), (32), using the symmetry of A_h, (10), and the inverse inequality (8), it follows that

$$\begin{aligned}
\|\partial_t \boldsymbol{e}_h\|_{-1} &\le Ch\|\partial_t \boldsymbol{e}_h\|_0 + C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0 \\
&= Ch\|A_h^{1/2}A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0 + C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0 \\
&= Ch\|\nabla(A_h^{-1/2}\partial_t \boldsymbol{e}_h)\|_0 + C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0 \\
&\le C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0.
\end{aligned} \tag{33}$$

Next, a bound for $\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0$ will be derived. Projecting the error equation (25) onto the discretely divergence-free space $V_h^{\rm div}$ and using integration by parts, one gets

$$\begin{aligned}
(\partial_t \boldsymbol{e}_h, \boldsymbol{v}_h) &+ \nu(\nabla \boldsymbol{e}_h, \nabla \boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \alpha \boldsymbol{e}_h, \boldsymbol{v}_h) + \mu(\nabla\cdot \boldsymbol{e}_h, \nabla\cdot \boldsymbol{v}_h) \\
&= (\partial_t(\boldsymbol{u}-\boldsymbol{s}_h), \boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \alpha(\boldsymbol{u}-\boldsymbol{s}_h), \boldsymbol{v}_h) \\
&\quad + \mu(\nabla\cdot(\boldsymbol{u}-\boldsymbol{s}_h), \nabla\cdot \boldsymbol{v}_h) - (p-\pi_h, \nabla\cdot \boldsymbol{v}_h),
\end{aligned} \tag{34}$$

where π_h is the L^2 projection of p onto Q_h. Recalling definition (12) one has $(p-\pi_h, \nabla\cdot \boldsymbol{v}_h) = (D_h(p-\pi_h), \boldsymbol{v}_h)$, such that

$$\begin{aligned}
\partial_t \boldsymbol{e}_h = &-\nu A_h \boldsymbol{e}_h - \Pi_h^{\rm div}\big((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \alpha \boldsymbol{e}_h\big) - \mu B_h \boldsymbol{e}_h + \Pi_h^{\rm div}\big(\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\big) \\
&+ \Pi_h^{\rm div}\big((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \alpha(\boldsymbol{u}-\boldsymbol{s}_h)\big) + \mu B_h(\boldsymbol{u}-\boldsymbol{s}_h) \\
&- D_h(p-\pi_h).
\end{aligned} \tag{35}$$

With (11), the Cauchy–Schwarz inequality, (6), and (10), one obtains for all $\boldsymbol{v}_h \in V_h^{\mathrm{div}}$

$$
\begin{aligned}
\|A_h^{-1/2} B_h \boldsymbol{v}_h\|_0 &= \sup_{\boldsymbol{w}_h \in V_h^{\mathrm{div}}, \boldsymbol{w}_h \neq 0} \frac{|(B_h \boldsymbol{v}_h, A_h^{-1/2} \boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_0} \\
&= \sup_{\boldsymbol{w}_h \in V_h^{\mathrm{div}}, \boldsymbol{w}_h \neq 0} \frac{|(\nabla \cdot \boldsymbol{v}_h, \nabla \cdot (A_h^{-1/2} \boldsymbol{w}_h))|}{\|\boldsymbol{w}_h\|_0} \\
&\leq \sup_{\boldsymbol{w}_h \in V_h^{\mathrm{div}}, \boldsymbol{w}_h \neq 0} \frac{\|\nabla \cdot \boldsymbol{v}_h\|_0 \|\nabla \cdot (A_h^{-1/2} \boldsymbol{w}_h)\|_0}{\|\boldsymbol{w}_h\|_0} \\
&\leq \sup_{\boldsymbol{w}_h \in V_h^{\mathrm{div}}, \boldsymbol{w}_h \neq 0} \frac{\|\nabla \cdot \boldsymbol{v}_h\|_0 \|\nabla (A_h^{-1/2} \boldsymbol{w}_h)\|_0}{\|\boldsymbol{w}_h\|_0} \\
&= \sup_{\boldsymbol{w}_h \in V_h^{\mathrm{div}}, \boldsymbol{w}_h \neq 0} \frac{\|\nabla \cdot \boldsymbol{v}_h\|_0 \|\boldsymbol{w}_h\|_0}{\|\boldsymbol{w}_h\|_0} = \|\nabla \cdot \boldsymbol{v}_h\|_0. \qquad (36)
\end{aligned}
$$

The same argument applied to $\|A_h^{-1/2} D_h(p - \pi_h)\|_0$ gives

$$
\|A_h^{-1/2} D_h(p - \pi_h)\|_0 \leq \|p - \pi_h\|_0. \qquad (37)
$$

For $\boldsymbol{g} \in L^2(\Omega)^d$, the definition (13) and the symmetry of A_h allows to write $(A_h^{-1/2} \Pi_h^{\mathrm{div}} \boldsymbol{g}, \boldsymbol{v}_h) = (\boldsymbol{g}, A_h^{-1/2} \boldsymbol{v}_h)$ for all $\boldsymbol{v}_h \in V_h^{\mathrm{div}}$. Taking $\boldsymbol{v}_h = A_h^{-1/2} \Pi_h^{\mathrm{div}} \boldsymbol{g} \in V_h^{\mathrm{div}}$ in this relation and applying (10) yields

$$
\|A_h^{-1/2} \Pi_h^{\mathrm{div}} \boldsymbol{g}\|_0^2 \leq \|\boldsymbol{g}\|_{-1} \|\nabla (A_h^{-1/2} A_h^{-1/2} \Pi_h^{\mathrm{div}} \boldsymbol{g})\|_0 = \|\boldsymbol{g}\|_{-1} \|A_h^{-1/2} \Pi_h^{\mathrm{div}} \boldsymbol{g}\|_0
$$

and, hence,

$$
\|A_h^{-1/2} \Pi_h^{\mathrm{div}} \boldsymbol{g}\|_0 \leq \|\boldsymbol{g}\|_{-1} \quad \forall \boldsymbol{g} \in L^2(\Omega)^d. \qquad (38)
$$

Next, $A_h^{-1/2}$ is applied to (35). Using (36), (37) and (38), one gets

$$
\begin{aligned}
&\|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0 \\
&\leq \nu \|A_h^{1/2} \boldsymbol{e}_h\|_0 + \|(\boldsymbol{b} \cdot \nabla) \boldsymbol{e}_h + \alpha \boldsymbol{e}_h\|_{-1} + \mu \|\nabla \cdot \boldsymbol{e}_h\|_0 + \|\partial_t (\boldsymbol{u} - \boldsymbol{s}_h)\|_{-1} \qquad (39) \\
&\quad + \|(\boldsymbol{b} \cdot \nabla)(\boldsymbol{u} - \boldsymbol{s}_h) + \alpha (\boldsymbol{u} - \boldsymbol{s}_h)\|_{-1} + \mu \|\nabla \cdot (\boldsymbol{u} - \boldsymbol{s}_h)\|_0 + \|p - \pi_h\|_0.
\end{aligned}
$$

Taking the square of (39) and integrating on $(0, t)$ yields

$$
\begin{aligned}
&\int_0^t \|A_h^{-1/2} \partial_s \boldsymbol{e}_h(s)\|_0^2 \, ds \\
&\leq C \left(\int_0^t \nu^2 \|A_h^{1/2} \boldsymbol{e}_h(s)\|_0^2 \, ds + \int_0^t \|((\boldsymbol{b} \cdot \nabla) \boldsymbol{e}_h + \alpha \boldsymbol{e}_h)(s)\|_{-1}^2 \, ds \right. \\
&\quad + \mu^2 \int_0^t \|(\nabla \cdot \boldsymbol{e}_h)(s)\|_0^2 \, ds + \int_0^t \|\partial_s (\boldsymbol{u} - \boldsymbol{s}_h)(s)\|_{-1}^2 \, ds \\
&\quad + \int_0^t \|(p - \pi_h)(s)\|_0^2 \, ds + \int_0^t \|((\boldsymbol{b} \cdot \nabla)(\boldsymbol{u} - \boldsymbol{s}_h) + \alpha (\boldsymbol{u} - \boldsymbol{s}_h))(s)\|_{-1}^2 \, ds \\
&\quad \left. + \mu^2 \int_0^t \|\nabla \cdot (\boldsymbol{u} - \boldsymbol{s}_h)(s)\|_0^2 \, ds \right). \qquad (40)
\end{aligned}
$$

It will be proved that all the terms on the right-hand-side of (40) are $\mathcal{O}(h^{2k})$. To this end, it will be assumed as before that the initial error $\|\boldsymbol{e}_h(0)\|_0$ is of order $\mathcal{O}(h^k)$. Then, the desired asymptotic behavior is obtained for the first and third terms directly from (28). Comparing with the bound in (28), an extra factor $\mu = \mathcal{O}(1)$ multiplies the third term.

For the second term in (40), the definition of the $H^{-1}(\Omega)$ norm, integrating by parts and Poincaré's inequality leads to

$$\int_0^t \|((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \alpha\boldsymbol{e}_h)(s)\|_{-1}^2\, ds \le C(1+\alpha^2)\int_0^t \|\boldsymbol{e}_h(s)\|_0^2\, ds,$$

such that the desired order of convergence can be again deduced from (28). Concerning $\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1}$, the definition of the $H^{-1}(\Omega)$ norm and Poincaré's inequality are applied to bound this term by $C\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_0$. Now, (22) is applied (see Remark 1) and with hypothesis (27), the estimate for $\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1}$ is $\mathcal{O}(h^k)$. Once this term is bounded, it is clear that the integral of its square is also bounded

$$\int_0^t \|\partial_s(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_{-1}^2\, ds \le Ch^{2k}\|\partial_t\boldsymbol{u}\|_{L^2(0,T;H^k)}^2.$$

To bound $\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{s}_h)\|_0$, estimate (22) is used again. Arguing as for the second term, one obtains

$$\begin{aligned}&\int_0^t \|((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h)+\alpha(\boldsymbol{u}-\boldsymbol{s}_h))(s)\|_{-1}^2\, ds\\ &\le C(1+\alpha^2)\int_0^t \|(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_0^2\, ds.\end{aligned}$$

The bound for this term is concluded by applying (22). Combining the estimates for (40) with (33), and recalling that $\alpha = 1/T$, it follows that

$$\int_0^t \|\partial_s(\boldsymbol{e}_h)(s)\|_{-1}^2\, ds = \mathcal{O}(h^{2k}). \tag{41}$$

Theorem 2 *Let the assumptions of Theorem 1 hold, let $\nu \le 1$ and $l = k-1$, then*

$$\begin{aligned}\|p-p_h\|_{L^2(0,t;L^2)}^2 &\le C\,(1+\mu)\,h^{2k}\left(\|\boldsymbol{u}_0\|_k^2+\|\boldsymbol{u}(t)\|_k^2\right)\\ &+ Ch^{2k}\Big((1+\mu)\,(T+\mu)\|\boldsymbol{u}\|_{L^2(0,t;H^{k+1})}^2 + T(1+\mu)\|\partial_t\boldsymbol{u}\|_{L^2(0,t;H^k)}^2\Big)\\ &+ C\left(1+\mu^{-1}\right)h^{2l+2}\|p\|_{L^2(0,t;H^{l+1})}^2,\end{aligned} \tag{42}$$

where $C = C\left(\beta_0^{-1}, \|\boldsymbol{b}\|_{L^\infty(0,t;L^\infty)}\right)$.

Proof Using the discrete inf-sup condition (7) and (25), one obtains

$$\begin{aligned}\beta_0\|p_h-\pi_h\|_0 \le{}& \nu\|\nabla \boldsymbol{e}_h\|_0 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h+\alpha\boldsymbol{e}_h\|_{-1} + \|\partial_t\boldsymbol{e}_h\|_{-1} + \mu\|\nabla\cdot\boldsymbol{e}_h\|_0\\ &+\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1} + \|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h)+\alpha(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1}\\ &+\mu\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{s}_h)\|_0 + \|p-\pi_h\|_0 + \|l_h\|_0.\end{aligned}$$

Squaring both sides of this inequality and integrating on $(0,t)$ one has

$$\begin{aligned}&\beta_0^2\int_0^t \|(p_h-\pi_h)(s)\|_0^2\,ds\\ &\le C\left(\int_0^t \nu^2\|\nabla\boldsymbol{e}_h(s)\|_0^2\,ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h+\alpha\boldsymbol{e}_h)(s)\|_{-1}^2\,ds\right.\\ &\quad+\int_0^t \|\partial_s(\boldsymbol{e}_h)(s)\|_{-1}^2\,ds + \mu^2\int_0^t \|(\nabla\cdot\boldsymbol{e}_h)(s)\|_0^2\,ds\\ &\quad+\int_0^t \|\partial_s(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_{-1}^2\,ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h)+\alpha(\boldsymbol{u}-\boldsymbol{s}_h))(s)\|_{-1}^2\,ds\\ &\quad\left.+\mu^2\int_0^t \|\nabla\cdot(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_0^2\,ds + \int_0^t \|(p-\pi_h)(s)\|_0^2\,ds + \int_0^t \|l_h(s)\|_0^2\right).\end{aligned}$$

Arguing exactly as for the estimates of the right-hand side of (40), using estimates (41) for $\int_0^t \|\partial_s(\boldsymbol{e}_h)(s)\|_{-1}^2\,ds$, (23) to bound the last term, and finally the triangle inequality, then (42) is proved.

Remark 5 Neither the analysis of this section nor the analysis of the fully discrete case in next section require the assumption $Q_h \subset H^1(\Omega)$.

In addition, the analysis works for inf-sup stable divergence-free pairs of finite element spaces, like the Scott–Vogelius pair on barycenter-refined grids, without grad-div stabilization. Proving the error bound for the velocity with such elements, the term $(\nabla p, \boldsymbol{e}_h) = -(p, \nabla\cdot\boldsymbol{e}_h)$ in (25) vanishes since $\nabla\cdot\boldsymbol{e}_h = 0$. To get the error bound for the pressure, the term $(p-\pi_h, \nabla\cdot\boldsymbol{v}_h)$ vanishes when (34) is projected onto V_h^{div}. Thus, in the case of stable divergence-free elements estimates with ν-independent constant (for a sufficiently smooth solution) are also achieved.

4 Fully discrete cases

This section studies fully discrete cases. First, in Section 4.1, the backward Euler scheme is considered as temporal discretization and then BDF2 in Section 4.2. Finally, the Crank–Nicolson scheme is analyzed in Section 4.3. Error estimates for both velocity and pressure errors are derived.

Consider a decomposition of the time interval $[0,T]$ with equidistant steps τ such that $0 = t_0 < t_1 \ldots < t_N = T$ and $t_n = t_{n-1} + \tau$, $n = 1, \ldots, N$.

4.1 Backward Euler method

The backward Euler method, together with an inf-sup stable finite element discretization, applied to (1) reads as follows: Find $(\widetilde{\boldsymbol{U}}_h^n, \widetilde{P}_h^n) \in V_h \times Q_h$ such that for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$\left(\frac{\widetilde{\boldsymbol{U}}_h^n - \widetilde{\boldsymbol{U}}_h^{n-1}}{\tau}, \boldsymbol{v}_h\right) + \nu(\nabla\widetilde{\boldsymbol{U}}_h^n, \nabla\boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)\widetilde{\boldsymbol{U}}_h^n, \boldsymbol{v}_h^n) - (\nabla\cdot\boldsymbol{v}_h, \widetilde{P}_h^n)$$
$$+(\nabla\cdot\widetilde{\boldsymbol{U}}_h^n, q_h) + \mu(\nabla\cdot\widetilde{\boldsymbol{U}}_h^n, \nabla\cdot\boldsymbol{v}_h) = (\tilde{\boldsymbol{f}}^n, \boldsymbol{v}_h), \quad n = 1, \ldots, N, \tag{43}$$

where $\widetilde{\boldsymbol{U}}_h^0$ is a finite element approximation of $\tilde{\boldsymbol{u}}_0$. Using the notation $(\boldsymbol{U}_h^n, P_h^n) = e^{-\alpha t_n}(\widetilde{\boldsymbol{U}}_h^n, \widetilde{P}_h^n)$, a direct calculation yields

$$\begin{aligned} e^{-\alpha t_n}(\widetilde{\boldsymbol{U}}_h^n - \widetilde{\boldsymbol{U}}_h^{n-1}) &= \boldsymbol{U}_h^n - \boldsymbol{U}_h^{n-1} + \alpha^*\tau\boldsymbol{U}_h^{n-1} \\ &= \boldsymbol{U}_h^n - \boldsymbol{U}_h^{n-1} + \alpha^*\tau\boldsymbol{U}_h^n - \alpha^*\tau(\boldsymbol{U}_h^n - \boldsymbol{U}_h^{n-1}), \end{aligned}$$

where $\alpha^* = (1 - e^{-\alpha\tau})/\tau = \alpha + C\tau$. Using this relation in (43) it follows that $(\boldsymbol{U}_h^n, P_h^n)$ satisfies

$$\begin{aligned} &\left(\frac{\boldsymbol{U}_h^n - \boldsymbol{U}_h^{n-1}}{\tau}, \boldsymbol{v}_h\right) + \nu(\boldsymbol{U}_h^n, \nabla\boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{U}_h^n, \boldsymbol{v}_h^n) + (\alpha^*\boldsymbol{U}_h^n, \boldsymbol{v}_h) \\ &\quad - \alpha^*(\boldsymbol{U}_h^n - \boldsymbol{U}_h^{n-1}, \boldsymbol{v}_h) - (\nabla\cdot\boldsymbol{v}_h, P_h^n) + (\nabla\cdot\boldsymbol{U}_h^n, q_h) + \mu(\nabla\cdot\boldsymbol{U}_h^n, \nabla\cdot\boldsymbol{v}_h) \\ &= (\boldsymbol{f}^n, \boldsymbol{v}_h), \quad n = 1, \ldots, N, \end{aligned} \tag{44}$$

for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$. Let $\boldsymbol{s}_h^n = \boldsymbol{s}_h(t_n)$ be the solution of the discrete Stokes problem (17) with right-hand side (21) corresponding to t_n and denote $\boldsymbol{e}_h^n = \boldsymbol{U}_h^n - \boldsymbol{s}_h^n \in V_h^{\mathrm{div}}$. Subtracting the equation for $\boldsymbol{s}_h^n$ from (44) gives

$$\begin{aligned} &\left(\frac{\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}}{\tau}, \boldsymbol{v}_h\right) + \nu(\nabla\boldsymbol{e}_h^n, \nabla\boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h^n, \boldsymbol{v}_h) + (\alpha^*\boldsymbol{e}_h^n, \boldsymbol{v}_h) \\ &\quad -(\nabla\cdot\boldsymbol{v}_h, P_h^n - l_h^n) + (\nabla\cdot\boldsymbol{e}_h^n, q_h) + \mu(\nabla\cdot\boldsymbol{e}_h^n, \nabla\cdot\boldsymbol{v}_h) - \alpha^*(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}, \boldsymbol{v}_h) \\ &= \left(\partial_t\boldsymbol{s}_h^n - \frac{\boldsymbol{s}_h^n - \boldsymbol{s}_h^{n-1}}{\tau}, \boldsymbol{v}_h\right) + (\partial_t(\boldsymbol{u}^n - \boldsymbol{s}_h^n), \boldsymbol{v}_h) \\ &\quad +((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^n - \boldsymbol{s}_h^n) + (\alpha\boldsymbol{u}^n - \alpha^*\boldsymbol{s}_h^{n-1}), \boldsymbol{v}_h) \\ &\quad +\mu(\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{s}_h^n), \nabla\cdot\boldsymbol{v}_h) + (\nabla p^n, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in V_h,\ q_h \in Q_h. \end{aligned} \tag{45}$$

A direct calculation shows that

$$(\boldsymbol{e}_h^n - \boldsymbol{e}_n^{n-1}, \boldsymbol{e}_h^n) = \frac{1}{2}\|\boldsymbol{e}_h^n\|_0^2 - \frac{1}{2}\|\boldsymbol{e}_h^{n-1}\|_0^2 + \frac{1}{2}\|\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\|_0^2.$$

Using this relation, taking $\boldsymbol{v}_h = \boldsymbol{e}_h^n$ in (45), using (4), $\boldsymbol{e}_h^n \in V_h^{\mathrm{div}}$, integration by parts, the Cauchy–Schwarz and Young's inequality yields

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\|\boldsymbol{e}_h^n\|_0^2 - \|\boldsymbol{e}_h^{n-1}\|_0^2 + \|\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\|_0^2\right) + \nu\|\nabla \boldsymbol{e}_h^n\|_0^2 + \alpha^*\|\boldsymbol{e}_h^n\|_0^2 + \mu\|\nabla\cdot\boldsymbol{e}_h^n\|_0^2 \\
&\quad\le \alpha^*(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}, \boldsymbol{e}_h^n) + \frac{1}{\alpha^*}\left\|\partial_t \boldsymbol{s}_h^n - \frac{\boldsymbol{s}_h^n - \boldsymbol{s}_h^{n-1}}{\tau}\right\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^n\|_0^2 \\
&\qquad + \frac{1}{\alpha^*}\|\partial_t(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^n\|_0^2 \qquad (46) \\
&\qquad + \frac{1}{\alpha^*}\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^n - \boldsymbol{s}_h^n) + (\alpha\boldsymbol{u}^n - \alpha^*\boldsymbol{s}_h^{n-1})\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^n\|_0^2 \\
&\qquad + \mu\|\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_0^2 + \frac{\mu}{4}\|\nabla\cdot\boldsymbol{e}_h^n\|_0^2 + \frac{1}{\mu}\|p^n - \pi_h^n\|_0^2 + \frac{\mu}{4}\|\nabla\cdot\boldsymbol{e}_h^n\|_0^2.
\end{aligned}
$$

With the notations

$$
\begin{aligned}
T_1^n &= \frac{1}{\alpha^*}\|\partial_t(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_0^2, \quad T_2^n = \frac{1}{\alpha^*}\left\|\partial_t \boldsymbol{s}_h^n - \frac{\boldsymbol{s}_h^n - \boldsymbol{s}_h^{n-1}}{\tau}\right\|_0^2, \\
T_3^n &= \frac{1}{\alpha^*}\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^n - \boldsymbol{s}_h^n) + (\alpha\boldsymbol{u}^n - \alpha^*\boldsymbol{s}_h^{n-1})\|_0^2, \\
T_4^n &= \mu\|\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_0^2 + \frac{1}{\mu}\|p^n - \pi_h^n\|_0^2,
\end{aligned}
$$

estimate (46) can be written in the form

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\|\boldsymbol{e}_h^n\|_0^2 - \|\boldsymbol{e}_h^{n-1}\|_0^2 + \|\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\|_0^2\right) + \nu\|\nabla \boldsymbol{e}_h^n\|_0^2 + \frac{\mu}{2}\|\nabla\cdot\boldsymbol{e}_h^n\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^n\|_0^2 \\
&\qquad \le \alpha^*(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}, \boldsymbol{e}_h^n) + T_1^n + T_2^n + T_3^n + T_4^n.
\end{aligned}
$$

Observing that

$$
\alpha^*(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}, \boldsymbol{e}_h^n) \le \frac{\|\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\|_0^2}{2\tau} + \frac{\tau}{2}(\alpha^*)^2\|\boldsymbol{e}_h^n\|_0^2,
$$

and assuming

$$
\frac{\tau(\alpha^*)^2}{2} < \frac{\alpha^*}{8} \quad \Longleftrightarrow \quad \tau < \frac{1}{4\alpha^*} = \frac{\tau}{4(1 - e^{-\alpha\tau})}, \qquad (47)
$$

one gets

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\|\boldsymbol{e}_h^n\|_0^2 - \|\boldsymbol{e}_h^{n-1}\|_0^2\right) + \nu\|\nabla \boldsymbol{e}_h^n\|_0^2 + \frac{\mu}{2}\|\nabla\cdot\boldsymbol{e}_h^n\|_0^2 + \frac{\alpha^*}{8}\|\boldsymbol{e}_h^n\|_0^2 \\
&\qquad \le T_1^n + T_2^n + T_3^n + T_4^n. \qquad (48)
\end{aligned}
$$

Observe that (47) holds if $\tau < \log(4/3)T$, since $\alpha = 1/T$.

After summation of the discrete times $j=1,\dots,n$, one obtains

$$\begin{aligned}&\|\boldsymbol{e}_h^n\|_0^2+\tau\sum_{j=1}^n\left(2\nu\|\nabla\boldsymbol{e}_h^j\|_0^2+\mu\|\nabla\cdot\boldsymbol{e}_h^j\|_0^2+\frac{\alpha^*}{4}\|\boldsymbol{e}_h^j\|_0^2\right)\\&\le\|\boldsymbol{e}_h^0\|_0^2+2\tau\sum_{j=1}^n\left(T_1^j+T_2^j+T_3^j+T_4^j\right).\end{aligned}\tag{49}$$

The terms on the right-hand side have to be bounded. Applying (22) together with Remark 1 and recalling that $\alpha^*\approx\alpha=1/T$ yields

$$T_1^j\le CTh^{2k}\|\partial_t\boldsymbol{u}(t_j)\|_k^2,\qquad j=1,\dots,N.\tag{50}$$

To bound T_2^j, a sequence of inequalities is used applying a Taylor series expansion with reminder in integral form, the Cauchy–Schwarz inequality, the triangle inequality, Remark 1 for the second derivative, and the Poincaré–Friedrichs inequality (5)

$$\begin{aligned}T_2^j&\le CT\left\|\partial_t\boldsymbol{s}_h^j-\frac{\boldsymbol{s}_h^j-\boldsymbol{s}_h^{j-1}}{\tau}\right\|_0^2=CT\frac{1}{\tau^2}\left\|\int_{t_{j-1}}^{t_j}(t-t_{j-1})\partial_{tt}\boldsymbol{s}_h\ dt\right\|_0^2\\&\le CT\frac{1}{\tau^2}\left(\int_{t_{j-1}}^{t_j}(t-t_{j-1})^2\ dt\right)\int_{t_{j-1}}^{t_j}\|\partial_{tt}\boldsymbol{s}_h\|_0^2\ dt\\&\le CT\tau\int_{t_{j-1}}^{t_j}\|\partial_{tt}\boldsymbol{s}_h\|_0^2\ dt\le CT\tau\int_{t_{j-1}}^{t_j}\|\partial_{tt}\boldsymbol{u}\|_1^2\ dt,\quad j=1,\dots,N.\end{aligned}\tag{51}$$

The first part of T_3^j is bounded by applying (22)

$$\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^j-\boldsymbol{s}_h^j)\|_0^2\le Ch^{2k}\|\boldsymbol{u}(t_j)\|_{k+1}^2,\qquad j=1,\dots,N.\tag{52}$$

To bound $\|(\alpha\boldsymbol{u}^n-\alpha^*\boldsymbol{s}_h^{n-1})\|_0^2$ one can proceed as follows. Denoting $\tilde{\boldsymbol{s}}_h(t)=e^{\alpha t}\boldsymbol{s}_h(t)$, a direct calculation leads to the representation

$$\begin{aligned}\alpha^*\boldsymbol{s}_h^{n-1}&=e^{-\alpha t_n}\frac{\tilde{\boldsymbol{s}}_h^n-\tilde{\boldsymbol{s}}_h^{n-1}}{\tau}-\frac{\boldsymbol{s}_h^n-\boldsymbol{s}_h^{n-1}}{\tau}=e^{-\alpha t_n}\partial_t\tilde{\boldsymbol{s}}_h^n-\partial_t\boldsymbol{s}_h^n+T_5^n+T_6^n\\&=\alpha\boldsymbol{s}_h^n+T_5^n+T_6^n,\end{aligned}$$

where

$$T_5^n=e^{-\alpha t_n}\left(\frac{\tilde{\boldsymbol{s}}_h^n-\tilde{\boldsymbol{s}}_h^{n-1}}{\tau}-\partial_t\tilde{\boldsymbol{s}}_h^n\right),\quad T_6^n=\partial_t\boldsymbol{s}_h^n-\frac{\boldsymbol{s}_h^n-\boldsymbol{s}_h^{n-1}}{\tau}.$$

Since $\|T_5^n+T_6^n\|_0$ can be bounded similarly to T_2^n, for $n=1,\dots,N$, one gets

$$\begin{aligned}&\|(\alpha\boldsymbol{u}^n-\alpha^*\boldsymbol{s}_h^{n-1})\|_0^2\\&\le\|\alpha(\boldsymbol{u}^n-\boldsymbol{s}_h^n)\|_0^2+C\tau\left(e^{-2\alpha t_n}\int_{t_{n-1}}^{t_n}\|\partial_{tt}\tilde{\boldsymbol{u}}\|_1^2\ dt+\int_{t_{n-1}}^{t_n}\|\partial_{tt}\boldsymbol{u}\|_1^2\ dt\right)\end{aligned}\tag{53}$$

Together with (52) it follows that, for $j = 1, \ldots, N$,

$$T_3^j \le CTh^{2k}\|\boldsymbol{u}(t_j)\|_{k+1}^2 + CT\tau \int_{t_{j-1}}^{t_j} \left(\|\partial_{tt}\tilde{\boldsymbol{u}}\|_1^2 + \|\partial_{tt}\boldsymbol{u}\|_1^2\right)\, dt.$$

Finally, applying (22) gives

$$\mu\|\nabla\cdot(\boldsymbol{u}^j - \boldsymbol{s}_h^j)\|_0^2 \le C\mu h^{2k}\|\boldsymbol{u}(t_j)\|_{k+1}^2.$$

Likewise, using (14) yields

$$\frac{1}{\mu}\|p^j - \pi_h^j\|_0^2 \le C\frac{h^{2k}}{\mu}\|p(t_j)\|_k^2,$$

such that

$$T_4^j \le Ch^{2k}\left(\mu\|\boldsymbol{u}(t_j)\|_{k+1}^2 + \mu^{-1}\|p(t_j)\|_k^2\right), \qquad j = 1, \ldots, N. \tag{54}$$

Collecting all estimates, one gets from (49) for $n = 1, \ldots, N$,

$$\begin{aligned}
&\|\boldsymbol{e}_h^n\|_0^2 + \tau\sum_{j=1}^n\Big(2\nu\|\nabla\boldsymbol{e}_h^j\|_0^2 + \mu\|\nabla\cdot\boldsymbol{e}_h^j\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^j\|_0^2\Big) \\
&\le \|\boldsymbol{e}_h^0\|_0^2 + CT\tau^2\left(\|\partial_{tt}\tilde{\boldsymbol{u}}\|_{L^2(0,t_n;H^1)}^2 + \|\partial_{tt}\boldsymbol{u}\|_{L^2(0,t_n;H^1)}^2\right) \\
&\quad + Ch^{2k}(T+\mu)\tau\sum_{j=1}^n\|\boldsymbol{u}(t_j)\|_{k+1}^2 \\
&\quad + Ch^{2k}\left(\mu^{-1}\tau\sum_{k=1}^n\|p(t_j)\|_k^2 + T\tau\sum_{j=1}^n\|\partial_t\boldsymbol{u}(t_j)\|_k^2\right).
\end{aligned} \tag{55}$$

Theorem 3 *Let $(\tilde{\boldsymbol{u}}, \tilde{p}) \in V \times Q$ be the solution of (1) and let $(\widetilde{\boldsymbol{U}}_h^n, \widetilde{P}_h^n) \in V_h \times Q_h$ be the backward Euler approximation solving (43) for $n = 1, \ldots, N$. Assume that $\|\boldsymbol{e}_h^0\|_0 = \mathcal{O}(h^k)$, $l = k-1$, $\nu \le 1$, $\tau < \log(4/3)T$ (so that (47) holds) and that the solution is sufficiently regular such that all norms appearing in the formulation of this theorem are well defined. Then, the following error estimate holds*

$$\begin{aligned}
&\left\|\tilde{\boldsymbol{u}}(t_n) - \widetilde{\boldsymbol{U}}_h^n\right\|_0^2 + \tau\sum_{j=1}^n\left(\nu\left\|\nabla\left(\tilde{\boldsymbol{u}}(t_j) - \widetilde{\boldsymbol{U}}_h^j\right)\right\|_0^2 + \mu\left\|\nabla\cdot\left(\tilde{\boldsymbol{u}}(t_j) - \widetilde{\boldsymbol{U}}_h^j\right)\right\|_0^2\right) \\
&\le Ch^{2k}\left(\|\tilde{\boldsymbol{u}}_0\|_k^2 + \|\tilde{\boldsymbol{u}}(t_n)\|_k^2\right) + CT\tau^2\|\tilde{\boldsymbol{u}}\|_{H^2(0,t_n;H^1)}^2 \\
&\quad + Ch^{2k}(T+\mu)\tau\sum_{j=1}^n\|\tilde{\boldsymbol{u}}(t_j)\|_{k+1}^2 \\
&\quad + Ch^{2k}\left(T\tau\sum_{j=1}^n\|\partial_t\tilde{\boldsymbol{u}}(t_j)\|_k^2 + \mu^{-1}\tau\sum_{j=1}^n\|\tilde{p}(t_j)\|_k^2\right).
\end{aligned} \tag{56}$$

Proof Estimate (56) is obtained applying the triangle inequality and using (22) and (55).

The bound for the error in the pressure can be obtained along the lines of the proof for the time-continuous case. Considering (45) and simplifying the terms $(\alpha^* \boldsymbol{e}_h^n, \boldsymbol{v}_h) - \alpha^*(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}, \boldsymbol{v}_h)$ to $(\alpha^* \boldsymbol{e}_h^{n-1}, \boldsymbol{v}_h)$, then one obtains with the the discrete inf-sup condition (7)

$$\begin{aligned}
&\beta_0 \|P_h^n - \pi_h^n\|_0 \\
&\le \nu \|\nabla \boldsymbol{e}_h^n\|_0 + \|\boldsymbol{b}\cdot\nabla \boldsymbol{e}_h^n + \alpha^* \boldsymbol{e}_h^{n-1}\|_{-1} + \left\| \frac{\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}}{\tau} \right\|_{-1} + \mu \|\nabla\cdot \boldsymbol{e}_h^n\|_0 \\
&\quad + \|\partial_t(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_{-1} + \left\| \partial_t \boldsymbol{s}_h^n - \frac{\boldsymbol{s}_h^n - \boldsymbol{s}_h^{n-1}}{\tau} \right\|_{-1} + \|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_{-1} \\
&\quad + \|\alpha \boldsymbol{u}^n - \alpha^* \boldsymbol{s}_h^{n-1}\|_{-1} + \mu \|\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_0 + \|p^n - \pi_h^n\|_0 + \|l_h^n\|_0, \qquad (57)
\end{aligned}$$

and, consequently

$$\begin{aligned}
&\beta_0^2 \tau \sum_{j=1}^n \|P_h^j - \pi_h^j\|_0^2 \\
&\le C \Bigg[\nu^2 \tau \sum_{j=1}^n \|\nabla \boldsymbol{e}_h^j\|_0^2 + \tau \sum_{j=1}^n \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h^j\|_{-1}^2 + \tau \sum_{j=1}^n \|\alpha^* \boldsymbol{e}_h^{j-1}\|_0^2 \\
&\quad + \tau \sum_{j=1}^n \left\| \frac{\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1}}{\tau} \right\|_{-1}^2 + \mu^2 \tau \sum_{j=1}^n \|\nabla\cdot \boldsymbol{e}_h^j\|_0^2 + \tau \sum_{j=1}^n \|\partial_t(\boldsymbol{u}^j - \boldsymbol{s}_h^j)\|_{-1}^2 \\
&\quad + \tau \sum_{j=1}^n \left\| \partial_t \boldsymbol{s}_h^j - \frac{\boldsymbol{s}_h^j - \boldsymbol{s}_h^{j-1}}{\tau} \right\|_{-1}^2 + \tau \sum_{j=1}^n \|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^j - \boldsymbol{s}_h^j)\|_{-1}^2 \\
&\quad + \tau \sum_{j=1}^n \|\alpha \boldsymbol{u}^j - \alpha^* \boldsymbol{s}_h^{j-1}\|_{-1}^2 + \tau\mu^2 \sum_{j=1}^n \|\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_0^2 \\
&\quad + \tau \sum_{j=1}^n \|p^j - \pi_h^j\|_0^2 + \tau \sum_{j=1}^n \|l_h^j\|_0^2 \Bigg]. \qquad (58)
\end{aligned}$$

The first, third, and fifth terms on the right-hand side above are already bounded in (55). Since the bound $\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h^j\|_{-1}^2 \le C\|\boldsymbol{e}_h^j\|_0^2$ is satisfied, one can also apply (55) to estimate this term. For terms ranging from the sixth to the ninth on the right-hand side of (58), one can first bound $\|\cdot\|_{-1}$ by $\|\cdot\|_0$ and then apply the bounds (50) to (53). To bound the tenth term on the right-hand side above, (54) is used. For the last two terms one can apply (14) and (23). Altogether, to conclude the estimate, it only remains to bound the fourth term on the right-hand side of (58). The estimate of this term follows with the same arguments as in the time-continuous case. First, one applies

$$\left\| \frac{\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1}}{\tau} \right\|_{-1} \le C \left\| A_h^{-1/2} \left(\frac{\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1}}{\tau} \right) \right\|_0$$

and then one derives the estimate

$$
\begin{aligned}
&\left\| A_h^{-1/2}\left(\frac{\boldsymbol{e}_h^j-\boldsymbol{e}_h^{j-1}}{\tau}\right)\right\|_0 \\
&\le \nu\|A_h^{1/2}\boldsymbol{e}_h^j\|_0+\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h^j+\alpha^*\boldsymbol{e}_h^{j-1}\|_{-1}+\mu\|\nabla\cdot\boldsymbol{e}_h^j\|_0 \\
&\quad+\|\partial_t(\boldsymbol{u}^j-\boldsymbol{s}_h^j)\|_{-1}+\left\|\partial_t\boldsymbol{s}_h^n-\frac{\boldsymbol{s}_h^n-\boldsymbol{s}_h^{n-1}}{\tau}\right\|_{-1}+\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^j-\boldsymbol{s}_h^j)\|_{-1} \\
&\quad+\|(\alpha\boldsymbol{u}^j-\alpha^*\boldsymbol{s}_h^{j-1})\|_{-1}+\mu\|\nabla\cdot\boldsymbol{s}_h^j\|_0+\|p^j-\pi_h^j\|_0.
\end{aligned}
$$

From this inequality, one gets the bound for the fourth term on the right-hand side of (58) in the same way as for the other terms.

Theorem 4 *Let the assumptions of Theorem 3 hold, then the following error estimate for the pressure is valid*

$$
\begin{aligned}
\tau\sum_{j=1}^n\left\|\widetilde{P}_h^j-\tilde{p}(t_j)\right\|^2 &\le Ch^{2k}\|\tilde{\boldsymbol{u}}_0\|_k^2+CT(1+\mu)\tau^2\|\tilde{\boldsymbol{u}}\|_{H^2(0,t_n;H^1)}^2 \\
&\quad+Ch^{2k}(T+\mu)(1+\mu)\tau\sum_{j=1}^n\|\tilde{\boldsymbol{u}}(t_j)\|_{k+1}^2 \qquad (59)\\
&\quad+Ch^{2k}\Big(T(1+\mu)\tau\sum_{j=1}^n\|\partial_t\tilde{\boldsymbol{u}}(t_j)\|_k^2+(1+\mu^{-1})\tau\sum_{j=1}^n\|\tilde{p}(t_j)\|_k^2\Big).
\end{aligned}
$$

4.2 BDF2

As in the case of the backward Euler method, $(\widetilde{\boldsymbol{U}}_h^n,\widetilde{P}_h^n)$ denotes the fully discrete approximation at time t_n, $n=2,\ldots,N$, which satisfies for BDF2

$$
\begin{aligned}
&\left(\frac{1}{\tau}\Big(D+\frac{1}{2}D^2\Big)\widetilde{\boldsymbol{U}}_h^n,\boldsymbol{v}_h\right)+\nu(\nabla\widetilde{\boldsymbol{U}}_h^n,\nabla\boldsymbol{v}_h)+((\boldsymbol{b}\cdot\nabla)\widetilde{\boldsymbol{U}}_h^n,\boldsymbol{v}_h)-(\nabla\cdot\boldsymbol{v}_h,\widetilde{P}_h^n) \\
&\quad+(\nabla\cdot\widetilde{\boldsymbol{U}}_h^n,q_h)+\mu(\nabla\cdot\widetilde{\boldsymbol{U}}_h^n,\nabla\cdot\boldsymbol{v}_h)=(\tilde{\boldsymbol{f}}^n,\boldsymbol{v}_h). \qquad (60)
\end{aligned}
$$

Here, D is the backward difference for a sequence $(y_n)_{n=1}^N$, i.e., $Dy_n=y_n-y_{n-1}$, $n=1,\ldots,N$. Note that $D^2y_n=y_n-2y_{n-1}+y_{n-2}$, for $n=2,\ldots,N$, such that

$$
\left(D+\frac{1}{2}D^2\right)\widetilde{\boldsymbol{U}}_h^n=\frac{3}{2}\widetilde{\boldsymbol{U}}_h^n-2\widetilde{\boldsymbol{U}}_h^{n-1}+\frac{1}{2}\widetilde{\boldsymbol{U}}_h^{n-2},\qquad n=2,\ldots,N.
$$

It will be assumed that $\widetilde{\boldsymbol{U}}_h^1$ is obtained by one step with the backward Euler method and that $\widetilde{\boldsymbol{U}}_h^0$ is an appropriate finite element approximation of $\tilde{\boldsymbol{u}}_0$.

Recall that $(\boldsymbol{U}_h^n, P_h^n) = e^{-\alpha t_n}(\widetilde{\boldsymbol{U}}_h^n, \widetilde{P}_h^n)$, such that a straightforward calculation yields

$$
\begin{aligned}
e^{-\alpha t_n}\Big(\frac{3}{2}\widetilde{\boldsymbol{U}}_h^n - 2\widetilde{\boldsymbol{U}}_h^{n-1} + \frac{1}{2}\widetilde{\boldsymbol{U}}_h^{n-2}\Big) =& \Big(\frac{3}{2}\boldsymbol{U}_h^n - 2\boldsymbol{U}_h^{n-1} + \frac{1}{2}\boldsymbol{U}_h^{n-2}\Big)\\
&+ \tau\Big(2\alpha^*\boldsymbol{U}_h^{n-1} - \frac{1}{2}\alpha^{**}\boldsymbol{U}_h^{n-2}\Big), \qquad (61)
\end{aligned}
$$

where $\alpha^* = (1 - e^{-\alpha\tau})/\tau$, as in the case of the backward Euler method, and $\alpha^{**} = (1 - e^{-2\alpha\tau})/\tau$. For the last two terms on the right-hand side one can write

$$
\begin{aligned}
&2\alpha^*\boldsymbol{U}_h^{n-1} - \frac{1}{2}\alpha^{**}\boldsymbol{U}_h^{n-2}\\
&= \alpha^*\boldsymbol{U}_h^n - \alpha^*\left(\boldsymbol{U}_h^n - 2\boldsymbol{U}_h^{n-1} + \boldsymbol{U}_h^{n-2}\right) + \Big(\alpha^* - \frac{1}{2}\alpha^{**}\Big)\boldsymbol{U}_h^{n-2}\\
&= \alpha^*\boldsymbol{U}_h^n - \alpha^* D^2\boldsymbol{U}_h^n + \Big(\alpha^* - \frac{1}{2}\alpha^{**}\Big)\boldsymbol{U}_h^{n-2}.
\end{aligned}
$$

Now, observe that $\tau(\alpha^* - \alpha^{**}/2) = 1/2 - e^{-\alpha\tau} + 1/2e^{-\alpha 2\tau}$ that is, a second backward difference of $e^{-\alpha t}$ at $t = 0$, and, thus

$$
\tau\left(\alpha^* - \frac{1}{2}\alpha^{**}\right) = \beta\tau^2, \qquad (62)
$$

where $\beta = e^{-\alpha\xi}\alpha^2/2$ for some $\xi \in (0, 2\tau)$. Thus, (61) can be written in the form

$$
\begin{aligned}
e^{-\alpha t_n}\Big(\frac{3}{2}\widetilde{\boldsymbol{U}}_h^n - 2\widetilde{\boldsymbol{U}}_h^{n-1} + \frac{1}{2}\widetilde{\boldsymbol{U}}_h^{n-2}\Big) =& \Big(\frac{3}{2}\boldsymbol{U}_h^n - 2\boldsymbol{U}_h^{n-1} + \frac{1}{2}\boldsymbol{U}_h^{n-2}\Big) \qquad (63)\\
&+ \tau\left(\alpha^*\boldsymbol{U}_h^n - \alpha^* D^2\boldsymbol{U}_h^n + \beta\tau\boldsymbol{U}_h^{n-2}\right).
\end{aligned}
$$

Then, arguing as in the case of the backward Euler method, one finds for BDF2 that $\boldsymbol{e}_h^n = \boldsymbol{U}_h^n - \boldsymbol{s}_h^n \in V_h^{\mathrm{div}}$ satisfies

$$
\begin{aligned}
&\left(\frac{1}{\tau}\Big(D + \frac{1}{2}D^2\Big)\boldsymbol{e}_h^n, \boldsymbol{v}_h\right) + \nu(\nabla\boldsymbol{e}_h^n, \nabla\boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h^n, \boldsymbol{v}_h) + (\alpha^*\boldsymbol{e}_h^n, \boldsymbol{v}_h)\\
&\quad -(\nabla\cdot\boldsymbol{v}_h, P_h^n - l_h^n) + (\nabla\cdot\boldsymbol{e}_h^n, q_h) + \mu(\nabla\cdot\boldsymbol{e}_h^n, \nabla\cdot\boldsymbol{v}_h)\\
&\quad -\alpha^*(D^2\boldsymbol{e}_h^n, \boldsymbol{v}_h) + \beta\tau(\boldsymbol{e}_h^{n-2}, \boldsymbol{v}_h)\\
&= \left((\partial_t\boldsymbol{s}_h^n) - \frac{1}{\tau}\Big(D + \frac{1}{2}D^2\Big)\boldsymbol{s}_h^n, \boldsymbol{v}_h\right) + (\partial_t(\boldsymbol{u}^n - \boldsymbol{s}_h^n), \boldsymbol{v}_h) \qquad (64)\\
&\quad +((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^n - \boldsymbol{s}_h^n) + (\alpha\boldsymbol{u}^n - \alpha^*(\boldsymbol{s}_h^n - D^2\boldsymbol{s}_h^n) - \beta\tau\boldsymbol{s}_h^{n-2}, \boldsymbol{v}_h)\\
&\quad +\mu(\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{s}_h^n), \nabla\cdot\boldsymbol{v}_h) + (\nabla p^n, \boldsymbol{v}_h), \quad \forall\boldsymbol{v}_h \in V_h,\ q_h \in Q_h.
\end{aligned}
$$

Observe that there are the following differences between (64) and (45). In the first term on both left- and right-hand sides, the first divided differences appearing in (45) are replaced by $(D + D^2/2)/\tau$. Also, the last term on the left-hand side in (45), $(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}, \boldsymbol{v}_h)$, is replaced by the last two terms on the left-hand side in (64). The rest of the terms in (45) is the same as in

(64). Finally, in the fourth term on the right-hand side in (45), the difference $\alpha\boldsymbol{u}^n - \alpha^*\boldsymbol{s}_h^{n-1}$, is replaced by $\alpha\boldsymbol{u}^n - \alpha^*(\boldsymbol{s}_h^n - D^2\boldsymbol{s}_h^n) - \beta\tau\boldsymbol{s}_h^{n-2}$ in (64).

Due to the similarities between both expressions, now only the terms which are different to (45) will be considered in detail. Taking $\boldsymbol{v}_h = \boldsymbol{e}_h^n$ in (64), a direct calculation reveals that

$$\Big(\Big(D + \frac{1}{2}D^2\Big)\boldsymbol{e}_h^n, \boldsymbol{e}_h^n\Big) = \frac{1}{4}\|\boldsymbol{e}_h^n\|_0^2 + \frac{1}{4}\|\boldsymbol{e}_h^n + D\boldsymbol{e}_h^n\|_0^2 + \frac{1}{4}\|D^2\boldsymbol{e}_h^n\|_0^2 - \frac{1}{4}\|\boldsymbol{e}_h^{n-1}\|_0^2 - \frac{1}{4}\|\boldsymbol{e}_h^{n-1} + D\boldsymbol{e}_h^{n-1}\|_0^2. \quad (65)$$

The last two terms on the left-hand side of (64) can be estimated from below as follows

$$\begin{aligned} &-\alpha^*(D^2\boldsymbol{e}_h^n, \boldsymbol{e}_h^n) + \beta\tau(\boldsymbol{e}_h^{n-2}, \boldsymbol{e}_h^n) \\ &\geq -\frac{\|D^2\boldsymbol{e}_h^n\|_0^2}{4\tau} - \tau\Big((\alpha^*)^2 + \frac{\beta}{2}\Big)\|\boldsymbol{e}_h^n\|_0^2 - \tau\frac{\beta}{2}\|\boldsymbol{e}_h^{n-2}\|_0^2 \\ &\geq -\frac{\|D^2\boldsymbol{e}_h^n\|_0^2}{4\tau} - \frac{\alpha^*}{16}\|\boldsymbol{e}_h^n\|_0^2 - \frac{\alpha^*}{16}\|\boldsymbol{e}_h^{n-2}\|_0^2, \end{aligned} \quad (66)$$

where the last inequality is true only if τ is sufficiently small, i.e., if

$$\tau\Big(\alpha^* + \frac{\beta}{2\alpha^*}\Big) \leq \frac{1}{16} \quad (67)$$

holds. Since in view of (62), $\tau\beta/\alpha^* = 1 - \alpha^{**}/(2\alpha^*) = 1 - (1 + e^{-\alpha\tau})/2 = (1 - e^{-\alpha\tau})/2$, one finds that (67) holds if $5(1 - e^{-\alpha\tau}) \leq 1/4$, which, since $\alpha = 1/T$ is assumed, holds if $\tau < \log(20/19)T$.

For the fourth term on the right-hand side of (64), using (63) with $\boldsymbol{U}_h^j$ and $\tilde{\boldsymbol{U}}_h^j$ replaced by $\boldsymbol{s}_h^j$ and $\tilde{\boldsymbol{s}}_h^j = e^{\alpha t_j}\boldsymbol{s}_h^j$, respectively, for $j = n-2, n-1, n$, one can write

$$\alpha^*(\boldsymbol{s}_h^n - D^2\boldsymbol{s}_h^n) + \beta\tau\boldsymbol{s}_h^{n-2} = e^{-\alpha t_n}\frac{1}{\tau}\Big(D + \frac{1}{2}D^2\Big)\tilde{\boldsymbol{s}}_h^n - \frac{1}{\tau}\Big(D + \frac{1}{2}D^2\Big)\boldsymbol{s}_h^n.$$

Thus, it is

$$\begin{aligned} \alpha\boldsymbol{u}^n - \alpha^*(\boldsymbol{s}_h^n - D^2\boldsymbol{s}_h^n) - \beta\tau\boldsymbol{s}_h^{n-2} &= \alpha\boldsymbol{u}^n - e^{-\alpha t_n}\partial_t\tilde{\boldsymbol{s}}_h^n + \partial_t\boldsymbol{s}_h^n + T_5^n + T_6^n \\ &= \alpha(\boldsymbol{u}^n - \boldsymbol{s}_h^n) + T_5^n + T_6^n, \end{aligned}$$

where

$$T_5^n = e^{-\alpha t_n}\partial_t(\tilde{\boldsymbol{s}}_h^n) - e^{-\alpha t_n}\frac{1}{\tau}\Big(D + \frac{1}{2}D^2\Big)\tilde{\boldsymbol{s}}_h^n, \quad T_6^n = \frac{1}{\tau}\Big(D + \frac{1}{2}D^2\Big)\boldsymbol{s}_h^n - \partial_t\boldsymbol{s}_h^n. \quad (68)$$

Consequently, when taking $\boldsymbol{v}_h = \boldsymbol{e}_h^n$ in (64) the fourth term on the right-hand side can be bounded in the following way

$$(\alpha\boldsymbol{u}^n - \alpha^*(\boldsymbol{s}_h^n - D^2\boldsymbol{s}_h^n) - \beta\tau\boldsymbol{s}_h^{n-2}, \boldsymbol{e}_h^n) \leq \frac{\alpha^*}{16}\|\boldsymbol{e}_h^n\|_0^2 + 8\frac{\alpha^2}{\alpha^*}\|\boldsymbol{u}^n - \boldsymbol{s}_h^n\|_0^2 + T_{56}^n, \quad (69)$$

where

$$T_{56}^n = \frac{8}{\alpha^*}\|T_5^n + T_6^n\|_0^2. \tag{70}$$

Then, arguing as in the case of the backward Euler method and using (65), (66), and (69), one gets for BDF2, instead of (48),

$$\frac{1}{4\tau}\Big(\|\boldsymbol{e}_h^n\|_0^2 + \|\boldsymbol{e}_h^n + D\boldsymbol{e}_h^n\|_0^2 - \|\boldsymbol{e}_h^{n-1}\|_0^2 - \|\boldsymbol{e}_h^{n-1} + D\boldsymbol{e}_h^{n-1}\|_0^2\Big) + \nu\|\nabla\boldsymbol{e}_h^n\|_0^2$$
$$+ \frac{\mu}{2}\|\nabla\cdot\boldsymbol{e}_h^n\|_0^2 + \frac{\alpha^*}{8}\|\boldsymbol{e}_h^n\|_0^2 \le \frac{\alpha^*}{16}\|\boldsymbol{e}_h^{n-2}\|_0^2 + T_1^n + \hat{T}_2^n + \hat{T}_3^n + T_4^n + T_{56}^n, \tag{71}$$

where T_1^n and T_4^n are as in the case of the backward Euler method,

$$\hat{T}_2^n = \frac{1}{\alpha^*}\left\|\partial_t\boldsymbol{s}_h^n - \frac{1}{\tau}\Big(D + \frac{1}{2}D^2\Big)\boldsymbol{s}_h^n\right\|_0^2,$$
$$\hat{T}_3^n = \frac{1}{\alpha^*}\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_0^2 + 8\frac{\alpha^2}{\alpha^*}\|\boldsymbol{u}^n - \boldsymbol{s}_h^n\|_0^2,$$

and T_{56}^n is defined in (68) and (70).

Multiplying (71) by 4τ, summing up, and rearranging terms gives

$$\begin{aligned}\|\boldsymbol{e}_h^n\|_0^2 &+ \tau\sum_{j=2}^n\big(4\nu\|\nabla\boldsymbol{e}_h^j\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^j\|_0^2 + 2\mu\|\nabla\cdot\boldsymbol{e}_h^j\|_0^2\big)\\ &\le \|\boldsymbol{e}_h^1\|_0^2 + \|\boldsymbol{e}_h^1 + D\boldsymbol{e}_h^1\|_0^2 + \tau\frac{\alpha^*}{4}\big(\|\boldsymbol{e}_h^0\|_0^2 + \|\boldsymbol{e}_h^1\|_0^2\big)\\ &\quad + 4\tau\sum_{j=2}^n(T_1^j + \hat{T}_2^j + \hat{T}_3^j + T_4^j + T_{56}^j).\end{aligned} \tag{72}$$

A direct calculation shows that

$$\begin{aligned}&\|\boldsymbol{e}_h^1\|_0^2 + \|\boldsymbol{e}_h^1 + D\boldsymbol{e}_h^1\|_0^2 + \tau\frac{\alpha^*}{4}\left(\|\boldsymbol{e}_h^0\|_0^2 + \|\boldsymbol{e}_h^1\|_0^2\right)\\ &\quad\le \left(7 + \tau\frac{\alpha^*}{4}\right)\|\boldsymbol{e}_h^1\|_0^2 + \left(3 + \tau\frac{\alpha^*}{4}\right)\|\boldsymbol{e}_h^0\|_0^2 \le C\left(\|\boldsymbol{e}_h^1\|_0^2 + \|\boldsymbol{e}_h^0\|_0^2\right).\end{aligned} \tag{73}$$

In view of (52) and (22) one has that

$$\hat{T}_3^j \le CTh^{2k}\|\boldsymbol{u}(t_j)\|_{k+1}^2, \quad j = 2,\ldots,N. \tag{74}$$

A Taylor series expansion yields for $\hat{T}_2^j$

$$\hat{T}_2^j \le C\frac{T}{\tau^2}\left\|\int_{t_{j-2}}^{t_j}\left(2(t - t_{j-1})_+^2 - \frac{1}{2}(t - t_{j-2})^2\right)\partial_{ttt}s_h(t)\;dt\right\|^2,$$

$j = 2, \ldots, N$, where $x_+ = \max(x, 0)$ for $x \in \mathbb{R}$. With the Cauchy–Schwarz inequality, one obtains for $j = 2, \ldots, N$,

$$\begin{aligned}\hat{T}_2^j &\le C\frac{T}{\tau^2}\left(\int_{t_{j-2}}^{t_j}\left(2(t-t_{j-1})_+^2 - \frac{1}{2}(t-t_{j-2})^2\right)^2\,dt\right)\int_{t_{j-2}}^{t_j}\|\partial_{ttt}\boldsymbol{s}_h(t)\|_0^2\,dt\\ &\le CT\tau^3\int_{t_{j-2}}^{t_j}\|\partial_{ttt}\boldsymbol{s}_h(t)\|_0^2\,dt \le CT\tau^3\int_{t_{j-2}}^{t_j}\|\partial_{ttt}\boldsymbol{u}(t)\|_1^2\,dt. \qquad (75)\end{aligned}$$

Since a similar bound is valid for T_{56}^j, and the bounds (50) and (54) on T_1^n and T_4^n computed in the case of backward Euler method also apply in the present case, from (72) – (75), it follows for $n = 2, \ldots, N$, that

$$\begin{aligned}&\|\boldsymbol{e}_h^n\|_0^2 + \tau\sum_{j=2}^n\Big(4\nu\|\nabla\boldsymbol{e}_h^j\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^j\|_0^2 + 2\mu\|\nabla\cdot\boldsymbol{e}_h^j\|_0^2\Big)\\ &\le C\left(\|\boldsymbol{e}_h^1\|_0^2 + \|\boldsymbol{e}_h^0\|_0^2\right) + CT\tau^4\left(\|\partial_{ttt}\boldsymbol{u}\|_{L^2(0,t_n;H^1)}^2 + \|\partial_{ttt}\tilde{\boldsymbol{u}}\|_{L^2(0,t_n;H^1)}^2\right)\\ &\quad + Ch^{2k}\Big((T+\mu)\tau\sum_{j=1}^n\|\boldsymbol{u}(t_j)\|_{k+1}^2 + T\tau\sum_{j=1}^n\|\partial_t\boldsymbol{u}(t_j)\|_k^2\Big) \qquad (76)\\ &\quad + Ch^{2k}\frac{\tau}{\mu}\sum_{j=1}^n\|p(t_j)\|_k^2.\end{aligned}$$

Finally, one has to show that $\left\|\boldsymbol{e}_h^1\right\|_0^2 = \mathcal{O}(h^{2k} + \tau^4)$. The first step is performed with the backward Euler scheme. To prove the needed order with respect to time, it will be exploited that the length of the first time interval is just τ. Starting from (45) with $n = 1$ and taking $\boldsymbol{v}_h = \boldsymbol{e}_h^1$, the first and third term on the right-hand side of (45) are estimated with the Cauchy–Schwarz and Young's inequality

$$\left(\partial_t\boldsymbol{s}_h^1 - \frac{\boldsymbol{s}_h^1 - \boldsymbol{s}_h^0}{\tau}, \boldsymbol{e}_h^1\right) \le 2\tau\left\|\partial_t\boldsymbol{s}_h^1 - \frac{\boldsymbol{s}_h^1 - \boldsymbol{s}_h^0}{\tau}\right\|_0^2 + \frac{1}{8\tau}\|\boldsymbol{e}_h^1\|_0^2,$$

and

$$\begin{aligned}&((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^1 - \boldsymbol{s}_h^1) + (\alpha\boldsymbol{u}^1 - \alpha^*\boldsymbol{s}_h^0), \boldsymbol{e}_h^1)\\ &\le CT\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^1 - \boldsymbol{s}_h^1)\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^1\|_0^2 + 2\tau\|\alpha\boldsymbol{u}^1 - \alpha^*\boldsymbol{s}_h^0\|_0^2 + \frac{1}{8\tau}\|\boldsymbol{e}_h^1\|_0^2.\end{aligned}$$

All other terms in (45) are bounded as in (46). Arguing in the same way as after (45), now instead of (48), one arrives at the estimate

$$\begin{aligned}\frac{1}{2\tau}\Big(\frac{1}{2}\|\boldsymbol{e}_h^1\|_0^2 - \|\boldsymbol{e}_h^0\|_0^2\Big) + \nu\|\nabla\boldsymbol{e}_h^1\|_0^2 + \frac{\mu}{2}\|\nabla\cdot\boldsymbol{e}_h^1\|_0^2 + \frac{\alpha^*}{8}\|\boldsymbol{e}_h^1\|_0^2&\\ \le T_1^1 + 2\tau\alpha^*T_2^1 + T_{31}^1 + 2\tau T_{32}^1 + T_4^1,&\end{aligned}$$

where

$$T_{31}^1 = \frac{1}{\alpha^*}\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^1 - \boldsymbol{s}_h^1)\|_0^2, \quad T_{32}^1 = \|\alpha\boldsymbol{u}^1 - \alpha^*\boldsymbol{s}_h^0\|_0^2.$$

Taking into account the bounds on T_1^1, T_2^1, T_4^1 given in (50), (51), and (54), respectively, as well as (52) and (53), one obtains

$$\begin{aligned}\|\boldsymbol{e}_h^1\|_0^2 &+ 2\tau\left(2\nu\|\nabla \boldsymbol{e}_h^1\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^1\|_0^2 + \mu\|\nabla\cdot\boldsymbol{e}_h^1\|_0^2\right)\\ &\le 2\|\boldsymbol{e}_h^0\|_0^2 + CTh^{2k}\left(\|\boldsymbol{u}(t_1)\|_{k+1}^2 + \|\partial_t\boldsymbol{u}(t_1)\|_k^2 + \|p(t_1)\|_k^2\right)\\ &\quad + CT\tau^3\int_0^\tau\left(\|\partial_{tt}\tilde{\boldsymbol{u}}\|_1^2 + \|\partial_{tt}\boldsymbol{u}\|_1^2\right)\,dt.\end{aligned}$$

The estimate of the last term on the right-hand side provides an additional factor τ from the length of the first interval

$$\begin{aligned}\tau^3\int_0^\tau \|\partial_{tt}\boldsymbol{u}\|_1^2\,dt &\le \tau^4\,\|\partial_{tt}\boldsymbol{u}\|_{L^\infty(0,\tau,H^1)}^2\\ &\le C\tau^4\Big(\big\|\partial_{tt}\boldsymbol{u}\big\|_{L^2(0,\tau,H^1)}^2 + \big\|\partial_{ttt}\boldsymbol{u}\big\|_{L^2(0,\tau,H^1)}^2\Big).\end{aligned}$$

In the second step $\|\cdot\|_{L^\infty(0,\tau,H^1)} \le C\|\cdot\|_{H^1(0,\tau,H^1)}$ was used, which is a consequence of the Sobolev inequality $\|\cdot\|_{L^\infty(0,\tau)} \le C\|\cdot\|_{H^1(0,\tau)}$. Analogously, one obtains

$$\begin{aligned}\tau^3\int_0^\tau \|\partial_{tt}\tilde{\boldsymbol{u}}\|_1^2\,dt &\le \tau^4\,\|\partial_{tt}\tilde{\boldsymbol{u}}\|_{L^\infty(0,\tau,H^1)}^2\\ &\le C\tau^4\Big(\big\|\partial_{tt}\tilde{\boldsymbol{u}}\big\|_{L^2(0,\tau,H^1)}^2 + \big\|\partial_{ttt}\tilde{\boldsymbol{u}}\big\|_{L^2(0,\tau,H^1)}^2\Big).\end{aligned}$$

Inserting the estimates for the first step into (76) yields for $n = 1,\ldots,N$,

$$\begin{aligned}\|\boldsymbol{e}_h^n\|_0^2 &+ \tau\sum_{j=2}^n\Big(4\nu\|\nabla\boldsymbol{e}_h^j\|_0^2 + \frac{\alpha^*}{4}\|\boldsymbol{e}_h^j\|_0^2 + 2\mu\|\nabla\cdot\boldsymbol{e}_h^j\|_0^2\Big)\\ &\le C\|\boldsymbol{e}_h^0\|_0^2 + CT\tau^4\Big(\|\partial_{ttt}\boldsymbol{u}\|_{L^2(0,t_n;H^1)}^2 + \|\partial_{ttt}\tilde{\boldsymbol{u}}\|_{L^2(0,t_n;H^1)}^2\Big)\\ &\quad + CT\tau^4\Big(\|\boldsymbol{u}\|_{H^3(0,\tau,H^1)}^2 + \big\|\tilde{\boldsymbol{u}}\big\|_{H^3(0,\tau,H^1)}^2\Big)\\ &\quad + Ch^{2k}\left((T+\mu)\tau\sum_{j=1}^n\|\boldsymbol{u}(t_j)\|_{k+1}^2 + T\tau\sum_{j=1}^n\|\partial_t\boldsymbol{u}(t_j)\|_k^2\right)\\ &\quad + Ch^{2k}\mu^{-1}\tau\sum_{j=1}^n\|p(t_j)\|_k^2. \end{aligned}\tag{77}$$

From this estimate, arguing as for the backward Euler scheme, one gets the following theorem.

Theorem 5 *Let $(\tilde{\boldsymbol{u}},\tilde{p}) \in V\times Q$ be the solution of (1), which should be sufficiently smooth such that all norms appearing in the formulation of this theorem are well defined, and let $(\widetilde{\boldsymbol{U}}_h^n, \widetilde{P}_h^n) \in V_h\times Q_h$ be the BDF2 approximation solving (60) for $n = 2,\ldots,N$. Assume that $\|\boldsymbol{e}_h^0\| = \mathcal{O}(h^k)$, $l = k-1$, $\nu \le 1$,*

$\tau \le \log(20/19)T$ *(so that* (67) *holds) and that the backward Euler method is used for the first step. Then, the following error estimate holds*

$$\begin{aligned}
&\big\|\widetilde{\boldsymbol{U}}_h^n - \tilde{\boldsymbol{u}}(t_n)\big\|_0^2 + \tau \sum_{j=1}^n \Big(\nu \big\|\nabla\big(\widetilde{\boldsymbol{U}}_h^j - \tilde{\boldsymbol{u}}(t_j)\big)\big\|_0^2 + \mu \big\|\nabla\cdot\big(\widetilde{\boldsymbol{U}}_h^j - \tilde{\boldsymbol{u}}(t_j)\big)\big\|_0^2\Big) \\
&\quad \le Ch^{2k}\big(\|\tilde{\boldsymbol{u}}_0\|_k^2 + \|\tilde{\boldsymbol{u}}(t_n)\|_k^2\big) + CT\tau^4\|\tilde{\boldsymbol{u}}\|^2_{H^3(0,t_n;H^1)} \\
&\qquad + Ch^{2k}\left((T+\mu)\tau\sum_{j=1}^n \|\tilde{\boldsymbol{u}}(t_j)\|_{k+1}^2 + T\tau\sum_{j=1}^n \|\partial_t \tilde{\boldsymbol{u}}(t_j)\|_k^2\right) \\
&\qquad + Ch^{2k}\mu^{-1}\tau\sum_{j=1}^n \|\tilde{p}(t_j)\|_k^2. \qquad (78)
\end{aligned}$$

The estimate for the pressure error is performed also along the same lines as for the backward Euler scheme. It starts from (64). For convenience, the term $(\alpha^*\boldsymbol{e}_h^n, \boldsymbol{v}_h) - \alpha^*(D^2\boldsymbol{e}_h^n, \boldsymbol{v}_h) + \beta\tau(\boldsymbol{e}_h^{n-2}, \boldsymbol{v}_h)$ in (64) is expressed in the form

$$(\alpha^*\boldsymbol{e}_h^n, \boldsymbol{v}_h) - \alpha^*(D^2\boldsymbol{e}_h^n, \boldsymbol{v}_h) + \beta\tau(\boldsymbol{e}_h^{n-2}, \boldsymbol{v}_h) = \left(2\alpha^*\boldsymbol{e}_h^{n-1} - \frac{1}{2}\alpha^{**}\boldsymbol{e}_h^{n-2}, \boldsymbol{v}_h\right).$$

Then the arguments used for the backward Euler method are valid for BDF2 if one replaces the occurrences of $(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1})/\tau$, $(\boldsymbol{s}_h^n - \boldsymbol{s}_h^{n-1})/\tau$, $\alpha^*\boldsymbol{e}_h^{n-1}$, and $\alpha^*\boldsymbol{s}_h^{n-1}$ by $(D + D^2/2)/\tau\boldsymbol{e}_h^n$, $(D + D^2/2)\boldsymbol{s}_h^n/\tau$, $2\alpha^*\boldsymbol{e}_h^{n-1} - \alpha^{**}\boldsymbol{e}_h^{n-2}/2$, and $\alpha^*(\boldsymbol{s}_h^n - D^2\boldsymbol{s}_h^n) - \beta\tau\boldsymbol{s}_h^{n-2}$, respectively. Thus, instead of (57), for the BDF2 one gets

$$\begin{aligned}
\beta_0\|P_h^n - \pi_h^n\|_0 \le{}& \Big\|\frac{1}{\tau}\big(D + \frac{1}{2}D^2\big)\boldsymbol{e}_h^n\Big\|_{-1} + \nu\|\nabla\boldsymbol{e}_h^n\|_0 + \|\boldsymbol{b}\cdot\nabla\boldsymbol{e}_h^n\|_{-1} \\
&+ \big\|2\alpha^*\boldsymbol{e}_h^{n-1} - \frac{1}{2}\alpha^{**}\boldsymbol{e}_h^{n-2}\big\|_{-1} + \mu\|\nabla\cdot\boldsymbol{e}_h^n\|_0 + \|\partial_t(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_{-1} \\
&+ \Big\|\partial_t\boldsymbol{s}_h^n - \frac{1}{\tau}\big(D + \frac{1}{2}D^2\big)\boldsymbol{s}_h^n\Big\|_{-1} + \big\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\big\|_{-1} \\
&+ \big\|\alpha\boldsymbol{u}^n - \alpha^*(\boldsymbol{s}_h^n - D^2\boldsymbol{s}_h^n) - \beta\tau\boldsymbol{s}_h^{n-2}\big\|_{-1} \qquad (79) \\
&+ \mu\|\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{s}_h^n)\|_0 + \|p^n - \pi_h^n\|_0 + \|l_h^n\|_0.
\end{aligned}$$

Since $\|\boldsymbol{b}\cdot\nabla\boldsymbol{e}_h^n\|_{-1} \le C\,\|\boldsymbol{e}_h^n\|_0$ and $\alpha^{**} = \alpha^*(1 + e^{-\alpha\tau}) \le 2\alpha^*$, it follows that

$$\|\boldsymbol{b}\cdot\nabla\boldsymbol{e}_h^n\|_{-1} + \left\|2\alpha^*\boldsymbol{e}_h^{n-1} - \frac{1}{2}\alpha^{**}\boldsymbol{e}_h^{n-2}\right\|_{-1} \le C\left(\|\boldsymbol{e}_h^n\|_0 + \big\|\boldsymbol{e}_h^{n-1}\big\|_0 + \big\|\boldsymbol{e}_h^{n-2}\big\|_0\right).$$

Using also the fact that $\|\cdot\|_{-1} \le C\,\|\cdot\|_0$, the truncation errors, i.e., the terms ranging from the sixth to the tenth on the right-hand side of (79), can be bounded as in the estimate for the velocity. Applying in addition (77) yields

$$\begin{aligned}
&\beta_0^2\tau\sum_{j=2}^n \big\|P_h^j - \pi_h^j\big\|_0^2 \\
&\quad \le C\tau\sum_{j=2}^n \Big\|\frac{1}{\tau}\big(D + \frac{1}{2}D^2\big)\boldsymbol{e}_h^j\Big\|_{-1}^2 + C\tau\sum_{j=2}^n\big(\|p^n - \pi_h^n\|_0^2 + \|l_h^n\|_0^2\big) + T_7^n,
\end{aligned}$$

where T_7^n can be bounded by the right-hand side of (77). Since the second sum on the right-hand side can be bounded by applying (14) and (23), it only remains to bound the first sum. Again, arguing as in the case of the backward Euler method, $\left\|\tau^{-1}(D+D^2/2)\boldsymbol{e}_h^n\right\|_{-1}$ can be bounded in terms of (the square of) those terms on the right-hand side in (79) ranging from the second until the eleventh. It follows that, for $n=2,\ldots,N$, one finds

$$
\begin{aligned}
&\beta_0^2\tau\sum_{j=2}^{n}\left\|P_h^j-\pi_h^j\right\|_0^2\\
&\le C\|\boldsymbol{e}_h^0\|_0^2+CT(1+\mu)\tau^4\Big(\|\boldsymbol{u}\|_{H^3(0,\tau,H^1)}^2+\left\|\tilde{\boldsymbol{u}}\right\|_{H^3(0,\tau,H^1)}^2\Big)\\
&\quad+CT(1+\mu)\tau^4\left(\|\partial_{ttt}\boldsymbol{u}\|_{L^2(0,t_n;H^1)}^2+\|\partial_{ttt}\tilde{\boldsymbol{u}}\|_{L^2(0,t_n;H^1)}^2\right)\\
&\quad+Ch^{2k}\left((T+\mu)(1+\mu)\tau\sum_{j=1}^{n}\|\tilde{\boldsymbol{u}}(t_j)\|_{k+1}^2+T(1+\mu)\tau\sum_{j=1}^{n}\|\partial_t\boldsymbol{u}(t_j)\|_k^2)\right)\\
&\quad+Ch^{2k}(1+\mu^{-1})\tau\sum_{j=1}^{n}\|p(t_j)\|_k^2.
\end{aligned}
$$

Now, the estimate for the pressure follows with the same arguments as before.

Theorem 6 *Let the assumptions of Theorem 5 hold. Then, the following bound holds for the approximation to the pressure using BDF2*

$$
\begin{aligned}
&\tau\sum_{j=2}^{n}\left\|\widetilde{P}_h^j-\tilde{p}(t_j)\right\|_0^2\\
&\le Ch^{2k}\|\tilde{\boldsymbol{u}}_0\|_k^2+CT(1+\mu)\tau^4\|\tilde{\boldsymbol{u}}\|_{H^3(0,t_n;H^1)}^2\\
&\quad+Ch^{2k}\left((T+\mu)(1+\mu)\tau\sum_{j=1}^{n}\|\tilde{\boldsymbol{u}}(t_j)\|_{k+1}^2+T(1+\mu)\tau\sum_{j=1}^{n}\|\tilde{\partial}_t\boldsymbol{u}(t_j)\|_k^2\right)\\
&\quad+Ch^{2k}(1+\mu^{-1})\tau\sum_{j=1}^{n}\|\tilde{p}(t_j)\|_k^2.
\end{aligned}
\tag{80}
$$

4.3 Crank–Nicolson scheme

The Crank–Nicolson scheme is perhaps the most popular time integrator among the schemes considered here. The analysis of this scheme is more involved than the analysis of the two other methods. For instance, it is more difficult to estimate the convective term, which will be performed here with a special family of test functions, see (95).

The Crank–Nicolson method, together with an inf-sup stable finite element discretization, applied to (1) reads as follows: Find $(\widetilde{\boldsymbol{U}}_h^n,\widetilde{P}_h^{n-1/2})\in V_h\times Q_h$

such that for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$
\begin{aligned}
&\left(\frac{\widetilde{\boldsymbol{U}}_h^n - \widetilde{\boldsymbol{U}}_h^{n-1}}{\tau}, \boldsymbol{v}_h\right) + \nu\left(\nabla\frac{\widetilde{\boldsymbol{U}}_h^n + \widetilde{\boldsymbol{U}}_h^{n-1}}{2}, \nabla\boldsymbol{v}_h\right) \\
&\quad + \left((\boldsymbol{b}^{n-1/2}\cdot\nabla)\frac{\widetilde{\boldsymbol{U}}_h^n + \widetilde{\boldsymbol{U}}_h^{n-1}}{2}, \boldsymbol{v}_h\right) - (\nabla\cdot\boldsymbol{v}_h, \widetilde{P}_h^{n-1/2}) \\
&\quad + \left(\nabla\cdot\frac{\widetilde{\boldsymbol{U}}_h^n + \widetilde{\boldsymbol{U}}_h^{n-1}}{2}, q_h + \mu\nabla\cdot\boldsymbol{v}_h\right) \qquad (81)\\
&= \left(\frac{\tilde{\boldsymbol{f}}^n + \tilde{\boldsymbol{f}}^{n-1}}{2}, \boldsymbol{v}_h\right),
\end{aligned}
$$

$n = 1, \ldots, N$, where $\widetilde{\boldsymbol{U}}_h^0$ is a finite element approximation of $\tilde{\boldsymbol{u}}_0$, $t_{n-1/2} = t_n - \tau/2$, $n = 1, \ldots, N$, and for a function $v = v(t)$, the notation $v^j = v(t_j)$ is used. To simplify notation, in the sequel the superscript in the convection field will be omitted. It will be assumed that $(\nabla\cdot\tilde{\boldsymbol{U}}_h^0, q_h) = 0$ for all $q_h \in Q_h$, so that taking $\boldsymbol{v}_h = \boldsymbol{0}$ in (81), it follows that

$$
(\nabla\cdot\tilde{\boldsymbol{U}}_h^n, q_h) = 0, \quad \forall q_h \in Q_h, \qquad n = 0, \ldots, N. \tag{82}
$$

For $\boldsymbol{g} = \tilde{\boldsymbol{f}} - \partial_t\tilde{\boldsymbol{u}} - (\boldsymbol{b}\cdot\nabla)\tilde{\boldsymbol{u}} - \nabla\tilde{p}$, let $(\tilde{\boldsymbol{u}}, 0)$ be the solution of (16), and let $(\tilde{\boldsymbol{s}}_h, \tilde{l}_h)$ be the corresponding Galerkin approximation in $V_h \times Q_h$. Using $\nabla\cdot\tilde{\boldsymbol{u}}^n = 0$ and integration by parts, one finds with a straightforward calculation that the error $\tilde{\boldsymbol{e}}_h^n = \widetilde{\boldsymbol{U}}_h^n - \tilde{\boldsymbol{s}}_h^n \in V_h^{\mathrm{div}}$, $n = 0, \ldots, N$, satisfies for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$,

$$
\begin{aligned}
&\left(\frac{\tilde{\boldsymbol{e}}_h^n - \tilde{\boldsymbol{e}}_h^{n-1}}{\tau}, \boldsymbol{v}_h\right) + \nu\left(\nabla\frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2}, \nabla\boldsymbol{v}_h\right) + \left((\boldsymbol{b}\cdot\nabla)\frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2}, \boldsymbol{v}_h\right) \\
&\quad -(\nabla\cdot\boldsymbol{v}_h, \tilde{P}_h^{n-1/2} - \tilde{\pi}_h^{n-1/2}) + \left(\nabla\cdot\frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2}, q_h + \mu\nabla\cdot\boldsymbol{v}_h\right) \\
&= (\boldsymbol{r}_{123}^{n-1/2}, \boldsymbol{v}_h) + (r_{45}^{n-1/2}, \nabla\cdot\boldsymbol{v}_h), \quad n = 1, \ldots, N, \qquad (83)
\end{aligned}
$$

where $\tilde{\pi}_h^{n-1/2}$ denotes the orthogonal projection of $\tilde{p}(t_{n-1/2})$ onto Q_h,

$$
\begin{aligned}
\boldsymbol{r}_{123}^{n-1/2} &= \boldsymbol{r}_1^{n-1/2} + \boldsymbol{r}_2^{n-1/2} + \boldsymbol{r}_3^{n-1/2}, \\
r_{45}^{n-1/2} &= r_4^{n-1/2} + r_5^{n-1/2},
\end{aligned}
$$

and

$$\begin{aligned}
\boldsymbol{r}_1^{n-1/2} =& \frac{\partial_t \tilde{\boldsymbol{u}}^n + \partial_t \tilde{\boldsymbol{u}}^{n-1}}{2} - \frac{\tilde{\boldsymbol{u}}^n - \tilde{\boldsymbol{u}}^{n-1}}{\tau}, \\
\boldsymbol{r}_2^{n-1/2} =& \frac{\tilde{\boldsymbol{u}}^n - \tilde{\boldsymbol{u}}^{n-1}}{\tau} - \frac{\tilde{\boldsymbol{s}}_h^n - \tilde{\boldsymbol{s}}_h^{n-1}}{\tau}, \\
\boldsymbol{r}_3^{n-1/2} =& -(\boldsymbol{b}\cdot\nabla)\frac{\tilde{\boldsymbol{s}}_h^n + \tilde{\boldsymbol{s}}_h^{n-1}}{2} + (\boldsymbol{b}\cdot\nabla)\frac{\tilde{\boldsymbol{u}}^n + \tilde{\boldsymbol{u}}^{n-1}}{2}, \\
r_4^{n-1/2} =& -\mu\nabla\cdot\frac{\tilde{\boldsymbol{s}}_h^n + \tilde{\boldsymbol{s}}_h^{n-1}}{2} + \mu\nabla\cdot\frac{\tilde{\boldsymbol{u}}^n + \tilde{\boldsymbol{u}}^{n-1}}{2}, \\
r_5^{n-1/2} =& -\frac{\tilde{l}_h^n + \tilde{l}_h^{n-1}}{2} - \frac{\tilde{p}^n + \tilde{p}^{n-1}}{2} + \tilde{\pi}_h^{n-1/2}.
\end{aligned}$$

Since $(\nabla\cdot\tilde{\boldsymbol{s}}_h^n, q_h) = 0$ for all $q_h \in Q_h$, and in view of (82), it follows that

$$(\nabla\cdot\tilde{\boldsymbol{e}}_h^n, q_h) = 0, \qquad \forall q_h \in Q_h, \qquad n = 0,\ldots,N. \tag{84}$$

Using similar arguments to those applied in (51) and (75), one shows that

$$\|\boldsymbol{r}_1^{n-1/2}\|_0 = \tau^{3/2}\,\|\partial_{ttt}\tilde{\boldsymbol{u}}\|_{L^2(t_{n-1},t_n,L^2)}\,. \tag{85}$$

With (22) and Remark 1, one obtains

$$\|\boldsymbol{r}_2^{n-1/2}\|_0 = \left\| \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (\partial_t\tilde{\boldsymbol{u}} - \partial_t\tilde{\boldsymbol{s}}_h)\,dt \right\| \le C\frac{h^k}{\tau^{1/2}}\|\partial_t\tilde{\boldsymbol{u}}\|_{L^2(t_{n-1},t_n,H^k)} \tag{86}$$

and

$$\|\boldsymbol{r}_3^{n-1/2}\|_0 + \|r_4^{n-1/2}\|_0 \le C(1+\mu)h^k(\|\tilde{\boldsymbol{u}}^n\|_{k+1} + \|\tilde{\boldsymbol{u}}^{n-1}\|_{k+1}). \tag{87}$$

Applying a Taylor series expansion and (23) gives

$$\|r_5^{n-1/2}\|_0 = C\left(h^k\sum_{j=0}^{1}(\|\tilde{\boldsymbol{u}}^{n-j}\|_{k+1} + \|\tilde{p}^{n-j}\|_k) + \tau^{3/2}\|\partial_{tt}\tilde{p}\|_{L^2(t_{n-1},t_n,L^2)} \right). \tag{88}$$

The following formulae are valid for $a,b,u,v\in\mathbb{R}$

$$\begin{aligned}
au - bv &= \frac{a+b}{2}(u-v) + (a-b)\frac{u+v}{2}, \\
au + bv &= \frac{a+b}{2}(u+v) + (a-b)\frac{u-v}{2}.
\end{aligned}$$

By applying them, one obtains with straightforward calculations

$$\begin{aligned}
e^{-\alpha t_{n-1/2}}\frac{\tilde{\boldsymbol{e}}_h^n - \tilde{\boldsymbol{e}}_h^{n-1}}{\tau} &= (1+\beta\tau^2)\frac{\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}}{\tau} + \alpha^*\frac{\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1}}{2}, \\
e^{-\alpha t_{n-1/2}}\frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2} &= (1+\beta\tau^2)\frac{\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1}}{2} + \frac{\tau}{2}\alpha^*\frac{\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}}{2},
\end{aligned} \tag{89}$$

where

$$\alpha^* = \frac{2\sinh(\alpha\tau/2)}{\tau}, \qquad \beta = \frac{\cosh(\alpha\tau/2)-1}{\tau^2},$$

and, as before, $\boldsymbol{e}^n = e^{-\alpha t_n}\tilde{\boldsymbol{e}}^n$, $n = 0,\ldots,N$. Observe that $\alpha^* \to \alpha$ and $\beta \to \alpha^2/8$, as $\tau \to 0$.

Now, the terms on the left-hand side of (83) will be considered. Setting

$$\gamma_n = \tau e^{-\alpha t_{n-1/2}}, \tag{90}$$

it follows that

$$\begin{aligned}
\gamma_n\left(\frac{\tilde{\boldsymbol{e}}_h^n - \tilde{\boldsymbol{e}}_h^{n-1}}{\tau}, \boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1}\right) &= (1+\beta\tau^2)(\|\boldsymbol{e}_h^n\|_0^2 - \|\boldsymbol{e}_h^{n-1}\|_0^2) \\
&\quad + \frac{\tau}{2}\alpha^*\|\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1}\|_0^2, \qquad (91)\\
\gamma_n\nu\left(\nabla\frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2}, \nabla(\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1})\right) &= \frac{\tau}{2}(1+\beta\tau^2)\nu\|\nabla(\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1})\|_0^2 \\
&\quad + \frac{\tau^2}{4}\alpha^*\nu(\|\nabla\boldsymbol{e}_h^n\|_0^2 - \|\nabla\boldsymbol{e}_h^{n-1}\|_0^2),\\
\gamma_n\mu\left(\nabla\cdot\frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2}, \nabla\cdot(\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1})\right) &= \frac{\tau}{2}(1+\beta\tau^2)\mu\|\nabla\cdot(\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1})\|_0^2 \qquad (92)\\
&\quad + \frac{\tau^2}{4}\alpha^*\mu(\|\nabla\cdot\boldsymbol{e}_h^n\|_0^2 - \|\nabla\cdot\boldsymbol{e}_h^{n-1}\|_0^2),\\
\gamma_n\left((\boldsymbol{b}\cdot\nabla)\frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2}, \boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1}\right) &= \frac{\tau^2}{2}\alpha^*((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h^n, \boldsymbol{e}_h^{n-1}). \qquad (93)
\end{aligned}$$

The last term has to be canceled by an appropriate choice of the test function. To this end, inner products with $\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}$ will be considered. One has

$$\gamma_n\left((\boldsymbol{b}\cdot\nabla)\frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2}, \boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\right) = -\tau(1+\beta\tau^2)((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h^n, \boldsymbol{e}_h^{n-1}). \tag{94}$$

Now, the test function $\boldsymbol{v}_h \in V_h^{\mathrm{div}}$ in (83) is chosen as a linear combination of $(\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1})$ and $(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1})$ such that the terms on the right-hand sides of (93) and (94) cancel each other. For this purpose, use $q_h = 0$ and define

$$\boldsymbol{v}_h = \gamma_n\left((\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1} + \rho(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1})\right), \quad \rho = \frac{\alpha^*\tau}{2(1+\beta\tau^2)}. \tag{95}$$

Direct calculations show

$$\rho\gamma_n\left(\frac{\tilde{\boldsymbol{e}}_h^n-\tilde{\boldsymbol{e}}_h^{n-1}}{\tau},\boldsymbol{e}_h^n-\boldsymbol{e}_h^{n-1}\right) = \frac{\tau}{2}\alpha^*(\|\boldsymbol{e}_h^n-\boldsymbol{e}_h^{n-1}\|_0^2) + \frac{\tau}{2}\alpha^*\rho(\|\boldsymbol{e}_h^n\|_0^2-\|\boldsymbol{e}_h^{n-1}\|_0^2), \quad (96)$$

$$\rho\gamma_n\nu\left(\nabla\frac{\tilde{\boldsymbol{e}}_h^n+\tilde{\boldsymbol{e}}_h^{n-1}}{2},\nabla(\boldsymbol{e}_h^n-\boldsymbol{e}_h^{n-1})\right) = \frac{\tau^2}{4}\alpha^*\nu(\|\nabla\boldsymbol{e}_h^n\|_0^2-\|\nabla\boldsymbol{e}_h^{n-1}\|_0^2) + \frac{\tau^2}{4}\alpha^*\rho\nu(\|\nabla(\boldsymbol{e}_h^n-\boldsymbol{e}_h^{n-1})\|_0^2), \quad (97)$$

$$\rho\gamma_n\mu\left(\nabla\cdot\frac{\tilde{\boldsymbol{e}}_h^n+\tilde{\boldsymbol{e}}_h^{n-1}}{2},\nabla\cdot(\boldsymbol{e}_h^n-\boldsymbol{e}_h^{n-1})\right) = \frac{\tau^2}{4}\alpha^*\mu(\|\nabla\cdot\boldsymbol{e}_h^n\|_0^2-\|\nabla\cdot\boldsymbol{e}_h^{n-1}\|_0^2) + \frac{\tau^2}{4}\alpha^*\rho\mu\|\nabla\cdot(\boldsymbol{e}_h^n-\boldsymbol{e}_h^{n-1})\|_0^2. \quad (98)$$

With the Cauchy–Schwarz inequality and Young's inequality, one obtains for $\boldsymbol{v}_h$ defined in (95)

$$|(\boldsymbol{r}_{123}^{n-1/2},\boldsymbol{v}_h)| \le \frac{\tau}{4}\alpha^*\|\boldsymbol{e}_h^n+\boldsymbol{e}_h^{n-1}\|_0^2+\frac{\tau}{4}\alpha^*(\|\boldsymbol{e}_h^n-\boldsymbol{e}_h^{n-1}\|_0^2) + \tau C_1\|e^{-\alpha t_{n-1/2}}\boldsymbol{r}_{123}^{n-1/2}\|_0^2, \quad (99)$$

where

$$C_1=\frac{1}{\alpha^*}+\frac{\alpha^*\tau^2}{4(1+\beta\tau^2)^2}, \quad (100)$$

such that the first two terms on the right-hand side in (99) cancel with one half of the corresponding terms in (91) and (96). In the same way, one gets

$$|(r_{45}^{n-1/2},\nabla\cdot\boldsymbol{v}_h)| \le (1+\beta\tau^2)\frac{\tau}{4}\mu\|\nabla\cdot(\boldsymbol{e}_h^n+\boldsymbol{e}_h^{n-1})\|_0^2 + \frac{(\alpha^*)^2\tau^3}{16(1+\beta\tau^2)}\mu\|\nabla\cdot(\boldsymbol{e}_h^n-\boldsymbol{e}_h^{n-1})\|_0^2 + \tau C_2\|e^{-\alpha t_{n-1/2}}r_{45}^{n-1/2}\|_0^2, \quad (101)$$

where

$$C_2=\frac{2}{\mu(1+\beta\tau^2)}, \quad (102)$$

so that the first two terms on the right-hand side in (101) cancel with one half of the sum of the corresponding terms in (92) and (98). The pressure term on the left-hand side of (83) vanishes since $\boldsymbol{v}_h\in V_h^{\mathrm{div}}$ because of (84).

Taking $(\boldsymbol{v}_h, q_h)$ as specified in (95), summing from 1 to n, and ignoring some non-negative terms on the left-hand side yields

$$
\begin{aligned}
&(1+\beta\tau^2)\left(\|\boldsymbol{e}_h^n\|_0^2 + \frac{\tau}{2}\sum_{j=1}^{n}\Big(\nu\|\nabla(\boldsymbol{e}_h^j+\boldsymbol{e}_h^{j-1})\|_0^2 + \frac{\mu}{2}\|\nabla\cdot(\boldsymbol{e}_h^j+\boldsymbol{e}_h^{j-1})\|_0^2\Big)\right)\\
&\quad\le (1+\beta\tau^2)\left\|\boldsymbol{e}_h^0\right\|_0^2 + E_0^2 \qquad (103)\\
&\quad\quad +\tau\sum_{n=1}^{m}\left(C_1\|e^{-\alpha t_{j-1/2}}\boldsymbol{r}_{123}^{j-1/2}\|_0^2 + C_2\|e^{-\alpha t_{j-1/2}}r_{45}^{j-1/2}\|_0^2\right),
\end{aligned}
$$

where C_1 and C_2 are the constants in (100) and (102) and

$$
E_0^2 = \alpha^*\frac{\tau^2}{4}\left(2\nu\left\|\nabla\boldsymbol{e}_h^0\right\|_0^2 + 2\mu\left\|\nabla\cdot\boldsymbol{e}_h^0\right\|_0^2 + \frac{\alpha^*}{1+\beta\tau^2}\left\|\boldsymbol{e}_h^0\right\|_0^2\right).
$$

For $\alpha = 1/T$ and $\tau \le T$, it is a simple calculation to check that $C_1 \le CT$ and $C_2 \le C\mu^{-1}$ where $C \le 2$. In view of (85) – (88) and since $e^{-\alpha t_{j-1/2}} \le 1$, one obtains

$$
\begin{aligned}
&\tau\sum_{j=1}^{n}\Big(C_1\|e^{-\alpha t_{j-1/2}}\boldsymbol{r}_{123}^{j-1/2}\|_0^2 + C_2\|e^{-\alpha t_{j-1/2}}r_{45}^{j-1/2}\|_0^2\Big)\\
&\le C\Bigg\{\tau^4\left(T\left\|\partial_{ttt}\tilde{\boldsymbol{u}}\right\|_{L^2(0,t_n,L^2)}^2 + \mu^{-1}\left\|\partial_{tt}\tilde{p}\right\|_{L^2(0,t_n,L^2)}^2\right) \qquad (104)\\
&\quad +Th^{2k}\|\partial_t\tilde{\boldsymbol{u}}\|_{L^2(0,t_n,H^k)}^2\\
&\quad +\left[(1+\mu)^2(T+\mu^{-1})+\mu^{-1}\right]h^{2k}\tau\sum_{j=0}^{n}\|\tilde{\boldsymbol{u}}^j\|_{k+1}^2 + \mu^{-1}h^{2k}\tau\sum_{j=0}^{n}\|\tilde{p}^j\|_k^2\Bigg\}.
\end{aligned}
$$

For $\alpha = 1/T$ and $\tau \le T$, one has $\alpha^* \le 2\alpha\sinh(1/2) \le C$ and $1+\beta\tau^2 \le \cosh(1/2) \le C$. Then, it follows from (103) and (104) that

$$
\begin{aligned}
&\|\boldsymbol{e}_h^n\|_0^2 + \frac{\tau}{2}\sum_{j=1}^{n}\left(\nu\|\nabla(\boldsymbol{e}_h^j+\boldsymbol{e}_h^{j-1})\|_0^2 + \frac{\mu}{2}\|\nabla\cdot(\boldsymbol{e}_h^j+\boldsymbol{e}_h^{j-1})\|_0^2\right) \qquad (105)\\
&\le \|\boldsymbol{e}_h^0\|_0^2 + C\Bigg\{\tau^2(\nu\|\nabla\boldsymbol{e}_h^0\|_0^2 + \mu\|\nabla\cdot\boldsymbol{e}_h^0\|_0^2 + \|\boldsymbol{e}_h^0\|_0^2)\\
&\quad +\tau^4\left(T\left\|\partial_{ttt}\tilde{\boldsymbol{u}}\right\|_{L^2(0,t_n,L^2)}^2 + \mu^{-1}\left\|\partial_{tt}\tilde{p}\right\|_{L^2(0,t_n,L^2)}^2\right)\\
&\quad +Th^{2k}\|\partial_t\tilde{\boldsymbol{u}}\|_{L^2(0,t_n,H^k)}^2\\
&\quad +\left[(1+\mu)^2(T+\mu^{-1})+\mu^{-1}\right]h^{2k}\tau\sum_{j=0}^{n}\|\tilde{\boldsymbol{u}}^j\|_{k+1}^2 + \mu^{-1}h^{2k}\tau\sum_{j=0}^{n}\|\tilde{p}^j\|_k^2\Bigg\}.
\end{aligned}
$$

Applying the triangle inequality and (22), the following result is obtained.

Theorem 7 *Let $(\tilde{\boldsymbol{u}}, \tilde{p}) \in V \times Q$ be the solution of (1), which should be sufficiently smooth such that all norms appearing in the formulation of this theorem are well defined, and let $(\widetilde{\boldsymbol{U}}_h^n, \widetilde{P}_h^n) \in V_h \times Q_h$ be the Crank–Nicolson approximation solving (81) for $n = 1, \ldots, N$. Let $\alpha = 1/T$ and $\mu > 0$, assume that $(\nabla \cdot \tilde{\boldsymbol{U}}_h^0, q_h) = 0$ for all $q_h \in Q_h$, and that for some positive constant C it is $\|\boldsymbol{e}_h^0\| + h\|\nabla \boldsymbol{e}_h^0\| \le Ch^{k+1}\|\boldsymbol{u}_0\|_{k+1}$, $l = k-1$, $\nu \le 1$, and $\tau \le T$. Then, there exist a positive constant $C > 0$ independent of ν, μ, τ, and T such that the following error estimate holds for $n = 1, \ldots, N$,*

$$
\begin{aligned}
&\left\|\tilde{\boldsymbol{u}}(t_n) - \widetilde{\boldsymbol{U}}_h^n\right\|_0^2 + \tau \sum_{j=1}^n \nu \left\| \nabla \left(\frac{\tilde{\boldsymbol{u}}(t_j) + \tilde{\boldsymbol{u}}(t_{j-1})}{2} - \frac{\widetilde{\boldsymbol{U}}_h^j + \widetilde{\boldsymbol{U}}_h^{j-1}}{2} \right) \right\|_0^2 \\
&\quad + \tau \sum_{j=1}^n \mu \left\| \nabla \cdot \left(\frac{\tilde{\boldsymbol{u}}(t_j) + \tilde{\boldsymbol{u}}(t_{j-1})}{2} - \frac{\widetilde{\boldsymbol{U}}_h^j + \widetilde{\boldsymbol{U}}_h^{j-1}}{2} \right) \right\|_0^2 \\
&\le Ch^{2k} \Bigg\{ (1+\mu)\tau^2 \|\tilde{\boldsymbol{u}}_0\|_{k+1}^2 + \|\tilde{\boldsymbol{u}}(t_n)\|_k^2 + T\|\partial_t \tilde{\boldsymbol{u}}\|_{L^2(0,t_n,H^k)}^2 \\
&\quad + \left[T(1+\mu)^2 + \mu + \mu^{-1}\right] \tau \sum_{j=0}^n \|\tilde{\boldsymbol{u}}^j\|_{k+1}^2 + \mu^{-1}\tau \sum_{j=0}^n \|\tilde{p}^j\|_k^2 \Bigg\} \\
&\quad + C\tau^4 \left\{ T \left\|\partial_{ttt}\tilde{\boldsymbol{u}}\right\|_{L^2(0,t_n,L^2)}^2 + \mu^{-1} \left\|\partial_{tt}\tilde{p}\right\|_{L^2(0,t_n,L^2)}^2 \right\}. \qquad (106)
\end{aligned}
$$

Since μ and μ^{-1} appear on the right-hand side of (106), one finds again that $\mu = \mathcal{O}(1)$ is the appropriate asymptotic choice.

For estimating the errors in the pressure, denote $P_h^{n-1/2} = e^{-\alpha t_{n-1/2}} \tilde{P}_h^{n-1/2}$ and $\pi_h^{n-1/2} = e^{-\alpha t_{n-1/2}} \tilde{\pi}_h^{n-1/2}$. Some of the terms that have been ignored on the left-hand side in (103) are those arising from (97) and (98), together with one half of the sum of those in (91) and (96). If they are included in the previous computation, after dividing by $(1 + \beta\tau^2)$, one gets also that

$$
\begin{aligned}
&\frac{\alpha^*}{4(1+\beta\tau^2)} \tau \sum_{j=1}^n \Big(\|\boldsymbol{e}_h^j + \boldsymbol{e}_h^{j-1}\|_0^2 + \|\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1}\|_0^2 \Big) \\
&\quad + \rho^2 \frac{\tau}{4} \sum_{j=1}^n \big(\nu \|\nabla(\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1})\|_0^2 + \mu \|\nabla \cdot (\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1})\|_0^2 \big) \le R, \qquad (107)
\end{aligned}
$$

where R is the right-hand side of (105) and ρ is defined in (95). Multiplying (83) by γ^n from (90) and using the inf-sup condition (7) yields with a straightforward calculation

$$
\beta_0 \tau \|P_h^{n-1/2} - \pi_h^{n-1/2}\|_0 \le \left\| \gamma_n \frac{\tilde{\boldsymbol{e}}_h^n - \tilde{\boldsymbol{e}}_h^{n-1}}{\tau} \right\|_{-1} + R_0, \qquad (108)
$$

where

$$R_0 = \nu \left\| \gamma_n \nabla \frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2} \right\|_0 + C \, \|\boldsymbol{b}\|_\infty \left\| \gamma_n \frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2} \right\|_0$$
$$+ \mu \left\| \gamma_n \nabla \cdot \frac{\tilde{\boldsymbol{e}}_h^n + \tilde{\boldsymbol{e}}_h^{n-1}}{2} \right\|_0 + \|\gamma_n \boldsymbol{r}_{123}^{n-1/2}\|_{-1} + \|\gamma_n r_{45}^{n-1/2}\|_0. \qquad (109)$$

Since $\tilde{\boldsymbol{e}}_h^n - \tilde{\boldsymbol{e}}_h^{n-1} \in V_h^{\text{div}}$, the first term on the right-hand side of (108) can be bounded, applying the same arguments as in (33) and (10), by $C\|A_h^{-1/2}\gamma_n(\tilde{\boldsymbol{e}}_h^n - \tilde{\boldsymbol{e}}_h^{n-1})/\tau\|_0$. Taking $\boldsymbol{v} = \boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}$ in (83), applying the same estimates that led to (108), and applying Poincaré's inequality it is easy to show that the first term on the right-hand side of (108) is bounded by CR_0, so that

$$\tau\|P_h^{n-1/2} - \pi_h^{n-1/2}\|_0 \leq \beta_0^{-1} C R_0.$$

Squaring both sides of this inequality, dividing by τ, and using (89) to express the first three terms on the right-hand side of (109) in terms of $\boldsymbol{e}_h^n$ and $\boldsymbol{e}_h^{n-1}$ gives with a direct calculation

$$\begin{aligned}\tau\|P_h^{n-1/2} - \pi_h^{n-1/2}\|_0^2 \leq C\Big\{ & \tau(1+\beta\tau^2)^2\Big(\nu^2\|\nabla(\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1})\|_0^2 \\ & + \|\boldsymbol{b}\|_\infty^2 \, \|\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1}\|_0^2 + \mu^2\|\nabla \cdot (\boldsymbol{e}_h^n + \boldsymbol{e}_h^{n-1})\|_0^2\Big) \\ & + \tau\Big(\|e^{-\alpha t_{n-1/2}}\boldsymbol{r}_{123}^{n-1/2}\|_0^2 + \|e^{-\alpha t_{n-1/2}} r_{45}^{n-1/2}\|_0^2\Big) \\ & + \tau^3(\alpha^*)^2\Big(\nu^2\|\nabla(\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1})\|_0^2 + \|\boldsymbol{b}\|_\infty^2 \, \|\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\|_0^2 \\ & + \mu^2\|\nabla \cdot (\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1})\|_0^2\Big)\Big\}, \end{aligned} \qquad (110)$$

where $\|\cdot\|_{-1}\| \leq C \, \|\cdot\|_0$ has been used. Terms of the same form as on the right-hand side of (110) were estimated in (104), (105), and (107). Applying these estimates and summing from 1 to n gives

$$\tau \sum_{j=1}^{n} \|P_h^{j-1/2} - \pi_h^{j-1/2}\|_0^2 = \mathcal{O}(h^{2k} + \tau^4).$$

Using the triangle inequality and the estimate for the L^2 projection leads to the following theorem.

Theorem 8 *Let the assumptions of Theorem 7 hold. Then, there exists a constant $C > 0$ such that the following bound holds for the approximation to* *the pressure computed with the Crank–Nicolson method* (81)

$$\tau \sum_{j=1}^{n} \left\| \widetilde{P}_h^{j-1/2} - \tilde{p}(t_{j-1/2}) \right\|_0^2 \leq Ch^{2k} \Bigg\{ \tau \sum_{j=1}^{n} \|\tilde{p}(t_{j-1/2})\|_k^2 + (1+\mu)^2\tau^2 \|\tilde{\boldsymbol{u}}_0\|_{k+1}^2$$
$$+ T(1+\mu)\|\partial_t \tilde{\boldsymbol{u}}\|_{L^2(0,t_n,H^k)}^2 + (\mu^{-1}+\mu)\tau \sum_{j=0}^{n} \|\tilde{p}^j\|_k^2$$
$$+ \left[T(1+\mu)^3 + (\mu^{-1}+\mu)(1+\mu^2)\right] \tau \sum_{j=0}^{n} \|\tilde{\boldsymbol{u}}^j\|_{k+1}^2 \Bigg\} \tag{111}$$
$$+ C\tau^4(1+\mu)\left\{T\, \|\partial_{ttt}\tilde{\boldsymbol{u}}\|_{L^2(0,t_n,L^2)}^2 + \mu^{-1}\, \|\partial_{tt}\tilde{p}\|_{L^2(0,t_n,L^2)}^2 \right\}.$$

5 Numerical Studies

This section will present a few numerical results which support the error estimates from Section 4. Comparisons of numerical results with and without grad-div stabilization can be already found in the literature, e.g., see [10], and will be omitted here for the sake of brevity. To this end, the Oseen problem (2) was considered in $\Omega = (0,1)^2$ and in the time interval $[0,5]$ with different parameters ν and α and with the prescribed solution

$$\boldsymbol{u} = \cos(t)\begin{pmatrix} \sin(\pi x - 0.7)\sin(\pi y + 0.2) \\ \cos(\pi x - 0.7)\cos(\pi y + 0.2) \end{pmatrix},$$
$$p = \cos(t)(\sin(x)\cos(y) + (\cos(1) - 1)\sin(1)).$$

The right-hand side, the Dirichlet boundary condition, and the initial condition were chosen in accordance to the prescribed solution.

For all simulations, the Taylor–Hood pair of finite elements P_2/P_1 on uniform grids was used (triangles with diagonals from bottom left to top right). The coarsest grid in space (level 3) consisted of 128 mesh cells (578 velocity degrees of freedom, 81 pressure degrees of freedom). On this grid, the length of the time step τ_0 was used. Refining the spatial grid once uniformly, the length of the time step was reduced by the factor of four for the backward Euler scheme and the factor of two for BDF2 and the Crank–Nicolson scheme. With this approach, one expects second order convergence for the terms on the left-hand side of the error estimates (56), (59), (78), (80), (106), and (111). Results will be presented for $\tau_0 = 0.05$ for the backward Euler scheme and $\tau_0 = 0.5$ for BDF2 and the Crank–Nicolson method. With these choices two situations are illustrated: in the results of the backward Euler scheme the spatial error is dominant and in the results of the other schemes, the temporal error is of more importance. Qualitatively the same results were obtained for all schemes in the respective other situation.

Numerical studies concerning the choice of the grad-div stabilization parameter for the steady-state Oseen equations and the Taylor–Hood element

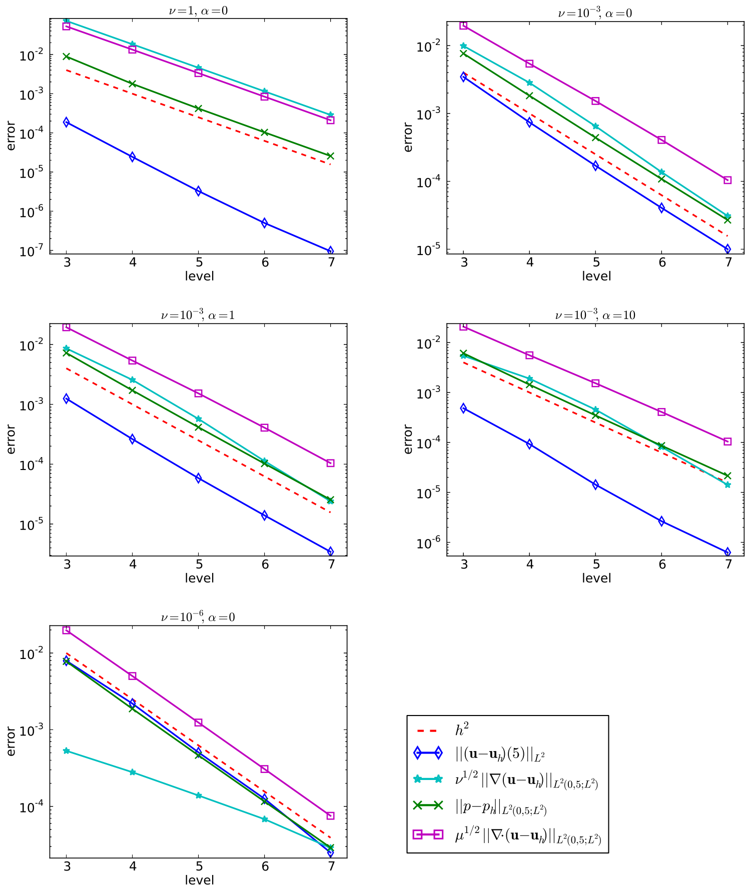

Fig. 1 Backward Euler scheme, error reduction of the errors from estimates (56) and (59), several parameters.

Q_2/Q_1 can be found in [20]. In all studies, the best choice of this parameter was below 1, approximately one order. Because of these results and also based on our own experience, the value of the stabilization parameter was set to be $\mu = 0.25$ in the simulations. All simulations were performed with the code MooNMD [18].

Results for the backward Euler scheme are presented in Figure 1. The second order convergence can be seen for all errors but $\nu^{1/2}\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(0,5;L^2)}$ in the case $\nu = 10^{-6}$ and $\alpha = 0$, where the order of error reduction increases with increasing level but it is not yet two. Note that the error bound (56) is for the linear combination of three errors for the velocity. For small ν, this combination is dominated by $\mu^{1/2}\|\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(0,5;L^2)}$ such that this linear combination converges with the predicted order. The independence of the error of the divergence and the pressure on ν can be observed very well.

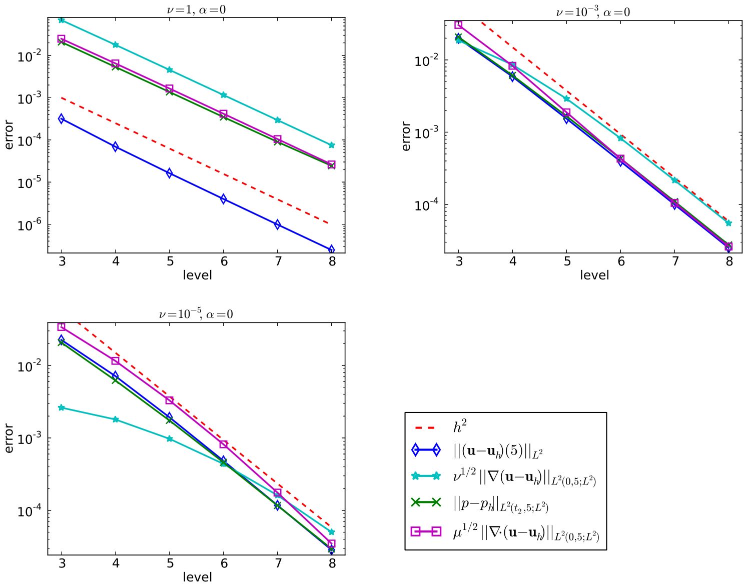

Fig. 2 BDF2, error reduction of the errors from estimates (78) and (80), several parameters.

Numerical results for the BDF2 scheme can be seen in Figure 2. The behavior for $\alpha > 0$ compared with $\alpha = 0$ is very similar to the backward Euler scheme such that the presentation of the corresponding results is omitted. To illustrate the behavior for small ν, results for $\nu = 10^{-5}$, $\alpha = 0$ are presented here. One can observe clearly the reduced order convergence of $\nu^{1/2}\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(0,5;L^2)}$ on coarser grids and the tendency to become second order on finer grids. All other errors converge at least of second order already on coarser grids. Again, the pressure error and the error of the divergence are independent of ν.

For the results obtained with the Crank–Nicolson scheme, see Figure 3, the same comments apply as for the results computed with BDF2.

6 Conclusions and future research

This paper studied the effect of grad-div stabilization added to the Galerkin method for the transient Oseen equations. The error analysis was performed for the continuous-in-time case and several fully discrete cases (backward Euler method, BDF2 formula, Crank–Nicolson schemes). Optimal convergence of the L^2 norms of the divergence of the velocity and the pressure were proved for sufficiently smooth solutions with error constants independent of the viscosity.

A change of variable allowed to transform the original equations into new ones having a non-vanishing reaction term. Thanks to this change, the analysis could be performed equally well for dissipative methods, such as the backward Euler method and BDF2, and for the non-dissipative Crank–Nicolson scheme.

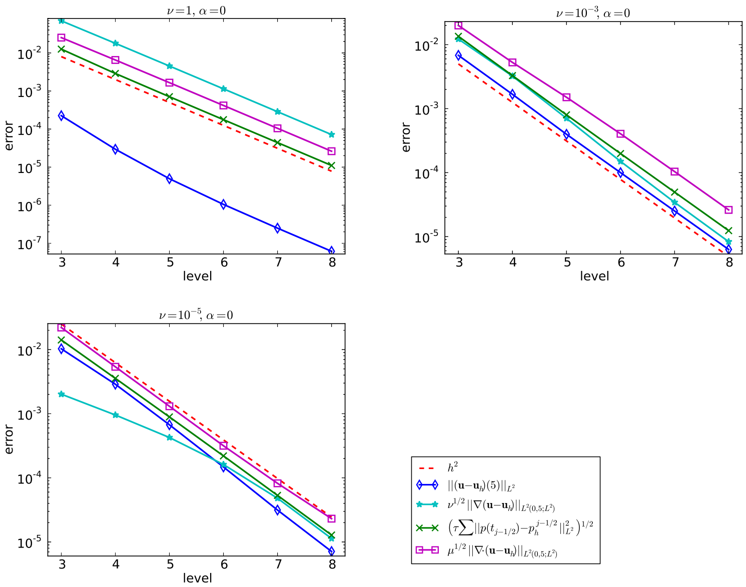

Fig. 3 Crank–Nicolson, error reduction of the errors from estimates (106) and (111), several parameters.

Discontinuous pressure approximations are covered by the analysis. In particular, the analysis is valid for the case of inf-sup stable divergence-free mixed finite elements.

The extension of the analysis of this paper to the Navier–Stokes equations is part of the current research of the authors [11].

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